



HAL
open science

The Boussinesq system with non-smooth boundary conditions : existence, relaxation and topology optimization.

Alexandre Vieira, Pierre-Henri Cocquet

► To cite this version:

Alexandre Vieira, Pierre-Henri Cocquet. The Boussinesq system with non-smooth boundary conditions : existence, relaxation and topology optimization.. 2022. hal-03207923v2

HAL Id: hal-03207923

<https://hal.science/hal-03207923v2>

Preprint submitted on 10 Jan 2022 (v2), last revised 15 Sep 2022 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 **THE BOUSSINESQ SYSTEM WITH NON-SMOOTH BOUNDARY**
2 **CONDITIONS : EXISTENCE, RELAXATION AND TOPOLOGY**
3 **OPTIMIZATION. ***

4 ALEXANDRE VIEIRA[†] AND PIERRE-HENRI COCQUET^{*‡}

5 **Abstract.** In this paper, we tackle a topology optimization problem which consists in finding
6 the optimal shape of a solid located inside a fluid that minimizes a given cost function. The motion
7 of the fluid is modeled thanks to the Boussinesq system which involves the unsteady Navier-Stokes
8 equation coupled to a heat equation. In order to cover several models presented in the literature, we
9 choose a non-smooth formulation for the outlet boundary conditions. This paper aims at proving
10 existence of solutions to the resulting equations, along with the study of a relaxation scheme of the
11 non-smooth conditions. A second part covers the topology optimization problem itself for which
12 we proved the existence of optimal solutions and provides the definition of first order necessary
13 optimality conditions.

14 **Key words.** Non-smooth boundary conditions, topology optimization, relaxation scheme, di-
15 rectional do-nothing boundary conditions

16 **AMS subject classifications.** 49K20, 49Q10, 76D03, 76D55

17 **1. Introduction.**

18 *Directional do-nothing conditions.* For many engineering applications, simula-
19 tions of flows coupled with the temperature are useful for predicting the behaviour
20 of physical designs before their manufacture, reducing the cost of the development
21 of new products. The relevance of the model and the adequacy with the experiment
22 therefore become important [17, 42, 48]. In this paper, we choose to model the flow
23 with the Boussinesq system which involves the Navier-Stokes equations coupled with
24 an energy equation. In most mathematical papers analyzing this model [9, 29, 49],
25 homogeneous Dirichlet boundary conditions are considered on the whole boundary.
26 This simplifies the mathematical analysis of the incompressible Navier-Stokes equa-
27 tion since the non-linear term vanishes after integrating by part hence simplifying the
28 derivation of a priori estimates [8, 22, 28, 49].

29 However, several applications use different boundary conditions that model inlet,
30 no-slip and outlet conditions [1]. Unlike the inlet and the no-slip conditions, the
31 outlet conditions are more subject to modelling choices. A popular one consists in
32 using a do-nothing outlet condition (see e.g. [7, 26, 27, 35, 47, 50]) which naturally
33 comes from integration by parts when defining a weak formulation of the Navier-
34 Stokes equations. However, since this outlet condition does not deal with re-entering
35 flows, several papers use a non-smooth outlet boundary conditions for their numerical
36 simulations (see e.g. [5, 24]). A focus on non-smooth outflow conditions when the
37 temperature appears can be found in [13, 24, 43, 45].

38 In particular, directional do-nothing (DDN) boundary conditions are non-smooth
39 conditions that become popular. The idea is originally described in [14], and several

*We thank Franck Boyer for his precious advice.

Funding: All the authors are supported by the "Agence Nationale de la Recherche" (ANR),
Project O-TO-TT-FU number ANR-19-CE40-0011.

[†]Physique et Ingénierie Mathématique pour l'Énergie et l'Environnement (PIMENT), Univer-
sité de la Réunion, 2 rue Joseph Wetzell, 97490 Sainte-Clotilde, France. (Alexandre.Vieira@univ-reunion.fr)

[‡]Laboratoire des Sciences de l'Ingénieur Appliquées à la Mécanique et au Génie Electrique
(SIAME), E2S-UPPA, Université de Pau et des Pays de l'Adour, 64000 Pau, France (Pierre-Henri.Cocquet@univ-pau.fr)

40 other mathematical studies followed [5, 10, 12]. These conditions were considered
41 especially for turbulent flows. In this situation, the flow may alternatively exit and
42 re-enter the domain. These directional boundary conditions tries to capture this
43 phenomenon, while limiting the reflection. It is worth noting that other boundary
44 conditions can be used, namely the so-called local/global Bernouilli boundary condi-
45 tions [13, 24, 45]. The latter implies the do-nothing boundary condition is satisfied for
46 exiting fluid and that both the normal velocity gradient and the total pressure vanish
47 for re-entering fluid. Nevertheless, in this paper, we are going to use non-smooth DDN
48 boundary condition since they are easier to impose though a variational formulation.

49 Concerning the mathematical study of Boussinesq system with directional do-
50 nothing conditions, the literature is rather scarce. To the best of our knowledge, we
51 only found [6, 16], where the steady case is studied in depth, but the unsteady case only
52 presents limited results. Indeed, while [16, p. 16, Theorem 3.2] gives existence and
53 uniqueness of a weak solution with additional regularity to the steady-state Boussinesq
54 system involving non-smooth boundary conditions at the inlet, it requires the source
55 terms and the physical constants like for example the Reynolds number to be small
56 enough. We emphasize that these limitations comes from the proof which relies on a
57 fixed-point strategy. The first aim of this paper will then be to fill that gap by proving
58 existence and, in a two-dimensional setting, uniqueness of solutions for the unsteady
59 Boussinesq system with non-smooth DDN boundary condition at the outlet.

60 *Topology optimization.* On top of the previous considerations, this paper aims at
61 using these equations in a topology optimization (TO) framework. In fluid mechanics,
62 the term *topology optimization* refers to the problem of finding the shape of a solid
63 located inside a fluid that optimizes a given physical effect. There exist various
64 mathematical methods to deal with such problems that fall into the class of PDE-
65 constrained optimization, such as the topological asymptotic expansion [3, 15, 41] or
66 the shape optimization method [25, 39, 40]. In this paper, we choose to locate the
67 solid thanks to a penalization term added in the unsteady Navier-Stokes equations,
68 as exposed in [4]. However, the binary function introduced in [4] is usually replaced
69 by a smooth approximation, referred as *interpolation function* [45], in order to be
70 used in gradient-based optimization algorithms. We refer to the review papers [1,
71 23] for many references that deal with numerical resolution of TO problems applied
72 to several different physical settings. However, as noted in [1, Section 4.7], most
73 problems tackling topology optimization for flows only focus on steady flows, and
74 time-dependant approaches are still rare. Furthermore, to the best of our knowledge,
75 no paper is dedicated to the mathematical study of unsteady TO problems involving
76 DDN boundary conditions, even though they are already used in numerical studies
77 [13, 24, 43, 45]. Therefore, the second goal of this paper will be to prove existence
78 of optimal solution to a TO problem involving Boussinesq system with non-smooth
79 DDN boundary conditions at the outlet.

80 *First order optimality conditions.* As hinted above, a gradient based method is
81 often used in order to compute an optimal solution of a TO problem. However, the
82 introduction of the non-smooth DDN boundary conditions implies that the control-
83 to-state mapping is no longer differentiable. The literature presents several ways to
84 deal with such PDE-constrained optimization problems. Most focus on elliptic equa-
85 tions, using subdifferential calculus [18, 31, 20] or as the limit of relaxation schemes
86 [19, 36, 46]. We may also cite [38] for a semilinear parabolic case and [51] which
87 involves the Maxwell equations. We emphasize that using directly a subdifferential
88 approach presents several drawbacks: the subdifferential of composite functions may
89 be hardly computed, and the result may be hardly enlightening nor used [18]. We will

90 therefore use a differentiable relaxation approach, as studied in [46]. First, we will
 91 be able to use standard first order necessary optimality conditions since the relaxed
 92 control-to-state mapping will be smooth. A convergence analysis will let us design
 93 necessary optimality condition for the non-smooth problem. Secondly, we find this
 94 approach more enlightening, as it may be used as a numerical scheme for solving the
 95 TO problem.

96 **1.1. Problem settings.** Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded open set with
 97 Lipschitz boundary whose outward unitary normal is \mathbf{n} . We assume the fluid occupies a
 98 region $\Omega_f \subset \Omega$ and that a solid fills a region Ω_s such that $\Omega = \Omega_f \cup \Omega_s$. The penalized
 99 Boussinesq approximation (see e.g. [45] for the steady case) of the Navier-Stokes
 100 equations coupled to convective heat transfer reads:

$$\begin{aligned}
 & \nabla \cdot \mathbf{u} = 0, \\
 101 \quad (1.1) \quad & \partial_t \theta + \nabla \cdot (\mathbf{u}\theta) - \nabla \cdot (Ck(\alpha)\nabla\theta) = 0, \quad \text{a.e. in } \Omega \\
 & \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - A\Delta\mathbf{u} + \nabla p - B\theta e_y + h(\alpha)\mathbf{u} = f, \\
 & \mathbf{u}(0) = u_0(\alpha), \quad \theta(0) = \theta_0(\alpha),
 \end{aligned}$$

102 where \mathbf{u} denotes the velocity of the fluid, p the pressure and θ the temperature (all
 103 dimensionless), $u_0(\alpha), \theta_0(\alpha)$ are initial conditions. In (1.1), $A = \text{Re}^{-1}$ with Re being
 104 the Reynolds number, $B = \text{Ri}$ is the Richardson number and $C = (\text{Re Pr})^{-1}$ where Pr
 105 is the Prandtl number. In a topology optimization problem, it is classical to introduce
 106 a function $\alpha : x \in \Omega \mapsto \alpha(x) \in \mathbb{R}^+$ as optimization parameter (see e.g. [1, 23]). The
 107 function $h(\alpha)$ then penalizes the flow in order to mimic the presence of a solid:

- 108 • if $h \equiv 0$, then one retrieves the classical Boussinesq approximation.
- 109 • if, for some $s > s_0$ and large enough α_{\max} , $h : s \in [0, \alpha_{\max}] \mapsto h(s) \in [0, \alpha_{\max}]$
 110 is a smooth function such that $h(s) = 0$ for $s \leq s_0$ and $h(s) = \alpha_{\max}$ for $s \geq s_0$,
 111 one retrieves the formulations used in topology optimization [1, 9, 45]. In the
 112 sequel, we work in this setting since we wish to study a TO problem.

113 Since the classical Boussinesq problem is retrieved when $h(\alpha) = 0$, the fluid zones
 114 $\Omega_f \subset \Omega$ and the solid ones $\Omega_s \subset \Omega$ can be defined as $\Omega_s := \{x \in \Omega \mid \alpha(x) < s_0\}$, $\Omega_f :=$
 115 $\{x \in \Omega \mid \alpha(x) > s_0\}$, where $\alpha_{\max} > 0$ is large enough to ensure the velocity \mathbf{u} is small
 116 enough for the Ω_s above to be considered as a solid (see [4, Corollary 4.1]). The
 117 function $k(\alpha) : x \in \Omega \mapsto k(\alpha(x))$ is the dimensionless diffusivity defined as $k(\alpha)|_{\Omega_f} = 1$
 118 and $k(\alpha)|_{\Omega_s} = k_s/k_f$ with k_s and k_f are respectively the diffusivities of the solid and
 119 the fluid. We also assume that k is a smooth regularization of $(k_s/k_f)\mathbf{1}_{\Omega_s} + \mathbf{1}_{\Omega_f}$. In
 120 this framework, α is thus defined as a parameter function, which will let us control
 121 the distribution of the solid in Ω .

122 Let us now specify the boundary conditions. Assume $\partial\Omega = \Gamma$ is Lipschitz and
 123 is split into three parts: $\Gamma = \Gamma_w \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$. Here, Γ_w are the walls, Γ_{in} the
 124 inlet/entrance and Γ_{out} is the exit/outlet of the computational domain. Let β be
 125 a function defined on Γ_{out} and define: $\forall x \in \mathbb{R} : x^+ = \text{pos}(x) = \max(0, x), x^- =$
 126 $\text{neg}(x) = \max(0, -x), x = x^+ - x^-$. Inspired by [14], we supplement (1.1) with the
 127 following boundary conditions:

$$\begin{aligned}
 & \underline{\text{On } \Gamma_{\text{in}}} : \quad \mathbf{u} = \mathbf{u}_{\text{in}}, \quad \theta = 0, \\
 & \underline{\text{On } \Gamma_w} : \quad \mathbf{u} = 0, \quad Ck\partial_n\theta = \phi, \\
 128 \quad (1.2) \quad & \underline{\text{On } \Gamma_{\text{out}}} : \quad A\partial_n\mathbf{u} - \mathbf{n}p = A\partial_n\mathbf{u}^{\text{ref}} - \mathbf{n}p^{\text{ref}} - \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} - \mathbf{u}^{\text{ref}}), \\
 & \quad Ck\partial_n\theta + \beta(\mathbf{u} \cdot \mathbf{n})^-\theta = 0,
 \end{aligned}$$

129 with $\phi \in L^2(0, T; L^2(\Gamma_w))$, $f \in L^2(0, T; (H^1(\Omega))')$, $\mathbf{u}_{\text{in}} \in L^2(0, T; H_{00}^{1/2}(\Gamma_{\text{in}}))$, $\partial_n =$
130 $\mathbf{n} \cdot \nabla$ and $(\mathbf{u}^{\text{ref}}, p^{\text{ref}})$ denotes a reference solution. As stated in [30], this nonlinear
131 condition is physically meaningful: if the flow is outward, we impose the constraint
132 coming from the selected reference flow ; if it is inward, we need to control the increase
133 of energy, so, according to Bernoulli's principle, we add a term that is quadratic with
134 respect to velocity.

135 **Weak formulation.** To define a weak formulation of (1.1)-(1.2), we introduce
136 $V^u = \{\mathbf{u} \in H^1(\Omega)^d; \nabla \cdot \mathbf{u} = 0, \mathbf{u}|_{\Gamma_{\text{in}} \cup \Gamma_w} = 0\}$, and define H^u as the closure of V^u
137 in $(L^2(\Omega))^d$. Similarly, we define $V^\theta = \{\theta \in H^1(\Omega); \theta|_{\Gamma_{\text{in}}} = 0\}$, and $H^\theta = L^2(\Omega)$.
138 We identify H^u and H^θ with their dual, and denote by $(V^u)'$ (resp. $(V^\theta)'$) the dual
139 of V^u (resp. V^θ). Multiplying (1.1)-(1.2) with $\varphi \in V^\theta$ and integrating by parts, the
140 result reads as:

$$141 \quad \int_{\Omega} \partial_t \theta \varphi - \int_{\Omega} \theta \mathbf{u} \cdot \nabla \varphi + \int_{\Omega} Ck \nabla \theta \cdot \nabla \varphi + \int_{\Gamma} (\theta(\mathbf{u} \cdot \mathbf{n}) - Ck \partial_n \theta) \varphi = 0,$$

142 for all $\varphi \in V^\theta$. From (1.2), the boundary term reduces to:

$$\begin{aligned} \int_{\Gamma} (\theta(\mathbf{u} \cdot \mathbf{n}) - Ck \partial_n \theta) \varphi &= - \int_{\Gamma_w} \phi \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta(\mathbf{u} \cdot \mathbf{n})^-) \theta \varphi \\ &\quad - \int_{\Gamma_{\text{out}}} (\beta \theta (\mathbf{u} \cdot \mathbf{n})^- + Ck \partial_n \theta) \varphi \\ 143 \quad &= - \int_{\Gamma_w} \phi \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta(\mathbf{u} \cdot \mathbf{n})^-) \theta \varphi, \end{aligned}$$

144 and the weak form of the heat transfer equation is then

$$\begin{aligned} 145 \quad (\text{WF.1}) \quad &\int_{\Omega} \partial_t \theta \varphi - \int_{\Omega} \theta \mathbf{u} \cdot \nabla \varphi + \int_{\Omega} Ck \nabla \theta \cdot \nabla \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta(\mathbf{u} \cdot \mathbf{n})^-) \theta \varphi \\ &= \int_{\Gamma_w} \phi \varphi. \end{aligned}$$

For the Navier-Stokes equations, we are going to use the next formula to replace the
inertial term $(\mathbf{u} \cdot \nabla) \mathbf{u}$ by a symmetric one which helps to get a priori estimates (see
also [11, 14]). For all $\Psi \in V^u$, the latter is given as

$$\int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \Psi = \frac{1}{2} \int_{\Omega} ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \Psi - ((\mathbf{u} \cdot \nabla) \Psi) \cdot \mathbf{u} + \frac{1}{2} \int_{\partial \Omega} (\mathbf{u} \cdot \mathbf{n}) (\mathbf{u} \cdot \Psi).$$

148 Multiplying (1.1) by $\Psi \in V^u$, integrating by parts and using the boundary conditions,
146 the weak formulation of the Navier-Stokes system is then defined as
149

$$\begin{aligned} &\int_{\Omega} \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2} \{((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \Psi - ((\mathbf{u} \cdot \nabla) \Psi) \cdot \mathbf{u}\} + A \nabla \mathbf{u} : \nabla \Psi + h(\alpha) \mathbf{u} \cdot \Psi \\ 150 \quad (\text{WF.2}) \quad &- \int_{\Omega} B \theta \cdot \vec{e}_y \cdot \Psi + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{u} \cdot \mathbf{n})^+ (\mathbf{u} \cdot \Psi) \\ &= \int_{\Omega} f \cdot \Psi + \int_{\Gamma_{\text{out}}} (A \partial_n \mathbf{u}^{\text{ref}} - \mathbf{n} p^{\text{ref}}) \cdot \Psi + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{u} \cdot \mathbf{n})^- (\mathbf{u}^{\text{ref}} \cdot \Psi) \end{aligned}$$

151 for all $\Psi \in V^u$. A weak solution to (1.1)-(1.2) is then defined as $(\mathbf{u}, \theta) \in L^2(0, T; V^u) \times$
152 $L^2(0, T; V^\theta)$ satisfying the weak formulations (WF).

153 **1.2. The topology optimization problem.** A goal of this paper is to analyze
 154 the next topology optimization problem

$$155 \quad \begin{aligned} & \min \mathcal{J}(\alpha, \mathbf{u}, \theta) \\ \text{(OPT)} \quad & \text{s.t.} \quad \begin{cases} (\mathbf{u}, \theta) \text{ solution of (WF) parametrized by } \alpha, \\ \alpha \in \mathcal{U}_{ad}, \end{cases} \end{aligned}$$

156 where \mathcal{J} is a given cost function. For some $\kappa > 0$, we set $\mathcal{U}_{ad} = \{\alpha \in \text{BV}(\Omega)$
 157 $: 0 \leq \alpha(x) \leq \alpha_{\max} \text{ a.e. on } \Omega, |D\alpha|(\Omega) \leq \kappa\}$ where $\text{BV}(\Omega)$ stands for functions of
 158 bounded variations. As exposed in [2], the weak-* convergence in $\text{BV}(\Omega)$ is defined as
 159 follows: $(\alpha_\varepsilon)_\varepsilon \subset \text{BV}(\Omega)$ weakly-* converges to $\alpha \in \text{BV}(\Omega)$ if (α_ε) strongly converges
 160 to α in $L^1(\Omega)$ and $(D\alpha_\varepsilon)$ weakly-* converges to $D\alpha$ in Ω , meaning:

$$161 \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nu dD\alpha_\varepsilon = \int_{\Omega} \nu dD\alpha, \quad \forall \nu \in C_0(\Omega),$$

162 where $C_0(\Omega)$ denotes the closure, in the sup norm, of the set of real continuous
 163 functions with compact support over Ω . We choose \mathcal{U}_{ad} as a subset of $\text{BV}(\Omega)$ since
 164 it is a nice way to approximate piecewise constant functions, which is close to the
 165 desired solid distribution.

166 **REMARK 1.1.** *The set \mathcal{U}_{ad} have been used for instance in [21, 50] and have the*
 167 *property that any sequence $(\alpha_n)_n \subset \mathcal{U}_{ad}$ is bounded in $\text{BV}(\Omega)$ and thus have a subse-*
 168 *quence that converges strongly in $L^1(\Omega)$ toward some $\alpha \in \mathcal{U}_{ad}$. It then has a further*
 169 *subsequence that converges almost everywhere in Ω toward α and thus $h(\alpha_n)$ and $k(\alpha_n)$*
 170 *converge respectively toward $h(\alpha)$ and $k(\alpha)$. The last statement is going to be useful*
 171 *to prove some smoothness result on the control-to-state mapping $\alpha \mapsto (\mathbf{u}(\alpha), \theta(\alpha))$. In*
 172 *addition, we emphasize we may actually replace the above \mathcal{U}_{ad} by any Banach space*
 173 *\mathcal{B}_{ad} for which any $(\alpha_n)_n \subset \mathcal{B}_{ad}$ has a subsequence that converges toward some $\alpha \in \mathcal{B}_{ad}$*
 174 *strongly in $L^p(\Omega)$ for $p \geq 1$.*

175 It is classical for these problems to compute first order optimality conditions
 176 (see e.g. [34, 44]). This approach needs smoothness of the control-to-state mapping.
 177 However, the presence of the non-differentiable function $\text{neg}(x) = x^-$ makes this
 178 approach impossible. Therefore, we adopt a smoothing approach, as studied in [36,
 179 46], and we approximate the neg function with a C^1 positive approximation, denoted
 180 neg_ε . We suppose this approximation satisfies the following assumptions:

- 181 **(A1)** $\forall s \in \mathbb{R}, \text{neg}_\varepsilon(s) \geq \text{neg}(s)$.
- 182 **(A2)** $\forall s \in \mathbb{R}, -1 \leq \text{neg}'_\varepsilon(s) \leq 0$.
- 183 **(A3)** neg_ε converges to neg uniformly over \mathbb{R} .
- 184 **(A4)** for every $\delta > 0$, the sequence $(\text{neg}'_\varepsilon)_{\varepsilon > 0}$ converges uniformly to 0 on $[\delta, +\infty)$
 185 and uniformly to -1 on $(-\infty, -\delta]$ as $\varepsilon \rightarrow 0$.

186 As presented in [46], we may choose:

$$187 \quad (1.3) \quad \text{neg}_\varepsilon(s) = \begin{cases} s^- & \text{if } |s| \geq \frac{\varepsilon}{2}, \\ \frac{1}{2} \left(\frac{1}{2} - \frac{s}{\varepsilon} \right)^3 \left(\frac{3\varepsilon}{2} + s \right) & \text{if } |s| < \frac{\varepsilon}{2}. \end{cases}$$

We also introduce the notation

$$\text{pos}_\varepsilon(s) = s + \text{neg}_\varepsilon(s).$$

188 Remark that, owing to the mean value theorem, **(A2)**-**(A3)** implies that, for all
 189 $x \in \mathbb{R}$ and for ε small enough

$$190 \quad (1.4) \quad |\text{neg}_\varepsilon(x)| \leq |x| + O(\varepsilon).$$

191 We redefine (WF) with an approximation of s^- and s^+ , which gives:

$$192 \quad (\text{WFe.1}) \quad \int_{\Omega} \partial_t \theta_{\varepsilon} \varphi - \int_{\Omega} \theta_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u}_{\varepsilon} \cdot \mathbf{n}) + \beta \text{neg}_{\varepsilon}(\mathbf{u}_{\varepsilon} \cdot \mathbf{n})) \theta_{\varepsilon} \varphi \\ + \int_{\Omega} Ck \nabla \theta_{\varepsilon} \cdot \nabla \varphi = \int_{\Gamma_w} \phi \varphi.$$

193

$$194 \quad (\text{WFe.2}) \quad \int_{\Omega} \partial_t \mathbf{u}_{\varepsilon} \cdot \Psi + \frac{1}{2} \{ ((\mathbf{u}_{\varepsilon} \cdot \nabla) \mathbf{u}_{\varepsilon}) \cdot \Psi - ((\mathbf{u}_{\varepsilon} \cdot \nabla) \Psi) \cdot \mathbf{u}_{\varepsilon} \} + A \nabla \mathbf{u}_{\varepsilon} : \nabla \Psi \\ + \int_{\Omega} h(\alpha) \mathbf{u}_{\varepsilon} \cdot \Psi - B \theta_{\varepsilon} \cdot \vec{e}_y \cdot \Psi + \frac{1}{2} \int_{\Gamma_{\text{out}}} \text{pos}_{\varepsilon}(\mathbf{u}_{\varepsilon} \cdot \mathbf{n}) (\mathbf{u}_{\varepsilon} \cdot \Psi) \\ = \int_{\Omega} f \cdot \Psi + \int_{\Gamma_{\text{out}}} (A \partial_n \mathbf{u}^{\text{ref}} - \mathbf{n} p^{\text{ref}}) \cdot \Psi + \frac{1}{2} \int_{\Gamma_{\text{out}}} \text{neg}_{\varepsilon}(\mathbf{u}_{\varepsilon} \cdot \mathbf{n}) (\mathbf{u}^{\text{ref}} \cdot \Psi)$$

195 for all $(\Psi, \varphi) \in V^u \times V^{\theta}$.

196 We then define the approximate optimal control problem:

$$197 \quad (\text{OPTe}) \quad \min \mathcal{J}(\alpha_{\varepsilon}, \mathbf{u}_{\varepsilon}, \theta_{\varepsilon}) \\ \text{s.t.} \quad \begin{cases} (\mathbf{u}_{\varepsilon}, \theta_{\varepsilon}) \text{ solution of (WFe.1) - (WFe.2) parametrized by } \alpha_{\varepsilon}, \\ \alpha_{\varepsilon} \in \mathcal{U}_{ad}. \end{cases}$$

198 As it will be made clear later, the control-to-state mapping in (WFe.1)-(WFe.2) is
199 smooth, which will let us derive first order conditions.

200 **1.3. Plan of the paper.** The rest of this introduction is dedicated to the pre-
201 sentation of some notations used in this article and some important results of the
202 literature. The core of this paper is organized in two sections. First, we will prove the
203 existence of solutions to (WFe), which will let us prove, with a compactness argument,
204 the existence of solutions to (WF). We then focus on the two dimensional case, where
205 we prove uniqueness of the solutions along with stronger convergence results. This
206 is an extension of the work done by [14], where only the pressure and the velocity
207 were considered, and to [6, 16], where the steady case was studied in depth, but the
208 results concerning the unsteady case were obtained using restrictive assumptions. We
209 then study the approximate optimal control problem (OPTe), for which we will derive
210 first order conditions. We conclude this paper with the convergence of the optimality
211 conditions of (OPTe), which let us design first order conditions of (OPT).

212 *Notations.* We set $a \lesssim b$ if there exists a constant $C(\Omega) > 0$ depending only on
213 Ω such that $a \leq C(\Omega)b$. Denote:

- 214 • $\mathcal{A} : V^u \rightarrow (V^u)'$ defined by $\langle \mathcal{A} \mathbf{u}, \mathbf{v} \rangle_{(V^u)', V^u} = A \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v}$,
- 215 • $\mathcal{B} : V^u \times V^u \rightarrow (V^u)'$ defined by $\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} -$
216 $(\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v}$,
- 217 • $\mathcal{T} : V^{\theta} \rightarrow (V^u)'$ defined by $\langle \mathcal{T} \theta, \mathbf{v} \rangle_{(V^u)', V^u} = \int_{\Omega} B \theta e_y \cdot \mathbf{v}$,
- 218 • $\mathcal{P} : V^u \times V^u \rightarrow (V^u)'$ defined by $\langle \mathcal{P}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{pos}(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{w})$,
- 219 • $\mathcal{P}_{\varepsilon} : V^u \times V^u \rightarrow (V^u)'$ given by $\langle \mathcal{P}_{\varepsilon}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{pos}_{\varepsilon}(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot$
220 $\mathbf{w})$.
- 221 • $\mathcal{N} : V^u \times V^u \rightarrow (V^u)'$ defined by $\langle \mathcal{N}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{neg}(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{w})$,
- 222 • $\mathcal{N}_{\varepsilon} : V^u \times V^u \rightarrow (V^u)'$ given by $\langle \mathcal{N}_{\varepsilon}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{neg}_{\varepsilon}(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot$
223 $\mathbf{w})$.

- 224 • $\mathcal{C}(\alpha) : V^\theta \rightarrow (V^\theta)'$ defined by $\langle \mathcal{C}(\alpha)\theta, \varphi \rangle_{(V^\theta)', V^\theta} = \int_\Omega Ck(\alpha)\nabla\theta \cdot \nabla\varphi$,
- 225 • $\mathcal{D} : V^u \times V^\theta \rightarrow (V^\theta)'$ defined by $\langle \mathcal{D}(\mathbf{u}, \theta), \varphi \rangle_{(V^\theta)', V^\theta} = \int_\Omega \theta \mathbf{u} \cdot \nabla\varphi$,
- 226 • $\mathcal{M} : V^u \times V^\theta \rightarrow (V^\theta)'$ defined by $\langle \mathcal{M}(\mathbf{u}, \theta), \varphi \rangle_{(V^\theta)', V^\theta} = \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) +$
- 227 $\beta \text{neg}(\mathbf{u} \cdot \mathbf{n}))\theta\varphi$,
- 228 • $\mathcal{M}_\varepsilon : V^u \times V^\theta \rightarrow (V^\theta)'$ defined by $\langle \mathcal{M}(\mathbf{u}, \theta), \varphi \rangle_{(V^\theta)', V^\theta} = \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) +$
- 229 $\beta \text{neg}_\varepsilon(\mathbf{u} \cdot \mathbf{n}))\theta\varphi$,

230 We will also denote by σ^{ref} the element of $(V^u)'$ defined by $\langle \sigma^{\text{ref}}, \mathbf{w} \rangle_{(V^u)', V^u} =$
 231 $\int_{\Gamma_{\text{out}}} (A\partial_n \mathbf{u}^{\text{ref}} - p^{\text{ref}} \mathbf{n}) \cdot \mathbf{w}$, $h(\alpha) : V^u \rightarrow (V^u)'$ the function defined by $\langle h(\alpha)\mathbf{u}, \mathbf{v} \rangle_{(V^u)', V^u}$
 232 $= \int_\Omega h(\alpha)\mathbf{u} \cdot \mathbf{v}$, and ϕ the element of $(V^\theta)'$ defined by $\langle \phi, \varphi \rangle_{(V^\theta)', V^\theta} = \int_{\Gamma_{\text{out}}} \phi\varphi$.

233 *Results from the literature.* We now recall two results that will be heavily used
 234 throughout this paper.

235 **PROPOSITION 1.2.** (*[11, Proposition III.2.35]*) *Let Ω be a Lipschitz domain of \mathbb{R}^d*
 236 *with compact boundary. Let $p \in [1, +\infty]$ and $q \in [p, p^*]$, where p^* is the critical*
 237 *exponent associated with p , defined as:*

$$238 \quad \begin{cases} \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} & \text{for } p < d, \\ p^* \in [1, +\infty[& \text{for } p = d, \\ p^* = +\infty & \text{for } p > d. \end{cases}$$

239 *Then, there exists a positive constant C such that, for any $u \in W^{1,p}(\Omega)$:*

$$240 \quad \|u\|_{L^q(\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1 + \frac{d}{q} - \frac{d}{p}} \|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p} - \frac{d}{q}}.$$

241 **PROPOSITION 1.3.** (*[11, Theorem III.2.36]*) *Let Ω be a Lipschitz domain of \mathbb{R}^d*
 242 *with compact boundary, and $1 < p < d$. Then for any $r \in [p, \frac{p(d-1)}{d-p}]$, there exists a*
 243 *positive constant C such that, for any $u \in W^{1,p}(\Omega)$:*

$$244 \quad \|u|_{\partial\Omega}\|_{L^r(\partial\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1 - \frac{d}{p} + \frac{d-1}{r}} \|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p} - \frac{d-1}{r}}.$$

245 *In the case $p = d$, the previous result holds true for any $r \in [p, +\infty[$.*

246 **2. Existence of solutions.** In this section, we will focus on proving the existence of solutions to (WFe) and prove their convergence toward the ones of (WF).

248 We make the following assumptions throughout this paper:

249 **ASSUMPTIONS 2.1.** • *The source term $f \in L^2(0, T; (H^1(\Omega))')$.*

250 • *$(\mathbf{u}^{\text{ref}}, p^{\text{ref}})$ are such that:*

$$251 \quad \begin{cases} \mathbf{u}^{\text{ref}} \in L^r(0, T; (H^1(\Omega))^d) \cap L^\infty(0, T; (L^2(\Omega))^d) \\ \text{with } r = 2 \text{ if } d = 2 \text{ and } r = 4 \text{ if } d = 3, \\ \nabla \cdot \mathbf{u}^{\text{ref}} = 0, \\ \partial_t \mathbf{u}^{\text{ref}} \in L^2(0, T; (L^2(\Omega))^d), \\ \mathbf{u}^{\text{ref}} = 0 \text{ on } \Gamma_w \\ \mathbf{u}^{\text{ref}} = \mathbf{u}_{in} \text{ on } \Gamma_{in}. \end{cases}$$

252 *and $A\partial_n \mathbf{u}^{\text{ref}} - p^{\text{ref}} \mathbf{n} \in L^2(0, T; H^{-\frac{1}{2}}(\partial\Omega))$.*

253 • *There exists k_{\min} such that $k(x) \geq k_{\min} > 0$ and $h(x) \geq 0$ for a.e. $x \in \Omega$.*

- 254 • The initial condition \mathbf{u}_0 (resp. θ_0) is a Fréchet-differentiable function from
255 \mathcal{U}_{ad} to V^u (resp. V^θ). Furthermore, for all $\alpha \in \mathcal{U}_{ad}$, $\mathbf{u}_0(\alpha)|_{\Gamma_{in}} = \mathbf{u}_{in}(0)$,
256 $\mathbf{u}_0(\alpha)|_{\Gamma_w} = 0$, and $\theta_0(\alpha)|_{\Gamma_{in}} = 0$.
- 257 • $\beta \in L^\infty(0, T; L^\infty(\Gamma_{out}))$ such that $\beta(t, x) \geq \frac{1}{2}$, for a.e. $(t, x) \in [0, T] \times \Gamma_{out}$.

258 **2.1. Existence in dimension 2 or 3.** In this part, we work with a fixed $\varepsilon > 0$
259 and a given α_ε in \mathcal{U}_{ad} .

260 To prove the existence of solutions to (WFe), we follow the classical Fedeo-
261 Galerkin method as used in [14, 37, 49]. By construction, V^u and V^θ are separable.
262 Therefore, both admit a countable Hilbert basis $(w_k^u)_k$ and $(w_k^\theta)_k$. Let us construct
263 an approximate problem, which will converge to a solution of the original problem
264 (WFe). Denote by V_n^u (resp. V_n^θ) the space spanned by $(w_k^u)_{k \leq n}$ (resp. $(w_k^\theta)_{k \leq n}$).
265 We consider the following Galerkin approximated problem:

266 find $t \mapsto \mathbf{v}_n(t) \in V_n^u$ and $t \mapsto \theta_n(t) \in V_n^\theta$ such that, defining $\mathbf{u}_n = \mathbf{v}_n + \mathbf{u}^{ref}$, (\mathbf{u}_n, θ_n)
267 satisfy (WFe) for all $t \in [0, T]$ and for all $(\Psi, \varphi) \in V_n^u \times V_n^\theta$.

268 As done in [49], such (\mathbf{u}_n, θ_n) exist. We now prove that these solutions are
269 bounded uniformly with respect to n and ε :

270 **PROPOSITION 2.2.** *There exist positive constants $c_1^\theta, c_2^\theta, c_1^\Psi$ and c_2^Ψ , independent*
271 *of ε and n , such that:*

$$272 \quad (2.1) \quad \sup_{[0, T]} \|\theta_n\|_{L^2(\Omega)} \leq c_1^\theta, \quad 276 \quad (2.3) \quad \sup_{[0, T]} \|\mathbf{v}_n\|_{L^2(\Omega)} \leq c_1^\Psi,$$

$$273 \quad (2.2) \quad \int_0^T \|\nabla \theta_n\|_{L^2(\Omega)}^2 \leq c_2^\theta, \quad 278 \quad (2.4) \quad \int_0^T \|\nabla \mathbf{v}_n\|_{L^2(\Omega)}^2 \leq c_2^\Psi.$$

279 *Proof.* Taking $\varphi_n = \theta_n$ in (WFe.1) and integrating by part give:

$$280 \quad \frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 - \frac{1}{2} \int_{\Gamma_{out}} \theta_n^2 (\mathbf{u}_n \cdot \mathbf{n}) + \int_{\Omega} Ck |\nabla \theta_n|^2$$

$$281 \quad + \int_{\Gamma_{out}} ((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n})) \theta_n^2 = \int_{\Gamma_w} \phi \theta_n.$$

281 Since $\beta \geq \frac{1}{2}$ and using assumption (A1), one has on Γ_{out} :

$$282 \quad ((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n})) \theta_n^2 - \frac{1}{2} (\mathbf{u}_n \cdot \mathbf{n}) \theta_n^2 \geq \frac{1}{2} ((\mathbf{u}_n \cdot \mathbf{n}) + \text{neg}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n})) \theta_n^2$$

$$\geq \frac{1}{2} \text{pos}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n}) \theta_n^2 \geq 0.$$

283 Therefore: $\frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 + Ck_{\min} \|\nabla \theta_n\|_{L^2(\Omega)}^2 \leq \|\phi\|_{L^2(\Gamma_w)} \|\theta_n\|_{L^2(\Gamma_w)}$. Using the con-
284 tinuity of the trace operator and Young's inequality, one proves that there exists a
285 positive constant $C(\Omega)$ such that, for any $\nu > 0$:

$$286 \quad \frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 + Ck_{\min} \|\nabla \theta_n\|_{L^2(\Omega)}^2 \leq \frac{1}{2\nu} \|\phi\|_{L^2(\Gamma_w)}^2 + \frac{C(\Omega)\nu}{2} (\|\theta_n\|_{L^2(\Omega)}^2 + \|\nabla \theta_n\|_{L^2(\Omega)}^2).$$

287 Taking ν small enough, we are left with:

$$288 \quad \frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 \leq \frac{1}{2\nu} \|\phi\|_{L^2(\Gamma_w)}^2 + \frac{C(\Omega)\nu}{2} \|\theta_n\|_{L^2(\Omega)}^2.$$

289 Integrating this equation and using Gronwall's lemma then give (2.1) and (2.2).
 290 Now, take $\Psi_n = \mathbf{v}_n$ in (WFe.2). After some calculations, one gets:

$$\begin{aligned}
 & \frac{d}{dt} |\mathbf{v}_n|^2 + A |\nabla \mathbf{v}_n|^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}} \text{pos}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n}) |\mathbf{v}_n|^2 + \int_{\Omega} h |\mathbf{v}_n|^2 \\
 291 \quad & = \int_{\Omega} f_\theta \cdot \mathbf{v}_n - \int_{\Omega} \partial_t \mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n - A \int_{\Omega} \nabla \mathbf{u}^{\text{ref}} : \nabla \mathbf{v}_n - \int_{\Omega} h \mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n \\
 & \quad - \int_{\Omega} (\mathbf{u}_n \cdot \nabla) \mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n + \int_{\Gamma_{\text{out}}} (A \partial_n \mathbf{u}^{\text{ref}} - \mathbf{n} p^{\text{ref}}) \mathbf{v}_n
 \end{aligned}$$

292 where $f_\theta = f + B\theta_n e_y$. First, using (2.2), one has $\|f_\theta\|_{(H^u)'} \leq \|f\|_{(H^u)'} + Bc_1^\theta$. Sec-
 293 ondly, (A1) gives that $\int_{\Gamma_{\text{out}}} \text{neg}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n}) |\mathbf{v}_n|^2 \geq 0$. Following then the same pattern
 294 of proof as in [14, Proposition 2], one proves (2.3) and (2.4). \square

295 Following [11, 49], we need to bound the fractional derivatives of the solution in
 296 order to prove some convergence results. For any real-valued function f defined on
 297 $[0, T]$, define by \tilde{f} the extension by 0 of f to the whole real line \mathbb{R} , and by $\mathcal{F}(\tilde{f})$
 298 the Fourier transform of \tilde{f} , which we define as: $\mathcal{F}(\tilde{f})(\tau) = \int_{\mathbb{R}} \tilde{f}(t) e^{-it\tau} dt$. Using the
 299 Hausdorff-Young inequality [11, Theorem II.5.20] we can prove the

300 PROPOSITION 2.3. For all $\sigma \in [0, \frac{1}{6}]$, there exists a constant $C(\sigma) > 0$ indepen-
 301 dent of ε and n such that:

$$302 \quad (2.5) \quad \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F} \left(\widetilde{\theta}_n \right) \right\|_{(L^2(\Omega))^d}^2 \leq C(\sigma),$$

303

$$304 \quad (2.6) \quad \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F} \left(\widetilde{\mathbf{u}}_n \right) \right\|_{L^2(\Omega)}^2 \leq C(\sigma).$$

305 *Proof.* We emphasize that (2.6) is proved if (2.5) holds by using [11, Proposition
 306 VII.1.3] by replacing f by $f_\theta = f + B\theta e_y$. The proof of (2.5) consists in adapting the
 307 one of [11, Proposition VII.1.3] and is thus omitted. \square

308 Combining the two previous results, we now have the following existence theorem
 309 for (WFe).

310 THEOREM 2.4. For all $(\mathbf{v}_0, \theta_0) \in H^u \times H^\theta$ and all $T > 0$, there exists $\mathbf{v}_\varepsilon \in$
 311 $L^\infty(0, T; H^u) \cap L^2(0, T; V^u)$, $\theta_\varepsilon \in L^\infty(0, T; H^\theta) \cap L^2(0, T; V^\theta)$ solution of (WFe) such
 312 that, defining $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{u}^{\text{ref}}(0)$ and $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{u}^{\text{ref}}$, one has for all $(\Psi, \varphi) \in V^u \times V^\theta$:
 313 $(\int_{\Omega} \mathbf{u}_\varepsilon \cdot \Psi)(0) = \int_{\Omega} \mathbf{u}_0 \cdot \Psi$, $(\int_{\Omega} \theta_\varepsilon \varphi)(0) = \int_{\Omega} \theta_0 \varphi$. Moreover, one has $\mathbf{v}'_\varepsilon = \frac{d\mathbf{v}_\varepsilon}{dt} \in$
 314 $L^{\frac{4}{3}}(0, T; (V^u)')$ and $\theta'_\varepsilon \in L^{\frac{4}{3}}(0, T; (V^\theta)')$.

315 *Proof.* The proof of existence is similar to part (iv) of the proof of [49, Theorem
 316 3.1] and the proof of [11, Proposition VII.1.4], where estimates (2.1)-(2.4) and (2.5)-
 317 (2.6) are used in a compactness argument.

318 We only add the proof that (\mathbf{u}_n, θ_n) converges to a solution of (WFe.1). Using
 319 (2.1), (2.2), (2.5) and [49, Theorem 2.2], one shows that, up to a subsequence, θ_n
 320 strongly converges to an element θ_ε of $L^2(0, T; H^\theta)$, weakly converges in $L^2(0, T; V^\theta)$,
 321 and weak- \star converges in $L^\infty(0, T; L^2(\Omega))$. These results imply that θ_n strongly con-
 322 verges to θ_ε in $L^2(0, T; L^2(\Gamma))$ thanks to Proposition 1.3. The only technical points
 323 which need more details are the non-linear terms in (WFe.1). Using the strong con-
 324 vergence of \mathbf{u}_n to \mathbf{u}_ε in $L^2(0, T; H^u)$ proved in [49, Eq (3.41)], one proves that $(\theta_n \mathbf{u}_n)$

325 strongly converges to $\theta_\varepsilon \mathbf{u}_\varepsilon$ in $L^1(0, T; L^2(\Omega))$. Furthermore, notice that:

$$\begin{aligned}
\int_0^T \|(\mathbf{u}_n \cdot \mathbf{n})\theta_n\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} &\leq \int_0^T \|\mathbf{u}_n\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} \|\theta_n\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} \\
326 \qquad \qquad \qquad &\leq C \int_0^T \|\mathbf{u}_n\|_{L^2(\Omega)}^{\frac{1}{3}} \|\theta_n\|_{L^2(\Omega)}^{\frac{1}{3}} \|\mathbf{u}_n\|_{H^1(\Omega)} \|\theta_n\|_{H^1(\Omega)} \\
&\leq C \|\mathbf{u}_n\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{3}} \|\theta_n\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{3}} \\
&\qquad \qquad \qquad \|\mathbf{u}_n\|_{L^2(0, T; H^1(\Omega))} \|\theta_n\|_{L^2(0, T; H^1(\Omega))}.
\end{aligned}$$

327 This inequality together with (2.1)-(2.4) proves that $((\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$ is bounded in
328 $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma))$, which is reflexive. Therefore, it proves that, up to a subsequence,
329 there exists a weak limit κ_1 in $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma))$ of $((\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$. A simple adapta-
330 tion of the above reasoning proves that $(\text{neg}_\varepsilon(\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$ weakly converges to some
331 κ_2 in $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma))$. Using the strong convergence of θ_n in $L^2(0, T; L^2(\Gamma))$, [11,
332 Proposition II.2.12] implies that:

$$333 \quad ((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_n \cdot \mathbf{n}))\theta_n \rightharpoonup ((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}))\theta_\varepsilon \text{ in } L^{\frac{4}{3}}(0, T; L^1(\Gamma))$$

334 obtained using the uniform Lipschitz continuity with respect to ε of $s \in \mathbb{R} \mapsto \text{neg}_\varepsilon(s)$.
335 By uniqueness of the limit in the sense of distribution, we can identify $\kappa_1 + \beta\kappa_2$ with
336 $((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}))\theta_\varepsilon$. Therefore, $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ is a solution of (WF.1).

337 The convergence of the weak derivative with respect to time of \mathbf{v}_ε in $L^{\frac{4}{3}}(0, T;$
338 $(V^u)')$ is proved in [11, Proposition V.1.3]. Concerning the weak derivative with
339 respect to time of θ_ε , remark that, for all $\varphi \in V^\theta$ with $\varphi \neq 0$:

$$\begin{aligned}
340 \quad \frac{\langle \partial_t \theta_n, \varphi \rangle_{(V^\theta)', V^\theta}}{\|\varphi\|_{V^\theta}} &= \frac{1}{\|\varphi\|_{V^\theta}} (\langle \mathcal{D}(\mathbf{u}_n, \theta_n), \varphi \rangle_{(V^\theta)', V^\theta} - \langle \mathcal{C}(\alpha_\varepsilon)\theta_n, \varphi \rangle_{(V^\theta)', V^\theta} \\
&\quad - \langle \mathcal{M}_\varepsilon(\mathbf{u}_n, \theta_n), \varphi \rangle_{(V^\theta)', V^\theta} + \langle \phi, \varphi \rangle_{(V^\theta)', V^\theta}).
\end{aligned}$$

341 One easily derives the following inequalities:

$$\begin{aligned}
342 \quad \frac{\langle \mathcal{D}(\mathbf{u}_n, \theta_n), \varphi \rangle_{(V^\theta)', V^\theta}}{\|\varphi\|_{V^\theta}} &\lesssim (\|\theta_n\|_{L^2(\Omega)} \|\mathbf{u}_n\|_{L^2(\Omega)})^{\frac{1}{4}} (\|\theta_n\|_{H^1(\Omega)} \|\mathbf{u}_n\|_{H^1(\Omega)})^{\frac{3}{4}}, \\
343 \quad \frac{\langle \mathcal{C}(\alpha_\varepsilon)\theta_n, \varphi \rangle_{(V^\theta)', V^\theta}}{\|\varphi\|_{V^\theta}} &\lesssim \|\nabla \theta_n\|_{L^2(\Omega)} \lesssim \|\theta_n\|_{H^1(\Omega)}, \\
344 \quad \frac{\langle \mathcal{M}_\varepsilon(\mathbf{u}_n, \theta_n), \varphi \rangle_{(V^\theta)', V^\theta}}{\|\varphi\|_{V^\theta}} &\lesssim \|\mathbf{u}_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|\mathbf{u}_n\|_{H^1(\Omega)}^{\frac{1}{2}} \|\theta_n\|_{H^1(\Omega)}.
\end{aligned}$$

345 Since (\mathbf{u}_n) is bounded in $L^2(0, T; H^1(\Omega)^d) \cap L^\infty(0, T; L^2(\Omega)^d)$ (the same goes for (θ_n)),
346 these inequalities prove that $(\partial_t \theta_n)_n$ is bounded in $L^{\frac{4}{3}}(0, T; (V^\theta)')$. Therefore, $(\partial_t \theta_n)_n$
347 weakly converges in $L^{\frac{4}{3}}(0, T; (V^\theta)')$. By continuity of the weak derivative with respect
348 to time, this weak limit needs to be $\partial_t \theta_\varepsilon$. \square

349 We now use the existence of solutions to the approximate problem (WFe) to prove
350 existence of solutions to the limit problem (WF), along with the convergence of the
351 approximate solutions to those of (WF).

352 **THEOREM 2.5.** *Let $(\alpha_\varepsilon) \subset \mathcal{U}_{ad}$ and $\alpha \in \mathcal{U}_{ad}$ such that $\alpha_\varepsilon \xrightarrow{*} \alpha$ in $BV(\Omega)$. Define*
353 *by $(\mathbf{v}_\varepsilon, \theta_\varepsilon)$ a solution of (WFe) parametrized by α_ε , and define $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{u}^{ref}$. Then,*
354 *there exists $(\mathbf{v}, \theta) \in L^\infty(0, T; H^u) \cap L^2(0, T; V^u) \times L^\infty(0, T; H^\theta) \cap L^2(0, T; V^\theta)$ such*
355 *that, defining $\mathbf{u} = \mathbf{v} + \mathbf{u}^{ref}$, up to a subsequence, we have*

- 356 • $\mathbf{u}_\varepsilon \xrightarrow{*} \mathbf{u}$ in $L^\infty(0, T; H^u)$ and $\theta_\varepsilon \xrightarrow{*} \theta$ in $L^\infty(0, T; H^\theta)$,
- 357 • $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $L^2(0, T; V^u)$ and in $L^2(0, T; (L^6(\Omega))^d)$,
- 358 • $\theta_\varepsilon \rightharpoonup \theta$ in $L^2(0, T; V^\theta)$ and in $L^2(0, T; L^6(\Omega))$,
- 359 • $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $L^4(0, T; (L^2(\Gamma))^d)$ and $\theta_\varepsilon \rightharpoonup \theta$ in $L^4(0, T; L^2(\Gamma))$,
- 360 • $\mathbf{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{u}$ in $L^2(0, T; (L^2(\Omega))^d)$ and $\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \theta$ in $L^2(0, T; L^2(\Omega))$,
- 361 • $\mathbf{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{u}$ in $L^2(0, T; (L^2(\Gamma))^d)$ and $\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \theta$ in $L^2(0, T; L^2(\Gamma))$,
- 362 • $\partial_t \mathbf{u}_\varepsilon \rightharpoonup \partial_t \mathbf{u}$ in $L^{\frac{4}{3}}(0, T; (V^u)')$ and $\partial_t \theta_\varepsilon \rightharpoonup \partial_t \theta$ in $L^{\frac{4}{3}}(0, T; (V^\theta)')$.

363 Furthermore, (\mathbf{v}, θ) is a solution to (WF) parametrized by α .

364 *Proof.* Using (2.1)-(2.4) and (2.5)-(2.6), we prove that there exists \mathbf{u} and θ such
 365 that all the convergences above are verified in the same manner as in [11, Proposition
 366 VII.1.4].

367 Let us prove first that \mathbf{u} is a solution of (WF.2) parametrized by α and θ .

- 368 • With the same pattern of proof as in Theorem 2.4, one proves immediately
 369 that $(\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \rightharpoonup (\mathbf{u} \cdot \nabla) \mathbf{u}$ in $L^1(0, T; (L^1(\Omega))^d)$, and $(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \mathbf{u}_\varepsilon^{\text{ref}} \rightharpoonup (\mathbf{u} \cdot$
 370 $\mathbf{n}) \mathbf{u}^{\text{ref}}$ in $L^4(0, T; (L^{\frac{4}{3}}(\Gamma))^d)$.
- 371 • Regarding the penalization term:

$$372 \quad \begin{aligned} \|h(\alpha_\varepsilon) \mathbf{u}_\varepsilon - h(\alpha) \mathbf{u}\|_{L^2(0, T; L^2(\Omega)^d)}^2 &\lesssim \|h\|_\infty^2 \|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^2(0, T; L^2(\Omega)^d)}^2 \\ &+ \int_0^T \int_\Omega (h(\alpha_\varepsilon) - h(\alpha))^2 |\mathbf{u}|^2. \end{aligned}$$

373 Since $\alpha_\varepsilon \rightarrow \alpha$ strongly in $L^1(\Omega)$, $h(\alpha_\varepsilon) \rightarrow h(\alpha)$ pointwise in Ω up to a
 374 subsequence (not relabeled). Lebesgue dominated convergence theorem then
 375 implies: $h(\alpha_\varepsilon) \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} h(\alpha) \mathbf{u}$ in $L^2(0, T; (L^2(\Omega))^d)$.

- 376 • Concerning the boundary terms, we only consider the term with the approx-
 377 imation of the pos function. First, we claim that there exists γ such that
 378 $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \mathbf{u}_\varepsilon \rightharpoonup \gamma$ in $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma)^d)$. Notice that, for ε large enough and
 379 using (1.4), we have:

$$380 \quad \begin{aligned} (2.7) \quad \int_0^T \|\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \mathbf{u}_\varepsilon\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} &\lesssim \int_0^T \left(\|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} + C \right) \|\mathbf{u}_\varepsilon\|_{L^{\frac{3}{2}}(\Gamma)}^{\frac{4}{3}} \\ &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{8}{3}} + \left(\int_0^T \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{8}{3}} \right)^{\frac{1}{2}}. \end{aligned}$$

In addition, from Proposition 1.3, we have

$$\|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{8}{3}} \lesssim \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{3}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^2.$$

381 Since \mathbf{u}_ε is bounded in $L^\infty(0, T; (L^2(\Omega))^d)$ and $L^2(0, T; (H^1(\Omega))^d)$ as proved
 382 in Proposition 2.2, we see that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \mathbf{u}_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma)^d)$
 383 uniformly in ε . Since this Banach space is reflexive, it proves the claimed weak
 384 convergence.

- 385 • Let us now prove that γ can be identified with $(\mathbf{u} \cdot \mathbf{n})^+ \mathbf{u}$. First, since $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$
 386 strongly in $L^2(0, T; L^2(\Gamma)^d)$, $\text{pos}_\varepsilon(\cdot) \rightarrow (\cdot)^+$ uniformly and $|\text{neg}'_\varepsilon(\cdot)| \leq 1$, one
 387 proves that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) - \text{pos}_\varepsilon(\mathbf{u} \cdot \mathbf{n}) \rightarrow 0$ and $\text{pos}_\varepsilon(\mathbf{u} \cdot \mathbf{n}) \rightarrow (\mathbf{u} \cdot \mathbf{n})^+$ in
 388 $L^2(0, T; L^2(\Gamma))$. Therefore, $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \rightarrow (\mathbf{u} \cdot \mathbf{n})^+$ in $L^2(0, T; L^2(\Gamma))$. Then,
 389 the weak convergence of \mathbf{u}_ε in $L^4(0, T; L^2(\Gamma)^d)$ and [11, Proposition II.2.12]

390 implies that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \mathbf{u}_\varepsilon \rightharpoonup (\mathbf{u} \cdot \mathbf{n})^+ \mathbf{u}$ weakly in $L^{\frac{4}{3}}(0, T; L^1(\Gamma)^d)$. Using
 391 [11, Proposition II.2.9], we argue that $\gamma = (\mathbf{u} \cdot \mathbf{n})^+ \mathbf{u}$.

392 • Regarding $\partial_t \mathbf{u}_\varepsilon$, remark that:

$$393 \quad \begin{aligned} \|\partial_t \mathbf{u}_\varepsilon\|_{(V^u)'} &\lesssim \|\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)\|_{(V^u)'} + \|\mathcal{A}\mathbf{u}_\varepsilon\|_{(V^u)'} + \|h(\alpha)\mathbf{u}_\varepsilon\|_{(V^u)'} + \|\mathcal{T}\theta_\varepsilon\|_{(V^u)'} \\ &\quad + \|\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)\|_{(V^u)'} + \|\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}})\|_{(V^u)'} + \|f + \sigma^{\text{ref}}\|_{(V^u)'} \end{aligned}$$

394 We now bound each term depending on ε :

- 395 – Since the Stokes operator is continuous, $\|\mathcal{A}\mathbf{u}_\varepsilon\|_{(V^u)'} \lesssim \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ and
 396 therefore, $\mathcal{A}\mathbf{u}_\varepsilon$ is bounded in $L^2(0, T; (V^u)')$.
- 397 – Obviously, $\|h(\alpha)\mathbf{u}_\varepsilon\|_{(V^u)'} \leq \|h\|_\infty \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}$ and therefore, $h(\alpha)\mathbf{u}_\varepsilon$ is
 398 bounded in $L^\infty(0, T; (V^u)')$.
- 399 – $\|\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}})\|_{(V^u)'} \lesssim \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ and therefore, $\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}})$ is bounded
 400 in $L^2(0, T; (V^u)')$.

401 We are left with the boundary term \mathcal{P}_ε and the non linear term \mathcal{B} . Concerning
 402 \mathcal{B} , remark that :

$$403 \quad \forall \Psi \in V^u, \langle \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon), \Psi \rangle_{(V^u)', V^u} = - \int_\Omega (\mathbf{u}_\varepsilon \cdot \nabla) \Psi \cdot \mathbf{u}_\varepsilon + \frac{1}{2} \int_\Gamma (\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \Psi).$$

404 The first term can be treated as in [49, Lemma 3.1] while the second one on
 405 the boundary needs more details.

406 Let $0 \neq \Psi \in V^u$. Since the proof is similar in dimension 2, we will only focus
 407 on the dimension $d = 3$. Using Holder's inequality and Proposition 1.3, we
 408 obtain:

$$409 \quad \frac{\int_{\Gamma_{\text{out}}} |(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \Psi)|}{\|\Psi\|_{V^u}} \lesssim \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^{\frac{3}{2}}.$$

410 Therefore:

$$411 \quad \int_0^T \left(\sup_{\Psi \in V^u \setminus \{0\}} \frac{\int_{\Gamma_{\text{out}}} |(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \Psi)|}{\|\Psi\|_{V^u}} \right)^{\frac{4}{3}} \lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{2}{3}} \|\mathbf{u}_\varepsilon\|_{L^2(0, T; H^1(\Omega))}.$$

412 This proves that $(\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0, T; (V^u)')$. We prove analogously
 413 that $(\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0, T; (V^u)')$. These bounds prove
 414 that $(\partial_t \mathbf{u}_\varepsilon)$ is bounded in $L^{\frac{4}{3}}(0, T; (V^u)')$, and by continuity of the time deri-
 415 vative, we argue that $(\partial_t \mathbf{u}_\varepsilon)$ weakly converges to $\partial_t \mathbf{u}$ in $L^{\frac{4}{3}}(0, T; (V^u)')$.

416 Concerning θ , the convergence is largely proved in the same way as in Theorem 2.4.
 417 The only difference concerns the convergence of $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \theta_\varepsilon$ to $(\mathbf{u} \cdot \mathbf{n})^- \theta$, which
 418 is proved in the same manner as (2.7). All these convergence results let us say that
 419 (\mathbf{u}, θ) is a solution to (WF) in the distribution sense. \square

420 **2.2. Further results in dimension 2.** It is notably known that the solution
 421 of the Navier-Stokes equations with homogeneous Dirichlet boundary conditions are
 422 unique in dimension 2. We prove here that uniqueness still holds with the boundary
 423 conditions (1.2). Denote $\mathbb{X}^u = L^2(0, T; V^u) \cap L^4(0, T; H^u)$ and $\mathbb{X}^\theta = L^2(0, T; V^\theta) \cap$
 424 $L^4(0, T; H^\theta)$. These space are endowed with the norm:

$$425 \quad \|\mathbf{u}\|_{\mathbb{X}^u} = \max\{\|\mathbf{u}\|_{L^2(0, T; V^u)}, \|\mathbf{u}\|_{L^\infty(0, T; H^u)}\},$$

426 and the same definition follows for $\|\cdot\|_{\mathbb{X}^\theta}$.

427 LEMMA 2.6. Assume $d = 2$. Then the solution $(\mathbf{v}_\varepsilon, \theta_\varepsilon)$ of (WFe) is such that:

428
$$\partial_t \mathbf{v}_\varepsilon \in (\mathbb{X}^u)', \quad \partial_t \theta_\varepsilon \in (\mathbb{X}^\theta)'$$

429 *Proof.* The proof being similar, we will only focus on $\partial_t \mathbf{u}_\varepsilon$. First, remark that:

430
$$\partial_t \mathbf{u}_\varepsilon = -\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - \mathcal{A}\mathbf{u}_\varepsilon - h(\alpha)\mathbf{u}_\varepsilon + \mathcal{T}\theta_\varepsilon - \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}}) + f + \sigma^{\text{ref}}.$$

431 Due to the fact that $\mathbf{u}_\varepsilon \in \mathbb{X}^u$ and $\theta_\varepsilon \in \mathbb{X}^\theta$, it is straightforward to prove that $\mathcal{A}\mathbf{u}_\varepsilon$,
432 $h(\alpha)\mathbf{u}_\varepsilon$, $\mathcal{T}\theta_\varepsilon$, $\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}})$ and $f + \sigma^{\text{ref}}$ are in $(\mathbb{X}^u)'$. Concerning \mathcal{B} , we use once again
433 the identity:

434
$$\forall \Psi \in V^u, \langle \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon), \Psi \rangle_{(V^u)', V^u} = - \int_\Omega (\mathbf{u}_\varepsilon \cdot \nabla) \Psi \cdot \mathbf{u}_\varepsilon + \frac{1}{2} \int_\Gamma (\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \Psi),$$

435 and only focus on the boundary part.

436 Let $\Psi \in \mathbb{X}^u$. Notice that, using Proposition 1.3:

437
$$\begin{aligned} \int_0^T \int_\Gamma (\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \Psi) &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|_{L^2(\Gamma)} \|\mathbf{u}_\varepsilon\|_{L^4(\Gamma)} \|\Psi\|_{L^4(\Gamma)} \\ &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{3}{4}} \|\Psi\|_{L^2(\Omega)}^{\frac{1}{2}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^{\frac{5}{4}} \|\Psi\|_{H^1(\Omega)}^{\frac{1}{2}} \\ &\lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{3}{4}} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}^{\frac{5}{4}} \|\Psi\|_{L^4(0,T;L^2(\Omega))}^{\frac{1}{2}} \|\Psi\|_{L^2(0,T;H^1(\Omega))}^{\frac{1}{2}} \\ &\lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{3}{4}} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}^{\frac{5}{4}} \|\Psi\|_{\mathbb{X}^u} \end{aligned}$$

438 This proves that $\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)$ is in $(\mathbb{X}^u)'$. Similar computations for $\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)$ show that
439 $\partial_t \mathbf{u}_\varepsilon \in (\mathbb{X}^u)'$. \square

440 We may now prove uniqueness of the solution. We only sketch the proof.

441 PROPOSITION 2.7. Let $d = 2$. Then, the solution $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ of (WFe) is unique.

442 *Sketch of proof* Let $(\mathbf{u}_{\varepsilon_1}, \theta_{\varepsilon_1})$ and $(\mathbf{u}_{\varepsilon_2}, \theta_{\varepsilon_2})$ be two solutions of (WF.1)-(WF.2).
443 Define $\mathbf{u} = \mathbf{v} = \mathbf{u}_{\varepsilon_1} - \mathbf{u}_{\varepsilon_2}$ and $\theta = \theta_{\varepsilon_1} - \theta_{\varepsilon_2}$. Slightly adapting the proof in [11,
444 Section VII.1.2.5], one proves that:

445 (2.8)
$$\frac{d}{dt} |\mathbf{v}|_{L^2(\Omega)}^2 + A |\nabla \mathbf{v}|_{L^2(\Omega)}^2 \lesssim g^v(t) |\mathbf{v}|_{L^2(\Omega)}^2 + B |\theta|_{L^2(\Omega)}^2 + \nu^v |\nabla \mathbf{v}|_{L^2(\Omega)}^2$$

446 where ν^v is a positive constant and g^v is a function in $L^1([0, T])$.

447 Testing the differential equation verified by θ with θ and using Lemma A.2, it
448 proves that:

449
$$\begin{aligned} \frac{d}{dt} |\theta|_{L^2(\Omega)}^2 + 2C \int_\Omega k |\nabla \theta|^2 + \int_{\Gamma_{\text{out}}} \theta^2 \left(\frac{1}{2} (\mathbf{u}_{\varepsilon_1} \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon (\mathbf{u}_{\varepsilon_1} \cdot \mathbf{n}) \right) \\ = - \int_{\Gamma_{\text{out}}} \left(\beta (\text{neg}_\varepsilon (\mathbf{u}_{\varepsilon_1} \cdot \mathbf{n}) - \text{neg}_\varepsilon (\mathbf{u}_{\varepsilon_2} \cdot \mathbf{n})) + \frac{1}{2} (\mathbf{u} \cdot \mathbf{n}) \right) \theta_{\varepsilon_2} \theta. \end{aligned}$$

450 With a similar proof as the one of Proposition 2.2, we can prove that, on Γ_{out} ,
451 $\theta^2 \left(\frac{1}{2} (\mathbf{u}_1 \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon (\mathbf{u}_1 \cdot \mathbf{n}) \right) \geq 0$. Therefore, using (A3), one has:

452 (2.9)
$$\frac{d}{dt} |\theta|_{L^2(\Omega)}^2 + 2C \int_\Omega k |\nabla \theta|^2 \lesssim \left(|\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2} \right) |\mathbf{u} \cdot \mathbf{n}|_{L^3(\Gamma_{\text{out}})} |\theta_{\varepsilon_2}|_{L^3(\Gamma_{\text{out}})} |\theta|_{L^3(\Gamma_{\text{out}})}.$$

453 Using Sobolev embeddings and Young inequality, we prove:

$$\begin{aligned}
& \left(|\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2} \right) |\mathbf{u} \cdot \mathbf{n}|_{L^3(\Gamma_{\text{out}})} |\theta_{\varepsilon 2}|_{L^3(\Gamma_{\text{out}})} |\theta|_{L^3(\Gamma_{\text{out}})} \\
454 & \lesssim \left(|\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2} \right)^3 \frac{|\theta_{\varepsilon 2}|_{L^2(\Omega)} |\nabla \theta_{\varepsilon 2}|_{L^2(\Omega)}^2}{2(\nu^\theta)^3} (|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2) \\
& + \frac{(\nu^\theta)^{\frac{3}{2}}}{2} \left(|\nabla \mathbf{u}|_{L^2(\Omega)}^2 + |\nabla \theta|_{L^2(\Omega)}^2 \right),
\end{aligned}$$

455 where ν^θ is a positive constant. Therefore, summing (2.8) and (2.9) gives $\frac{d}{dt} (|\mathbf{u}|_{L^2(\Omega)}^2 +$
456 $|\theta|_{L^2(\Omega)}^2) \lesssim \max(g_1^\nu, g^\theta) (|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2)$, with g_1^ν and g^θ integrable. Therefore,
457 applying Gronwall's lemma and noticing that $|\mathbf{u}(0)|_{L^2(\Omega)}^2 + |\theta(0)|_{L^2(\Omega)}^2 = 0$, one shows
458 that $\mathbf{u} = 0$ and $\theta = 0$. \square

459 Note that we may also prove that, for $d = 2$, the solution (\mathbf{u}, θ) of (WF) is unique,
460 and that $\partial_t \mathbf{u} \in (\mathbb{X}^u)'$, $\partial_t \theta \in (\mathbb{X}^\theta)'$. We can also state stronger convergence (compared
461 to the ones stated in Theorem 2.5) in dimension 2. These results will be useful in the
462 analysis of the optimisation problems.

463 Denote $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\varepsilon$ and $\bar{\theta} = \theta - \theta_\varepsilon$. The variational formulation verified by $(\bar{\mathbf{u}}, \bar{\theta})$
464 reads as: for all $(\Psi, \varphi) \in V^u \times V^\theta$:

$$\begin{aligned}
(2.10a) \quad & 0 = \langle \partial_t \bar{\mathbf{u}} + \mathcal{A} \bar{\mathbf{u}} + h(\alpha) \bar{\mathbf{u}}, \Psi \rangle_{(V^u)'} + \langle (h(\alpha) - h(\alpha_\varepsilon)) \mathbf{u}_\varepsilon, \Psi \rangle_{(V^u)'} + \\
465 & \frac{1}{2} \langle \mathcal{P}(\mathbf{u}, \bar{\mathbf{u}}) + \mathcal{P}(\mathbf{u}, \mathbf{u}_\varepsilon) - \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon), \Psi \rangle_{(V^u)'} + \langle \mathcal{T} \bar{\theta}, \Psi \rangle_{(V^u)'} \\
& - \frac{1}{2} \langle \mathcal{N}(\mathbf{u}, \mathbf{u}^{\text{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}}), \Psi \rangle_{(V^u)'} + \langle \mathcal{B}(\mathbf{u}, \bar{\mathbf{u}}) + \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}_\varepsilon), \Psi \rangle_{(V^u)'} ,
\end{aligned}$$

466

$$\begin{aligned}
(2.10b) \quad & 0 = \langle \partial_t \bar{\theta}, \varphi \rangle_{(V^\theta)'} - \langle \mathcal{D}(\mathbf{u}, \bar{\theta}) + \mathcal{D}(\bar{\mathbf{u}}, \theta_\varepsilon), \varphi \rangle_{(V^\theta)'} \\
467 & + \langle (\mathcal{C}(\alpha) - \mathcal{C}(\alpha_\varepsilon)) \theta + \mathcal{C}(\alpha_\varepsilon) \bar{\theta}, \varphi \rangle_{(V^\theta)'} \\
& + \langle \mathcal{M}(\mathbf{u}, \theta) + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon), \varphi \rangle_{(V^\theta)'} .
\end{aligned}$$

468 We now bound some of the terms above in the following lemma. The proof is
469 omitted since it mainly relies on Proposition 1.2, Theorem 1.3 and Holder's inequality.

470 LEMMA 2.8. *Suppose $d = 2$. Denote $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\varepsilon$ and $\bar{\theta} = \theta - \theta_\varepsilon$. Let $C_\varepsilon =$
471 $\sup_{s \in \mathbb{R}} |\text{neg}_\varepsilon(s) - s^-|$. Owing to (A1), one has $C_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$. The following inequalities
472 are then valid:*

$$\begin{aligned}
1. \quad & (2.11) \\
& \langle \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}_\varepsilon), \bar{\mathbf{u}} \rangle_{(V^u)'} \lesssim \|\bar{\mathbf{u}}\|_{L^2(\Omega)} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)} \\
473 & + \left(\|\bar{\mathbf{u}}\|_{L^2(\Omega)} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^3 \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)} \right)^{\frac{1}{2}} .
\end{aligned}$$

2.

$$(2.12) \quad \text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \leq |\bar{\mathbf{u}} \cdot \mathbf{n}| + C_\varepsilon$$

475

$$\begin{aligned}
(2.13a) \quad & \int_{\Gamma_{out}} (\text{pos}(\mathbf{u} \cdot \mathbf{n}) - \text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})) \mathbf{u}_\varepsilon \cdot \bar{\mathbf{u}} \\
& \lesssim \left(\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} + C_\varepsilon \right) \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \\
& \quad \times \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}}.
\end{aligned}$$

477

$$\begin{aligned}
(2.13b) \quad & \int_{\Gamma_{out}} (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})) \mathbf{u}^{ref} \cdot \bar{\mathbf{u}} \\
& \lesssim \left(\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} + C_\varepsilon \right) \|\mathbf{u}^{ref}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\mathbf{u}^{ref}\|_{H^1(\Omega)}^{\frac{1}{2}} \\
& \quad \times \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}}.
\end{aligned}$$

478

3.

$$(2.14) \quad \int_{\Omega} \theta_\varepsilon \bar{\mathbf{u}} \cdot \nabla \bar{\theta} \lesssim \|\theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\theta_\varepsilon\|_{H^1(\Omega)}^{\frac{1}{2}} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \bar{\theta}\|_{L^2(\Omega)}.$$

479

4.

$$(2.15a) \quad \int_{\Gamma_{out}} (\bar{\mathbf{u}} \cdot \mathbf{n}) \theta_\varepsilon \bar{\theta} \lesssim \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} \|\theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\bar{\theta}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\theta}\|_{L^2(\Omega)}^{\frac{3}{4}}.$$

480

481

$$\begin{aligned}
(2.15b) \quad & \int_{\Gamma_{out}} (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})) \theta_\varepsilon \bar{\theta} \lesssim \left(\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} + C_\varepsilon \right) \|\theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \\
& \quad \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\bar{\theta}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\theta}\|_{L^2(\Omega)}^{\frac{3}{4}}.
\end{aligned}$$

482

483 **COROLLARY 2.9.** *Suppose $d = 2$. Under the assumptions of [Theorem 2.5](#), $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$*
484 *strongly in $L^\infty(0, T; L^2(\Omega)^2)$ and $\theta_\varepsilon \rightarrow \theta$ strongly in $L^\infty(0, T; L^2(\Omega))$.*

485 *Proof.* Since $d = 2$, one has $\partial_t \bar{\mathbf{u}} \in (\mathbb{X}^u)'$ and we may choose $\Psi = \bar{\mathbf{u}}(t)$ for fixed t
486 in [\(2.10a\)](#). After rearranging the terms, and using [Lemma A.2](#), we obtain:

$$\begin{aligned}
& \frac{d}{dt} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + 2A \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} h(\alpha) |\bar{\mathbf{u}}|^2 + \int_{\Gamma_{out}} \text{pos}(\mathbf{u} \cdot \mathbf{n}) |\bar{\mathbf{u}}|^2 = \\
& \quad - 2 \langle (h(\alpha) - h(\alpha_\varepsilon)) \mathbf{u}_\varepsilon, \bar{\mathbf{u}} \rangle_{(V^u)', V^u} - \int_{\Omega} B \bar{\theta} e_y \cdot \bar{\mathbf{u}} \\
& \quad - \langle \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}_\varepsilon), \bar{\mathbf{u}} \rangle_{(V^u)', V^u} + \int_{\Gamma_{out}} (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})) \mathbf{u}^{ref} \cdot \bar{\mathbf{u}} \\
& \quad - \int_{\Gamma_{out}} (\text{pos}(\mathbf{u} \cdot \mathbf{n}) - \text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})) \mathbf{u}_\varepsilon \cdot \bar{\mathbf{u}}.
\end{aligned}$$

487

488 Therefore, [\(2.11\)](#), [\(2.13\)](#), [Proposition 1.3](#) and Young's inequality imply there exists
489 $C_1 > 0$ independent of ε such that:

$$\begin{aligned}
(490) \quad & \frac{d}{dt} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + C_1 \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \lesssim \|\bar{\theta}\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2 \\
& \quad + g_1^u \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + (g_2^u)^{\frac{4}{3}} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{2}{3}},
\end{aligned}$$

491 where $g_1^u = \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^2 + \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^2 \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^2 + \|\mathbf{u}^{\text{ref}}\|_{L^2(\Omega)}^2 \|\mathbf{u}^{\text{ref}}\|_{H^1(\Omega)}^2$ and $g_2^u =$
 492 $C_\varepsilon^2 (\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)} + \|\mathbf{u}^{\text{ref}}\|_{L^2(\Omega)} \|\mathbf{u}^{\text{ref}}\|_{H^1(\Omega)})$. Using once again Young's inequality
 493 ity, one has:

$$494 \quad (2.16) \quad \frac{d}{dt} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + C_1 \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \lesssim \|\bar{\theta}\|_{L^2(\Omega)}^2 + (1 + g_1^u) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \\ + 2 \int_{\Omega} |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2 + g_2^u.$$

495 We now move back to (2.10b) and choose $\varphi = \bar{\theta}$, which gives, after some manip-
 496 ulation:

$$497 \quad \frac{1}{2} \frac{d}{dt} \|\bar{\theta}\|_{L^2(\Omega)}^2 + C \int_{\Omega} k(\alpha_\varepsilon) |\nabla \bar{\theta}|^2 + \int_{\Gamma_{\text{out}}} \left(\frac{1}{2} (\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}(\mathbf{u} \cdot \mathbf{n}) \right) \bar{\theta}^2 \\ = \int_{\Omega} \theta_\varepsilon \bar{\mathbf{u}} \cdot \nabla \bar{\theta} - C \int_{\Omega} (k(\alpha) - k(\alpha_\varepsilon)) \nabla \theta \cdot \nabla \bar{\theta} \\ - \int_{\Gamma_{\text{out}}} [((\bar{\mathbf{u}} \cdot \mathbf{n}) + \beta (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})))] \theta_\varepsilon \bar{\theta}.$$

498 As shown in Proposition 2.2, $\int_{\Gamma_{\text{out}}} (\frac{1}{2} (\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}(\mathbf{u} \cdot \mathbf{n})) \bar{\theta}^2$ is positive. Therefore,
 499 using (2.15), Proposition 1.3 and Young's inequality, one proves that there exist $C_3 >$
 500 $0, C_4 > 0$, such that:

$$501 \quad (2.17) \quad \frac{d}{dt} \|\bar{\theta}\|_{L^2(\Omega)}^2 + C_3 \|\nabla \bar{\theta}\|_{L^2(\Omega)}^2 \lesssim \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + C_4 \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \\ + \left(C \int_{\Omega} (k(\alpha) - k(\alpha_\varepsilon))^2 |\nabla \theta|^2 \right) + g_1^\theta \|\bar{\theta}\|_{L^2(\Omega)}^2 + g_2^\theta,$$

502 where $g_1^\theta = 1 + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\theta_\varepsilon\|_{H^1(\Omega)}^2$, $g_2^\theta = C_\varepsilon^2 \|\theta_\varepsilon\|_{L^2(\Omega)} \|\theta_\varepsilon\|_{H^1(\Omega)}$.

503 Summing (2.16) and (2.17) and choosing C_4 small enough, there exists $C^* > 0$
 504 such that:

$$505 \quad (2.18) \quad \frac{d}{dt} (\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\bar{\theta}\|_{L^2(\Omega)}^2) + C^* (\|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\nabla \bar{\theta}\|_{L^2(\Omega)}^2) \lesssim g_2^u + g_2^\theta \\ + (1 + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 + g_1^u) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + (g_1^\theta + 1) \|\bar{\theta}\|_{L^2(\Omega)}^2 \\ + \int_{\Omega} (k(\alpha) - k(\alpha_\varepsilon))^2 |\nabla \theta|^2 + \int_{\Omega} |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2.$$

506 We now introduce the following functions

$$507 \quad a_\varepsilon^u = (1 + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 + g_1^u), \quad b_\varepsilon^u = \int_{\Omega} |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2 + g_2^u, \\ 508 \quad a_\varepsilon^\theta = (1 + g_1^\theta), \quad b_\varepsilon^\theta = \int_{\Omega} (k(\alpha) - k(\alpha_\varepsilon))^2 |\nabla \theta|^2 + g_2^\theta.$$

509 Since \mathbf{u} and \mathbf{u}_ε both belong to $L^2(0, T; H^1(\Omega)^2) \cap L^\infty(0, T; L^2(\Omega)^2)$ (the same holds
 510 for θ and θ_ε), $a_\varepsilon^u, b_\varepsilon^u, a_\varepsilon^\theta, b_\varepsilon^\theta$ are integrable, and so are $a_\varepsilon = \max(a_\varepsilon^u, a_\varepsilon^\theta)$ and $b_\varepsilon =$
 511 $b_\varepsilon^u + b_\varepsilon^\theta$. Grönwall's lemma proves that for all $t \in [0, T]$, $\|\bar{\mathbf{u}}(t)\|_{L^2(\Omega)}^2 + \|\bar{\theta}(t)\|_{L^2(\Omega)}^2 \leq$
 512 $\left(\int_0^t b_\varepsilon(s) ds \right) \exp \left(\int_0^t a_\varepsilon(s) ds \right)$. Since $a_\varepsilon \geq 0$ and $b_\varepsilon \geq 0$, $t \mapsto \left(\int_0^t b_\varepsilon(s) ds \right)$ and $t \mapsto$

513 $\exp\left(\int_0^t a_\varepsilon(s)ds\right)$ are non-decreasing and we have

$$514 \quad (2.19) \quad \sup_{t \in [0, T]} (\|\bar{\mathbf{u}}(t)\|_{L^2(\Omega)} + \|\bar{\theta}(t)\|_{L^2(\Omega)}) \leq \left(\int_0^T b_\varepsilon(s)ds\right)^{\frac{1}{2}} \exp\left(\frac{1}{2} \int_0^T a_\varepsilon(s)ds\right).$$

515 Since, on one hand, $\alpha_\varepsilon \rightarrow \alpha$ in $L^1(\Omega)$ and α_ε is independent of time, and on the other
516 hand, $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$ strongly in $L^2(0, T; L^2(\Omega))$, Lebesgue's dominated convergence gives a
517 subsequence (ε_k) such that:

$$(2.20) \\ 518 \quad \int_0^T \int_\Omega |h(\alpha) - h(\alpha_{\varepsilon_k})|^2 |\mathbf{u}_{\varepsilon_k}|^2 \xrightarrow{k \rightarrow +\infty} 0, \quad \int_0^T \int_\Omega |k(\alpha) - k(\alpha_{\varepsilon_k})|^2 |\nabla \theta|^2 \xrightarrow{k \rightarrow +\infty} 0.$$

519 Notice that, owing to the convergence of \mathbf{u}_ε and θ_ε , $\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ and
520 $\|\theta_\varepsilon\|_{L^2(\Omega)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)}$ are bounded w.r.t ε in $L^1([0, T])$. Therefore, since $C_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$, it

521 proves that $\int_0^T (g_2^u + g_2^\theta) \xrightarrow{\varepsilon_k \rightarrow +\infty} 0$. Gathering the previous convergence results then

522 ensure that $\int_0^T b_{\varepsilon_k}(s)ds \xrightarrow{k \rightarrow +\infty} 0$. In addition, thanks to [Theorem 2.5](#), we show that

523 $\int_0^T a_\varepsilon(s)ds$ is bounded w.r.t. ε . Therefore, it proves that $\|\mathbf{u} - \mathbf{u}_{\varepsilon_k}\|_{L^\infty(0, T; L^2(\Omega))} +$
524 $\|\theta - \theta_{\varepsilon_k}\|_{L^\infty(0, T; L^2(\Omega))} \xrightarrow{k \rightarrow +\infty} 0$. \square

525 **COROLLARY 2.10.** *Suppose $d = 2$. Under the assumptions of [Theorem 2.5](#), $\nabla \mathbf{u}_\varepsilon$*
526 *$\rightarrow \nabla \mathbf{u}$ strongly in $L^2(0, T; L^2(\Omega)^2)$ and $\nabla \theta_\varepsilon \rightarrow \nabla \theta$ strongly in $L^2(0, T; L^2(\Omega))$.*

527 *Proof.* Move back to (2.18). We integrate each side of the inequality:

$$528 \quad \int_0^T \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\nabla \bar{\theta}\|_{L^2(\Omega)}^2 \lesssim F_\varepsilon^{u, \theta} + \int_0^T (g_1^u + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 + 1) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \\ + \int_0^T (g_1^\theta + 1) \|\bar{\theta}\|_{L^2(\Omega)}^2,$$

529 with

$$530 \quad F_\varepsilon^{u, \theta} = \|\mathbf{u}_0(\alpha_\varepsilon) - \mathbf{u}_0(\alpha)\|_{L^2(\Omega)}^2 + \|\theta_0(\alpha_\varepsilon) - \theta_0(\alpha)\|_{L^2(\Omega)}^2 + \int_0^T (g_2^u + g_2^\theta) \\ 531 \quad + \int_0^T \int_\Omega |k(\alpha) - k(\alpha_\varepsilon)|^2 |\nabla \theta|^2 + \int_0^T \int_\Omega |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2.$$

532 • From [Assumptions 2.1](#), the initial conditions are continuous with respect to
533 α and thus the two first terms in $F_\varepsilon^{u, \theta}$ goes to 0 as $\varepsilon \rightarrow 0$.

534 • The third, fourth and fifth terms in $F_\varepsilon^{u, \theta}$ have been already treated (see (2.20)).

535 • We now prove convergence for the term $g_1^u \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$. The main problem concerns

536 the term $\int_0^T (1 + \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^2) \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$. First, remark that $(\mathbf{u}_\varepsilon)_\varepsilon$

537 is bounded in $L^\infty(0, T; L^2(\Omega)^2)$ Secondly, as proved in [Theorem 2.5](#), up to

538 a subsequence, \mathbf{u}_ε weakly converges to \mathbf{u} in $L^2(0, T; H^1(\Omega))$ and $\bar{\mathbf{u}} \rightarrow 0$ in
539 $L^\infty(0, T; L^2(\Omega))$. Therefore, the whole term converges to 0.

540 • Concerning the other terms in g_1^u , they are all independent of ε , and we
541 mainly use the fact that $\|\bar{\mathbf{u}}\|_{L^2(\Omega)} \rightarrow 0$ in $L^\infty([0, T])$.

542 • We may do the same proof concerning $\int_0^T \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$ and

543 $\int_0^T \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\theta_\varepsilon\|_{H^1(\Omega)}^2 \|\bar{\theta}\|_{L^2(\Omega)}^2$.

544 Therefore, $\int_0^T (1 + \|\theta_{\varepsilon_k}\|_{L^2(\Omega)} \|\nabla \theta_{\varepsilon_k}\|_{L^2(\Omega)}^2 + g_1) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \xrightarrow{\varepsilon_k \rightarrow 0} 0$ and
 545 $\int_0^T (g_1^\theta + 1) \|\bar{\theta}\|_{L^2(\Omega)}^2 \xrightarrow{\varepsilon_k \rightarrow 0} 0$. It eventually proves that $\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon_k})\|_{L^2(0,T;L^2(\Omega))} +$
 546 $\|\nabla(\theta - \theta_{\varepsilon_k})\|_{L^2(0,T;L^2(\Omega))} \xrightarrow{k \rightarrow +\infty} 0$.

547 Owing to Urysohn's subsequence principle and the uniqueness of the solution to
 548 (WF), we actually obtain that the whole sequence $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ strongly converges toward
 549 (\mathbf{u}, θ) . \square

550 **REMARK 2.11.** *If $\alpha_\varepsilon = \alpha$, then the next estimate holds*

$$551 \quad \sup_{t \in [0, T]} \left(\|\mathbf{u}(t) - \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)} + \|\theta(t) - \theta_\varepsilon(t)\|_{L^2(\Omega)} \right)$$

$$552 \quad + \left(\int_0^T \|\nabla \bar{\mathbf{u}}(t) - \nabla \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)}^2 + \|\nabla \theta(t) - \nabla \theta_\varepsilon(t)\|_{L^2(\Omega)}^2 \right)^{1/2} = O(C_\varepsilon).$$

553 *The convergence of $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ toward (\mathbf{u}, θ) as $\varepsilon \rightarrow 0$ thus has the same rate as the one*
 554 *of neg_ε toward neg .*

555 **3. First order necessary conditions for the non-smooth optimization**
 556 **problem.** We now begin the analysis of the optimization problems (OPT) and
 557 (OPTe). Let us detail first some assumptions made on the objective functional:

558 **ASSUMPTIONS 3.1.** \bullet *For $d = 2$, \mathcal{J} is lower semi-continuous with respect to*
 559 *the (weak-*, strong, strong, strong) topology of $\mathcal{U}_{ad} \times L^2(0, T; V^u) \times L^2(0, T;$*
 560 *V^θ).*
 561 \bullet *In dimension 3, \mathcal{J} is either lower semi-continuous with respect to the (weak-*
 562 **, strong, strong) topology of $\mathcal{U}_{ad} \times L^2(0, T; H^u) \times L^2(0, T; H^\theta)$, or lower*
 563 *semi-continuous with respect to the (weak-*, weak, weak) topology of $\mathcal{U}_{ad} \times$*
 564 *$L^2(0, T; V^u) \times L^2(0, T; V^\theta)$.*

565 Remark that these assumptions exclude terminal costs, but these could be easily
 566 added by considering functionals continuous in time with respect to the topology of
 567 $L^\infty([0, T])$.

568 The existence of solutions to (OPTe) and (OPT) is rather classical and we refer
 569 for instance to [21, 32, 34]. We state a first result that let us see that a solution of
 570 (OPT) can be approximated by (OPTe).

571 **THEOREM 3.2.** *Assume Assumptions 3.1 is verified. Let $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ be a globally*
 572 *optimal solution of (OPTe). Then $(\alpha_\varepsilon^*) \subset \mathcal{U}_{ad}$ is a bounded sequence. Furthermore,*
 573 *there exists $(\alpha^*, \mathbf{u}^*, \theta^*) \in \mathcal{U}_{ad} \times L^2(0, T; V^u) \times L^2(0, T; V^\theta)$ such that a subsequence*
 574 *of $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ converges to $(\alpha^*, \mathbf{u}^*, \theta^*)$ in the topology of Assumptions 3.1, and for*
 575 *all $(\alpha, \mathbf{u}, \theta)$ in $\mathcal{U}_{ad} \times L^2(0, T; V^u) \times L^2(0, T; V^\theta)$: $\mathcal{J}(\alpha^*, \mathbf{u}^*, \theta^*) \leq \mathcal{J}(\alpha, \mathbf{u}, \theta)$. Hence,*
 576 *any accumulation point of $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ is a globally optimal solution of (OPT).*

577 *Proof.* The proof can be adapted from [21, Theorem 15] or [32, Theorem 3]. \square

578 However, the fact that this only concerns global solutions may appear restrictive.
 579 Under an additional assumption, we can state a slightly stronger result.

580 **COROLLARY 3.3.** *Assume Assumptions 3.1 hold. Let α^* be a local strict solution*
 581 *of (OPT), meaning that there exists $\rho > 0$ such that $\mathcal{J}(\alpha^*, \mathbf{u}^*, \theta^*) < \mathcal{J}(\alpha, \mathbf{u}, \theta)$ for*
 582 *all α such that $\|\alpha^* - \alpha\|_{BV} < \rho$. Then, there exists a family of local solution (α_ε^*) of*
 583 *(OPTe) such that (α_ε^*) converges weak-* to α^* .*

584 *Proof.* Similar to [36, Theorem 3.14]. \square

585 **3.1. First order necessary conditions for (OPTe).** From now on, we set
 586 $d = 2$ in order to have uniqueness of solution of (WFe). We make the following
 587 assumption on the cost function:

588 ASSUMPTIONS 3.4. Assume $d = 2$ and \mathcal{J} is Fréchet-differentiable.

589 We define the sets $W^u(0, T) = \{\mathbf{u} \in \mathbb{X}^u; \partial_t \mathbf{u} \in (\mathbb{X}^u)'\}$, and $W^\theta(0, T) = \{\theta \in$
 590 $\mathbb{X}^\theta; \partial_t \theta \in (\mathbb{X}^\theta)'\}$. Write, in $(\mathbb{X}^u)' \times (\mathbb{X}^\theta)'$, the equation (WFe) as $e(\mathbf{u}_\varepsilon, \theta_\varepsilon, \alpha_\varepsilon) = 0$,
 591 where $e : W^u(0, T) \times W^\theta(0, T) \times \mathcal{U}_{\text{ad}} \rightarrow (\mathbb{X}^u)' \times (\mathbb{X}^\theta)' \times H^u \times H^\theta$ is defined as:

$$592 \quad e(\mathbf{u}_\varepsilon, \theta_\varepsilon, \alpha_\varepsilon) = \begin{pmatrix} \partial_t \mathbf{u}_\varepsilon + \mathcal{A}\mathbf{u}_\varepsilon + \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + h(\alpha_\varepsilon)\mathbf{u}_\varepsilon \\ + \frac{1}{2}\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - \frac{1}{2}\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}}) - f - \sigma^{\text{ref}} \\ \partial_t \theta_\varepsilon - \mathcal{D}(\mathbf{u}_\varepsilon, \theta_\varepsilon) + \mathcal{C}(\alpha_\varepsilon)\theta_\varepsilon + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon) - \phi \\ \mathbf{u}_\varepsilon(0, \cdot) - \mathbf{u}_0(\alpha_\varepsilon) \\ \theta_\varepsilon(0, \cdot) - \theta_0(\alpha_\varepsilon) \end{pmatrix}.$$

593 The operators \mathcal{P}_ε , \mathcal{N}'_ε and \mathcal{M}_ε are Fréchet differentiable with the same smoothness
 594 as the approximation neg_ε . Their derivatives with respect to \mathbf{u}_ε are denoted by
 595 $d_u \mathcal{P}_\varepsilon : W^u(0, T)^2 \rightarrow \mathcal{L}(W^u(0, T), (\mathbb{X}^u)'),$ $\mathcal{N}'_\varepsilon : W^u(0, T)^2 \rightarrow \mathcal{L}(W^u(0, T), (\mathbb{X}^u)'),$
 596 $d_u \mathcal{M}_\varepsilon : W^u(0, T) \times W^\theta(0, T) \rightarrow \mathcal{L}(W^u(0, T), (\mathbb{X}^\theta)'),$ defined by:

$$597 \quad d_u \mathcal{P}_\varepsilon(\mathbf{u}, \mathbf{u})\mathbf{v} = \mathcal{P}_\varepsilon(\mathbf{u}, \mathbf{v}) + \mathcal{P}'_\varepsilon(\mathbf{u}, \mathbf{u})\mathbf{v},$$

$$598$$

$$599 \quad \langle \mathcal{N}'_\varepsilon(\mathbf{u}, \mathbf{w})\mathbf{v}, \Psi \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{neg}'_\varepsilon(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})\mathbf{w} \cdot \Psi.$$

$$600$$

$$601 \quad \langle d_u \mathcal{M}_\varepsilon(\mathbf{u}, \theta)\mathbf{v}, \varphi \rangle_{(V^\theta)', V^\theta} = \int_{\Gamma_{\text{out}}} (1 + \beta \text{neg}'_\varepsilon(\mathbf{u} \cdot \mathbf{n}))(\mathbf{v} \cdot \mathbf{n})\theta \varphi,$$

602 where $\mathcal{P}'_\varepsilon(\mathbf{u}, \mathbf{w})$ is defined by:

$$603 \quad \langle \mathcal{P}'_\varepsilon(\mathbf{u}, \mathbf{w})\mathbf{v}, \Psi \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{pos}'_\varepsilon(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})\mathbf{w} \cdot \Psi.$$

604 Furthermore, these operators are bounded, as proved in the following lemma:

605 LEMMA 3.5. Given $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ solution of (WFe):

$$606 \quad \|d_u \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)\mathbf{v}\|_{(\mathbb{X}^u)'} \lesssim (\|\mathbf{u}_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{4}} \|\mathbf{u}_\varepsilon\|_{L^2(0, T; H^1(\Omega))}^{\frac{3}{4}} + C_\varepsilon)$$

$$607 \quad \|\mathbf{v}\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{2}} \|\mathbf{v}\|_{L^2(0, T; H^1(\Omega))}^{\frac{1}{2}},$$

$$608 \quad \|\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}})\mathbf{v}\|_{(\mathbb{X}^u)'} \lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{4}} \|\mathbf{v}\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{2}}$$

$$609 \quad \|\mathbf{u}_\varepsilon\|_{L^2(0, T; H^1(\Omega))}^{\frac{3}{4}} \|\mathbf{v}\|_{L^2(0, T; H^1(\Omega))}^{\frac{1}{2}},$$

$$610 \quad \|d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon)\mathbf{v}\|_{(\mathbb{X}^\theta)'} \lesssim \|\theta_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{4}} \|\mathbf{v}\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{2}}$$

$$611 \quad \|\theta_\varepsilon\|_{L^2(0, T; H^1(\Omega))}^{\frac{3}{4}} \|\mathbf{v}\|_{L^2(0, T; H^1(\Omega))}^{\frac{1}{2}}.$$

611 *Proof.* The proof is similar to the proof of Lemma 2.6. Thanks to (A2), we obtain
 612 also:

$$613 \quad \langle \mathcal{P}'_\varepsilon(\mathbf{u}, \mathbf{u})\mathbf{v}, \Psi \rangle_{(V^u)', V^u} \lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{4}} \|\mathbf{v}\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{2}}$$

$$\|\mathbf{u}_\varepsilon\|_{L^2(0, T; H^1(\Omega))}^{\frac{3}{4}} \|\mathbf{v}\|_{L^2(0, T; H^1(\Omega))}^{\frac{1}{2}} \|\Psi\|_{\mathbb{X}^u}.$$

614 Analogously, using (A4), ones proves that there exists $C_\varepsilon > 0$ such that:

$$615 \int_0^T \langle \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{v}), \Psi \rangle_{(V^u)', V^u} \lesssim (\|\mathbf{u}_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{4}} \|\mathbf{u}_\varepsilon\|_{L^2(0, T; H^1(\Omega))}^{\frac{3}{4}} + C_\varepsilon) \\ \|\mathbf{v}\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{2}} \|\mathbf{v}\|_{L^2(0, T; H^1(\Omega))}^{\frac{1}{2}} \|\Psi\|_{\mathbb{X}^u}.$$

616 Adding the two inequalities and dividing by $\|\Psi\|_{H^1(\Omega)}$ concludes the proof. The proof
617 of the second and third inequalities being similar, they are thus omitted. \square

618 Using the results of [34, Section 1.8.2], one shows easily that e is Fréchet differ-
619 entiable w.r.t. $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$, with derivative given by:

$$620 e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon}(\alpha_\varepsilon) \begin{pmatrix} \mathbf{v} \\ \ell \end{pmatrix} = \begin{pmatrix} \partial_t \mathbf{v} + \mathcal{A} \mathbf{v} + \mathcal{B}(\mathbf{v}, \mathbf{u}_\varepsilon) + \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{v}) + h(\alpha_\varepsilon) \mathbf{v} + \mathcal{T} \ell \\ \quad \quad \quad + \frac{1}{2} d_u \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \mathbf{v} - \frac{1}{2} \mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}}) \mathbf{v} \\ \partial_t \ell - \mathcal{D}(\mathbf{u}_\varepsilon, \ell) - \mathcal{D}(\mathbf{v}, \theta_\varepsilon) + \mathcal{C}(\alpha_\varepsilon) \ell + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \ell) \\ \quad \quad \quad + d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon) \mathbf{v} \\ \mathbf{v}(0, \cdot) \\ \ell(0, \cdot) \end{pmatrix}.$$

621 For defining first order conditions (see [34]), a question of interest is to determine
622 if, for all $g = (g^u, g^\theta, \mathbf{v}_0, \ell_0) \in (\mathbb{X}^u)' \times (\mathbb{X}^\theta)' \times H^u \times H^\theta$, the following linearized
623 equation

$$624 (3.1) \quad e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon}(\alpha_\varepsilon) \begin{pmatrix} \mathbf{v} \\ \ell \end{pmatrix} = g$$

625 admits a solution $(\mathbf{v}, \ell) \in W^u(0, T) \times W^\theta(0, T)$.

626 THEOREM 3.6. *For all $\alpha_\varepsilon \in \mathcal{U}_{ad}$, Eq. (3.1) admits a unique solution. Therefore,*
627 *$e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon}(\alpha_\varepsilon)$ is invertible.*

628 *Sketch of proof.* Using Lemma 3.5, the proof can be adapted from Theorem 2.5
629 and [33, Appendix A2]. Uniqueness is proved as for Proposition 2.7 (see also [33,
630 Appendix A2]). \square

631 A consequence of Theorem 3.6 is that for all $G = (g_1, g_2) \in W^u(0, T)' \times W^\theta(0, T)'$,
632 the following adjoint equation admits a unique solution $\Lambda_\varepsilon = (\lambda_\varepsilon^u, \lambda_\varepsilon^\theta, \lambda_\varepsilon^{u_0}, \lambda_\varepsilon^{\theta_0}) \in$
633 $\mathbb{X}^u \times \mathbb{X}^\theta \times H^u \times H^\theta$:

$$634 (3.2) \quad (e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon}(\alpha_\varepsilon))^* \Lambda_\varepsilon = G,$$

635 where $(e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon}(\alpha_\varepsilon))^*$ denotes the adjoint operator of $e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon}(\alpha_\varepsilon)$.

636 After some calculations, equation (3.2) is equivalent to solve, for all $(\mathbf{v}, \ell) \in$
637 $W^u(0, T) \times W^\theta(0, T)$, the following variational problem:
(3.3)

$$\begin{aligned} & \langle -\partial_t \lambda_\varepsilon^u + \mathcal{A} \lambda_\varepsilon^u + \frac{1}{2} ((\nabla \mathbf{u}_\varepsilon)^\top \lambda_\varepsilon^u - (\nabla \lambda_\varepsilon^u)^\top \mathbf{u}_\varepsilon) - \mathcal{B}(\mathbf{u}_\varepsilon, \lambda_\varepsilon^u) + h(\alpha_\varepsilon) \lambda_\varepsilon^u - \mathcal{D}_1(\theta_\varepsilon) \lambda_\varepsilon^\theta \\ & + \frac{1}{2} \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \lambda_\varepsilon^u) + \frac{1}{2} (\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - \mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}}))^* \lambda_\varepsilon^u \\ & + (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda_\varepsilon^\theta, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)} \\ & + \langle \mathbf{v}(0, \cdot), \lambda_\varepsilon^{u_0} \rangle_H \\ & = \langle g_1, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)}, \\ & \langle -\partial_t \lambda_\varepsilon^\theta + \mathcal{T}^* \lambda_\varepsilon^u + \mathcal{C}(\alpha_\varepsilon) \lambda_\varepsilon^\theta - \mathcal{D}_2(\mathbf{u}_\varepsilon) \lambda_\varepsilon^\theta + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon)^* \lambda_\varepsilon^\theta, \ell \rangle_{W^\theta(0, T)', W^\theta(0, T)} \\ & = \langle g_2, \ell \rangle_{W^\theta(0, T)', W^\theta(0, T)} \end{aligned}$$

639 where $\langle \mathcal{D}(\theta, \mathbf{u}), \varphi \rangle = \langle \mathcal{D}_1(\theta)\varphi, \mathbf{u} \rangle = \langle \mathcal{D}_2(\mathbf{u})\varphi, \theta \rangle$, $\langle \mathcal{M}_\varepsilon(\mathbf{u})\theta, \varphi \rangle = \langle \mathcal{M}_\varepsilon(\mathbf{u})\varphi, \theta \rangle$
640 $= \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u} \cdot \mathbf{n}))\theta\varphi$. This equation, in turn, is the weak formulation
641 of:

$$\begin{aligned}
& -\partial_t \lambda_\varepsilon^{\mathbf{u}} - A\Delta \lambda_\varepsilon^{\mathbf{u}} + h(\alpha_\varepsilon)\lambda_\varepsilon^{\mathbf{u}} + (\nabla \mathbf{u}_\varepsilon)^\top \lambda_\varepsilon^{\mathbf{u}} - (\mathbf{u}_\varepsilon \cdot \nabla)\lambda_\varepsilon^{\mathbf{u}} - \theta_\varepsilon \nabla \lambda_\varepsilon^\theta = g_1 \\
& \nabla \cdot \lambda_\varepsilon^{\mathbf{u}} = 0, \\
& -\partial_t \lambda_\varepsilon^\theta + B\lambda_\varepsilon^{\mathbf{u}} \cdot e_y - \nabla \cdot (Ck(\alpha_\varepsilon)\nabla \lambda_\varepsilon^\theta) - \nabla \cdot (\mathbf{u}_\varepsilon \lambda_\varepsilon^\theta) = g_2 \\
& \lambda_\varepsilon^{\mathbf{u}}|_{\Gamma_w \cup \Gamma_{\text{in}}} = 0, \\
& \lambda_\varepsilon^\theta|_{\Gamma_{\text{in}}} = 0, \\
642 \quad (3.4a) \quad \partial_n \lambda_\varepsilon^\theta|_{\Gamma_w} = 0, \\
& A\partial_n \lambda_\varepsilon^{\mathbf{u}}|_{\Gamma_{\text{out}}} = \frac{1}{2}(\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) + (\mathbf{u}_\varepsilon \cdot \mathbf{n}))\lambda_\varepsilon^{\mathbf{u}} + (1 + \beta\mu_\varepsilon)\theta_\varepsilon \lambda_\varepsilon^\theta \mathbf{n} \\
& \quad + \frac{1}{2}\mu_\varepsilon((\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \lambda_\varepsilon^{\mathbf{u}})\mathbf{n}, \\
& Ck(\alpha_\varepsilon)\partial_n \lambda_\varepsilon^\theta + \beta\lambda_\varepsilon^\theta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})|_{\Gamma_{\text{out}}} = 0 \\
& \lambda_\varepsilon^{\mathbf{u}}(T) = 0, \lambda_\varepsilon^\theta(T) = 0,
\end{aligned}$$

643

$$644 \quad (3.4b) \quad \mu_\varepsilon = \text{neg}'_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})$$

645 and, as shown in a similar fashion in [33], $\lambda_\varepsilon^{\mathbf{u}^0} = \lambda_\varepsilon^{\mathbf{u}}(0, \cdot)$, $\lambda_\varepsilon^{\theta^0} = \lambda_\varepsilon^\theta(0, \cdot)$. Further-
646 more, we can argue that the weak solution $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta)$ of (3.4) are in $L^\infty(0, T; L^2(\Omega)^2) \times$
647 $L^\infty(0, T; L^2(\Omega))$, as done in [Theorem 2.4](#).

648 An other consequence of [Theorem 3.6](#) is that we can apply [34, Corollary 1.3]
649 which states that at any local solution $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon^*, \theta_\varepsilon^*)$ of (OPTe), the following optimality
650 conditions hold:

651 **THEOREM 3.7.** *Let α_ε^* be an optimal solution of (OPTe) with associated states*
652 *$(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*)$. Then there exist adjoint states $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta) \in \mathbb{X}^{\mathbf{u}} \times \mathbb{X}^\theta$ such that, denoting*
653 *$(\lambda_\varepsilon^{\mathbf{u}^0}, \lambda_\varepsilon^{\theta^0}) = (\lambda_\varepsilon^{\mathbf{u}}(0, \cdot), \lambda_\varepsilon^\theta(0, \cdot))$ and $\Lambda_\varepsilon = (\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta, \lambda_\varepsilon^{\mathbf{u}^0}, \lambda_\varepsilon^{\theta^0})$:*

$$\begin{aligned}
& e(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) = 0, \\
& \mathcal{J}'_{\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*}(\alpha_\varepsilon^*) + (e_{\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*}(\alpha_\varepsilon^*))^* \Lambda_\varepsilon = 0, \\
654 \quad (3.5) \quad \left\langle \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*))^* \Lambda_\varepsilon, \alpha - \alpha_\varepsilon^* \right\rangle_{\mathcal{U}'_{ad}, \mathcal{U}_{ad}} \geq 0, \quad \forall \alpha \in \mathcal{U}_{ad}, \\
& \alpha_\varepsilon \in \mathcal{U}_{ad}.
\end{aligned}$$

655 **REMARK 3.8.** *As stated in [34, Eq. (1.89)], since e and \mathcal{J} are Fréchet dif-*
656 *ferentiable, the mapping $\alpha_\varepsilon \mapsto \hat{\mathcal{J}}(\alpha_\varepsilon) = \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon)$ is Fréchet differentiable, and*
657 *$\hat{\mathcal{J}}'(\alpha_\varepsilon) = \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*))^* \Lambda_\varepsilon$, which reads as:*

$$\begin{aligned}
658 \quad (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*))^* \Lambda_\varepsilon &= \int_0^T (h'(\alpha_\varepsilon)\mathbf{u}_\varepsilon \cdot \lambda_\varepsilon^{\mathbf{u}} + Ck'(\alpha_\varepsilon)\nabla \theta_\varepsilon \cdot \nabla \lambda_\varepsilon^\theta) \\
& \quad + \mathbf{u}'_0(\alpha_\varepsilon) \cdot \lambda_\varepsilon^{\mathbf{u}^0} + \theta'_0(\alpha_\varepsilon)\lambda_\varepsilon^{\theta^0}.
\end{aligned}$$

659 **3.2. Limit adjoint system.** To conclude this paper, we will now study the
660 convergence, as $\varepsilon \rightarrow 0$, of the adjoint states $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta)$ to functions $(\lambda^{\mathbf{u}}, \lambda^\theta)$. The main
661 difficulty concerns the multiplier μ_ε defined in (3.4b). We will prove that at the limit,

662 μ is defined thanks to the convex-hull of the Heaviside function $H : \mathbb{R} \rightarrow [0, 1]$, given
 663 by:

$$664 \quad (3.6) \quad H(u) = \begin{cases} \{0\} & \text{if } u < 0, \\ \{1\} & \text{if } u > 0, \\ [0, 1] & \text{if } u = 0. \end{cases}$$

665 As we will prove in this section, these limit adjoint states $(\lambda^{\mathbf{u}}, \lambda^\theta)$ let us define neces-
 666 sary conditions of optimality for the unrelaxed problem (OPT).

667 **LEMMA 3.9.** *Let $(\alpha_\varepsilon) \subset \mathcal{U}_{ad}$ and $\alpha \in \mathcal{U}_{ad}$ such that $\alpha_\varepsilon \xrightarrow{*} \alpha$. Define by $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta)$ a*
 668 *weak solution of (3.4) parametrized by α_ε . Then, there exists $\lambda^{\mathbf{u}} \in L^\infty(0, T; H^u) \cap$*
 669 *$L^2(0, T; V^u)$, $\lambda^\theta \in L^\infty(0, T, H^\theta) \cap L^2(0, T; V^\theta)$ such that, up to a subsequence:*

- 670 • $\lambda_\varepsilon^{\mathbf{u}} \rightarrow \lambda^{\mathbf{u}}$ in $L^\infty(0, T; (L^2(\Omega))^2)$ and $\lambda_\varepsilon^\theta \rightarrow \lambda^\theta$ in $L^\infty(0, T; L^2(\Omega))$,
- 671 • $\lambda_\varepsilon^{\mathbf{u}} \xrightarrow{\varepsilon \rightarrow 0} \lambda^{\mathbf{u}}$ in $L^2(0, T; (H^1(\Omega))^2)$ and $\lambda_\varepsilon^\theta \xrightarrow{\varepsilon \rightarrow 0} \lambda^\theta$ in $L^2(0, T; (H^1(\Omega)))$,
- 672 • $\lambda_\varepsilon^{\mathbf{u}} \xrightarrow{\varepsilon \rightarrow 0} \lambda^{\mathbf{u}}$ in $L^2(0, T; (L^2(\Gamma))^2)$ and $\lambda_\varepsilon^\theta \xrightarrow{\varepsilon \rightarrow 0} \lambda^\theta$ in $L^2(0, T; (L^2(\Gamma)))$.

673 *Furthermore, there exists $\mu \in L^\infty([0, T] \times \Gamma_{out})$ defined by $-\mu \in H(-\mathbf{u} \cdot \mathbf{n})$ a.e. in Γ_{out}*
 674 *such that $(\lambda^{\mathbf{u}}, \lambda^\theta)$ is a weak solution to (3.4a) parametrized by α and μ , replacing*
 675 *$\text{neg}_\varepsilon(\cdot)$ (resp. $\text{pos}_\varepsilon(\cdot)$) by $\text{neg}(\cdot)$ (resp. $\text{pos}(\cdot)$).*

676 *Proof.* The proof is very similar to the ones presented in section 2.

- 677 • In a similar manner as for Proposition 2.2 and Proposition 2.3, one shows
 678 that, for all $\sigma \in [0, \frac{1}{6})$, there exist constants $c_\lambda^\theta(\sigma)$ and $c_\lambda^u(\sigma)$, independent of
 679 ε , such that:

$$680 \quad \sup_{[0, T]} \|\lambda_\varepsilon^{\mathbf{u}}\|_{L^2(\Omega)} + \int_0^T \|\nabla \lambda_\varepsilon^{\mathbf{u}}\|_{L^2(\Omega)} + \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F} \left(\widetilde{\lambda_\varepsilon^{\mathbf{u}}} \right) \right\|_{L^2(\Omega)} d\tau \leq c_\lambda^u(\sigma),$$

681

$$682 \quad \sup_{[0, T]} \|\lambda_\varepsilon^\theta\|_{L^2(\Omega)} + \int_0^T \|\nabla \lambda_\varepsilon^\theta\|_{L^2(\Omega)} + \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F} \left(\widetilde{\lambda_\varepsilon^\theta} \right) \right\|_{L^2(\Omega)} d\tau \leq c_\lambda^\theta(\sigma).$$

- 683 • These bounds prove a weaker set of convergence in the same manner as in
 684 Theorem 2.5. Since once again, we set $d = 2$, one proves the strong conver-
 685 gence stated above as in Corollary 2.9.

686 We only need to prove that $(\lambda^{\mathbf{u}}, \lambda^\theta)$ is a weak solution to (3.4a). The terms $\langle (\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon,$
 687 $\mathbf{u}_\varepsilon))^* \lambda_\varepsilon^{\mathbf{u}} \rangle_{W^u(0, T)', W^u(0, T)}$ and $\langle (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda_\varepsilon^\theta, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)}$ need a more
 688 thorough examination. We start with the first term for which we have

$$689 \quad \langle (\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v} \rangle_{W^u} = \int_0^T \int_{\Gamma_{out}} \text{pos}'_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) (\mathbf{u}_\varepsilon \cdot \lambda_\varepsilon^{\mathbf{u}}) \mathbf{n} \cdot \mathbf{v}.$$

690 In the same spirit as in [18, Proof of Lemma 4.3], we prove that up to a subsequence
 691 (not relabeled) one has $\text{neg}'_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \xrightarrow{*} \mu$ in $L^\infty([0, T] \times \Gamma_{out})$, and such that $-1 \leq$
 692 $\mu \leq 0$ a.e. in Γ_{out} and

$$693 \quad \mu = -1 \text{ a.e. in } \{\mathbf{u} \cdot \mathbf{n} < 0\}, \quad \mu = 0 \text{ a.e. in } \{\mathbf{u} \cdot \mathbf{n} > 0\}.$$

694 Furthermore, due to the convergence presented above, $\mathbf{u}_\varepsilon \cdot \lambda_\varepsilon^{\mathbf{u}} \rightarrow \mathbf{u} \cdot \lambda^{\mathbf{u}}$ in $L^1(0, T;$
 695 $L^1(\Gamma_{out}))$. Therefore, it proves that:

$$696 \quad \langle (\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)} \rightarrow \int_0^T \int_{\Gamma_{out}} (1 + \mu) (\mathbf{u} \cdot \lambda^{\mathbf{u}}) \mathbf{n} \cdot \mathbf{v}.$$

697 Similarly, we have that:

$$698 \quad \langle (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda_\varepsilon^\theta, \mathbf{v} \rangle_{W^u(0,T)', W^u(0,T)} \rightarrow \int_0^T \int_{\Gamma_{\text{out}}} (1 + \beta\mu) (\mathbf{v} \cdot \mathbf{n}) \theta \lambda^\theta.$$

699 All other terms in (3.3) can be dealt with as in the proof of [Theorem 2.5](#). Therefore,
700 $(\lambda^\mathbf{u}, \lambda^\theta)$ is a weak solution to (3.4a) parametrized by α and μ . \square

701 We may now prove the final result of this paper ; namely the necessary optimality
702 conditions of (OPT).

703 **THEOREM 3.10.** *Let α^* be an optimal solution of (OPT) with associated state*
704 *\mathbf{u}^*, θ^* . Then there exist a multiplier $\mu \in L^\infty([0, T] \times \Gamma_{\text{out}})$ and adjoint states $(\lambda^\mathbf{u}, \lambda^\theta) \in$*
705 *$\mathbb{X}^u \times \mathbb{X}^\theta$ solution of (3.4a) such that, denoting $(\lambda^{\mathbf{u}_0}, \lambda^{\theta_0}) =$*
706 *$(\lambda^\mathbf{u}(0, \cdot), \lambda^\theta(0, \cdot))$ and $\Lambda = (\lambda^\mathbf{u}, \lambda^\theta, \lambda^{\mathbf{u}_0}, \lambda^{\theta_0})$:*

$$707 \quad \langle \mathcal{J}'_{\alpha^*}(\mathbf{u}^*, \theta^*) + (e_{\alpha^*}(\mathbf{u}^*, \theta^*))' \Lambda, \alpha - \alpha^* \rangle_{\mathcal{U}_{\text{ad}}, \mathcal{U}_{\text{ad}}} \geq 0, \quad \forall \alpha \in \mathcal{U}_{\text{ad}}.$$

708 *Proof.* The proof follows the lines of [18, Theorem 4.4]. Denote by S_ε the solution
709 operator which associates to α the solution of the relaxed equations (WFe) and by S
710 the solution operator which to α associates the solution of (WF). For some $\rho > 0$,
711 consider the auxiliary optimal control problem:

$$712 \quad \begin{aligned} \min F_\varepsilon(\alpha_\varepsilon) &= \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon) + \frac{1}{2} \|\alpha^* - \alpha_\varepsilon\|_{L^2(\Omega)}^2 \\ \text{s.t.} \quad &\begin{cases} (\mathbf{u}_\varepsilon, \theta_\varepsilon) = S_\varepsilon(\alpha_\varepsilon), \\ \alpha_\varepsilon \in \mathcal{U}_{\text{ad}}, \\ \|\alpha_\varepsilon - \alpha^*\|_{L^2(\Omega)} \leq \rho. \end{cases} \end{aligned} \quad (3.7)$$

713 Since α_ε and α^* are both in \mathcal{U}_{ad} , they are both bounded in $L^\infty(\Omega)$ and therefore,
714 $\|\alpha^* - \alpha_\varepsilon\|_{L^2(\Omega)}$ is well defined. It is classical to show that (3.7) admits a global
715 minimizer $\alpha_\varepsilon^* \in \mathcal{U}_{\text{ad}}$.

716 Using (2.19) (but with $\alpha_\varepsilon \equiv \alpha$), one proves that (in the norm of the topology
717 given in [Assumptions 3.1](#) with $d = 2$):

$$718 \quad (3.8) \quad \|S(\alpha) - S_\varepsilon(\alpha)\| \lesssim C_\varepsilon, \quad \forall \alpha \in \mathcal{U}_{\text{ad}},$$

719 where C_ε has been defined in (2.12).

720 Note that due to the Fréchet-differentiability of \mathcal{J} supposed in [Assumptions 3.4](#)
721 and (3.8), it holds, for ε small enough:

$$722 \quad |\mathcal{J}(\alpha, S(\alpha)) - \mathcal{J}(\alpha, S_\varepsilon(\alpha))| \lesssim C_\varepsilon, \quad \forall \alpha \in \mathcal{U}_{\text{ad}}, \quad \|\alpha - \alpha^*\| \leq \rho.$$

723 We obtain as a consequence that $F_\varepsilon(\alpha^*) \lesssim C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*))$, and:

$$724 \quad F_\varepsilon(\alpha) \gtrsim -C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*)) + \frac{1}{2} \|\alpha - \alpha^*\|_{L^2(\Omega)}^2, \quad \forall \alpha \in \mathcal{U}_{\text{ad}}, \quad \|\alpha - \alpha^*\|_{L^2(\Omega)} \leq \rho.$$

725 Therefore, for all $\alpha \in \mathcal{U}_{\text{ad}}$ such that $\|\alpha - \alpha^*\|_{L^2(\Omega)} \leq \rho$:

$$726 \quad F_\varepsilon(\alpha^*) \lesssim C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*)) \lesssim C_\varepsilon + \mathcal{J}(\alpha, S(\alpha)) \lesssim 2C_\varepsilon + F_\varepsilon(\alpha).$$

727 Hence, for some constant C' , and denoting $C'_\varepsilon = C' C_\varepsilon$, one has the implication:

$$728 \quad \forall \alpha \in \mathcal{U}_{\text{ad}}, \quad 2C'_\varepsilon < \frac{1}{2} \|\alpha - \alpha^*\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \rho^2 \implies F_\varepsilon(\alpha^*) < F_\varepsilon(\alpha).$$

729 One has therefore the following necessary condition of optimality:

$$730 \quad (3.9) \quad \|\alpha_\varepsilon^* - \alpha^*\|_{L^2(\Omega)} \leq \sqrt{4C'_\varepsilon}.$$

731 Hence, for ε small enough, α_ε^* is in the ρ -ball around α^* ; therefore, α_ε^* is a local
732 solution of (OPTe). Using Theorem 3.7, one then proves that there exists adjoint
733 states $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta)$ solution of (3.4a) such that, for all $\alpha \in \mathcal{U}_{\text{ad}}$:

$$734 \quad (3.10) \quad \left\langle \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*))' \Lambda_\varepsilon, \alpha - \alpha_\varepsilon^* \right\rangle_{\mathcal{U}'_{\text{ad}}, \mathcal{U}_{\text{ad}}} + \langle \alpha_\varepsilon^* - \alpha^*, \alpha - \alpha_\varepsilon^* \rangle_{L^2(\Omega)} \geq 0.$$

735 From (3.9), one has $\alpha_\varepsilon^* \rightarrow \alpha^*$ strongly in $L^2(\Omega)$, and therefore, in $L^1(\Omega)$. Since
736 $(\alpha_\varepsilon^* - \alpha^*)_\varepsilon \subset \mathcal{U}_{\text{ad}}$, one has also $(\alpha_\varepsilon^* - \alpha^*)_\varepsilon$ bounded in $BV(\Omega)$. Hence, $\alpha_\varepsilon^* \xrightarrow{*} \alpha^*$ in
737 \mathcal{U}_{ad} . Using then Corollary 2.9, Assumptions 3.1 and Lemma 3.9, we can pass to the
738 limit in (3.10), which concludes this proof. \square

739 **Appendix A. Technical lemma.** Let $\mathbb{X} = L^2(0, T; H^1(\Omega)) \cap L^4(0, T; L^2(\Omega))$,
740 and denote by \mathbb{X}' the dual of \mathbb{X} with the following dual pairing: $\langle f, g \rangle_{\mathbb{X}', \mathbb{X}} =$
741 $\int_0^T \langle f(t), g(t) \rangle_{L^2(\Omega)}$. Denote $E_{\mathbb{X}} = \{\mathbf{u} \in \mathbb{X} | \mathbf{u}' = \frac{d\mathbf{u}}{dt} \in \mathbb{X}'\}$. We endow $E_{\mathbb{X}}$ with the
742 norm: $\|\mathbf{u}\|_{E_{\mathbb{X}}} = \|\mathbf{u}\|_{\mathbb{X}} + \|\mathbf{u}'\|_{\mathbb{X}'}$, where $\|\mathbf{u}\|_{\mathbb{X}} = \max\{\|\mathbf{u}\|_{L^2(0, T; H^1(\Omega))},$
743 $\|\mathbf{u}\|_{L^\infty(0, T; L^2(\Omega))}\}$. Finally, denote $\mathcal{D}(0, T; X)$ the set of infinitely differentiable func-
744 tions from $[0, T]$ to X with compact support in $[0, T]$.

745 LEMMA A.1. Let $\mathbf{u} \in E_{\mathbb{X}}$. There exists $(\mathbf{u}_n)_n \subset \mathcal{D}(0, T; H^1(\Omega))$ such that:

$$746 \quad \mathbf{u}_n \rightarrow \mathbf{u} \text{ in } L^2(0, T; H^1(\Omega)), \quad \mathbf{u}'_n \rightharpoonup \mathbf{u}' \text{ in } \mathbb{X}'.$$

747 *Proof.* From [11, Theorem II.2.26], one proves directly that there exists $(\mathbf{u}_n)_n \subset$
748 $\mathcal{D}(0, T; H^1(\Omega))$ such that $\mathbf{u}_n \rightarrow \mathbf{u}$ strongly in $L^2(0, T; H^1(\Omega))$.

749 For all $\varphi \in \mathcal{D}(0, T; H^1(\Omega))$, one has:

$$750 \quad \langle \mathbf{u}'_n, \varphi \rangle_{\mathbb{X}', \mathbb{X}} = -\langle \mathbf{u}_n, \varphi' \rangle_{\mathbb{X}, \mathbb{X}} \xrightarrow{n \rightarrow +\infty} -\langle \mathbf{u}, \varphi' \rangle_{\mathbb{X}, \mathbb{X}} = \langle \mathbf{u}', \varphi \rangle_{\mathbb{X}', \mathbb{X}}.$$

751 By the density result [11, Theorem II.2.26], we prove that:

$$752 \quad \forall \varphi \in \mathbb{X}, \langle \mathbf{u}'_n, \varphi \rangle_{\mathbb{X}', \mathbb{X}} \xrightarrow{n \rightarrow +\infty} \langle \mathbf{u}', \varphi \rangle_{\mathbb{X}', \mathbb{X}}. \quad \square$$

753 LEMMA A.2. Let $\mathbf{u}, \mathbf{v} \in E_{\mathbb{X}}$. Then, $t \mapsto \langle \mathbf{u}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)}$ is in $W^{1,1}([0, T])$ and
754 for all $t \in [0, T]$:

$$755 \quad \frac{d}{dt} \langle \mathbf{u}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)} = \left\langle \frac{d\mathbf{u}}{dt}(t), \mathbf{v}(t) \right\rangle_{L^2(\Omega)} + \left\langle \frac{d\mathbf{v}}{dt}(t), \mathbf{u}(t) \right\rangle_{L^2(\Omega)}.$$

756 *Proof.* Using Lemma A.1, the proof is a simple adaptation of [11, Theorem
757 II.5.12]. \square

758 REFERENCES

- 759 [1] J. Alexandersen and C. S. Andreasen. A review of topology optimisation for fluid-based prob-
760 lems. *Fluids*, 5(1):29, 2020.
761 [2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity*
762 *problems*, volume 254. Clarendon Press Oxford, 2000.

- 763 [3] S. Amstutz. The topological asymptotic for the Navier–Stokes equations. ESAIM: Control,
764 Optimisation and Calculus of Variations, 11(3):401–425, 2005.
- 765 [4] P. Angot, C.-H. Bruneau, and P. Fabrie. A penalization method to take into account obstacles
766 in incompressible viscous flows. Numerische Mathematik, 81(4):497–520, 1999.
- 767 [5] D. Arndt, M. Braack, and G. Lube. Finite elements for the Navier–Stokes problem with outflow
768 condition. In Numerical Mathematics and Advanced Applications ENUMATH 2015, pages
769 95–103. Springer, 2016.
- 770 [6] R. Arndt, A. N. Ceretani, and C. Rautenberg. On existence and uniqueness of solutions to a
771 Boussinesq system with nonlinear and mixed boundary conditions. Journal of Mathematical
772 Analysis and Applications, 490(1):124201, 2020.
- 773 [7] L. Baffico, C. Grandmont, and B. Maury. Multiscale modeling of the respiratory tract.
774 Mathematical Models and Methods in Applied Sciences, 20(01):59–93, 2010.
- 775 [8] J. Boland and W. Layton. Error analysis for finite element methods for steady natural convection
776 problems. Numerical Functional Analysis and Optimization, 11(5-6):449–483, 1990.
- 777 [9] T. Borrvall and J. Petersson. Topology optimization of fluids in Stokes flow. International
778 Journal for Numerical Methods in Fluids, 41(1):77–107, 2003.
- 779 [10] F. Boyer and P. Fabrie. Outflow boundary conditions for the incompressible non-homogeneous
780 Navier–Stokes equations. Discrete and Continuous Dynamical Systems-Series B, 7(2):pp–
781 219, 2007.
- 782 [11] F. Boyer and P. Fabrie. Mathematical Tools for the Study of the Incompressible Navier–Stokes
783 Equations and Related Models, volume 183. Springer Science & Business Media, 2012.
- 784 [12] M. Braack and P. B. Mucha. Directional do-nothing condition for the Navier–Stokes equations.
785 Journal of Computational Mathematics, pages 507–521, 2014.
- 786 [13] B. Brangeon, P. Joubert, and A. Bastide. Influence of the dynamic boundary conditions on
787 natural convection in an asymmetrically heated channel. International Journal of Thermal
788 Sciences, 95:64–72, 2015.
- 789 [14] C-H Bruneau and P Fabrie. New efficient boundary conditions for incompressible Navier–
790 Stokes equations: a well-posedness result. ESAIM: Mathematical Modelling and Numerical
791 Analysis, 30(7):815–840, 1996.
- 792 [15] F. Caubet, C. Conca Rosende, and M. Godoy. On the detection of several obstacles in 2D
793 Stokes flow: topological sensitivity and combination with shape derivatives. IPI : Inverse
794 Problems and Imaging, 2016.
- 795 [16] A. Ceretani and C. Rautenberg. The Boussinesq system with mixed non-smooth boundary
796 conditions and do-nothing boundary flow. Zeitschrift für angewandte Mathematik und
797 Physik, 70(1):1–24, 2019.
- 798 [17] N. Chami and A. Zoughaib. Modeling natural convection in a pitched thermosyphon system in
799 building roofs and experimental validation using particle image velocimetry. Energy and
800 buildings, 42(8):1267–1274, 2010.
- 801 [18] C. Christof, C. Meyer, S. Walther, and C. Clason. Optimal control of a non-smooth semilinear
802 elliptic equation. Mathematical Control & Related Fields, 8(1):247, 2018.
- 803 [19] C. Clason and K. Kunisch. A duality-based approach to elliptic control problems in non-reflexive
804 banach spaces. ESAIM: Control, Optimisation and Calculus of Variations, 17(1):243–266,
805 2011.
- 806 [20] C. Clason, K. Kunisch, and A. Rund. Optimal control of partial differential equations with
807 nonsmooth cost functionals. In Proceedings of the XXIV Congress on Differential Equations
808 and Applications / XIV Congress on Applied Mathematics, Cádiz, 2015.
- 809 [21] P.-H. Cocquet, S. Rizzo, and J. Salomon. Optimization of bathymetry for long waves with small
810 amplitude. SIAM Journal on Control and Optimization, 59(6):4429–4456, 2021.
- 811 [22] E. Colmenares, G. N. Gatica, and R. Oyarzúa. A posteriori error analysis of an augmented
812 fully-mixed formulation for the stationary Boussinesq model. Computers & Mathematics
813 with Applications, 77(3):693–714, 2019.
- 814 [23] T. Dbouk. A review about the engineering design of optimal heat transfer systems using
815 topology optimization. Applied Thermal Engineering, 112:841–854, 2017.
- 816 [24] G. Desrayaud, E. Chénier, A. Joulin, A. Bastide, B. Brangeon, J.P. Caltagirone, Y. Cherif,
817 R. Eymard, C. Garnier, S. Giroux-Julien, et al. Benchmark solutions for natural convection
818 flows in vertical channels submitted to different open boundary conditions. International
819 journal of thermal sciences, 72:18–33, 2013.
- 820 [25] F. Feppon, G. Allaire, F. Bordeu, J. Cortial, and C. Dapogny. Shape optimization of a coupled
821 thermal fluid–structure problem in a level set mesh evolution framework. SeMA Journal,
822 76(3):413–458, 2019.
- 823 [26] L. Formaggia, J.-F. Gerbeau, F. Nobile, and A. Quarteroni. Numerical treatment of defec-
824 tive boundary conditions for the Navier–Stokes equations. SIAM Journal on Numerical

- 825 Analysis, 40(1):376–401, 2002.
- 826 [27] J. Fouchet-Incaux. Artificial boundaries and formulations for the incompressible navier–stokes
827 equations: applications to air and blood flows. SeMA Journal, 64(1):1–40, 2014.
- 828 [28] G. Galdi. An introduction to the mathematical theory of the Navier–Stokes equations:
829 Steady-state problems. Springer Science & Business Media, 2011.
- 830 [29] H. Garcke, M. Hinze, C. Kahle, and K. F. Lam. A phase field approach to shape optimiza-
831 tion in Navier–Stokes flow with integral state constraints. Advances in Computational
832 Mathematics, 44(5):1345–1383, 2018.
- 833 [30] T. Goudon, S. Krell, and G. Lissoni. Ddfv method for Navier–Stokes problem with outflow
834 boundary conditions. Numerische Mathematik, 142(1):55–102, 2019.
- 835 [31] F. Harder and G. Wachsmuth. Comparison of optimality systems for the optimal control of
836 the obstacle problem. GAMM-Mitteilungen, 40(4):312–338, 2018.
- 837 [32] J. Haslinger and R. Mäkinen. On a topology optimization problem governed by two-dimensional
838 Helmholtz equation. Computational Optimization and Applications, 62(2):517–544, 2015.
- 839 [33] M. Hinze. Optimal and instantaneous control of the instationary Navier–Stokes equations.
840 Habilitationsschrift, Fachbereich Mathematik, Technische Universität, Berlin, 2002.
- 841 [34] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. Optimization with PDE constraints, vol-
842 ume 23. Springer Science & Business Media, 2008.
- 843 [35] S. Kračmar and J. Neustupa. Modeling of the unsteady flow through a channel with an artificial
844 outflow condition by the Navier–Stokes variational inequality. Mathematische Nachrichten,
845 291(11-12):1801–1814, 2018.
- 846 [36] K. Kunisch and D. Wachsmuth. Sufficient optimality conditions and semi-smooth Newton meth-
847 ods for optimal control of stationary variational inequalities. ESAIM: Control, Optimisation
848 and Calculus of Variations, 18(2):520–547, 2012.
- 849 [37] J.L. Lions. Quelques méthodes de résolution des problèmes aux limites non linéaires. Les Cours
850 de référence. Dunod, 2002.
- 851 [38] C. Meyer and L. Susu. Optimal control of nonsmooth, semilinear parabolic equations. SIAM
852 Journal on Control and Optimization, 55(4):2206–2234, 2017.
- 853 [39] B. Mohammadi and O. Pironneau. Shape optimization in fluid mechanics. Annual Review of
854 Fluid Mechanics, 36:255–279, 2004.
- 855 [40] B. Mohammadi and O. Pironneau. Applied shape optimization for fluids. Oxford University
856 Press, 2010.
- 857 [41] A. Novotny, J. Sokolowski, and A. Żochowski. Topological derivatives of shape function-
858 als. Part II: first-order method and applications. Journal of Optimization Theory and
859 Applications, 180(3):683–710, 2019.
- 860 [42] C. Popa, D. Ospir, S. Fohanno, and C. Chereches. Numerical simulation of dynamical aspects
861 of natural convection flow in a double-skin façade. Energy and Buildings, 50:229–233, 2012.
- 862 [43] D. Ramalingom, P.-H. Cocquet, and A. Bastide. Numerical study of natural convection in
863 asymmetrically heated channel considering thermal stratification and surface radiation.
864 Numerical Heat Transfer, Part A: Applications, 72(9):681–696, 2017.
- 865 [44] D. Ramalingom, P.-H. Cocquet, and A. Bastide. A new interpolation technique to deal with
866 fluid-porous media interfaces for topology optimization of heat transfer. Computers &
867 Fluids, 168:144–158, 2018.
- 868 [45] D. Ramalingom, P.-H. Cocquet, R. Maleck, and A. Bastide. A multi-objective optimization
869 problem in mixed and natural convection for a vertical channel asymmetrically heated.
870 Structural and Multidisciplinary Optimization, 60(5):2001–2020, 2019.
- 871 [46] A. Schiela and D. Wachsmuth. Convergence analysis of smoothing methods for optimal con-
872 trol of stationary variational inequalities with control constraints. ESAIM: Mathematical
873 Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique,
874 47(3):771–787, 2013.
- 875 [47] J. S. Simon and H. Notsu. A convective boundary condition for the navier-stokes equations:
876 Existence analysis and numerical implementations. arXiv preprint arXiv:2109.10025, 2021.
- 877 [48] C. Suárez, P. Joubert, J. Molina, and F. Sánchez. Heat transfer and mass flow correlations for
878 ventilated facades. Energy and Buildings, 43(12):3696–3703, 2011.
- 879 [49] R. Temam. Navier–Stokes equations: theory and numerical analysis, volume 343. American
880 Mathematical Soc., 2001.
- 881 [50] A. Vieira, A. Bastide, and P.-H. Cocquet. Topology optimization for steady-state anisothermal
882 flow targeting solid with piecewise constant thermal diffusivity. hal-02569142, version 2,
883 2020.
- 884 [51] I. Yousept. Optimal control of non-smooth hyperbolic evolution Maxwell equations in type-II
885 superconductivity. SIAM Journal on Control and Optimization, 55(4):2305–2332, 2017.