



**HAL**  
open science

## Distribution of generalized mex-related integer partitions

Kalyan Chakraborty, Chiranjit Ray

► **To cite this version:**

Kalyan Chakraborty, Chiranjit Ray. Distribution of generalized mex-related integer partitions. Hardy-Ramanujan Journal, 2021, Volume 43 - Special Commemorative volume in honour of Srinivasa Ramanujan - 2020, pp.122-128. 10.46298/hrj.2021.7425 . hal-03208509

**HAL Id: hal-03208509**

**<https://hal.science/hal-03208509v1>**

Submitted on 26 Apr 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Distribution of generalized mex-related integer partitions

Kalyan Chakraborty and Chiranjit Ray

**Abstract.** The minimal excludant or “mex” function for an integer partition  $\pi$  of a positive integer  $n$ ,  $mex(\pi)$ , is the smallest positive integer that is not a part of  $\pi$ . Andrews and Newman introduced  $\sigma mex(n)$  to be the sum of  $mex(\pi)$  taken over all partitions  $\pi$  of  $n$ . Ballantine and Merca generalized this combinatorial interpretation to  $\sigma_r mex(n)$ , as the sum of least  $r$ -gaps in all partitions of  $n$ . In this article, we study the arithmetic density of  $\sigma_2 mex(n)$  and  $\sigma_3 mex(n)$  modulo  $2^k$  for any positive integer  $k$ .

**Keywords.** Minimal excludant, Integer partition; Eta-quotients; Modular forms; Distribution.

**2010 Mathematics Subject Classification.** Primary: 05A17, 11P83, 11F11, 11F20.

## 1. Introduction and statement of results

In 2015, Fraenkel and Peled [FrPe15] defined the minimal excludant or “mex” function on a set  $S$  of positive integers as the least positive integer not in  $S$ . Perhaps the notion of the mex function was introduced in the 1930s, and best known for the applications in combinatorics and game theory [Gr39, Sp35].

A partition of a non-negative integer  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ . Let  $\pi$  be a partition of  $n$  and  $\mathcal{P}(n)$  be the set of all partitions of  $n$ . Recently, Andrews and Newman [AnNe19] considered the minimal excludant function applied to integer partitions. The minimal excludant of  $\pi$ , denoted  $mex(\pi)$ , is the smallest positive integer which is not a part of  $\pi$ . Thus if  $\pi$  is  $6 + 4 + 3 + 2 + 1$ , a partition of 16, then  $mex(\pi) = 5$ . For each positive integer  $n$ , we have

$$\sigma mex(n) := \sum_{\pi \in \mathcal{P}(n)} mex(\pi).$$

For example,  $\sigma mex(4) = 9$  with the relevant mex partitions being:  $mex(4) = 1$ ,  $mex(3 + 1) = 2$ ,  $mex(2 + 2) = 1$ ,  $mex(2 + 1 + 1) = 3$ , and  $mex(1 + 1 + 1 + 1) = 2$ . The generating function for  $\sigma mex(n)$  is given by

$$\sum_{n=0}^{\infty} \sigma mex(n) q^n = (-q; q)_{\infty}^2,$$

where the  $q$ -shifted factorial  $(a; q)_{\infty} := \prod_{n=1}^{\infty} (1 - aq^{n-1})$ ,  $|q| < 1$ . Let  $\mathcal{D}_2(n)$  be the set of partitions of  $n$  into distinct parts using two colors and let  $D_2(n) = |\mathcal{D}_2(n)|$ . In [AnNe19], the authors give two proofs of the following theorem.

**Theorem 1.1.** *Given an integer  $n > 0$ , we have*

$$\sigma mex(n) = D_2(n).$$

They also studied the parity of  $\sigma mex$  function and proved that  $\sigma mex(n)$  is almost always even and is odd exactly when  $n$  is of the form  $j(3j \pm 1)$ , where  $j$  is a non-negative integer.

In some literature, the minimal excludant of a partition  $\pi$  is referred to as the least gap or smallest gap of  $\pi$ . The  $r$ -gap of a partition  $\pi$  is the least positive integer that does not appear at least  $r$  times as a part of  $\pi$ . Let  $s_r(\pi)$  be the smallest part of the partition  $\pi$  appearing less than  $r$  times and  $S_r(n) = \sum_{\pi \in \mathcal{P}(n)} s_r(\pi)$ .

For example, all the 3-gaps in the partitions of 5 are:  $s_3(5) = 1$ ,  $s_3(4 + 1) = 1$ ,  $s_3(3 + 2) = 1$ ,  $s_3(3 + 1 + 1) = 1$ ,  $s_3(2 + 2 + 1) = 1$ ,  $s_3(2 + 1 + 1 + 1) = 2$ ,  $s_3(1 + 1 + 1 + 1 + 1) = 2$ . Therefore  $S_3(5) = \sum_{\pi \in \mathcal{P}(5)} s_3(\pi) = 9$ .

Ballantine and Merca [Ba, Ba20] generalized Theorem 1.1 to the sum  $S_r(n)$  of  $r$ -gaps in all partitions of  $n$ . To keep notation uniform, in this article we use  $\sigma_r mex(n)$  for  $S_r(n)$ . The generating function for  $\sigma_r mex(n)$  is given by

$$\sum_{n=0}^{\infty} \sigma_r mex(n) q^n = \frac{(q^{2r}; q^{2r})_{\infty}}{(q; q)_{\infty} (q^r; q^{2r})_{\infty}}. \tag{1.1}$$

In [Ba20], Ballantine and Merca proved the following identity.

**Theorem 1.2.** *For  $n \geq 0$  and  $r \geq 1$  we have*

$$\sum_{k=0}^{\infty} p(n - rT_k) = \sigma_r mex(n),$$

where  $p(n)$  counts the number of partition of  $n$  and  $T_k$  is the  $k$ -th triangular number.

Recently, Ray and Barman [RaBa20, Theorem 1.6], studied the divisibility of Uncu’s partition function  $\mathcal{EO}_u(n)$ . After some elementary calculations we observe that the generating function of  $\mathcal{EO}_u(2n)$  and  $\sigma mex(n)$  (or  $\sigma_1 mex(n)$ ) are the same. So using the result mentioned above,  $\sigma mex(n)$  is almost always divisible by  $2^k$  for any positive integer  $k$ .

A well-known conjecture of Parkin and Shanks [PS67], for integer partitions  $p(n)$ , states that the even and odd values of  $p(n)$  are equally distributed, that is,

$$\lim_{X \rightarrow \infty} \frac{\#\{0 \leq n \leq X : p(n) \equiv a \pmod{2}\}}{X} = \frac{1}{2},$$

where  $a \in \{0, 1\}$ . Little is known regarding this conjecture. In the following theorem we prove that  $\sigma_2 mex(n)$  and  $\sigma_3 mex(n)$  are almost always even. More generally we prove the following result.

**Theorem 1.3. (Main Theorem)** *Let  $k$  be a positive integer and  $r \in \{2, 3\}$ . Then*

$$\lim_{X \rightarrow +\infty} \frac{\#\{0 \leq n < X : \sigma_r mex(n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$

In other words for almost every non-negative integer  $n$  lying in an arithmetic progression, the integer  $\sigma_r mex(n)$  is a multiple of  $2^k$  where  $r \in \{2, 3\}$ .

## 2. Preliminaries

In this section, we recall some definitions and facts relating to the arithmetic of classical modular forms. For more details, one can consult [On04, Ko93]. Let  $\mathbb{H}$  denote the upper-half plane. The complex vector space of modular forms of weight  $\ell$  (a positive integer) with respect to a congruence subgroup  $\Gamma$  will be denoted by  $M_{\ell}(\Gamma)$ .

**Definition 2.1.** [On04, Definition 1.15] Let  $\chi$  be a Dirichlet character modulo  $N$  (a positive integer). Then a modular form  $f \in M_\ell(\Gamma_1(N))$  has Nebentypus character  $\chi$  if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all  $z \in \mathbb{H}$  and all  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$ . The space of such modular forms is denoted by  $M_\ell(\Gamma_0(N), \chi)$ . Here  $\Gamma_0(N)$  will be as usual the Hecke congruence subgroup of level  $N$ .

Recall that Dedekind's eta-function is defined by

$$\eta(z) := q^{1/24}(q; q)_\infty = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where  $q = e^{2\pi iz}$  and  $z \in \mathbb{H}$ . A function  $f(z)$  is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

where  $N$  is a positive integer and  $r_\delta$  is an integer.

We now recall two theorems from [On04, p. 18], which help us check the modularity of eta-quotients that show up in our study.

**Theorem 2.2.** [On04, Theorem 1.64] Suppose that  $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$  is an eta-quotient such that

$$\begin{aligned} \ell &= \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z}, \\ \sum_{\delta|N} \frac{N}{\delta} r_\delta &\equiv 0 \pmod{24}. \end{aligned}$$

Then

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$ . Here

$$\chi(d) := \left( \frac{(-1)^\ell \prod_{\delta|N} \delta^{r_\delta}}{d} \right).$$

Suppose that  $f$  is an eta-quotient satisfying the conditions of Theorem 2.2. If  $f$  is also holomorphic at all of the cusps of  $\Gamma_0(N)$ , then  $f \in M_\ell(\Gamma_0(N), \chi)$ . To check the holomorphicity at cusps of  $f(z)$  it suffices to check that the orders at the cusps are non-negative. The necessary criterion for determining orders of an eta-quotient at cusps is the following.

**Theorem 2.3.** [On04, Theorem 1.65] Let  $c, d$ , and  $N$  be positive integers with  $d | N$  and  $\gcd(c, d) = 1$ . If  $f(z)$  is an eta-quotient satisfying the conditions of Theorem 2.2 for  $N$ , then the order of vanishing of  $f(z)$  at the cusp  $\frac{c}{d}$  is

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d \delta}.$$

### 3. Proof of main results

In this section, we prove Theorem 1.3. We prove the following lemmas.

**Lemma 3.1.** *Let  $k$  be a positive integer and  $r \in \{2, 3\}$ . Then*

$$\frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}} \equiv \sum_{n=0}^{\infty} \sigma_r \text{mex}(n) q^{24n+3r-1} \pmod{2^k}.$$

*Proof.* From (1.1), the generating function of  $\sigma_r \text{mex}(n)$  is given by

$$\sum_{n=0}^{\infty} \sigma_r \text{mex}(n) q^n = \frac{(q^{2r}; q^{2r})_{\infty}^2}{(q; q)_{\infty} (q^r; q^r)_{\infty}}. \tag{3.2}$$

Consider

$$\mathcal{A}(z) = \prod_{n=1}^{\infty} \frac{(1 - q^{24rn})^2}{(1 - q^{48rn})} = \frac{\eta(24rz)^2}{\eta(48rz)}.$$

By the binomial theorem, for any positive integers  $r$  and  $k$  we have

$$(q^r; q^r)_{\infty}^{2^k} \equiv (q^{2r}; q^{2r})_{\infty}^{2^{k-1}} \pmod{2^k}.$$

Therefore,

$$\mathcal{A}^{2^{k-1}}(z) = \frac{\eta(24rz)^{2^k}}{\eta(48rz)^{2^{k-1}}} \equiv 1 \pmod{2^k}.$$

Define  $\mathcal{B}_{r,k}(z)$  by

$$\mathcal{B}_{r,k}(z) = \frac{\eta(48rz)^2}{\eta(24z)\eta(24rz)} \mathcal{A}^{2^{k-1}}(z).$$

Now, modulo  $2^k$ , we have

$$\begin{aligned} \mathcal{B}_{r,k}(z) &= \frac{\eta(48rz)^2}{\eta(24z)\eta(24rz)} \frac{\eta(24rz)^{2^k}}{\eta(48rz)^{2^{k-1}}} \\ &\equiv \frac{\eta(48rz)^2}{\eta(24z)\eta(24rz)} \\ &= q^{3r-1} \frac{(q^{48r}; q^{48r})_{\infty}^2}{(q^{24}; q^{24})_{\infty} (q^{24r}; q^{24r})_{\infty}}. \end{aligned} \tag{3.3}$$

Since

$$\mathcal{B}_{r,k}(z) = \frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}},$$

combining (3.2) and (3.3), we obtain the required result. □

**Lemma 3.2.** *Let  $k > 1$  be a positive integer and  $r \in \{2, 3\}$ . Then*

$$\mathcal{B}_{r,k}(z) = \frac{\eta(24rz)^{2^k-1}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}} \in M_{2^{k-2}}(\Gamma_0(L), \chi(\bullet)),$$

where

$$L = \begin{cases} 1152 & \text{if } r = 2, \\ 576 & \text{if } r = 3. \end{cases}$$

*Proof.* First, we use Theorem 2.2 and find the following:

1. The weight of the *eta*-quotient  $\mathcal{B}_{r,k}(z)$  is  $2^{k-2}$ .
2. Suppose the level of the *eta*-quotient  $\mathcal{B}_{r,k}(z)$  is  $48ru$ , where  $u$  is the smallest positive integer satisfying the following identity.

$$\frac{48ru}{24r}(2^k - 1) - \frac{48ru}{24} - \frac{48ru}{48r}(2^{k-1} - 2) \equiv 0 \pmod{24}$$

Equivalently, we have

$$u(3 \cdot 2^{k-1} - 2r) \equiv 0 \pmod{24}.$$

Since  $k > 1$ , we have  $u = 12$  if  $r = 2$  and  $u = 4$  if  $r = 3$ . Hence level of the *eta*-quotient  $\mathcal{B}_{r,k}(z)$  is

$$\begin{cases} 1152 & \text{if } r = 2, \\ 576 & \text{if } r = 3. \end{cases}$$

3. The Nebentypus character is

$$\chi(\bullet) = \left( \frac{(-1)^{2^{k-2}}(24r)^{2^{k-1}}(24)^{-1}(48r)^{-2^{k-1}+2}}{\bullet} \right).$$

By Theorem 2.3, the cusps of  $\Gamma_0(L)$  are given by  $\frac{c}{d}$  where  $d \mid L$  and  $\gcd(c, d) = 1$ . Now note that *eta*-quotient  $\mathcal{B}_{r,k}(z)$  is holomorphic at a cusp  $\frac{c}{d}$  if and only if

$$(2^k - 1) \frac{\gcd(d, 24r)^2}{24r} - \frac{\gcd(d, 24)^2}{24} - (2^{k-1} - 2) \frac{\gcd(d, 48r)^2}{48r} \geq 0.$$

Equivalently, if and only if

$$2(2^k - 1) \frac{\gcd(d, 24r)^2}{\gcd(d, 48r)^2} - 2r \frac{\gcd(d, 24)^2}{\gcd(d, 48r)^2} - (2^{k-1} - 2) \geq 0. \quad (3.4)$$

**Case (i).** When  $r = 2$  then the left side of (3.4) can be written as

$$2(2^k - 1) \frac{\gcd(d, 48)^2}{\gcd(d, 96)^2} - 4 \frac{\gcd(d, 24)^2}{\gcd(d, 96)^2} - (2^{k-1} - 2) \geq 0. \quad (3.5)$$

To check the positivity of (3.5), we have to find all the possible divisors of 1152. We define three sets as follows

$$\begin{aligned} \mathcal{H}_1 &= \{2^\alpha 3^\beta : 0 \leq \alpha \leq 3, 0 \leq \beta \leq 2\}, \\ \mathcal{H}_2 &= \{2^\alpha 3^\beta : \alpha = 4, 0 \leq \beta \leq 2\}, \\ \mathcal{H}_3 &= \{2^\alpha 3^\beta : 5 \leq \alpha \leq 7, 0 \leq \beta \leq 2\}. \end{aligned}$$

Note that  $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$  contains all positive divisors of 1152. In the following table we compute all necessary data to prove the positivity of (3.5).

Values of $d$ such that $d 1152$	$\frac{\gcd(d, 48)^2}{\gcd(d, 96)^2}$	$\frac{\gcd(d, 24)^2}{\gcd(d, 96)^2}$	Values of (3.5)
$d \in \mathcal{H}_1$	1	1	$2^{k-1}3 - 4$
$d \in \mathcal{H}_2$	1	1/4	$2^{k-1}3 - 1$
$d \in \mathcal{H}_3$	1/4	1/16	5/4

Since  $k > 1$ , it is clear from the above table that the quantities in (3.5) are always greater than equal to 0 for any positive integer  $d \mid 1152$ .

**Case (ii).** When  $r = 3$  then the left side of (3.4) can be written as

$$2(2^k - 1) \frac{\gcd(d, 72)^2}{\gcd(d, 144)^2} - 6 \frac{\gcd(d, 24)^2}{\gcd(d, 144)^2} - (2^{k-1} - 2) \geq 0. \tag{3.6}$$

To check the positivity of (3.6), we have to find all the possible divisors of 576. We define four sets as follows

$$\begin{aligned} \mathcal{G}_1 &= \{2^\alpha 3^\beta : 0 \leq \alpha \leq 3, 0 \leq \beta \leq 1\}, & \mathcal{G}_2 &= \{2^\alpha 3^\beta : 0 \leq \alpha \leq 3, \beta = 2\}, \\ \mathcal{G}_3 &= \{2^\alpha 3^\beta : 4 \leq \alpha \leq 6, 0 \leq \beta \leq 1\}, & \mathcal{G}_4 &= \{2^\alpha 3^\beta : 4 \leq \alpha \leq 6, \beta = 2\}. \end{aligned}$$

Note that  $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$  contains all positive divisors of 576. In the following table we compute all necessary data to prove the positivity of (3.6).

Values of $d$ such that $d 576$	$\frac{\gcd(d, 72)^2}{\gcd(d, 144)^2}$	$\frac{\gcd(d, 24)^2}{\gcd(d, 144)^2}$	Values of (3.6)
$d \in \mathcal{G}_1$	1	1	$2^{k-1}3 - 6$
$d \in \mathcal{G}_2$	1	1/9	$2^{k-1}3 - 2/3$
$d \in \mathcal{G}_3$	1/4	1/4	0
$d \in \mathcal{G}_4$	1/4	1/36	4/3

Since  $k > 1$ , it is clear from the above table that the quantities in (3.6) are always greater than equal to 0 for any positive integer  $d \mid 576$ .

Therefore, by **Case (i)** and **Case (ii)**, the eta-quotient  $\mathcal{B}_{r,k}(z)$ , where  $r \in \{2, 3\}$  and  $k > 1$ , are holomorphic at every cusp  $\frac{c}{d}$  and hence it is a modular form on  $\Gamma_0(L)$  with Nebentypus character  $\chi(\bullet)$ . This completes the proof of Lemma 3.2. □

We state the following result of Serre.

**Theorem 3.3.** [On04, Theorem 2.65] *Let  $k, m$  be positive integers. If  $f(z) \in M_k(\Gamma_0(N), \chi(\bullet))$  has the Fourier expansion  $f(z) = \sum_{n=0}^\infty c(n)q^n \in \mathbb{Z}[[q]]$ , then there is a constant  $\alpha > 0$  such that*

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod{m}\} = \mathcal{O}\left(\frac{X}{\log^\alpha X}\right).$$

*Proof of Theorem 1.3.* Suppose  $k > 1$  is a positive integer and  $r \in \{2, 3\}$ . From Lemma 3.2, we have

$$\mathcal{B}_{r,k}(z) = \frac{\eta(24rz)^{2^{k-1}}}{\eta(24z)\eta(48rz)^{2^{k-1}-2}} \in M_{2^{k-2}}(\Gamma_0(L), \chi(\bullet)).$$

Also the Fourier coefficients of the *eta*-quotient  $\mathcal{B}_{r,k}(z)$  are integers. So, by Theorem 3.3 and Lemma 3.1, we can find a constant  $\alpha > 0$  such that

$$\#\left\{n \leq X : \sigma_r \text{mex}(n) \not\equiv 0 \pmod{2^k}\right\} = \mathcal{O}\left(\frac{X}{\log^\alpha X}\right).$$

Hence

$$\lim_{X \rightarrow +\infty} \frac{\#\{n \leq X : \sigma_r \text{mex}(n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$

This completes the proof of Theorem 1.3.  $\square$

**Acknowledgement.** The second author has carried out this work at Harish-Chandra Research Institute, affiliated with Homi Bhabha National Institute (Department of Atomic Energy, India), as a Postdoctoral Fellow. We thank the anonymous referee for his/her thorough review and highly appreciate the comments and suggestions, which significantly contributed to improve this article.

## References

- [AnNe19] Andrews, G.E. and Newman, D., Partitions and the minimal excludant, *Ann. Comb.*, **23**, (2019), 249–254.
- [Ba20] Ballantine, C. and Merca, M., Bisected theta series, least  $r$ -gaps in partitions, and polygonal numbers, *Ramanujan J.*, **52**, (2020), 433–444.
- [Ba] Ballantine, C. and Merca, M., Combinatorial Proof of the Minimal Excludant Theorem, *Arxiv*, (arXiv:1908.06789).
- [FrPe15] Fraenkel, A.S. and Peled, U., Harnessing the unwieldy MEX function, *Games of no chance 4*, volume 63 of *Math. Sci. Res. Inst. Publ.*, 77–94. Cambridge Univ. Press, New York, 2015.
- [Gr39] Grundy, P. M., Mathematics and games, *Eureka*, **2**, (1939), 6–9.
- [Ko93] Koblitz, N., *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, second edition, 1993.
- [On04] Ono, K., *The web of modularity: arithmetic of the coefficients of modular forms and  $q$ -series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.
- [PS67] Parkin, T.R. and Shanks, D., On the distribution of parity in the partition function, *Math. Comp.*, **21**, (1967), 466–480.
- [RaBa20] Ray, C. and Barman, R., On Andrews’ integer partitions with even parts below odd parts, *J. Number Theory*, **215**, (2020), 321–338.
- [Sp35] R. Sprague, Über mathematische Kampfspiele, *Tohoku Mathematical Journal*, **41** (1935), 438–444.

### Kalyan Chakraborty

Department of Mathematics  
Kerala School of Mathematics  
Kozhikode – 673571  
Kerala, INDIA  
*e-mail*: director@ksom.res.in

### Chiranjit Ray

Department of Mathematics  
Harish-Chandra Research Institute  
Prayagraj – 211 019  
Uttar Pradesh, INDIA  
*e-mail*: chiranjitray.m@gmail.com