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Ali Aghaei, Nicolas Bochud, Giuseppe Rosi, Salah Naili. Wave propagation across a functionally graded interphase between soft and hard solids: Insight from a dynamic surface elasticity model. *Journal of the Mechanics and Physics of Solids*, 2021, 151, pp.104380. 10.1016/j.jmps.2021.104380 . hal-03209226

HAL Id: hal-03209226

<https://hal.science/hal-03209226>

Submitted on 27 Apr 2021

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Wave propagation across a functionally graded interphase between soft and hard solids: Insight from a dynamic surface elasticity model

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Abstract

Joining soft to hard materials is a challenging problem in modern engineering applications. In order to alleviate stress concentrations at the interface between materials with such a mismatch in mechanical properties, the use of functionally graded interphases is becoming more widespread in the design of the new generation of engineered composite materials. However, current macroscale models that aim at mimicking the mechanical behavior of such complex systems generally fail in incorporating the impact of microstructural details across the interphase because of computational burden. In this paper we propose to replace the thin, but yet finite, functionally graded interphase by a zero-thickness interface. This is achieved by means of an original model developed in the framework of surface elasticity, which accounts for both the elastic and inertial behavior of the actual interphase. The performance of the proposed equivalent model is evaluated in the context of elastic wave propagation, by comparing the calculated reflection coefficient to that obtained using different baseline models. Numerical results show that our dynamic surface elasticity model provides an accurate approximation of the reference interphase model over a broad frequency range. We demonstrate application of this modeling approach for the characterization of the graded tissue system at the tendon-to-bone interphase, which fulfills the challenging task of integrating soft to hard tissues over a submillimeter-wide region.

Keywords: Functionally graded interphase, Equivalent interface model, Enriched surface elasticity, Tendon-to-bone attachment, Elastic waves

1. Introduction

From a mechanical viewpoint, when materials having a dissimilar mechanical nature (*e.g.*, soft and hard) are attached together, they typically display highly non-uniform deformations upon loading, eventually leading to stress concentration at their abrupt interface, which in turn increase the failure probability. An appealing solution to improve the integration between soft and hard materials consists in designing functionally graded interphases,

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32 which are typically conceived as multilayers whose composition, microstructure and mate-
 33 rial properties gradually vary in space, in order to reduce mechanical stresses [1]. Naturally
 34 present in the human body [2], graded materials represent a source of inspiration that offers
 35 technological solutions for general engineering purposes as well as for biomedical applica-
 36 tions. This concerns for instance the skin, which is a complex multilayered system, where
 37 each layer has a specific and age-related biomechanical behavior [3], the cortex of long bones,
 38 which displays an increasing gradient of porosity from the periosteum to the endosteum that
 39 is, in turn, related to a gradual change in mechanical properties such as tensile strength and
 40 elasticity [4], and the so-called *entheses*, which are specialized interfacial regions of the
 41 musculoskeletal system that allow joining connective tissues, such as tendon, ligament or
 42 cartilage, to bone [5].

43 In particular, the tendon-to-bone interphase serves the challenging task of connecting
 44 two highly dissimilar tissues over a very small region, which is typically a few hundreds of
 45 micrometers wide (see Fig. 1a). This interphase has the remarkable ability to minimize stress
 46 concentration and related failure modes, with the possibility to withstand forces higher than
 47 the body weight for millions of loading cycles [6, 7]. These outstanding features, achieved
 48 by means of finely tuned gradients in structure, composition and biomechanical properties
 49 at different length scales (see Fig. 1b) [7, 8], are currently playing a significant role in the
 design of bioinspired interphases [9–11]. To unlock those mechanisms, computational models

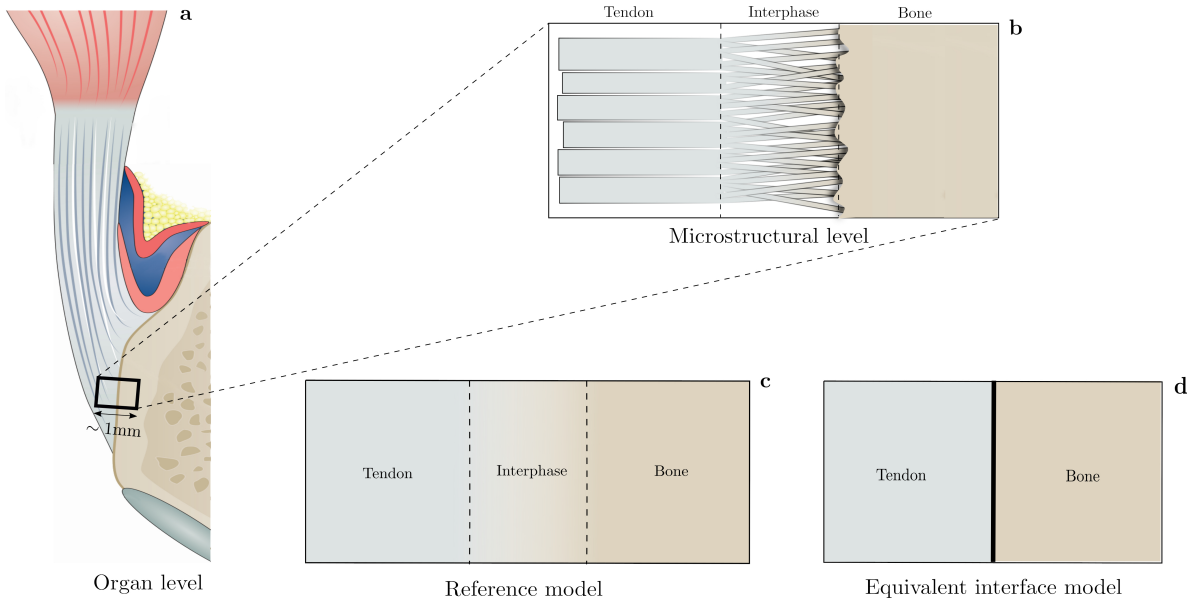


Figure 1: (a) Organ level illustration of the tendon-to-bone attachment at the insertion site corresponding to the Achilles tendon (image adapted from Ref. [12]); (b) Schematic of the attachment at the microstructural level, highlighting the gradients in composition and biomechanical properties across the tendon-to-bone interphase (image adapted from Ref. [8]); (c) Finite thickness interphase model with varying mechanical properties across the interphase; and (d) Equivalent interface model with specific properties.

50 were developed both to investigate fundamental anchoring strategies at the microstructural
 51

52 level [13–16] or to address applied orthopedic strategies at the organ level [17–19]. Notwith-
53 standing, from a modeling viewpoint, it is highly challenging to bridge the gap between these
54 two levels [20], and current models targeting reattachment procedures of the tendon-to-bone
55 interphase should be enriched by including a more detailed description of the microstruc-
56 ture [21, 22]. Indeed, to adequately capture the mechanical behavior of a graded interphase
57 layer, the optimal choice would be to consider its exact geometry and varying mechanical
58 properties across the two surrounding tissues [23]. However, this choice can be prohibitive
59 when dealing with complex heterogeneous interphases. In particular, the finite size of such
60 interphase, which is very small compared to that of the surrounding tissues, may cause
61 computational burden when mesh refinements are required for convergence purposes. In
62 contrast, a too basic model that would simplify the interphase to a large extent, or even
63 ignore it, would fail in capturing its mechanics. To face these limitations, a possible solu-
64 tion consists in replacing the finite heterogeneous interphase (see Fig. 1c) by an equivalent
65 interface with specific properties (see Fig. 1d) that retains the mechanical behavior of the
66 original medium over a certain range of validity.

67 It is commonly accepted that such modeling approach can be satisfactorily addressed
68 by enriching the equivalent model with additional fields [24]. Initial attempts proposed
69 to tackle this problem by replacing the interphase by an interface with null thickness and
70 purely elastic properties. A general framework for such general elastic interface model was
71 comprehensively described in [25, 26], and was subsequently extended to account for the
72 case where both the interphase and its surrounding media are anisotropic [27, 28] or the
73 case of an elasto-plastic interphase [29, 30]. In the case of dynamic problems, however, and
74 especially when dealing with wave propagation, a purely elastic surface model generally fails
75 in accounting for the local interactions between the mechanical perturbation (*e.g.*, a wave)
76 and the interphase, even if its dimensions are much smaller than the involved mechanical
77 characteristic length (*e.g.*, the wavelength). Within this context, the inertial behavior of the
78 interphase can have a considerable impact on its macroscopic dynamic response and should
79 thus be included into the modeling strategy. This observation was also at the basis of the
80 concept of a structural interface that possesses a finite thickness, which was introduced
81 in [31] and further developed in [32], where the role of its inertial properties was highlighted.
82 In the case of resonant meta-interfaces, an alternative approach was to obtain effective
83 jump conditions by applying a suitable homogenization process [33, 34]. Moreover, some
84 studies on elastic wave propagation showed that when the interphase is located between two
85 surrounding media with microstructure, inertial properties also play a pivotal role in the
86 modeling of the equivalent interface [35–37].

87 Following these recent findings, and in line with a former study by our group [38], we in-
88 troduce here an enriched equivalent interface model, whose properties are defined by means
89 of surface kinetic and potential energy densities. Furthermore, to account for the nonlin-
90 ear gradients in mechanical properties across the interphase found at the tendon-to-bone
91 attachment [39], we hypothesize that the displacement field can be approximated using a
92 piece-wise affine profile, whose characteristics depend upon an additional degree-of-freedom
93 within the interface. The performance of our modeling approach is evaluated by calculating
94 the frequency-dependent reflection coefficient of a plane pressure wave under normal inci-

95 dence. First, based on energetic concepts, the identification of the specific surface properties
 96 of the equivalent interface model is achieved by comparison with the reference interphase
 97 model. Second, an optimization procedure is conducted to investigate the impact of the
 98 additional degree-of-freedom. Third, the performance of the optimal equivalent model is
 99 compared to different models available in the literature, which typically serve as a baseline
 100 in finite element (FE) simulations at the organ scale. Our numerical results show that this
 101 enriched model with specific interface conditions provides a very accurate approximation
 102 of the reference model over a broad frequency range, thus outperforming more simplistic
 103 models that fail in capturing the complex dynamics of the interphase. As a by-product,
 104 the link between the position of the additional degree-of-freedom and the microstructural
 105 features of the interphase (*e.g.*, competing gradients in mineral content and collagen fibers
 106 organization) is also discussed, thus opening promising perspectives for characterizing the
 107 tendon-to-bone attachment status. Elastic waves indeed represent a relevant nondestructive
 108 means to probe the interphase quality, as they possess intrinsic sensitivity to the mechanical
 109 properties contributing to the tendon-to-bone attachment strength. Overall this modeling
 110 approach represents the first building block for developing more sophisticated models target-
 111 ing reattachment procedures at the organ scale that incorporate a more detailed description
 112 of properties at lower length scales.

113 The paper is structured as follows: Sect. 2 introduces the theoretical fundamentals of
 114 our modeling approach. The numerical results are then presented in Sect. 3. Finally, the
 115 strengths and limitations of the proposed model are discussed in Sect. 4.

116 2. Theory

117 Based on the variational principles [40–42], this section first presents the governing equa-
 118 tions and boundary conditions for the two models depicted in Fig. 1c–d: (1) A reference
 119 model, which consists in a finite thickness interphase with a gradient in mechanical proper-
 120 ties, surrounded by two homogeneous media (see Subsect. 2.1), and (2) an equivalent model
 121 in which the interphase is replaced by specific interface conditions between the two homo-
 122 geneous media (see Subsect. 2.2). This general framework is then reduced to the specific
 123 case of a pressure plane wave propagating under normal incidence (see Subsect. 2.3). Fi-
 124 nally, the strategy to identify the coefficients of the equivalent interface model is described
 125 in Subsect. 2.4.

126 2.1. Reference interphase model

127 Let us consider the Cartesian frame of reference with coordinates $\mathbf{R}(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, where
 128 O is the origin and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis for the space. The coordinates of a
 129 point M in \mathbf{R} are specified by (x_1, x_2, x_3) and the time is denoted by t . As depicted in Fig. 2,
 130 the reference model consists of two homogeneous half-spaces $\Omega^- = \{(x_1, x_2, x_3) \mid x_2, x_3 \in$
 131 $\mathbb{R} \text{ and } x_1 < -\frac{h}{2}\}$ and $\Omega^+ = \{(x_1, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \text{ and } x_1 > \frac{h}{2}\}$, separated by an
 132 heterogeneous interphase layer of thickness h , namely $\Omega^I = \{(x_1, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \text{ and } -$
 133 $\frac{h}{2} < x_1 < \frac{h}{2}\}$, where \mathbb{R} is the set of real numbers. In what follows, the equations of motion

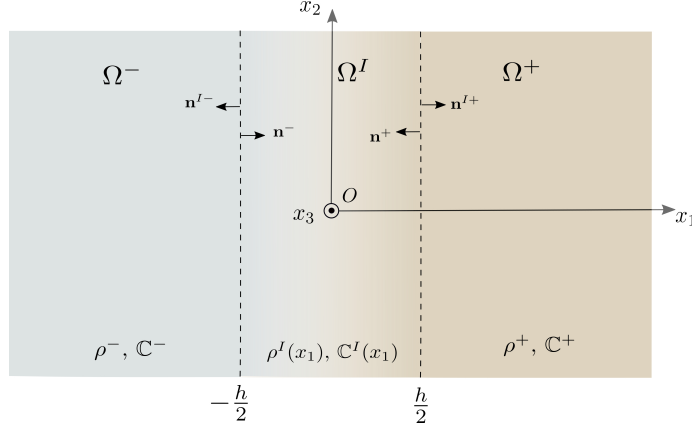


Figure 2: Modeling configuration of the reference model: A finite thickness interphase Ω^I with varying mechanical properties along the x_1 -direction is surrounded by two homogeneous half-spaces Ω^\pm . Superscripts $+$, $-$ and I denote the variables associated with the domains Ω^+ , Ω^- , and Ω^I , respectively.

134 are derived according to the least action principle. To this end, we introduce the action
 135 functional \mathcal{A}^{ref} of the considered system as in [36],

$$\mathcal{A}^{\text{ref}} = \int_{\Omega \times (0, t_f)} [T^+ + T^- + T^I - U^+ - U^- - U^I] dA, \quad (1)$$

where $\Omega = \Omega^\pm \cup \Omega^I$ spans the complete domain of interest, $(0, t_f)$ is a time interval and dA represents the differential volume element. The kinetic and potential energy densities per unit volume, T and U , associated with each domain of interest are defined as,

$$T^\pm = \frac{1}{2} \rho^\pm \dot{\mathbf{u}}^\pm \cdot \dot{\mathbf{u}}^\pm \quad \text{and} \quad U^\pm = \frac{1}{2} \boldsymbol{\sigma}^\pm : \boldsymbol{\epsilon}^\pm, \quad (2)$$

$$T^I = \frac{1}{2} \rho^I(x_1) \dot{\mathbf{u}}^I \cdot \dot{\mathbf{u}}^I \quad \text{and} \quad U^I = \frac{1}{2} \boldsymbol{\sigma}^I : \boldsymbol{\epsilon}^I, \quad (3)$$

136 where ρ^\pm are the mass densities of the domains Ω^\pm , and $\rho^I(x_1)$ is the mass density of the
 137 interphase Ω^I , which varies along the x_1 -direction. The displacement vector associated with
 138 each domain is denoted by \mathbf{u} , where the superimposed dot denotes the first derivative with
 139 respect to time. The stress and infinitesimal strain tensors associated with each domain
 140 are denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, respectively, the latter being defined as $\boldsymbol{\epsilon} = (1/2) (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$,
 141 whereby ∇ and the superscript T denote the gradient and transpose operators, respectively.
 142 The operators dot (\cdot) and colon ($:$) denote the scalar product of two vectors and the double
 143 contracted product of two tensors, respectively.

144 The least action principle implies that the variation of the action defined by Eq. (1)
 145 should verify that $\delta \mathcal{A}^{\text{ref}} = 0$. This condition, along with the consideration of kinematic
 146 constraints on the boundaries, *i.e.*, the continuity of displacements between the domains Ω^\pm

147 and Ω^I , leads to the following boundary value problem for the reference model,

$$\left\{ \begin{array}{l} \nabla \cdot \boldsymbol{\sigma}^\pm - \rho^\pm \ddot{\mathbf{u}}^\pm = \mathbf{0} \quad \forall M \in \Omega^\pm \\ \nabla \cdot \boldsymbol{\sigma}^I - \rho^I(x_1) \ddot{\mathbf{u}}^I = \mathbf{0} \quad \forall M \in \Omega^I \\ \left. \begin{array}{l} \boldsymbol{\sigma}^- \mathbf{n}^- + \boldsymbol{\sigma}^I \mathbf{n}^{I-} = \mathbf{0} \\ \mathbf{u}^- - \mathbf{u}^I = \mathbf{0} \end{array} \right\} \text{ for } x_1 = -\frac{h}{2}, \\ \left. \begin{array}{l} \boldsymbol{\sigma}^I \mathbf{n}^{I+} + \boldsymbol{\sigma}^+ \mathbf{n}^+ = \mathbf{0} \\ \mathbf{u}^I - \mathbf{u}^+ = \mathbf{0} \end{array} \right\} \text{ for } x_1 = \frac{h}{2} \end{array} \right. , \quad (4)$$

148 where $\nabla \cdot$ denote the divergence operator. The constitutive relation associated with each
 149 domain is defined in the frame of the linear elasticity, *i.e.*, $\boldsymbol{\sigma}^\pm = \mathbb{C}^\pm \boldsymbol{\epsilon}^\pm$ and $\boldsymbol{\sigma}^I = \mathbb{C}^I(x_1) \boldsymbol{\epsilon}^I$,
 150 whereby \mathbb{C}^\pm is the fourth-order elasticity tensor of the domains Ω^\pm , and $\mathbb{C}^I(x_1)$ is the fourth-
 151 order elasticity tensor of the interphase Ω^I , which varies with respect to the x_1 -direction.

152 2.2. Equivalent interface model

153 This section introduces the equivalent model based on surface elasticity, in which the
 154 finite thickness heterogeneous interphase is replaced by specific interface conditions. To this
 155 end, kinetic and potential energy densities of the interphase, T^I and U^I from Eq. (3), are
 156 now substituted by surface energy densities. The action functional \mathcal{A} for the equivalent
 157 system can thus be defined in a similar manner as in Sect. 2.1,

$$\mathcal{A} = \int_{\Omega \times (0, t_f)} [T^+ + T^- - U^+ - U^-] dA + \int_{\partial\Omega \times (0, t_f)} [T^S - U^S] dS, \quad (5)$$

where $\partial\Omega$ is the mid-surface of the domain Ω^I and dS represents the differential surface
 element, so that

$$T^S = \int_{-\frac{h}{2}}^{x_l} T^I dx_1 + \int_{x_l}^{\frac{h}{2}} T^I dx_1 \quad \text{and} \quad U^S = \int_{-\frac{h}{2}}^{x_l} U^I dx_1 + \int_{x_l}^{\frac{h}{2}} U^I dx_1, \quad (6)$$

where $-\frac{h}{2} < x_l < \frac{h}{2}$. Owing to the continuity of displacements at the boundaries, we can
 state that $\mathbf{u}^I(-\frac{h}{2}) = \mathbf{u}^-(-\frac{h}{2})$ and $\mathbf{u}^I(\frac{h}{2}) = \mathbf{u}^+(\frac{h}{2})$, where, for the sake of conciseness, the
 dependence of the fields on the spatial coordinates x_2 and x_3 , as well as on time t , is dropped.
 We furthermore assume that the integrals in Eq. (6) are quadratic forms of the displacement
 and velocity fields evaluated on the x_2x_3 -plane for x_1 equal to $-\frac{h}{2}$, x_l or $\frac{h}{2}$. Altogether, these
 hypotheses allow deriving a general form for the surface energy densities as,

$$T^S = \frac{1}{2} [\mathbf{m}^+ \dot{\mathbf{u}}^+ \cdot \dot{\mathbf{u}}^+ + \mathbf{m}_l \dot{\mathbf{u}}_l \cdot \dot{\mathbf{u}}_l + \mathbf{m}^- \dot{\mathbf{u}}^- \cdot \dot{\mathbf{u}}^- + 2\gamma_1 \dot{\mathbf{u}}^- \cdot \dot{\mathbf{u}}_l + 2\gamma_2 \dot{\mathbf{u}}_l \cdot \dot{\mathbf{u}}^+], \quad (7)$$

$$U^S = \frac{1}{2} \mathbf{K}_1 [\mathbf{u}_l - \mathbf{u}^-] \cdot [\mathbf{u}_l - \mathbf{u}^-] + \frac{1}{2} \mathbf{K}_2 [\mathbf{u}^+ - \mathbf{u}_l] \cdot [\mathbf{u}^+ - \mathbf{u}_l], \quad (8)$$

158 where $\mathbf{u}_l = \mathbf{u}^I(x_l)$. Note that, by abuse of notation, we dropped the dependence on space for
 159 all displacements. These surface energy densities, which are concentrated in the mid-surface

160 $\partial\Omega$ between the domains Ω^+ and Ω^- , can be interpreted by means of the general-
 161 mass system depicted in Fig. 3. Within this frame, the second-order tensors \mathbf{m}^\pm and \mathbf{m}_l
 162 represent masses concentrated at both sides of the interface and at position x_l , respectively;
 163 the second-order tensors \mathbf{K}_1 and \mathbf{K}_2 represent surface stiffnesses; and finally the second-
 164 order tensors γ_1 and γ_2 account for the kinetic interactions between the displacement fields.

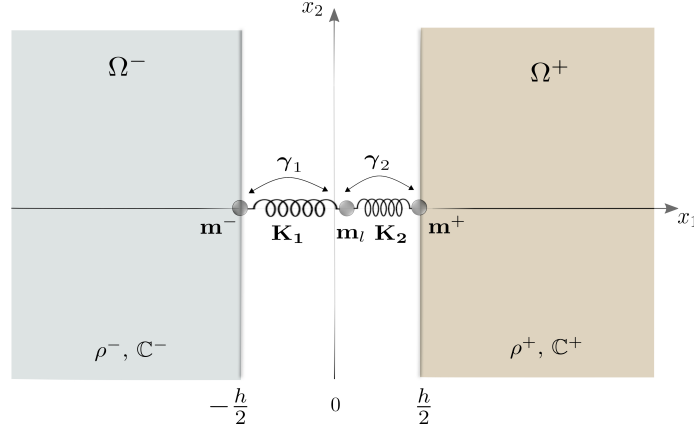


Figure 3: Modeling configuration of the equivalent model with specific interface conditions. The generalized spring-mass system illustrates the role played by the additional degree-of-freedom l to account for the dynamic interactions across the interphase.

165
 166 It is worth pointing out that, in the most general case, all these constitutive tensors
 167 are function of space and time (or frequency in the harmonic regime). Nevertheless, for
 168 the model to be exploitable, *i.e.*, to link these tensors to the physical characteristics of the
 169 interface rather than to its dynamic properties, a possibility is to turn them into constants.
 170 Here the calculation of these tensors will be achieved by selecting specific profiles for the
 171 displacement and velocity fields across the interphase, $\mathbf{u}^I(x_1)$ and $\dot{\mathbf{u}}^I(x_1)$, which fulfill the
 172 aforementioned constraints at the location x_1 given by the values $-\frac{h}{2}$, x_l or $\frac{h}{2}$. The natural
 173 consequence of this choice is that all surface energy terms will be considered as approxima-
 174 tions only.

Again, the variation of the action $\delta\mathcal{A}$ should verify that $\delta\mathcal{A} = 0$, thus leading to the following boundary value problem,

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma}^\pm - \rho^\pm \ddot{\mathbf{u}}^\pm = \mathbf{0}, & \forall M \in \Omega^\pm \\ \boldsymbol{\sigma}^- \mathbf{n}^- - \mathbf{K}_1(\mathbf{u}_l - \mathbf{u}^-) + \mathbf{m}^- \ddot{\mathbf{u}}^- + \gamma_1 \ddot{\mathbf{u}}_l = \mathbf{0} \\ \boldsymbol{\sigma}^+ \mathbf{n}^+ + \mathbf{K}_2(\mathbf{u}^+ - \mathbf{u}_l) + \mathbf{m}^+ \ddot{\mathbf{u}}^+ + \gamma_2 \ddot{\mathbf{u}}_l = \mathbf{0} \\ \mathbf{K}_1(\mathbf{u}_l - \mathbf{u}^-) - \mathbf{K}_2(\mathbf{u}^+ - \mathbf{u}_l) + \mathbf{m}_l \ddot{\mathbf{u}}_l + \gamma_1 \ddot{\mathbf{u}}^- + \gamma_2 \ddot{\mathbf{u}}^+ = \mathbf{0} \end{cases} \quad (9)$$

175 where the first equation retains the same form than that of Eq. (4), whereas the other three
 176 now account for the specific interface conditions, in which, by abuse of notation, the stress
 177 tensors are now stated as $\boldsymbol{\sigma}^- = \boldsymbol{\sigma}^I(-\frac{h}{2})$ and $\boldsymbol{\sigma}^+ = \boldsymbol{\sigma}^I(\frac{h}{2})$. It should be noted that this

178 boundary value problem could be easily implemented in a standard FE code, as it only
 179 involves a modification of the boundary conditions.

180 2.3. Pressure plane wave propagation under normal incidence

181 We now consider the problem of a plane elastic wave propagating under normal incidence
 182 across the considered functionally graded interphase. In this way, our modeling approach
 183 is reduced to an incident and a reflected longitudinal bulk wave in the domain Ω^- and a
 184 transmitted longitudinal bulk wave in the domain Ω^+ . By assuming a general harmonic
 185 solution for the interphase Ω^I , the general solution for a plane wave propagating along the
 186 x_1 -direction reads as

$$\begin{aligned} u^-(x_1, t) &= (A_i \exp(jk_p^- x_1) + A_r \exp(-jk_p^- x_1)) \exp(-j\omega t) \\ u^+(x_1, t) &= A_t \exp(j(k_p^+ x_1 - \omega t)) \\ u^I(x_1, t) &= g(x_1) \exp(-j\omega t) \end{aligned}, \quad (10)$$

187 where ω represents the angular frequency and j is the unit imaginary number. The variable
 188 $k_p^\pm = \omega/c_p^\pm$ denotes the wave number, which depends upon the longitudinal bulk wave
 189 velocity c_p^\pm in the domain Ω^\pm . The variables A_i , A_r and A_t denote the amplitudes of the
 190 incident, reflected and transmitted plane waves, respectively. The function $g(x_1)$ represents
 191 the unknown amplitude of the harmonic solution in the interphase, which accounts for the
 192 material heterogeneity along the x_1 -direction. Substituting the wave solutions from Eq.
 193 (10) into the boundary value problem from Eq. (4), and working through these analytical
 194 equations to remove the variables A_r and A_t , yield the following boundary value problem
 195 for the function $g(x_1)$ in strong form,

$$\begin{cases} (C_{11}^I(x_1)g_{,1}(x_1))_{,1} + \omega^2 \rho^I(x_1)g(x_1) = 0 \\ C_{11}^I\left(-\frac{h}{2}\right)g_{,1}\left(-\frac{h}{2}\right) + jk_p^- C_{11}^- g\left(-\frac{h}{2}\right) - 2jk_p^- C_{11}^- A_i \exp\left(-jk_p^- \frac{h}{2}\right) = 0 \\ C_{11}^I\left(\frac{h}{2}\right)g_{,1}\left(\frac{h}{2}\right) - jk_p^+ C_{11}^+ g\left(\frac{h}{2}\right) = 0 \end{cases}, \quad (11)$$

196 where the derivative with respect to x_1 is denoted by $(,1)$ and C_{11} is the stiffness coefficient
 197 along the x_1 -direction (in Voigt notation) associated with each domain. The first relation
 198 of Eq. (11) is an ordinary differential equation with respect to the coordinate x_1 , whereas
 199 the two last relations account for the boundary conditions at positions $x_1 = -\frac{h}{2}$ and $x_1 =$
 200 $\frac{h}{2}$, respectively. Note that such boundary value problem in strong form must be solved
 201 numerically.

In the same vein, the boundary value problem associated with the equivalent interface
 model (recall Eq. (9)) can be simplified as follows in the case of a pressure plane wave under

normal incidence,

$$\begin{cases} C_{11}^{\pm} u_{,11}^{\pm} - \rho^{\pm} \ddot{u}^{\pm} = 0, & \forall M \in \Omega^{\pm} \\ C_{11}^{-} u_{,1}^{-} - K_1(u_l - u^{-}) + m^{-} \ddot{u}^{-} + \gamma_1 \ddot{u}_l = 0 \\ -C_{11}^{+} u_{,1}^{+} + K_2(u^{+} - u_l) + m^{+} \ddot{u}^{+} + \gamma_2 \ddot{u}_l = 0 \\ K_1(u_l - u^{-}) - K_2(u^{+} - u_l) + m_l \ddot{u}_l + \gamma_1 \ddot{u}^{-} + \gamma_2 \ddot{u}^{+} = 0 \end{cases}, \quad (12)$$

202 where it should be noted that the second-order tensors of the surface energy terms are now
 203 all reduced to scalar coefficients. Furthermore, we hypothesize that the general solution for
 204 the displacement in the interface can be stated as

$$u_l(t) = B \exp(-j\omega t), \quad (13)$$

where B represents the unknown amplitude of the harmonic solution in the interface. Substituting Eq. (13), along with the wave solutions associated with the surrounding domains Ω^{\pm} from Eq. (10), into the boundary value problem from Eq. (12) leads to the following linear equation system

$$\begin{pmatrix} -jk_p^{-} C_{11}^{-} + K_1 - \omega^2 m^{-} & -K_1 - \omega^2 \gamma_1 & 0 \\ 0 & -K_2 - \omega^2 \gamma_2 & -jk_p^{+} C_{11}^{+} + K_2 - \omega^2 m^{+} \\ -K_1 - \omega^2 \gamma_1 & K_1 + K_2 - \omega^2 m_l & -K_2 - \omega^2 \gamma_2 \end{pmatrix} \begin{bmatrix} A_r \\ B \\ A_t \end{bmatrix} = \begin{pmatrix} -(jk_p^{-} C_{11}^{-} + K_1 - \omega^2 m^{-}) A_i \\ 0 \\ (K_1 + \omega^2 \gamma_1) A_i \end{pmatrix}, \quad (14)$$

205 whose solution allows determining the unknown amplitudes, A_r and A_t , of the reflected and
 206 transmitted waveforms. Note that, in contrast to the boundary value problem associated
 207 with the reference interphase model (recall Eq. (11)), the resulting system for the equivalent
 208 interface can be solved analytically.

209 2.4. Identification of the equivalent model coefficients

210 This section presents the adopted strategy for identifying the coefficients of the surface
 211 kinetic and internal energy densities, *i.e.*, m^{-} , m_l , m^{+} , γ_1 , γ_2 , K_1 , and K_2 . Towards this
 212 goal, we hypothesize that the displacement field is directed towards the surface normal, *i.e.*,
 213 $\mathbf{u}^I = u^I(x_1) \mathbf{e}_1$, and that it can be approximated using a piece-wise affine profile with respect
 214 to the coordinate x_1 (see Fig. 4).

In such a case, the kinetic and potential energy densities across the interphase (recall Eq. (3)) can be written as

$$T^I = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^I(x_1) (v^I(x_1))^2 dx_1 \quad \text{and} \quad U^I = \frac{1}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{11}^I(x_1) (u_{,1}^I(x_1))^2 dx_1, \quad (15)$$

215 where

$$u^I(x_1) = \begin{cases} \left(\frac{2(u_l - u^{-})}{h + 2x_l} \right) x_1 + \left(\frac{u_l h + 2u^{-} x_l}{h + 2x_l} \right) & \text{for } -\frac{h}{2} < x_1 < x_l \\ \left(\frac{2(u^{+} - u_l)}{h - 2x_l} \right) x_1 - \left(\frac{2u^{+} x_l - u_l h}{h - 2x_l} \right) & \text{for } x_l < x_1 < \frac{h}{2} \end{cases}, \quad (16)$$

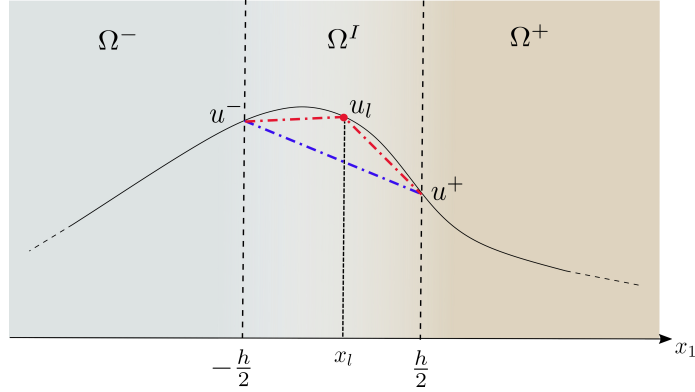


Figure 4: Illustration of the displacement field across the interphase for an arbitrarily selected frequency: Comparison between the exact profile (continuous black line) used in the reference model and the piece-wise affine profile (dashed red line) used in the equivalent interface model. The affine profile (dashed blue line) proposed in [38] is shown for comparison.

216 in which u^- , u_l and u^+ are the values of the displacement field $u^I(x_1)$ at the location x_1
 217 given by the values $-\frac{h}{2}$, x_l or $\frac{h}{2}$. Note that, since we assumed a general harmonic solution
 218 for the displacement field in the interphase, the velocity field $v^I(x_1)$ will have the same
 219 piece-wise form, modulo the constant complex factor $j\omega$, than that of Eq. (16), with v^- , v_l
 220 and v^+ being the values of the velocity field at the location x_1 given by the values $-\frac{h}{2}$, x_l
 221 or $\frac{h}{2}$. According to the pressure plane wave assumption, the kinetic and potential energies
 222 per unit surface, T^S and U^S , can now be written as

$$\begin{aligned}
 T^S &= \frac{1}{2} [m^+(v^+)^2 + m_l(v_l)^2 + m^-(v^-)^2 + 2\gamma_1 v^- v_l + 2\gamma_2 v_l v^+], \\
 U^S &= \frac{1}{2} K_1 (u_l - u^-)^2 + \frac{1}{2} K_2 (u^+ - u_l)^2,
 \end{aligned}
 \tag{17}$$

223 where the necessary conditions for the kinetic energy density, T^S , to be a positive definite
 224 quadratic form are that $m^\pm > 0$, $\gamma_2^2 < \frac{1}{2}m^+m_l$, and $\gamma_1^2 < \frac{1}{2}m^-m_l$. By inserting the dis-
 225 placement and velocity fields, $u^I(x_1)$ and $v^I(x_1)$, into Eq. (15), the outcome can then be
 226 compared to Eq. (17) by making use of the relations defined in Eq. (6), thus allowing for
 227 the identification of the coefficients belonging to the surface kinetic and potential energy

228 densities T^S and U^S as

$$\begin{aligned}
m^- &= 4 \int_{-\frac{h}{2}}^{x_l} \rho_I(x_1) \left(\frac{x_1 - x_l}{h + 2x_l} \right)^2 dx_1 \\
m^+ &= 4 \int_{x_l}^{\frac{h}{2}} \rho_I(x_1) \left(\frac{x_1 - x_l}{h - 2x_l} \right)^2 dx_1 \\
m_l &= \int_{-\frac{h}{2}}^{x_l} \rho_I(x_1) \left(\frac{h + 2x_1}{h + 2x_l} \right)^2 dx_1 + \int_{x_l}^{\frac{h}{2}} \rho_I(x_1) \left(\frac{h - 2x_1}{h - 2x_l} \right)^2 dx_1 \\
\gamma_1 &= 2 \int_{-\frac{h}{2}}^{x_l} \rho_I(x_1) \frac{(h + 2x_1)(-x_1 + x_l)}{(h + 2x_l)^2} dx_1 \\
\gamma_2 &= 2 \int_{x_l}^{\frac{h}{2}} \rho_I(x_1) \frac{(h - 2x_1)(x_1 - x_l)}{(h - 2x_l)^2} dx_1 \\
K_1 &= \frac{1}{x_l^2} \int_{-\frac{h}{2}}^{x_l} C_{11}^I(x_1) dx_1 \\
K_2 &= \frac{1}{(h - x_l)^2} \int_{x_l}^{\frac{h}{2}} C_{11}^I(x_1) dx_1.
\end{aligned} \tag{18}$$

229 In our earlier study [38], the equivalent interface model was associated with an affine dis-
230 placement field (dashed blue line in Fig. 4). To serve as a comparison with Eqs. (17)-(18),
231 the related expressions for the kinetic and potential energy densities per unit surface, along
232 with the identified coefficients, are briefly recalled in Appendix A.

233 3. Numerical results

234 The performance of the proposed enriched model is assessed by evaluating the frequency-
235 dependent reflection coefficient in a reflection/transmission problem between two homoge-
236 neous solid half-spaces, which are separated by a finite thickness and heterogeneous in-
237 terphase. First, we identify the specific surface properties of the equivalent interface by
238 comparison with the reference interphase. Second, we perform an optimization procedure to
239 investigate the impact of the position x_l of the additional degree-of-freedom on the modeling
240 performance. The optimal equivalent model is subsequently compared to different baselines.

241 Towards these goals, we consider here an interphase whose stiffness and density vary
242 along the x_1 -direction. This configuration is inspired by our recent multiscale modeling
243 results [39], which showed that the tendon-to-bone interphase can be seen as a continuous
244 functionally graded material. The selected model parameters for the two surroundings tis-
245 sues, *i.e.*, tendon Ω^- and bone Ω^+ , together with the finite thickness h of the interphase,
246 are summarized in Tab. 1.

247 The gradients in mechanical properties across the interphase Ω^I , which are used as
248 further input in the reference model, are depicted in Fig. 5. The effective stiffness variation
249 at the mesoscale is the result of competing gradients in mineral content and collagen fibers
250 organization at lower length scales. Indeed, it has been shown that the linear increase in

Mass density (g.cm ⁻³)		Stiffness coefficient (GPa)		Thickness (μm)
ρ^-	ρ^+	C_{11}^-	C_{11}^+	h
1.22	1.88	4.37	27.48	300

Table 1: Model parameters for the two surrounding tissues Ω^\pm and the interphase Ω^I .

251 mineral content caused a stiffening of the interphase that became significant beyond a certain
 252 percolation threshold (approximately located between 0 – 50 μm in Fig. 5a), whereas the
 253 decreasing collagen fiber organization across the interphase led to a reduced tissue stiffness
 254 along the main fibers direction [14, 39]. In contrast, the effective mass density evolves
 linearly between the two surrounding tissues (see Fig. 5b).

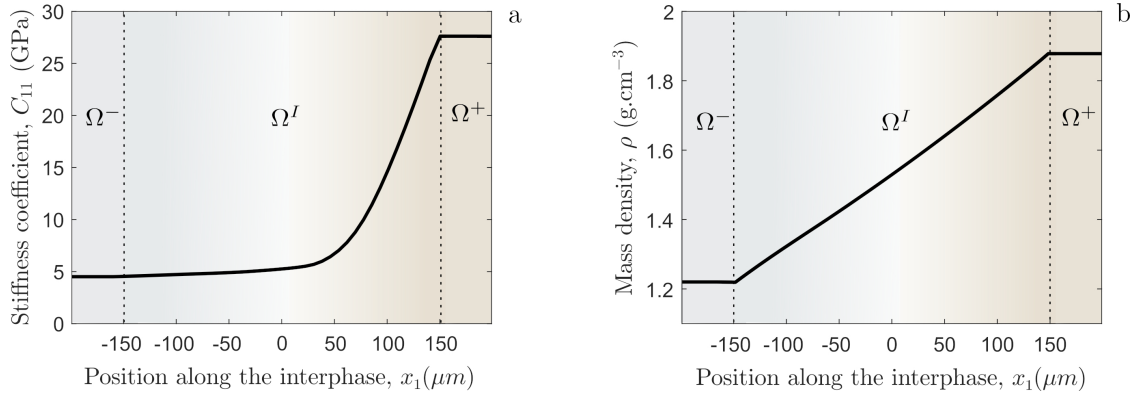


Figure 5: Graded mechanical properties across the tendon-to-bone interphase: (a) effective stiffness coefficient C_{11} and (b) effective mass density ρ . These trends are based upon the multiscale model reported in [39].

255

256 3.1. Evaluation of the equivalent interface model

To validate the proposed equivalent interface model, we first evaluate the reflection coefficient $R = |A_r/A_i|^2$ as a function of frequency, in which the amplitude A_i of the incident plane wave was set to unity. For a better interpretation, the reflection coefficient R was calculated as well as a function of the ratio between the average wavelength $\bar{\lambda}$ and the thickness h of the interphase. The average wavelength $\bar{\lambda}$ was defined as

$$\bar{\lambda} = \frac{\bar{c}_p^I}{f}, \quad \text{with } \bar{c}_p^I = \sqrt{\frac{\bar{C}_{11}^I}{\bar{\rho}^I}}, \quad (19)$$

257 where \bar{c}_p^I , \bar{C}_{11}^I and $\bar{\rho}^I$ are the average phase velocity, the average stiffness and the average
 258 mass density, respectively.

259 The reference model from Eq. (11) is solved using a standard FE code such as Comsol
 260 Multiphysics[®], from which the reflection coefficient, R^{ref} can be derived as

$$R^{\text{ref}} = \left| \frac{A_r}{A_i} \right|^2 = \left| \frac{g\left(-\frac{h}{2}\right) - A_i \exp\left(-jk_p^-\frac{h}{2}\right)}{A_i \exp\left(jk_p^-\frac{h}{2}\right)} \right|^2. \quad (20)$$

261 In contrast, the reflection coefficient associated with the equivalent interface model can be
 262 calculated analytically using Eq. (14). Note, however, that this enriched modeling approach
 263 now depends upon the position of the additional degree-of-freedom x_l , whose optimal lo-
 264 cation is *a priori* not known. To serve as an example, the coefficients of the kinetic and
 265 potential energy densities were first calculated for two arbitrarily selected values of x_l using
 266 Eq. (18). The numerical results are depicted in Fig. 6, in which the proposed model (dashed
 267 red line) with different values of x_l is compared to the reference interphase model (con-
 268 tinuous black line), to a homogeneous interphase Ω^I with averaged mechanical properties
 269 (dashed gray line), and to an abrupt transition (continuous gray line) corresponding to the
 case where the surrounding tissues Ω^\pm are placed directly in contact. As can be observed in

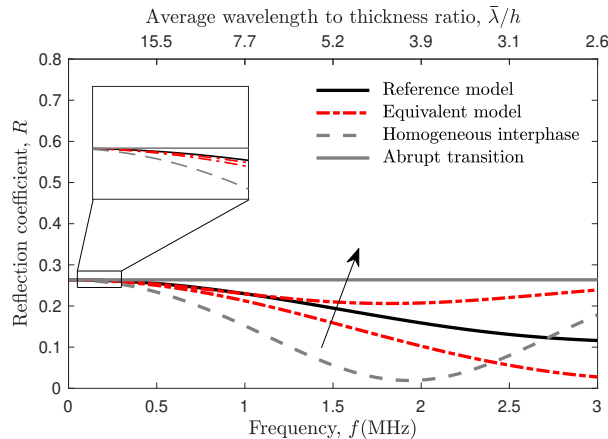


Figure 6: Frequency-dependent reflection coefficient R obtained using: (1) the reference model (continuous black line); (2) a homogeneous interphase (dashed gray line); (3) an abrupt transition (continuous gray line); and (4) the equivalent interface model (dashed red lines) for two arbitrarily selected values of x_l equal to -50 and $50 \mu\text{m}$. The black arrow indicates the resulting trend with respect to increasing values of x_l .

270 the enlargement of Fig. 6, for a low frequency regime below 0.3 MHz (*i.e.*, high ratio $\bar{\lambda}/h$),
 271 the equivalent interface model satisfactorily captures the dynamics of the reference model,
 272 independently of the position of the additional degree-of-freedom x_l . At an intermediate fre-
 273 quency regime around 0.3–2.5 MHz, the choice of x_l has a significant impact on the ability
 274 of our model to match the reference one, and thus account for the gradient in mechanical
 275 properties, which in turn reflect microstructural features at lower length scales. As expected,
 276 when the ratio of the average wavelength to the interphase thickness decreases towards val-
 277 ues below 3, the deviation from the reference model increases, and our approximated model
 278

279 fails in capturing the complex dynamics of the interphase, which is associated with stronger
 280 wave-interphase interactions. Nonetheless, it is worth mentioning that, overall, the devia-
 281 tion of this model from the reference ones is much lower than that obtained using a simple
 282 abrupt transition or a homogeneous interphase with averaged mechanical properties, which
 283 represent the commonly used baselines in FE simulations at the organ scale [43–45].

284 3.2. Optimization of the equivalent interface model

285 Second, an optimization procedure was conducted to further investigate the impact of the
 286 additional degree-of-freedom position x_l on the performance of the proposed model. To this
 287 end, the relative cumulative error on the reflection coefficient (denoted by err) between the
 288 one of the reference model (denoted by R^{ref}) and the one of the equivalent interface model
 289 (denoted by R) was calculated by sweeping the frequency and the position of x_l across the
 290 interphase as

$$\text{err}(x_l, f) = \frac{1}{N_{\text{max}}} \sum_{n=1}^{N_{\text{max}}} \left(\frac{R^{\text{ref}}(f_n) - R(x_l, f_n)}{R^{\text{ref}}(f_n)} \right)^2, \quad (21)$$

291 where N_{max} is an integer value corresponding to the maximal frequency position. Figure 7a
 292 represents the dynamics of the error as a function of the position of x_l and the frequency f .
 As can be observed, at a low frequency regime below 0.3 MHz, the equivalent interface model

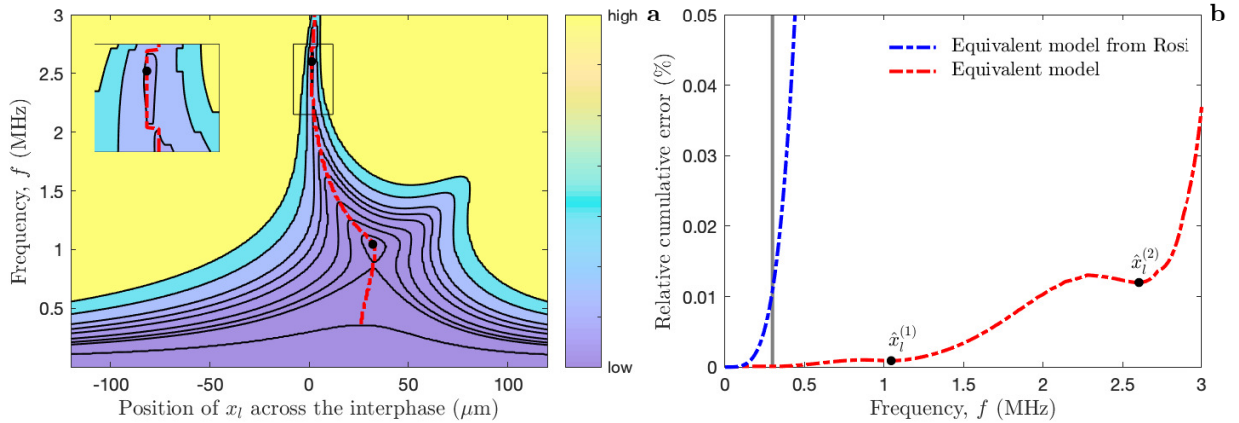


Figure 7: (a) Relative cumulative error on the reflection coefficient (Eq. (21)) between the one of the reference model and the one of the equivalent interface model as a function of the frequency and the position of the additional degree-of-freedom x_l across the interphase. The dashed red line represents the optimal x_l -path as a function of the frequency, whereas the black dots depict the local minima $\hat{x}_l^{(1)}$ and $\hat{x}_l^{(2)}$. The region around the second local minima is displayed in the insert. (b) Relative cumulative error along the optimal x_l -path as a function of the frequency (dashed red line). The error between the reference model and the model presented in [38] is shown for comparison (dashed blue line). The gray line represents the frequency threshold (0.3 MHz) below which the error is independent of the position x_l .

293 provides an accurate approximation of the reference model independently of the position of
 294 the additional degree-of-freedom x_l . Towards higher frequencies, the performance of the
 295 equivalent model slightly decreases, but there is a specific path along x_l (displayed as a
 296 dashed red line) for which the error remains reasonably low. Interestingly, there are two
 297

298 local minima located at around 1 and 2.6 MHz along this path (black dots), for which the
 299 optimal values of \hat{x}_l are equal to 32 and 1.2 μm , respectively (*i.e.*, thicknesses that fall within
 300 the range corresponding to the percolation threshold in Fig. 5a). A more advanced insight is
 301 provided in Fig. 7b, which depicts the relative cumulative error along the optimal x_l -path as
 302 a function of the frequency, thus clearly highlighting the presence of two local minima, *i.e.*,
 303 $x_l^{(1)}$ and $x_l^{(2)}$. It is worth mentioning that the error associated with $x_l^{(2)}$ at 2.6 MHz is of the
 304 same order than that obtained using the model from [38] at 0.3 MHz (low frequency regime
 305 represented as a continuous gray line). Therefore, fine-tuning the position of the additional
 306 degree-of-freedom x_l in our modeling approach allows enlarging nearly 8 times the validity
 307 range with respect to that model.

308 To further assess the performance of our enriched model, Fig. 8 now depicts the obtained
 309 frequency-dependent reflection coefficient R for the optimal position $x_l^{(2)}$ (dashed red line),
 310 which is compared to the reference model (continuous black line), an abrupt transition
 311 (continuous gray line), the equivalent model from [38] (dashed blue line) and an equivalent
 312 model possessing only elastic properties (dotted red line), that is to say non-inertial interface
 conditions (*i.e.*, $m^+ = m^- = m_l = 0 \text{ kg}\cdot\text{m}^{-2}$ and $\gamma_1 = \gamma_2 = 0 \text{ kg}\cdot\text{m}^{-2}$). As expected from the

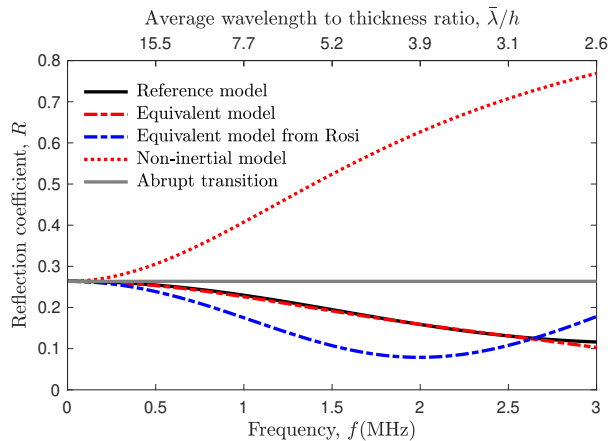


Figure 8: Frequency-dependent reflection coefficient R obtained using: (1) the reference model (continuous black line); (2) an abrupt transition (continuous gray line); (3) a non-inertial equivalent interface model (dotted red line); (4) the model (dashed blue line) associated with a displacement field with an affine profile (Ref. [38]); and (5) the equivalent interface model (dashed red lines) for an optimal value of x_l equal to 1.2 μm .

313 optimization, our equivalent model with specific interface conditions provides a very accurate
 314 approximation of the reference model over a broad frequency range up to around 2.6 MHz.
 315 It thus outperforms the results obtained with the equivalent model from [38], which still
 316 remains valid but for a much narrower frequency regime. In contrast, an equivalent model
 317 that only accounts for elastic interface conditions totally fails in capturing the complex
 318 dynamics of the interphase.
 319

320 4. Discussion

321 Computational modeling of functionally graded interphases between soft and hard mate-
322 rials has provided important insights on underlying anchoring mechanisms at the microscale
323 and on surgical repair strategies at the macroscale. To unravel additional mechanical char-
324 acteristics related to the functioning of such complex interphases, as well as to enhance the
325 characterization of bioinspired attachment procedures or tissue engineered implants, current
326 models should include a precise description of the graded mechanical behavior of the inter-
327 phase across several length scales. However, such modeling strategy raises several challenges
328 from a computational viewpoint, especially in the case of dynamic loading.

329 In this study, we proposed an equivalent model based on surface elasticity, in which the
330 finite thickness, continuous functionally graded interphase between the tendon and bone
331 can be substituted by a zero-thickness interface. This was achieved by replacing the kinetic
332 and potential energies of the interphase by specific surface energy density terms, whose
333 coefficients could be identified in a straightforward manner using known profiles for the
334 effective stiffness and density across the interphase. With respect to previously reported
335 models, this novel formulation introduces an additional degree-of-freedom, which allows
336 for a more refined description of the mechanical fields (*e.g.*, displacements and velocities)
337 across the interphase. The performance of the model was evaluated in the context of wave
338 propagation.

339 The main findings from this numerical study were as follows: (1) as the performance of
340 enriched models generally depends upon a proper selection of the kinematics across the inter-
341 phase, our numerical results showed that the displacement field could be adequately modeled
342 as a two-piece profile associated with an additional degree-of-freedom; (2) the assessment of
343 the frequency-dependent reflection coefficient showed that our optimized enriched model can
344 provide a very accurate approximation of the reference model over a broad frequency range
345 up to around 2.6 MHz, thus outperforming the results obtained with more simplistic mod-
346 els; and (3) this model has a much lower computational cost than the reference one, as the
347 interphase must not be incorporated (*i.e.*, meshed) explicitly. Overall, our numerical results
348 showed that the proposed approach is well-suited for the solution of a complete elastodynam-
349 ics problem in the frequency-domain, as it captures the complex mechanical behavior of the
350 tendon-to-bone interphase over a broad frequency range. Interestingly, all these significant
351 outcomes are reached based on the optimal position of the additional degree-of-freedom,
352 whose choice may depend on several parameters, among which the underlying behavior of
353 the gradient in mechanical properties across the interphase is thought to be a determinant
354 factor.

355 To test this hypothesis, we also investigated the impact of microstructural *damage* at
356 the tendon-bone interphase on the optimal position of the additional degree-of-freedom (see
357 Fig. 9). Clinical burden related to physical overloading, disuse or systemic diseases may
358 manifest as a delayed onset of mineralization and an increasing angular deviation from the
359 main collagen fibers direction across the tendon-to-bone insertion [46–48], which in turn lead
360 to modified graded mechanical properties at the macroscale [14, 39]. As can be observed, for
361 such a damaged configuration the gradient in stiffness displays a slightly shifted percolation

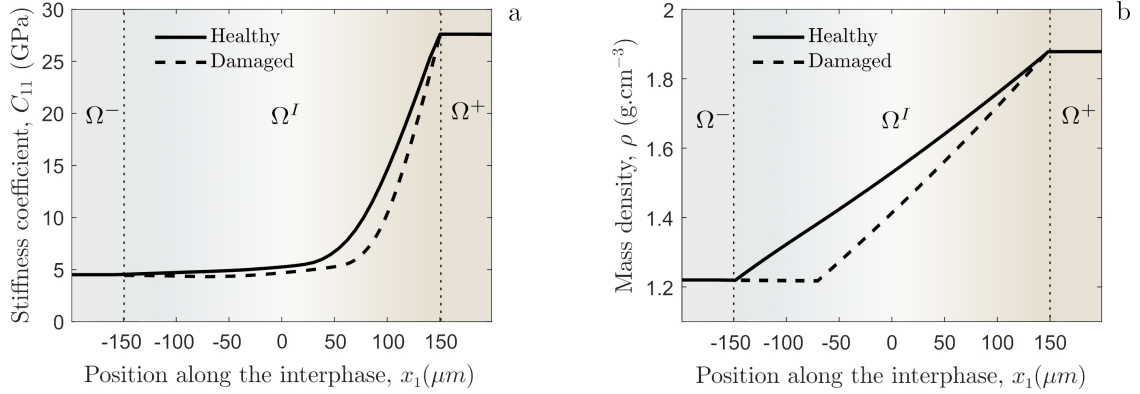


Figure 9: Graded mechanical properties across the tendon-to-bone interphase representative of a damaged configuration (dashed lines): (a) effective stiffness coefficient C_{11} and (b) effective mass density ρ . The nominal case from Fig. 5 is shown for comparison (continuous lines).

362 threshold with respect to the nominal case (see Fig. 9a), whereas the gradient in mass density
 363 now has a piece-wise profile (see Fig. 9b).

364 Using these modified graded profiles as input of our enriched model and applying the op-
 365 timization procedure described in Subsect. 3.2 yield different values for the optimal position
 366 \hat{x}_l and for the identified coefficients associated with the surface energy density terms (see
 367 Tab. 2). Interestingly, the optimal position \hat{x}_l has now raised to around 9 μm with respect
 368 to its nominal value, and this shift is of the same order than that related to the percolation
 369 threshold observed in the stiffness profile (see Fig. 9a). Beside the shift of \hat{x}_l , the identified
 370 coefficients are also impacted, but their interpretation is not straightforward and we post-
 371 pone this analysis to future works. Nevertheless, our model proves to be sensitive to slight

Case	Position \hat{x}_l μm	Surface mass densities			Kinetic interactions		Surface stiffnesses	
		m^-	m^+	m_l	γ_1	γ_2	K_1	K_2
		$\text{kg}\cdot\text{m}^{-2}$			$\text{kg}\cdot\text{m}^{-2}$		$\text{kg}\cdot(\text{s}\cdot\text{m})^{-2} \times 10^{13}$	
Healthy	1.2	0.0653	0.0881	0.1525	0.0346	0.0418	3.08	8.05
Damaged	10.3	0.0658	0.0818	0.1427	0.0339	0.0384	2.72	7.57

Table 2: Comparison between the optimal positions of the additional degree-of-freedom and the coefficients of the enriched model for the healthy and damaged cases.

371 changes of the graded properties across the interphase (see Fig. 10) and could potentially
 372 be used for characterization purposes.
 373

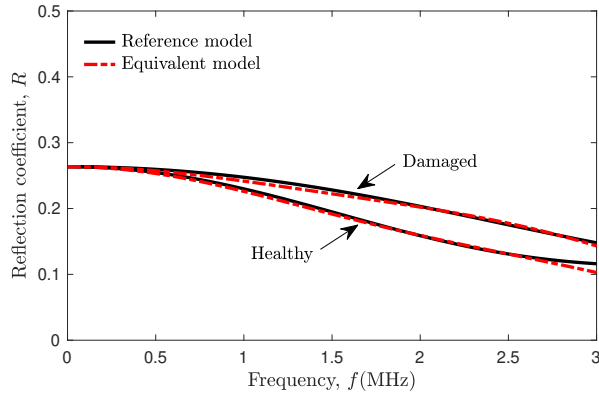


Figure 10: Comparison between the reference model (continuous black lines) and the optimal equivalent interface model (dashed red lines) for two different gradients in mechanical properties across the tendon-to-bone interphase (recall Fig. 9).

374 Although the proposed enriched framework was established to account for a general 3-D
375 formulation, the investigated configuration was subsequently reduced to the rather simple
376 case of a plane elastic wave propagating under normal incidence across a functionally graded
377 interphase, whose mechanical properties vary with respect to a single direction only. In a
378 such case, the reflection/transmission problem is limited to the analysis of longitudinal bulk
379 waves. Therefore, to further tackle the complexity of the tendon-to-bone attachment, whose
380 geometry, for instance at the Achilles tendon, is characterized by a shallow attachment angle
381 and an optimized shape of the outward splay, as well as by interdigitation mechanisms (recall
382 Fig. 1a–b) [49, 50], future modeling efforts should consider 2-D geometries using plane wave
383 propagating under oblique incidence (*i.e.*, both longitudinal and shear waves) or surface
384 waves [51]. As a further drawback, the performance of the enriched model was here only
385 assessed based on the analysis of the frequency-dependent reflection coefficient, thus only
386 displaying information on the amplitude spectrum of the propagating waves. However, the
387 phase of the waves plays a crucial role as well, especially for the analysis of time-domain
388 signals traveling across thin interphases. Further studies are warranted to precisely ad-
389 dress the impact of the equivalent modeling approach on the phase spectrum. Overall this
390 modeling approach represents the first critical step towards the development of more sophis-
391 ticated models targeting mechanical strategies for improving diagnostic and reattachment
392 procedures, which will not only be valuable for evaluating the tendon-to-bone insertion,
393 but also other multiscale and graded biological attachments such as the bone-implant inter-
394 phase [52, 53] and new generations of engineered composite materials [54].

395 5. Conclusion

396 The original equivalent interface model presented in this study aimed at mimicking the
397 elastodynamic behavior of a finite heterogeneous interphase, while reducing the overall com-
398 putational burden related to the consideration of its complete geometry. It has been shown
399 that, in the case of a dynamic experiment, *i.e.*, plane pressure waves under normal incidence,

400 the proposed enriched model remains valid over a broad frequency range and outperforms
 401 more simplistic models that typically serve as a baseline in FE simulations at the organ
 402 scale. The key feature of the model was the introduction of an additional degree of freedom
 403 in the interface, which allowed for a finer description of the kinematic fields within the in-
 404 terphase that accounted for both its inertial behavior and the kinetic interactions with the
 405 surrounding media. Furthermore, it has been shown that the model was sensitive to slight
 406 variations of the graded mechanical properties across the interphase, thus opening potential
 407 applications in the field of nondestructive testing and characterization. Finally, it should
 408 be noted that such modeling approach is straightforwardly extendable to account for more
 409 complex interphase geometries.

410 Acknowledgements

411 This work was partially funded by the BEST-AMUS project (IIN program, CNRS-
 412 INSIS), the ‘‘Support for research for newly appointed Associate Professors’’ and the ‘‘Bonus
 413 Qualité Recherche’’ (Faculté des Sciences et Technologie, Université Paris-Est Créteil).

414 Appendix A. Equivalent interface model used for comparison

Here we briefly recall the underlying equations of the equivalent interface model intro-
 duced in [38]. In that study, the displacement field was associated with an affine profile. As
 such, the general forms for the kinetic and potential energies per unit surface were stated as

$$T^S = \frac{1}{2} [m^+(v^+)^2 + m^-(v^-)^2 + 2\gamma v^+ v^-], \quad (\text{A.1})$$

$$U^S = \frac{1}{2} K (v^+ - v^-) \cdot (v^+ - v^-), \quad (\text{A.2})$$

where the coefficients to be identified are

$$m^\pm = \frac{M_0}{4} \pm \frac{M_1}{h} + \frac{M_2}{h^2}, \quad \gamma = \frac{M_0}{4} - \frac{M_2}{h^2}, \quad (\text{A.3})$$

in which M_α represents the α -th order moment of the density profile of the interphase

$$M_\alpha = \int_0^h x_1^\alpha \rho^I(x_1) dx_1, \quad \text{for } \alpha = 0, 1, 2. \quad (\text{A.4})$$

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