



**HAL**  
open science

# The Boussinesq system with non-smooth boundary conditions : existence, relaxation and topology optimization.

Alexandre Vieira, Pierre-Henri Cocquet

## ► To cite this version:

Alexandre Vieira, Pierre-Henri Cocquet. The Boussinesq system with non-smooth boundary conditions : existence, relaxation and topology optimization.. 2021. hal-03207923v1

**HAL Id: hal-03207923**

**<https://hal.science/hal-03207923v1>**

Preprint submitted on 26 Apr 2021 (v1), last revised 15 Sep 2022 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1     **THE BOUSSINESQ SYSTEM WITH NON-SMOOTH BOUNDARY**  
2     **CONDITIONS : EXISTENCE, RELAXATION AND TOPOLOGY**  
3     **OPTIMIZATION.**

4             ALEXANDRE VIEIRA\* AND PIERRE-HENRI COCQUET\*†

5     **Abstract.** In this paper, we tackle a topology optimization problem which consists in finding  
6 the optimal shape of a solid located inside a fluid that minimizes a given cost function. The motion  
7 of the fluid is modeled thanks to the Boussinesq system which involves the unsteady Navier-Stokes  
8 equation coupled to a heat equation. In order to cover several models presented in the literature, we  
9 choose a non-smooth formulation for the outlet boundary conditions and an optimization parameter  
10 of bounded variations. This paper aims at proving existence of solutions to the resulting equations,  
11 along with the study of a relaxation scheme of the non-smooth conditions. A second part covers  
12 the topology optimization problem itself for which we proved the existence of optimal solutions and  
13 provides the definition of first order necessary optimality conditions.

14     **Key words.** Non-smooth boundary conditions, topology optimization, relaxation scheme, di-  
15 rectional do-nothing boundary conditions

16     **AMS subject classifications.** 49K20, 49Q10, 76D03, 76D55

17     **1. Introduction.**

18     *Directional do-nothing conditions.* For many engineering applications, simula-  
19 tions of flows coupled with the temperature are useful for predicting the behaviour  
20 of physical designs before their manufacture, reducing the cost of the development  
21 of new products. The relevance of the model and the adequacy with the experiment  
22 therefore become important [16, 41, 46]. In this paper, we choose to model the flow  
23 with the Boussinesq system which involves the Navier-Stokes equations coupled with  
24 an energy equation. In most mathematical papers analyzing this model [8, 27, 47],  
25 homogeneous Dirichlet boundary conditions are considered on the whole boundary.  
26 This simplifies the mathematical analysis of the incompressible Navier-Stokes equa-  
27 tion since the non-linear term vanishes after integrating by part hence simplifying the  
28 derivation of a priori estimates [7, 21, 26, 47].

29     However, several applications use different boundary conditions that model inlet,  
30 no-slip and outlet conditions [1]. Unlike the inlet and the no-slip conditions, the  
31 outlet conditions are more subject to modelling choices. A popular choice consists in  
32 using a do-nothing outlet condition (see e.g. [25, 34, 48]) which naturally comes from  
33 integration by parts when defining a weak formulation of the Navier-Stokes equations.  
34 However, since this outlet condition can not deal with re-entering flows, several papers  
35 use a non-smooth outlet boundary conditions for their numerical simulations (see e.g.  
36 [5, 23]). A focus on non-smooth outflow conditions when the temperature appears  
37 can be found in [12, 23, 42, 43].

38     In particular, directional do-nothing (DDN) boundary conditions are non-smooth  
39 boundary conditions that become popular. The idea is originally described in [13],  
40 and several other mathematical studies followed [5, 9, 11]. These conditions were  
41 considered especially for turbulent flows. In this situation, the flow may alternatively

---

\*Physique et Ingénierie Mathématique pour l'Énergie et l'Environnement (PIMENT), Univer-  
sité de la Réunion, 2 rue Joseph Wetzell, 97490 Sainte-Clotilde, France. (Alexandre.Vieira@univ-  
reunion.fr)

†Laboratoire des Sciences de l'Ingénieur Appliquées à la Mécanique et au Génie Electrique  
(SIAME), E2S-UPPA, Université de Pau et des Pays de l'Adour, 64000 Pau, France (Pierre-  
Henri.Cocquet@univ-reunion.fr, Pierre-Henri.Cocquet@univ-pau.fr)

42 exit and re-enter the domain. These directional boundary conditions tries to capture  
43 this phenomenon, while limiting the reflection. It is worth noting that other boundary  
44 conditions can be used, namely the so-called local/global Bernouilli boundary condi-  
45 tions [12, 23, 43]. The latter implies the do-nothing boundary condition is satisfied  
46 for exiting fluid and that both the normal velocity gradient and the total pressure  
47 vanish for re-entering fluid. Nevertheless, in this paper, we are going to used non-  
48 smooth DDN boundary condition since they are easier to impose though a variational  
49 formulation.

50 Concerning the mathematical study of Boussinesq system with directional do-  
51 nothing conditions, the literature is rather scarce. To the best of our knowledge, we  
52 only found [6, 15], where the steady case is studied in depth, but the unsteady case  
53 only presents limited results. Indeed, while [15, p. 16, Theorem 3.2] gives existence  
54 and uniqueness of a weak solution with additional regularity to the Boussinesq system  
55 involving non-smooth boundary conditions at the inlet, it requires the source terms  
56 and the physical constants (e.g. Reynolds, Grashof numbers) to be small enough. We  
57 emphasize that these limitations comes from the proof which relies on a fixed-point  
58 strategy. The first aim of this paper will then be to fill that gap by proving existence  
59 and, in a two-dimensional setting, uniqueness of solutions for the unsteady Boussinesq  
60 system with non-smooth DDN boundary condition on the outlet.

61 *Topology optimization.* On top of the previous considerations, this paper aims at  
62 using these equations in a topology optimization (TO) framework. In fluid mechanics,  
63 the term *topology optimization* refers to the problem of finding the shape of a solid  
64 located inside a fluid that either minimizes or maximizes a given physical effect. There  
65 exist various mathematical methods to deal with such problems that fall into the  
66 class of PDE-constrained optimization, such as the topological asymptotic expansion  
67 [3, 14, 40] or the shape optimization method [24, 38, 39]. In this paper, we choose to  
68 locate the solid thanks to a penalization term added in the unsteady Navier-Stokes  
69 equations, as exposed in [4]. However, the binary function introduced in [4] is usually  
70 replaced by a smooth approximation, referred as *interpolation function* [43], in order  
71 to be used in gradient-based optimization algorithms. We refer to the review papers  
72 [1, 22] for many references that deal with numerical resolution of TO problems applied  
73 to several different physical settings. However, as noted in [1, Section 4.7], most  
74 problems tackling topology optimization for flows only focus on steady flows, and  
75 time-dependant approaches are still rare. Furthermore, to the best of our knowledge,  
76 no paper is dedicated to the mathematical study of unsteady TO problems involving  
77 DDN boundary conditions, even though they are already used in numerical studies  
78 [12, 23, 42, 43]. Therefore, a second goal of this paper will be to prove existence  
79 of optimal solution to a TO problem involving Boussinesq system with non-smooth  
80 DDN boundary conditions at the outlet.

81 *First order optimality conditions.* As hinted above, a gradient based method is  
82 often used in order to compute an optimal solution of a TO problem. However, the  
83 introduction of the non-smooth DDN boundary conditions implies that the control-  
84 to-state mapping is no longer differentiable. The literature presents several ways to  
85 deal with such PDE-constrained optimization problems. Most focus on elliptic equa-  
86 tions, using subdifferential calculus [17, 30, 19] or as the limit of relaxation schemes  
87 [18, 35, 45]. We may also cite [37] for a semilinear parabolic case and [49] which  
88 involves the Maxwell equations. We emphasize that using directly a subdifferential  
89 approach presents several drawbacks: the subdifferential of composite functions may  
90 be hardly computed, and the result may be hardly enlightening nor used [17]. We will  
91 therefore use a differentiable relaxation approach, as studied in [45]. First, we will

92 be able to use standard first order necessary optimality conditions since the relaxed  
 93 control-to-state mapping will be smooth. A convergence analysis will let us design  
 94 necessary optimality condition for the non-smooth problem. Secondly, we find this  
 95 approach more enlightening, as it may be used as a numerical scheme for solving the  
 96 TO problem.

97 **1.1. Problem settings.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$  be a bounded open set with  
 98 Lipschitz boundary whose outward unitary normal is  $\mathbf{n}$ . We assume the fluid occupies  
 99 a region  $\Omega_f \subset \Omega$  and that a solid is defined by a region  $\Omega_s$  such that  $\Omega = \Omega_f \cup \Omega_s$ .  
 100 The penalized Boussinesq approximation (see e.g. [43] for the steady case) of the  
 101 Navier-Stokes equations coupled to convective heat transfer reads:

$$\begin{aligned}
 & \nabla \cdot \mathbf{u} = 0, \\
 & \partial_t \theta + \nabla \cdot (\mathbf{u}\theta) - \nabla \cdot (Ck(\alpha)\nabla\theta) = 0, \quad \text{a.e. in } \Omega \\
 & \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - A\Delta\mathbf{u} + \nabla p - B\theta e_y + h(\alpha)\mathbf{u} = f, \\
 & \mathbf{u}(0) = u_0(\alpha), \quad \theta(0) = \theta_0(\alpha),
 \end{aligned}
 \tag{1.1}$$

103 where  $\mathbf{u}$  denotes the velocity of the fluid,  $p$  the pressure and  $\theta$  the temperature (all  
 104 dimensionless),  $u_0(\alpha), \theta_0(\alpha)$  are initial conditions. In (1.1),  $A = \text{Re}^{-1}$  with  $\text{Re}$  being  
 105 the Reynolds number,  $B = \text{Ri}$  is the Richardson number and  $C = (\text{RePr})^{-1}$  where  $\text{Pr}$   
 106 is the Prandtl number. In a topology optimization problem, it is classical to introduce  
 107 a function  $\alpha : x \in \Omega \mapsto \alpha(x) \in \mathbb{R}^+$  as optimization parameter (see e.g. [1, 22]). The  
 108 function  $h(\alpha)$  then penalizes the flow in order to mimic the presence of solid:

- 109 • if  $h \equiv 0$ , then one retrieves the classical Boussinesq approximation.
- 110 • if, for some  $s > s_0$  and large enough  $\alpha_{\max}$ ,  $h : s \in [0, \alpha_{\max}] \mapsto h(s) \in [0, \alpha_{\max}]$   
 111 is a smooth function such that  $h(s) \approx 0$  for  $s \leq s_0$  and  $h(s) \approx \alpha_{\max}$  for  $s \geq s_0$ ,  
 112 one retrieves the formulations used in topology optimization [1, 8, 43]. In the  
 113 sequel, we work in this setting since we wish to study a TO problem.

114 Since the classical Boussinesq problem is retrieved when  $h(\alpha) = 0$ , the fluid zones  
 115  $\Omega_f \subset \Omega$  and the solid ones  $\Omega_s \subset \Omega$  can be defined as  $\Omega_s := \{x \in \Omega \mid \alpha(x) < s_0\}$ ,  $\Omega_f :=$   
 116  $\{x \in \Omega \mid \alpha(x) > s_0\}$ , where  $\alpha_{\max} > 0$  is large enough to ensure the velocity  $\mathbf{u}$  is small  
 117 enough for the  $\Omega_s$  above to be considered as a solid. The function  $k(\alpha) : x \in \Omega \mapsto$   
 118  $k(\alpha(x))$  is the dimensionless diffusivity defined as  $k(\alpha)|_{\Omega_f} = 1$  and  $k(\alpha)|_{\Omega_s} = k_s/k_f$   
 119 with  $k_s$  and  $k_f$  are respectively the diffusivities of the solid and the fluid. We also  
 120 assume that  $k$  is a smooth regularization of  $(k_s/k_f)\mathbf{1}_{\Omega_s} + \mathbf{1}_{\Omega_f}$ . In this framework,  $\alpha$   
 121 is thus defined as a parameter function, which will let us control the distribution of  
 122 the solid in  $\Omega$ .

123 Let us now specify the boundary conditions. Assume  $\partial\Omega = \Gamma$  is Lipschitz and we  
 124 split it in three parts:  $\Gamma = \Gamma_w \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ . Here,  $\Gamma_w$  are the walls,  $\Gamma_{\text{in}}$  the inlet/entrance  
 125 and  $\Gamma_{\text{out}}$  is the exit/outlet of the computational domain. As exposed above, we would  
 126 like to rigorously study a non-smooth outlet boundary condition. Inspired by [13],  
 127 the following formulation tries to encapsulate these different approaches. Let  $\beta$  be  
 128 a function defined on  $\Gamma_{\text{out}}$  and define:  $\forall x \in \mathbb{R} : x^+ = \text{pos}(x) = \max(0, x), x^- =$   
 129  $\text{neg}(x) = \max(0, -x), x = x^+ - x^-$ . On top of (1.1), we impose the following boundary  
 130 conditions:

$$\begin{aligned}
 & \underline{\text{On } \Gamma_{\text{in}}} : \quad \mathbf{u} = \mathbf{u}_{\text{in}}, \quad \theta = 0, \\
 & \underline{\text{On } \Gamma_w} : \quad \mathbf{u} = 0, \quad Ck\partial_n\theta = \phi, \\
 & \underline{\text{On } \Gamma_{\text{out}}} : \quad A\partial_n\mathbf{u} - \mathbf{n}p = A\partial_n\mathbf{u}^{\text{ref}} - \mathbf{n}p^{\text{ref}} - \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} - \mathbf{u}^{\text{ref}}), \\
 & \quad Ck\partial_n\theta + \beta(\mathbf{u} \cdot \mathbf{n})^-\theta = 0,
 \end{aligned}
 \tag{1.2}$$

132 with  $\phi \in L^2(0, T; L^2(\Gamma_w))$ ,  $f \in L^2(0, T; H^{-1}(\Omega))$ ,  $\mathbf{u}_{\text{in}} \in L^2(0, T; H_{00}^{1/2}(\Gamma_{\text{in}}))$ ,  $\mathbf{n}$  de-  
 133 notes the normal vector to the boundary,  $\partial_n = \mathbf{n} \cdot \nabla$  and  $(\mathbf{u}^{\text{ref}}, p^{\text{ref}})$  denotes a reference  
 134 solution. As stated in [29], this nonlinear condition is physically meaningful: if the  
 135 flow is outward, we impose the constraint coming from the selected reference flow ;  
 136 if it is inward, we need to control the increase of energy, so, according to Bernoulli's  
 137 principle, we add a term that is quadratic with respect to velocity.

138 To define a weak formulation of (1.1)-(1.2), we introduce  $\mathcal{V}^u = \{\mathbf{u} \in \mathcal{C}^\infty(\Omega; \mathbb{R}^d);$   
 139  $\mathbf{u}|_{\Gamma_{\text{in}} \cup \Gamma_w} = 0\}$ , and define  $V^u$  (resp  $H^u$ ) as the closure of  $\mathcal{V}^u$  in  $(H^1(\Omega))^d$  (resp. in  
 140  $(L^2(\Omega))^d$ ). Similarly, we define  $\mathcal{V}^\theta = \{\theta \in \mathcal{C}^\infty(\Omega; \mathbb{R}); \theta|_{\Gamma_{\text{in}}} = 0\}$ , and  $V^\theta$  and  $H^\theta$   
 141 accordingly. A weak formulation of (1.1)-(1.2) then reads as:

$$142 \quad \int_{\Omega} \partial_t \theta \varphi - \int_{\Omega} \theta \mathbf{u} \cdot \nabla \varphi + \int_{\Omega} Ck \nabla \theta \cdot \nabla \varphi + \int_{\Gamma} (\theta(\mathbf{u} \cdot \mathbf{n}) - Ck \partial_n \theta) \varphi = 0,$$

143 for all  $\varphi \in H^\theta$ . However, from (1.2), we have:

$$\begin{aligned} \int_{\Gamma} (\theta(\mathbf{u} \cdot \mathbf{n}) - Ck \partial_n \theta) \varphi &= - \int_{\Gamma_w} \phi \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta(\mathbf{u} \cdot \mathbf{n})^-) \theta \varphi \\ &\quad - \int_{\Gamma_{\text{out}}} (\beta \theta(\mathbf{u} \cdot \mathbf{n})^- + Ck \partial_n \theta) \varphi \\ 144 \quad &= - \int_{\Gamma_w} \phi \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta(\mathbf{u} \cdot \mathbf{n})^-) \theta \varphi. \end{aligned}$$

145 Therefore:

(WF.1)

$$146 \quad \int_{\Omega} \partial_t \theta \varphi - \int_{\Omega} \theta \mathbf{u} \cdot \nabla \varphi + \int_{\Omega} Ck \nabla \theta \cdot \nabla \varphi + \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta(\mathbf{u} \cdot \mathbf{n})^-) \theta \varphi = \int_{\Gamma_w} \phi \varphi.$$

147 Doing similar computations with the Navier-Stokes system yield:

$$150 \quad (\text{WF.2}) \quad \int_{\Omega} q \nabla \cdot \mathbf{u} = 0, \quad \forall q \in L^2(\Omega),$$

151

$$\begin{aligned} &\int_{\Omega} (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \Psi + A \int_{\Omega} \nabla \mathbf{u} : \nabla \Psi - \int_{\Omega} B \theta e_y \cdot \Psi \\ 152 \quad (\text{WF.3}) \quad &- \int_{\Omega} p \nabla \cdot \Psi + \int_{\Omega} h \mathbf{u} \cdot \Psi + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{u} \cdot \mathbf{n})^- (\mathbf{u} - \mathbf{u}^{\text{ref}}) \cdot \Psi \\ &= \int_{\Omega} f \cdot \Psi + \int_{\Gamma_{\text{out}}} (A \partial_n \mathbf{u}^{\text{ref}} - \mathbf{n} p^{\text{ref}}) \cdot \Psi, \end{aligned}$$

153 for all  $\Psi \in H^u$ .

154 **1.2. The topology optimization problem.** A goal of this paper is to analyze  
 155 the next topology optimization problem

$$156 \quad (\text{OPT}) \quad \begin{aligned} &\min \mathcal{J}(\alpha, \mathbf{u}, \theta, p) \\ &\text{s.t.} \quad \begin{cases} (\mathbf{u}, \theta, p) \text{ solution of (WF) parametrized by } \alpha, \\ \alpha \in \mathcal{U}_{\text{ad}}, \end{cases} \end{aligned}$$

157 where  $\mathcal{J}$  is a given cost function and, for some  $\kappa > 0$ , we set  $\mathcal{U}_{\text{ad}} = \{\alpha \in \text{BV}(\Omega)$   
 158  $: 0 \leq \alpha(x) \leq \alpha_{\text{max}} \text{ a.e. on } \Omega, |D\alpha|(\Omega) \leq \kappa\}$ .  $\text{BV}(\Omega)$  stands for functions of bounded

159 variations, as exposed in [2]. We recall that the weak-\* convergence in  $BV(\Omega)$  is  
 160 defined as follows [2]:  $(\alpha_\varepsilon)_\varepsilon \subset BV(\Omega)$  weakly-\* converges to  $\alpha \in BV(\Omega)$  if  $(\alpha_\varepsilon)$   
 161 strongly converges to  $\alpha$  in  $L^1(\Omega)$  and  $(D\alpha_\varepsilon)$  weakly-\* converges to  $D\alpha$  in  $\Omega$ , meaning:

$$162 \quad \lim_{\varepsilon \rightarrow +\infty} \int_{\Omega} \nu dD\alpha_\varepsilon = \int_{\Omega} \nu dD\alpha, \quad \forall \nu \in C_0(\Omega),$$

163 where  $C_0(\Omega)$  denotes the closure, in the sup norm, of the set of real continuous  
 164 functions with compact support over  $\Omega$ . We choose  $\mathcal{U}_{ad}$  as a subset of  $BV(\Omega)$  since  
 165 it is a nice way to approximate piecewise constant functions, which is close to the  
 166 desired solid distribution.

167 It is classical for these problems to compute first order optimality conditions  
 168 (see e.g. [33, 44]). This approach needs smoothness of the control-to-state mapping.  
 169 However, the presence of the non-differentiable function  $\text{neg}(x) = x^-$  makes this  
 170 approach impossible. Therefore, we adopt a smoothing approach, as studied in [35,  
 171 45], and we approximate the  $\text{neg}$  function with a  $C^1$  positive approximation, denoted  
 172  $\text{neg}_\varepsilon$ . We suppose this approximation satisfies the following assumptions:

173 **(A1)**  $\forall x \in \mathbb{R}, \text{neg}_\varepsilon(x) \geq \text{neg}(x)$ .

174 **(A2)**  $\forall x \in \mathbb{R}, 0 \leq \text{neg}'_\varepsilon(x) \leq 1$ .

175 **(A3)**  $\text{neg}_\varepsilon$  converges to  $\text{neg}$  uniformly over  $\mathbb{R}$ .

176 **(A4)** for every  $\delta > 0$ , the sequence  $(\text{neg}'_\varepsilon)_{\varepsilon > 0}$  converges uniformly to 1 on  $[\delta, +\infty)$   
 177 and uniformly to 0 on  $(-\infty, -\delta]$  as  $\varepsilon \rightarrow +\infty$ .

178 As presented in [45], we may choose:

$$179 \quad (1.3) \quad \text{neg}_\varepsilon(x) = \begin{cases} x^- & \text{if } |x| \geq \frac{1}{2\varepsilon}, \\ \frac{1}{2} \left(\frac{1}{2} - \varepsilon x\right)^3 \left(\frac{3}{2\varepsilon} + x\right) & \text{if } |x| < \frac{1}{2\varepsilon}. \end{cases}$$

180 We thus redefine (WF) with an approximation of  $\text{neg}$ , which gives:

$$181 \quad (\text{WFe.1}) \quad \begin{aligned} & \int_{\Omega} (\partial_t \mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon) \cdot \Psi + A \int_{\Omega} \nabla \mathbf{u}_\varepsilon : \nabla \Psi - \int_{\Omega} B \theta e_y \cdot \Psi \\ & + \int_{\Omega} h(\alpha_\varepsilon) \mathbf{u}_\varepsilon \cdot \Psi - \int_{\Omega} p_\varepsilon \nabla \cdot \Psi + \frac{1}{2} \int_{\Gamma_{out}} \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) (\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^{\text{ref}}) \cdot \Psi \\ & = \int_{\Omega} f \cdot \Psi - \int_{\Gamma_{out}} (A \partial_n \mathbf{u}^{\text{ref}} + \mathbf{n} p^{\text{ref}}) \cdot \Psi, \end{aligned}$$

182

$$183 \quad (\text{WFe.2}) \quad \int_{\Omega} \mathbf{u}_\varepsilon \cdot \nabla q = 0,$$

184

$$185 \quad (\text{WFe.3}) \quad \begin{aligned} & \int_{\Omega} \partial_t \theta_\varepsilon \varphi - \int_{\Omega} \theta_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \varphi + \int_{\Omega} Ck \nabla \theta_\varepsilon \cdot \nabla \varphi \\ & + \int_{\Gamma_{out}} ((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})) \theta_\varepsilon \varphi = \int_{\Gamma_w} \phi \varphi, \end{aligned}$$

186 for all  $(\Psi, \varphi, q) \in H^u \times H^\theta \times L^2(\Omega)$ . We then define the approximate optimal control  
 187 problem:

$$188 \quad (\text{OPT}_\varepsilon) \quad \begin{aligned} & \min \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon) \\ & \text{s.t.} \quad \begin{cases} (\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon) \text{ solution of (WFe) parametrized by } \alpha_\varepsilon, \\ \alpha_\varepsilon \in \mathcal{U}_{ad}. \end{cases} \end{aligned}$$

189 As it will be made clear later, the control-to-state mapping in (WFe) is smooth, which  
 190 will let us derive first order conditions.

191 **1.3. Plan of the paper.** The rest of this introduction is dedicated to the pre-  
 192 sentation of some notations used in this article and some important results of the  
 193 literature. The core of this paper is organized in two sections. First, we will prove the  
 194 existence of solutions to (WFe), which will let us prove, with a compactness argument,  
 195 the existence of solutions to (WF). We then focus on the two dimensional case, where  
 196 we prove uniqueness of the solutions along with stronger convergence results. This  
 197 is an extension of the work done by [13], where only the pressure and the velocity  
 198 where considered, and to [6, 15], where the steady case was studied in depth, but the  
 199 results concerning the unsteady case were obtained using restrictive assumptions. We  
 200 then study the approximate optimal control problem (OPTe), for which we will derive  
 201 first order conditions. We conclude this paper with the convergence of the optimality  
 202 conditions of (OPTe), which let us design first order conditions of (OPT).

203 *Notations.* We denote by  $a \lesssim b$  if there exists a constant  $C(\Omega) > 0$  depending  
 204 only on  $\Omega$  such that  $a \leq C(\Omega)b$ . Denote:

- 205 •  $\mathcal{A} : V^u \rightarrow (V^u)'$  defined by  $\langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_{(V^u)', V^u} = A \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v}$ ,
- 206 •  $\mathcal{B} : V^u \times V^u \rightarrow (V^u)'$  defined by  $\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w}$ ,
- 207 •  $\mathcal{T} : V^{\theta} \rightarrow (V^u)'$  defined by  $\langle \mathcal{T}\theta, \mathbf{v} \rangle_{(V^u)', V^u} = \int_{\Omega} B\theta e_y \cdot \mathbf{v}$ ,
- 208 •  $\mathcal{P} : L^2(\Omega) \rightarrow (V^u)'$  defined by  $\langle \mathcal{P}p, \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Omega} p \nabla \cdot \mathbf{w}$ ,
- 209 •  $\mathcal{N} : V^u \times V^u \rightarrow (V^u)'$  defined by  $\langle \mathcal{N}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{neg}(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{w})$ ,
- 210 •  $\mathcal{N}_{\varepsilon} : V^u \times V^u \rightarrow (V^u)'$  given by  $\langle \mathcal{N}_{\varepsilon}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{neg}_{\varepsilon}(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{w})$ .
- 211 •  $\mathcal{C}(\alpha) : V^{\theta} \rightarrow (V^{\theta})'$  defined by  $\langle \mathcal{C}(\alpha)\theta, \varphi \rangle_{(V^{\theta})', V^{\theta}} = \int_{\Omega} Ck(\alpha) \nabla \theta \cdot \nabla \varphi$ ,
- 212 •  $\mathcal{D} : V^u \times V^{\theta} \rightarrow (V^{\theta})'$  defined by  $\langle \mathcal{D}(\mathbf{u}, \theta), \varphi \rangle_{(V^{\theta})', V^{\theta}} = \int_{\Omega} \theta \mathbf{u} \cdot \nabla \varphi$ ,
- 213 •  $\mathcal{M} : V^u \times V^{\theta} \rightarrow (V^{\theta})'$  defined by  $\langle \mathcal{M}(\mathbf{u}, \theta), \varphi \rangle_{(V^{\theta})', V^{\theta}} = \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}(\mathbf{u} \cdot \mathbf{n})) \theta \varphi$ ,
- 214 •  $\mathcal{M}_{\varepsilon} : V^u \times V^{\theta} \rightarrow (V^{\theta})'$  defined by  $\langle \mathcal{M}_{\varepsilon}(\mathbf{u}, \theta), \varphi \rangle_{(V^{\theta})', V^{\theta}} = \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}_{\varepsilon}(\mathbf{u} \cdot \mathbf{n})) \theta \varphi$ ,

215 By a slight abuse of notation, we will still denote by  $\sigma^{\text{ref}}$  the element of  $(V^u)'$  defined  
 216 by  $\langle \sigma^{\text{ref}}, \mathbf{w} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} (A \partial_n \mathbf{u}^{\text{ref}} - p^{\text{ref}} \mathbf{n}) \cdot \mathbf{w}$ ,  $h(\alpha) : V^u \rightarrow (V^u)'$  the function  
 217 defined by  $\langle h(\alpha)\mathbf{u}, \mathbf{v} \rangle_{(V^u)', V^u} = \int_{\Omega} h(\alpha) \mathbf{u} \cdot \mathbf{v}$ , and  $\phi$  the element of  $(V^{\theta})'$  defined by  
 218  $\langle \phi, \varphi \rangle_{(V^{\theta})', V^{\theta}} = \int_{\Gamma_{\text{out}}} \phi \varphi$ .

219 *Results from the literature.* We now recall two results from the literature that will  
 220 be heavily used throughout this paper.

221 **PROPOSITION 1.1.** ([10, Proposition III.2.35]) *Let  $\Omega$  be a Lipschitz domain of*  
 222  *$\mathbb{R}^d$  with compact boundary. Let  $p \in [1, +\infty]$  and  $q \in [p, p^*]$ , where  $p^*$  is the critical*  
 223 *exponent associated with  $p$ , defined as:*

$$224 \begin{cases} \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} & \text{for } p < d, \\ p^* \in [1, +\infty[ & \text{for } p = d, \\ p^* = +\infty & \text{for } p > d. \end{cases}$$

225 *Then, there exists a positive constant  $C$  such that, for any  $u \in W^{1,p}(\Omega)$ :*

$$226 \|u\|_{L^q(\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1 + \frac{d}{q} - \frac{d}{p}} \|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p} - \frac{d}{q}}.$$



230 PROPOSITION 1.2. ([10, Theorem III.2.36]) Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^d$   
 231 with compact boundary, and  $1 < p < d$ . Then for any  $r \in \left[ p, \frac{p(d-1)}{d-p} \right]$ , there exists a  
 232 positive constant  $C$  such that, for any  $u \in W^{1,p}(\Omega)$ :

$$233 \quad \|u|_{\partial\Omega}\|_{L^r(\partial\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1-\frac{d}{p}+\frac{d-1}{r}} \|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p}-\frac{d-1}{r}}.$$

234 In the case  $p = d$ , the previous result holds true for any  $r \in [p, +\infty[$ .

235 **2. Existence of solutions.** In this section, we will focus on proving the existence  
 236 of solutions to (WFe) and prove their convergence toward the ones of (WF).

237 We make the following assumptions throughout this paper:

238 ASSUMPTIONS 2.1. • The source term  $f \in L^2(0, T; H^{-1}(\Omega))$ .

239 •  $(\mathbf{u}^{ref}, p^{ref})$  are such that:

$$240 \quad \begin{cases} \mathbf{u}^{ref} \in L^r(0, T; (H^1(\Omega))^d) \cap L^\infty(0, T; (L^2(\Omega))^d) \\ \text{with } r = 2 \text{ if } d = 2 \text{ and } r = 4 \text{ if } d = 3, \\ \nabla \cdot \mathbf{u}^{ref} = 0, \\ \partial_t \mathbf{u}^{ref} \in L^2(0, T; (L^2(\Omega))^d), \\ \mathbf{u}^{ref} = \mathbf{u}_{in} \text{ on } \Gamma_{in}. \end{cases}$$

- 241 • There exists  $k_{min}$  such that  $k(x) \geq k_{min} > 0$  and  $h(x) \geq 0$  for a.e.  $x \in \Omega$ .
- 242 • The initial condition  $\mathbf{u}_0$  (resp.  $\theta_0$ ) is a Fréchet-differentiable function from
- 243  $\mathcal{U}_{ad}$  to  $V^u$  (resp.  $V^\theta$ ). Furthermore, for all  $\alpha \in \mathcal{U}_{ad}$ ,  $\mathbf{u}_0(\alpha)|_{\Gamma_{in}} = \mathbf{u}_{in}(0)$ ,
- 244  $\mathbf{u}_0(\alpha)|_{\Gamma_w} = 0$ ,  $\nabla \cdot \mathbf{u}_0(\alpha) = 0$  and  $\theta_0(\alpha)|_{\Gamma_{in}} = 0$ .
- 245 •  $\beta \in L^\infty(0, T; L^\infty(\Gamma_{out}))$  such that  $\beta(t, x) \geq \frac{1}{2}$ , for a.e.  $(t, x) \in [0, T] \times \Gamma_{out}$ .

246 **2.1. Existence in dimension 2 or 3.** In this part, we work with a fixed  $\varepsilon > 0$   
 247 and a given  $\alpha_\varepsilon$  in  $\mathcal{U}_{ad}$ .

248 In order to prove the existence of solutions to (WFe), we follow the classical Fedeo-  
 249 Galerkin method, as used in [13, 36, 47]. By construction,  $V^u$  and  $V^\theta$  are separable.  
 250 Therefore, both admit a countable Hilbert basis  $(w_k^u)_k$  and  $(w_k^\theta)_k$ . Let us construct  
 251 an approximate problem, which will converge to a solution of the original problem  
 252 (WFe). Denote by  $V_n^u$  (resp.  $V_n^\theta$ ) the space spanned by  $(w_k^u)_{k \leq n}$  (resp.  $(w_k^\theta)_{k \leq n}$ ).

253 We consider the following Galerkin approximated problem:

254 find  $t \mapsto \mathbf{v}_n(t) \in V_n^u$ ,  $t \mapsto p_n(t) \in L^2(\Omega)$  and  $t \mapsto \theta_n(t) \in V_n^\theta$  such that, defining  
 255  $\mathbf{u}_n = \mathbf{v}_n + \mathbf{u}^{ref}$ ,  $(\mathbf{u}_n, p_n, \theta_n)$  satisfy (WFe) for all  $t \in [0, T]$  and for all  $(\Psi, q, \varphi) \in$   
 256  $V_n^u \times L^2(\Omega) \times V_n^\theta$ .

257 As done in [47], we prove that such  $(\mathbf{u}_n, \theta_n, p_n)$  exist. We now prove that these  
 258 solutions are bounded with respect to  $n$  and  $\varepsilon$ :

259 PROPOSITION 2.2. There exist positive constants  $c_1^\theta$ ,  $c_2^\theta$ ,  $c_1^\Psi$  and  $c_2^\Psi$ , independent  
 260 of  $\varepsilon$  and  $n$ , such that:

$$261 \quad (2.1) \quad \sup_{[0, T]} \|\theta_n\|_{L^2(\Omega)} \leq c_1^\theta, \quad 265 \quad (2.3) \quad \sup_{[0, T]} \|\mathbf{v}_n\|_{L^2(\Omega)} \leq c_1^\Psi,$$

$$262 \quad (2.2) \quad \int_0^T \|\nabla \theta_n\|_{L^2(\Omega)}^2 \leq c_2^\theta, \quad 266 \quad (2.4) \quad \int_0^T \|\nabla \mathbf{v}_n\|_{L^2(\Omega)}^2 \leq c_2^\Psi.$$



268 *Proof.* Taking  $\varphi_n = \theta_n$  in (WFe.1) and integrating by part give:

$$269 \quad \begin{aligned} & \frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 - \frac{1}{2} \int_{\Gamma_{\text{out}}} \theta_n^2 (\mathbf{u}_n \cdot \mathbf{n}) + \int_{\Omega} Ck |\nabla \theta_n|^2 \\ & + \int_{\Gamma_{\text{out}}} ((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_{\varepsilon} (\mathbf{u}_n \cdot \mathbf{n})) \theta_n^2 = \int_{\Gamma_w} \phi \theta_n. \end{aligned}$$

270 Since  $\beta \geq \frac{1}{2}$  and using assumption (A1), one has on  $\Gamma_{\text{out}}$ :

$$271 \quad \begin{aligned} ((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_{\varepsilon} (\mathbf{u}_n \cdot \mathbf{n})) \theta_n^2 - \frac{1}{2} (\mathbf{u}_n \cdot \mathbf{n}) \theta_n^2 & \geq \frac{1}{2} ((\mathbf{u}_n \cdot \mathbf{n}) + \text{neg}_{\varepsilon} (\mathbf{u}_n \cdot \mathbf{n})) \theta_n^2 \\ & \geq \frac{1}{2} (\mathbf{u}_n \cdot \mathbf{n})^+ \theta_n^2 \geq 0. \end{aligned}$$

272 Therefore:  $\frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 + Ck_{\min} \|\nabla \theta_n\|_{L^2(\Omega)}^2 \leq \|\phi\|_{L^2(\Gamma_w)} \|\theta_n\|_{L^2(\Gamma_w)}$ . Using continuity  
273 of the trace operator and Young's inequality, one proves that there exists a positive  
274 constant  $C(\Omega)$  such that, for any  $\nu > 0$ :

$$275 \quad \frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 + Ck_{\min} \|\nabla \theta_n\|_{L^2(\Omega)}^2 \leq \frac{1}{2\nu} \|\phi\|_{L^2(\Gamma_w)}^2 + \frac{C(\Omega)\nu}{2} (\|\theta_n\|_{L^2(\Omega)}^2 + \|\nabla \theta_n\|_{L^2(\Omega)}^2).$$

276 Taking  $\nu$  small enough, we are left with:

$$277 \quad \frac{d}{dt} \|\theta_n\|_{L^2(\Omega)}^2 \leq \frac{1}{2\nu} \|\phi\|_{L^2(\Gamma_w)}^2 + \frac{C(\Omega)\nu}{2} \|\theta_n\|_{L^2(\Omega)}^2.$$

278 Integrating this equation and using Gronwall's lemma then give (2.1) and (2.2).

279 Now, take  $\Psi_n = \mathbf{v}_n$  in (WFe.3). After some calculations, one gets:

$$280 \quad \begin{aligned} & \frac{d}{dt} |\mathbf{v}_n|^2 + A |\nabla \mathbf{v}_n|^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}} \text{neg}_{\varepsilon} (\mathbf{u}_n \cdot \mathbf{n}) |\mathbf{v}_n|^2 + \int_{\Omega} h |\mathbf{v}_n|^2 \\ & = \int_{\Omega} f_{\theta} \cdot \mathbf{v}_n - \int_{\Omega} \partial_t \mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n - A \int_{\Omega} \nabla \mathbf{u}^{\text{ref}} : \nabla \mathbf{v}_n + \int_{\Omega} (\mathbf{u}_n \cdot \nabla) \mathbf{v}_n \cdot \mathbf{u}^{\text{ref}} \\ & - \int_{\Omega} h \mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n + \int_{\Gamma_{\text{out}}} (A \partial_n \mathbf{u}^{\text{ref}} - \mathbf{n} p^{\text{ref}}) \mathbf{v}_n \end{aligned}$$

281 where  $f_{\theta} = f + B\theta_n e_y$ . First, using (2.2), one has  $\|f_{\theta}\|_{(H^u)'} \leq \|f\|_{(H^u)'} + Bc_1^{\theta}$ .  
282 Secondly, using (A1) gives that  $\int_{\Gamma_{\text{out}}} \text{neg}_{\varepsilon} (\mathbf{u}_n \cdot \mathbf{n}) |\mathbf{v}_n|^2 \geq 0$  and following then the  
283 same pattern of proof as in [13, Proposition 2], one proves (2.3) and (2.4).  $\square$

284 Following [47, 10], we need to bound the fractional derivatives of the solution in  
285 order to prove some convergence results. For any real-valued function  $f$  defined on  
286  $[0, T]$ , define by  $\tilde{f}$  the extension by 0 of  $f$  to the whole real line  $\mathbb{R}$ , and by  $\mathcal{F}(\tilde{f})$   
287 the Fourier transform of  $\tilde{f}$ , which we define as:  $\mathcal{F}(\tilde{f})(\tau) = \int_{\mathbb{R}} \tilde{f}(t) e^{-it\tau} dt$ . Using the  
288 Hausdorff-Young inequality [10, Theorem II.5.20] we can prove the

289 PROPOSITION 2.3. For all  $\sigma \in [0, \frac{1}{6}]$ , there exists a constant  $C(\sigma) > 0$  indepen-  
290 dent of  $\varepsilon$  and  $n$  such that:

$$291 \quad (2.5) \quad \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F}(\tilde{\theta}_n) \right\|_{(L^2(\Omega))^d}^2 \leq C(\sigma),$$

292

$$293 \quad (2.6) \quad \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F}(\tilde{\mathbf{u}}_n) \right\|_{L^2(\Omega)}^2 \leq C(\sigma).$$

294 *Proof.* We emphasize that (2.6) is proved if (2.5) holds by using [10, Proposition  
 295 VII.1.3] by replacing  $f$  by  $f_\theta = f + B\theta e_y$ . The proof of (2.5) consists in adapting the  
 296 one of [10, Proposition VII.1.3] and is thus omitted.  $\square$

297 Combining the two previous results, we can now prove the following existence  
 298 theorem for (WFe).

299 **THEOREM 2.4.** *For all  $(\mathbf{v}_0, \theta_0) \in H^u \times H^\theta$  and all  $T > 0$ , there exists  $\mathbf{v}_\varepsilon \in$   
 300  $L^\infty(0, T; H^u) \cap L^2(0, T; V^u)$ ,  $\theta_\varepsilon \in L^\infty(0, T; H^\theta) \cap L^2(0, T; V^\theta)$  and  $p_\varepsilon \in W^{-1, \infty}(0, T;$   
 301  $L^2(\Omega))$  solution of (WFe) such that, defining  $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{u}^{ref}(0)$  and  $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{u}^{ref}$ ,  
 302 one has for all  $(\Psi, \varphi) \in V^u \times V^\theta$  such that  $\nabla \cdot \Psi = 0$ :  $(\int_\Omega \mathbf{u}_\varepsilon \cdot \Psi)(0) = \int_\Omega \mathbf{u}_0 \cdot \Psi$ ,  
 303  $(\int_\Omega \theta_\varepsilon \varphi)(0) = \int_\Omega \theta_0 \varphi$ . Moreover, one has  $\mathbf{v}'_\varepsilon = \frac{d\mathbf{v}_\varepsilon}{dt} \in L^{\frac{4}{3}}(0, T; (V^u)')$  and  $\theta'_\varepsilon \in$   
 304  $L^2(0, T; (V^\theta)')$ .*

305 *Proof.* The proof of existence is similar to part (iv) of the proof of [47, Theorem  
 306 3.1] and the proof of [10, Proposition VII.1.4], where estimates (2.1)-(2.4) and (2.5)-  
 307 (2.6) are used in a compactness argument.

308 We only add the proof that  $(\mathbf{u}_n, \theta_n)$  converges to a solution of (WFe.1). Using  
 309 (2.5) and [47, Theorem 2.2], one shows that, up to a subsequence,  $\theta_n$  strongly  
 310 converges to an element  $\theta_\varepsilon$  of  $L^2(0, T; H^\theta)$ . The only technical points which needs more  
 311 detail are the non-linear terms in (WFe.1). Using the strong convergence of  $\mathbf{u}_n$  to  $\mathbf{u}_\varepsilon$   
 312 in  $L^2(0, T; H^u)$  proved in [47, Eq (3.41)], one proves that  $(\theta_n \mathbf{u}_n)$  strongly converges  
 313 to  $\theta_\varepsilon \mathbf{u}_\varepsilon$  in  $L^1(0, T; L^2(\Omega))$ . Furthermore, notice that:

$$\begin{aligned} \int_0^T \|(\mathbf{u}_n \cdot \mathbf{n})\theta_n\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} &\leq \int_0^T \|\mathbf{u}_n\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} \|\theta_n\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} \\ &\leq C \int_0^T \|\mathbf{u}_n\|_{L^2(\Omega)}^{\frac{1}{3}} \|\theta_n\|_{L^2(\Omega)}^{\frac{1}{3}} \|\mathbf{u}_n\|_{H^1(\Omega)} \|\theta_n\|_{H^1(\Omega)} \\ &\leq C \|\mathbf{u}_n\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{3}} \|\theta_n\|_{L^\infty(0, T; L^2(\Omega))}^{\frac{1}{3}} \\ &\quad \|\mathbf{u}_n\|_{L^2(0, T; H^1(\Omega))} \|\theta_n\|_{L^2(0, T; H^1(\Omega))}. \end{aligned}$$

315 This inequality together with (2.1)-(2.4) proves that  $((\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$  is bounded in  
 316  $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma))$ , which is reflexive. Therefore, it proves that, up to a subsequence,  
 317 there exists a weak limit  $\kappa_1$  in  $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma))$  of  $((\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$ . A simple adapta-  
 318 tion of the above reasoning proves that  $(\text{neg}_\varepsilon(\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$  weakly converges to some  
 319  $\kappa_2$  in  $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma))$ . Using the strong convergence of  $\theta_n$  in  $L^2(0, T; L^2(\Omega))$ , [10,  
 320 Proposition II.2.12] implies that:

$$321 \quad ((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_n \cdot \mathbf{n}))\theta_n \rightharpoonup ((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}))\theta_\varepsilon \text{ in } L^{\frac{4}{3}}(0, T; L^1(\Gamma))$$

322 obtained using the continuity of  $x \in \mathbb{R} \mapsto \text{neg}_\varepsilon(x)$ . By uniqueness of the limit in  
 323 the sense of distribution, we can identify  $\kappa_1 + \beta\kappa_2$  with  $((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}))\theta_\varepsilon$ .  
 324 Therefore,  $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$  is a solution of (WFe.1).

325 The convergence of the weak derivative with respect to time of  $\mathbf{v}_\varepsilon$  in  $L^{\frac{4}{3}}(0, T;$   
 326  $(V^u)')$  is proved in [10, Proposition V.1.3]. Concerning the weak derivative with  
 327 respect to time of  $\theta_\varepsilon$ , it follows immediately from the fact that differentiation with  
 328 respect to time is continuous in the sense of distribution. Existence of the pressure  
 329  $p_\varepsilon$  follows from [10, Chapter V].  $\square$

330 We now use the existence of solutions to the approximate problem (WFe) to prove  
 331 existence of solutions to the limit problem (WF), along with the convergence of the  
 332 approximate solutions to the solutions of (WF).

333 THEOREM 2.5. Let  $(\alpha_\varepsilon) \subset \mathcal{U}_{ad}$  and  $\alpha \in \mathcal{U}_{ad}$  such that  $\alpha_\varepsilon \xrightarrow{*} \alpha$  in BV. Define by  
 334  $(\mathbf{v}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$  a solution of (WFe) parametrized by  $\alpha_\varepsilon$ , and define  $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{u}^{\text{ref}}$ . Then,  
 335 there exists  $(\mathbf{v}, \theta, p) \in L^\infty(0, T; H^u) \cap L^2(0, T; V^u) \times L^\infty(0, T, H^\theta) \cap L^2(0, T; V^\theta) \times$   
 336  $W^{-1, \infty}(0, T; L^2(\Omega))$  such that, defining  $\mathbf{u} = \mathbf{v} + \mathbf{u}^{\text{ref}}$ , up to a subsequence, we have

- 337 •  $\mathbf{u}_\varepsilon \xrightarrow{*} \mathbf{u}$  in  $L^\infty(0, T; H^u)$  and  $\theta_\varepsilon \xrightarrow{*} \theta$  in  $L^\infty(0, T; H^\theta)$ ,
- 338 •  $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$  in  $L^2(0, T; V^u)$  and in  $L^2(0, T; (L^6(\Omega)))$ ,
- 339 •  $\theta_\varepsilon \rightharpoonup \theta$  in  $L^2(0, T; V^\theta)$  and in  $L^2(0, T; (L^6(\Omega)))$ ,
- 340 •  $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$  in  $L^4(0, T; (L^2(\Gamma))^d)$  and  $\theta_\varepsilon \rightharpoonup \theta$  in  $L^4(0, T; (L^2(\Gamma)))$ ,
- 341 •  $\mathbf{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} \mathbf{u}$  in  $L^2(0, T; (L^2(\Omega))^d)$  and  $\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} \theta$  in  $L^2(0, T; (L^2(\Omega)))$ ,
- 342 •  $\mathbf{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} \mathbf{u}$  in  $L^2(0, T; (L^2(\Gamma))^d)$  and  $\theta_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} \theta$  in  $L^2(0, T; (L^2(\Gamma)))$ ,
- 343 •  $p_\varepsilon \rightharpoonup p$  in  $L^{\frac{4}{3}}(0, T; L^2(\Omega))$ .

344 Furthermore,  $(\mathbf{v}, \theta, p)$  is a solution to (WF) parametrized by  $\alpha$ .

345 *Proof.* Using (2.1)-(2.4) and (2.5)-(2.6), we prove that there exists  $\mathbf{u}$  and  $\theta$  such  
 346 that all the convergences above are verified in the same manner as in [10, Propo-  
 347 sition VII.1.4]. Let us prove first that  $\mathbf{u}$  is a solution of (WF.3) parametrized  
 348 by  $\alpha$  and  $\theta$ . With the same pattern of proof as in Theorem 2.4, one proves im-  
 349 mediately that  $(\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \rightharpoonup (\mathbf{u} \cdot \nabla) \mathbf{u}$  in  $L^1(0, T; (L^1(\Omega))^d)$ , and  $(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \mathbf{u}^{\text{ref}} \rightharpoonup$   
 350  $(\mathbf{u} \cdot \mathbf{n}) \mathbf{u}^{\text{ref}}$  in  $L^4(0, T; (L^{\frac{4}{3}}(\Gamma))^d)$ . Regarding the penalization term:

$$351 \quad \begin{aligned} \|h(\alpha_\varepsilon) \mathbf{u}_\varepsilon - h(\alpha) \mathbf{u}\|_{L^2(0, T; L^2(\Omega)^d)}^2 &\lesssim \|h\|_\infty^2 \|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^2(0, T; L^2(\Omega)^d)}^2 \\ &+ \int_0^T \int_\Omega (h(\alpha_\varepsilon) - h(\alpha))^2 |\mathbf{u}|^2. \end{aligned}$$

352 Since  $\alpha_\varepsilon \rightarrow \alpha$  strongly in  $L^1(\Omega)$ ,  $h(\alpha_\varepsilon) \rightarrow h(\alpha)$  pointwise in  $\Omega$  up to a subsequence  
 353 (which is not relabeled). Lebesgue dominated convergence theorem then implies:  
 354  $h(\alpha_\varepsilon) \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} h(\alpha) \mathbf{u}$  in  $L^2(0, T; (L^2(\Omega))^d)$ .

355 Concerning the boundary terms, we only consider the term with the approxima-  
 356 tion of the neg function. First, we claim that there exists  $\gamma$  such that  $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon +$   
 357  $\mathbf{u}^{\text{ref}}) \rightharpoonup \gamma$  in  $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma)^d)$ . Notice that, for  $\varepsilon$  large enough and using the proper-  
 358 ties of the neg approximation, we have:

$$(2.7) \quad \begin{aligned} \int_0^T \|\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon + \mathbf{u}^{\text{ref}})\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} &\lesssim \int_0^T \left( \|\mathbf{u}_\varepsilon \cdot \mathbf{n} + 1\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} \|\mathbf{u}_\varepsilon + \mathbf{u}^{\text{ref}}\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} \right) \\ &\lesssim \int_0^T \left( \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} + C \right) \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} \\ &+ \int_0^T \left( \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} + C \right) \|\mathbf{u}^{\text{ref}}\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{4}{3}} \\ &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{8}{3}} + 2 \left( \int_0^T \|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{8}{3}} \right)^{\frac{1}{2}} + C. \end{aligned}$$

360 In addition, from Proposition 1.2, we have  $\|\mathbf{u}_\varepsilon\|_{L^{\frac{8}{3}}(\Gamma)}^{\frac{8}{3}} \lesssim \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{8}{3}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ . Since  
 361  $\mathbf{u}_\varepsilon$  is bounded in  $L^\infty(0, T; (L^2(\Omega))^d)$  and in  $L^2(0, T; (H^1(\Omega))^d)$  as proved in Propo-  
 362 sition 2.2, we see that  $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon + \mathbf{u}^{\text{ref}})$  is bounded in  $L^{\frac{4}{3}}(0, T; L^{\frac{4}{3}}(\Gamma)^d)$  in-  
 363 dependently of  $\varepsilon$ . Since this Banach space is reflexive, it proves the claimed weak  
 364 convergence. Let us now prove that  $\gamma$  can be identified with  $(\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} + \mathbf{u}^{\text{ref}})$ .

365 First, since  $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$  strongly in  $L^2(0, T; L^2(\Gamma)^d)$  and  $\text{neg}_\varepsilon(\cdot) \rightarrow (\cdot)^-$  uniformly,  
 366 one proves that  $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \rightarrow (\mathbf{u} \cdot \mathbf{n})^-$  strongly in  $L^2(0, T; L^2(\Gamma))$ . Then, the  
 367 weak convergence of  $\mathbf{u}_\varepsilon$  in  $L^4(0, T; L^2(\Gamma)^d)$  and [10, Proposition II.2.12] implies that  
 368  $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon + \mathbf{u}^{\text{ref}}) \rightharpoonup (\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} + \mathbf{u}^{\text{ref}})$  weakly in  $L^{\frac{4}{3}}(0, T; L^1(\Gamma)^d)$ . Using [10,  
 369 Proposition II.2.9], we argue that  $\gamma = (\mathbf{u} \cdot \mathbf{n})^-(\mathbf{u} + \mathbf{u}^{\text{ref}})$ .

370 Regarding  $p_\varepsilon$ , we use an inf-sup condition as the one introduced in the proof of  
 371 [28, Theorem 5.1, eq. (5.14)], which states that

$$372 \quad (2.8) \quad \|p\|_{L^2(\Omega)} \lesssim \sup_{\Psi \in V} \frac{\int_{\Omega} p \nabla \cdot \Psi}{\|\Psi\|_{H^1(\Omega)}}.$$

373 Therefore, using (WFe.3), one shows that:

$$374 \quad \|p_\varepsilon\|_{L^2(\Omega)} \lesssim \|\partial_t \mathbf{u}_\varepsilon\|_{V'} + \|\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)\|_{V'} + \|\mathcal{A}\mathbf{u}_\varepsilon\|_{V'} + \|h(\alpha)\mathbf{u}_\varepsilon\|_{V'} + \|\mathcal{T}\theta\|_{V'} \\ + \|\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\|_{V'} + \|f\|_{V'} + \|\sigma^{\text{ref}}\|_{V'}.$$

375 We now bound each term depending on  $\varepsilon$ :

- 376 • Since the Stokes operator is continuous,  $\|\mathcal{A}\mathbf{u}_\varepsilon\|_{V'} \lesssim \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$  and therefore,  
 377  $\mathcal{A}\mathbf{u}_\varepsilon$  is bounded in  $L^2(0, T; V')$ .
- 378 • Using [10, Eq. (V.3)], we prove that  $\|\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)\|_{V'} \lesssim \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{2-\frac{d}{2}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^{\frac{d}{2}}$ ,  
 379 which in turn shows that  $\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)$  is bounded in  $L^{\frac{4}{d}}(0, T; V')$ .
- 380 • Obviously,  $\|h(\alpha)\mathbf{u}_\varepsilon\|_{V'} \leq \|h\|_\infty \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}$  and therefore,  $h(\alpha)\mathbf{u}_\varepsilon$  is bounded  
 381 in  $L^\infty(0, T; V')$ .

382 We are left with the boundary term. Let  $0 \neq \Psi \in V$ . In a similar manner as before  
 383 and using Proposition 1.2, there exists a constant  $C > 0$  such that:

$$384 \quad \frac{1}{\|\Psi\|_{H^1(\Omega)}} \int_{\Gamma_{\text{out}}} |\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \Psi| \lesssim \left( \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{3-d}{4}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^{\frac{d-1}{4}} + C \right)^2.$$

385 As proved before,  $\mathbf{u}_\varepsilon$  is bounded in  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ . Therefore,  $\mathbf{u}_\varepsilon$   
 386 is also bounded in  $L^{2-\frac{2}{d}}(0, T; H^1(\Omega))$ . Taking the supremum over  $\Psi$ , this proves that  
 387  $\mathcal{N}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)$  is bounded in  $L^{\frac{4}{d}}(0, T; V')$ . Finally, in a similar fashion as in [10, Lemma  
 388 V.1.6], the above bounds prove that  $\partial_t \mathbf{u}_\varepsilon$  is bounded in  $L^{\frac{4}{d}}(0, T; V')$ . These bounds  
 389 prove that  $(p_\varepsilon)$  is bounded in  $L^{\frac{4}{d}}(0, T; L^2(\Omega))$ , and therefore  $(p_\varepsilon)$  weakly converges to  
 390 some  $p$  in  $L^{\frac{4}{d}}(0, T; L^2(\Omega))$ .

391 Concerning  $\theta$ , the convergence is largely proved in the same way as in Theorem 2.4.  
 392 The only difference concerns the convergence of  $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})\theta_\varepsilon$  to  $(\mathbf{u} \cdot \mathbf{n})^-\theta$ , which  
 393 is proved in the same manner as (2.7). All these convergence results let us say that  
 394  $(\mathbf{u}, \theta, p)$  is a solution to (WF) in the distribution sense.  $\square$

395 **2.2. Further results in dimension 2.** It is notably known that the solution  
 396 of the Navier-Stokes equations with homogeneous Dirichlet boundary conditions are  
 397 unique in dimension 2. We prove here that uniqueness still holds with the boundary  
 398 conditions (1.2). We only sketch the proof.

399 **PROPOSITION 2.6.** *Let  $d = 2$ . Then, the solution  $(\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$  of (WFe) is unique.*

400 *Sketch of proof* First of all, note that uniqueness of  $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$  implies the uniqueness of  
 401  $p_\varepsilon$  via the De Rham Theorem [10, Theorem IV.2.4 and Chapter V].

402 Let  $(\mathbf{u}_{\varepsilon_1}, \theta_{\varepsilon_1})$  and  $(\mathbf{u}_{\varepsilon_2}, \theta_{\varepsilon_2})$  be two solutions of (WF.1)-(WF.3). Define  $\mathbf{u} = \mathbf{v} =$   
 403  $\mathbf{v}_{\varepsilon_1} - \mathbf{v}_{\varepsilon_2}$  and  $\theta = \theta_{\varepsilon_1} - \theta_{\varepsilon_2}$ . Slightly adapting the proof in [10, Section VII.1.2.5],

404 one proves that:

$$405 \quad (2.9) \quad \frac{d}{dt} |\mathbf{v}|_{L^2(\Omega)}^2 + A |\nabla \mathbf{v}|_{L^2(\Omega)}^2 \lesssim g^v(t) |\mathbf{v}|_{L^2(\Omega)}^2 + B |\theta|_{L^2(\Omega)}^2 + \nu^v |\nabla \mathbf{v}|_{L^2(\Omega)}^2$$

406 where  $\nu^v$  is a positive constant and  $g^v$  is a function in  $L^1([0, T])$ .

407 Testing the differential equation verified by  $\theta$  with  $\theta$  proves that:

$$408 \quad \begin{aligned} & \frac{d}{dt} |\theta|_{L^2(\Omega)}^2 + 2C \int_{\Omega} k |\nabla \theta|^2 + \int_{\Gamma_{\text{out}}} \theta^2 \left( \frac{1}{2} (\mathbf{u}_{\varepsilon_1} \cdot \mathbf{n}) + \beta \text{neg}_{\varepsilon} (\mathbf{u}_{\varepsilon_1} \cdot \mathbf{n}) \right) \\ & = - \int_{\Gamma_{\text{out}}} \left( \beta (\text{neg}_{\varepsilon} (\mathbf{u}_{\varepsilon_1} \cdot \mathbf{n}) - \text{neg}_{\varepsilon} (\mathbf{u}_{\varepsilon_2} \cdot \mathbf{n})) + \frac{1}{2} (\mathbf{u} \cdot \mathbf{n}) \right) \theta_{\varepsilon_2} \theta. \end{aligned}$$

409 With a similar proof as the one of [Proposition 2.2](#), we can prove that, on  $\Gamma_{\text{out}}$ ,  
410  $\theta^2 \left( \frac{1}{2} (\mathbf{u}_1 \cdot \mathbf{n}) + \beta \text{neg}_{\varepsilon} (\mathbf{u}_1 \cdot \mathbf{n}) \right) \geq 0$ . Therefore, using [\(A3\)](#), one has:

$$411 \quad (2.10) \quad \frac{d}{dt} |\theta|_{L^2(\Omega)}^2 + 2C \int_{\Omega} k |\nabla \theta|^2 \lesssim \left( |\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2} \right) |\mathbf{u} \cdot \mathbf{n}|_{L^3(\Gamma_{\text{out}})} |\theta_{\varepsilon_2}|_{L^3(\Gamma_{\text{out}})} |\theta|_{L^3(\Gamma_{\text{out}})}.$$

412 Using Sobolev embeddings and Young inequality, we prove:

$$413 \quad \begin{aligned} & \left( |\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2} \right) |\mathbf{u} \cdot \mathbf{n}|_{L^3(\Gamma_{\text{out}})} |\theta_{\varepsilon_2}|_{L^3(\Gamma_{\text{out}})} |\theta|_{L^3(\Gamma_{\text{out}})} \\ & \lesssim \left( |\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2} \right)^3 \frac{|\theta_{\varepsilon_2}|_{L^2(\Omega)} |\nabla \theta_{\varepsilon_2}|_{L^2(\Omega)}^2}{2(\nu^\theta)^3} (|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2) \\ & \quad + \frac{(\nu^\theta)^{\frac{3}{2}}}{2} \left( |\nabla \mathbf{u}|_{L^2(\Omega)}^2 + |\nabla \theta|_{L^2(\Omega)}^2 \right), \end{aligned}$$

414 where  $\nu^\theta$  is a positive constant. Therefore, summing [\(2.9\)](#) and [\(2.10\)](#) gives  $\frac{d}{dt} (|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2) \lesssim \max(g_1^v, g^\theta) (|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2)$ , with  $g_1^v$  and  $g^\theta$  integrable. Therefore,  
415 applying Gronwall's lemma and noticing that  $|\mathbf{u}(0)|_{L^2(\Omega)}^2 + |\theta(0)|_{L^2(\Omega)}^2 = 0$ , one shows  
416 that  $\mathbf{u} = 0$  and  $\theta = 0$ .  $\square$

417 Note that we may also prove that, for  $d = 2$ , the solution  $(\mathbf{u}, \theta, p)$  of [\(WF\)](#) is  
418 unique. We can also state stronger convergence (compared to the ones stated in  
419 [Theorem 2.5](#)) in dimension 2. These results will be useful in the analysis of the  
420 optimisation problems.  
421

422 **COROLLARY 2.7.** *Suppose  $d = 2$ . Under the assumptions of [Theorem 2.5](#),  $\mathbf{u}_\varepsilon \rightarrow$   
423  $\mathbf{u}$  strongly in  $L^\infty(0, T; L^2(\Omega)^2)$ ,  $\nabla \mathbf{u}_\varepsilon \rightarrow \nabla \mathbf{u}$  strongly in  $L^2(0, T; L^2(\Omega)^2)$ ,  $\theta_\varepsilon \rightarrow \theta$   
424 strongly in  $L^\infty(0, T; L^2(\Omega))$ ,  $\nabla \theta_\varepsilon \rightarrow \nabla \theta$  strongly in  $L^2(0, T; L^2(\Omega))$  and  $p_\varepsilon \rightarrow p$   
425 strongly in  $L^2(0, T; L^2(\Omega))$ .*

426 *Proof.* Denote  $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\varepsilon$ ,  $\bar{\theta} = \theta - \theta_\varepsilon$  and  $\bar{p} = p_\varepsilon - p$ . The variational formulation  
427 verified by  $(\bar{\mathbf{u}}, \bar{\theta}, \bar{p})$  reads as: for all  $\Psi \in V^u$ :

$$428 \quad (2.11a) \quad \begin{aligned} & -\langle \mathcal{P}\bar{p}, \Psi \rangle = \langle \partial_t \bar{\mathbf{u}} + \mathcal{A}\bar{\mathbf{u}} + h(\alpha)\bar{\mathbf{u}}, \Psi \rangle_{(V^u)', V^u} + \langle (h(\alpha) - h(\alpha_\varepsilon))\mathbf{u}_\varepsilon, \Psi \rangle_{(V^u)', V^u} + \\ & \quad \langle \mathcal{B}(\mathbf{u}, \mathbf{u}) - \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon), \Psi \rangle_{(V^u)', V^u} + \langle \mathcal{T}\bar{\theta}, \Psi \rangle_{(V^u)', V^u} \\ & \quad \frac{1}{2} \langle \mathcal{N}(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}), \Psi \rangle_{(V^u)', V^u}, \end{aligned}$$

429

$$430 \quad (2.11b) \quad 0 = \langle \nabla \cdot \bar{\mathbf{u}}, q \rangle_{L^2(\Omega)},$$

431

$$\begin{aligned}
& 0 = \langle \partial_t \bar{\theta}, \varphi \rangle_{(V^\theta)', V^\theta} - \langle \mathcal{D}(\mathbf{u}, \bar{\theta}) + \mathcal{D}(\bar{\mathbf{u}}, \theta_\varepsilon), \varphi \rangle_{(V^\theta)', V^\theta} \\
& \quad + \langle (\mathcal{C}(\alpha) - \mathcal{C}(\alpha_\varepsilon))\theta + \mathcal{C}(\alpha_\varepsilon)\bar{\theta}, \varphi \rangle_{(V^\theta)', V^\theta} \\
& \quad + \langle \mathcal{M}(\mathbf{u}, \theta) + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon), \varphi \rangle_{(V^\theta)', V^\theta}.
\end{aligned}
\tag{2.11c}$$

433 The following inequalities, valid for  $d = 2$ , will be useful throughout this proof:

434 • As proved in [10, Eq. (V.5)]:

$$\begin{aligned}
& (2.12) \\
& \frac{\langle \mathcal{B}(\mathbf{u}, \mathbf{u}) - \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon), \Psi \rangle_{(V^u)', V^u}}{\|\Psi\|_{H^1(\Omega)}} = \frac{\langle \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}) + \mathcal{B}(\mathbf{u}_\varepsilon, \bar{\mathbf{u}}), \Psi \rangle_{(V^u)', V^u}}{\|\Psi\|_{H^1(\Omega)}} \\
& \lesssim (\|\bar{\mathbf{u}}\|_{L^2(\Omega)} \|\mathbf{u}\|_{H^1(\Omega)} + \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}).
\end{aligned}
\tag{2.12}$$

436 • Concerning the boundary term in (2.11a):

$$\begin{aligned}
& \langle \mathcal{N}(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}), \Psi \rangle_{(V^u)', V^u} = \\
& \langle \mathcal{N}(\mathbf{u}, \bar{\mathbf{u}}), \Psi \rangle_{(V^u)', V^u} + \int_{\Gamma_{\text{out}}} (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}))(\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \Psi.
\end{aligned}
\tag{2.11a}$$

438 We now deal with each term separately. Concerning the first term, Young's  
439 inequality and Proposition 1.1 imply:

$$\langle \mathcal{N}(\mathbf{u}, \bar{\mathbf{u}}), \Psi \rangle_{(V^u)', V^u} \lesssim \|\bar{\mathbf{u}}\|_{H^1(\Omega)} \|\Psi\|_{H^1(\Omega)}.
\tag{2.13}$$

441 Owing to the Lipschitz behavior of the neg function, and the uniform con-  
442 vergence of  $\text{neg}_\varepsilon$  toward  $\text{neg}$  (see (A3)), there exists  $C_\varepsilon > 0$  such that:

$$\begin{aligned}
& \text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) = \text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \\
& \quad + \text{neg}(\mathbf{u}_\varepsilon \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \\
& \leq |\bar{\mathbf{u}} \cdot \mathbf{n}| + C_\varepsilon
\end{aligned}
\tag{2.14}$$

444 where  $C_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} 0$ . Therefore, using Proposition 1.2, we infer:

$$\begin{aligned}
& \int_{\Gamma_{\text{out}}} (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}))(\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \Psi \\
& \lesssim \|\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})\|_{L^4(\Gamma)} \|\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}\|_{L^2(\Gamma)} \|\Psi\|_{L^4(\Gamma)} \\
& \lesssim \left( \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} + C_\varepsilon \right) \|\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\|_{L^2(\Omega)}^{\frac{1}{2}} \\
& \quad \times \|\Psi\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \Psi\|_{L^2(\Omega)}^{\frac{3}{4}}.
\end{aligned}
\tag{2.15}$$

446 • The inequality proved in Proposition 1.1 shows that:

$$\int_{\Omega} \theta_\varepsilon \bar{\mathbf{u}} \cdot \nabla \bar{\theta} \lesssim \|\theta_\varepsilon\|_{H^1(\Omega)} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \bar{\theta}\|_{L^2(\Omega)}.
\tag{2.16}$$

448 • One will need also to bound the terms involving  $\mathbf{u}$  and  $\theta$  on the boundary.  
449 Using once again Proposition 1.1, one shows directly that:

$$\int_{\Gamma_{\text{out}}} (\bar{\mathbf{u}} \cdot \mathbf{n}) \theta_\varepsilon \bar{\theta} \lesssim \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} \|\theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^{\frac{1}{2}} \|\bar{\theta}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\theta}\|_{L^2(\Omega)}^{\frac{3}{4}}.
\tag{2.17}$$

451 With the same technique as for (2.15), one proves:

(2.18)

$$452 \int_{\Gamma_{\text{out}}} (\text{neg}(\bar{\mathbf{u}} \cdot \mathbf{n}) - \text{neg}_{\varepsilon}(\mathbf{u}_{\varepsilon} \cdot \mathbf{n})) \theta_{\varepsilon} \bar{\theta} \lesssim \left( \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{3}{4}} + C_{\varepsilon} \right) \|\theta_{\varepsilon}\|_{L^2(\Omega)}^{\frac{1}{2}} \\ \|\nabla \theta_{\varepsilon}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\bar{\theta}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\nabla \bar{\theta}\|_{L^2(\Omega)}^{\frac{3}{4}}.$$

453 Since  $d = 2$ , one has  $\bar{\mathbf{u}}' \in L^2(0, T; (V^u)')$  and we may choose  $\Psi = \bar{\mathbf{u}}(t)$  for fixed  $t$   
454 in (2.11a). Using the fact that  $\nabla \cdot \Psi = 0$  in this case, and after rearranging the terms,  
455 we obtain:

$$\begin{aligned} & \frac{d}{dt} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + 2A \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} h(\alpha) |\bar{\mathbf{u}}|^2 + \int_{\Gamma_{\text{out}}} \text{pos}(\mathbf{u} \cdot \mathbf{n}) |\bar{\mathbf{u}}|^2 = \\ & - 2 \langle (h(\alpha) - h(\alpha_{\varepsilon})) \mathbf{u}_{\varepsilon}, \bar{\mathbf{u}} \rangle_{(V^u)', V^u} - \langle \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}), \bar{\mathbf{u}} \rangle_{(V^u)', V^u} + \langle \mathcal{B}(\bar{\mathbf{u}}, \bar{\mathbf{u}}), \mathbf{u} \rangle_{(V^u)', V^u} \\ 456 & - \int_{\Omega} B \bar{\theta} e_y \cdot \bar{\mathbf{u}} - \int_{\Gamma_{\text{out}}} (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_{\varepsilon}(\mathbf{u}_{\varepsilon} \cdot \mathbf{n})) (\mathbf{u}_{\varepsilon} - \mathbf{u}^{\text{ref}}) \cdot \bar{\mathbf{u}} \\ & - \int_{\Gamma_{\text{out}}} (\bar{\mathbf{u}} \cdot \mathbf{n}) \mathbf{u}_{\varepsilon} \cdot \bar{\mathbf{u}} = 0. \end{aligned}$$

457 Therefore, (2.12), (2.15), Proposition 1.2 and Young's inequality imply there ex-  
458 ists  $C_1 > 0$  independent of  $\varepsilon$  such that:

$$459 \frac{d}{dt} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + C_1 \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \lesssim \|\bar{\theta}\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} |h(\alpha) - h(\alpha_{\varepsilon})|^2 |\mathbf{u}_{\varepsilon}|^2 \\ + g_1^u \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + (g_2^u)^{\frac{4}{5}} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^{\frac{2}{5}},$$

460 where  $g_1^u = 1 + \|\mathbf{u}\|_{H^1(\Omega)}^2 + \|\mathbf{u}\|_{L^2(\Omega)}^2 \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + 2 \|\mathbf{u}_{\varepsilon}\|_{L^2(\Omega)} \|\nabla \mathbf{u}_{\varepsilon}\|_{L^2(\Omega)}^2 +$   
461  $\|\mathbf{u}^{\text{ref}}\|_{L^2(\Omega)} \|\mathbf{u}^{\text{ref}}\|_{H^1(\Omega)}^2$  and  $g_2^u = C_{\varepsilon}^2 \|\mathbf{u}_{\varepsilon} - \mathbf{u}^{\text{ref}}\|_{L^2(\Omega)} \|\nabla(\mathbf{u}_{\varepsilon} - \mathbf{u}^{\text{ref}})\|_{L^2(\Omega)}$ .

462 Using once again Young's inequality, one has:

$$463 \frac{d}{dt} \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + C_1 \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \lesssim \|\bar{\theta}\|_{L^2(\Omega)}^2 + (1 + g_1^u) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \\ + 2 \int_{\Omega} |h(\alpha) - h(\alpha_{\varepsilon})|^2 |\mathbf{u}_{\varepsilon}|^2 + g_2^u.$$

464 We now move back to (2.11c) and choose  $\varphi = \bar{\theta}$ , which gives, after some manip-  
465 ulation:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{\theta}\|_{L^2(\Omega)}^2 + C \int_{\Omega} k(\alpha_{\varepsilon}) |\nabla \bar{\theta}|^2 + \int_{\Gamma_{\text{out}}} \left( \frac{1}{2} (\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}(\mathbf{u} \cdot \mathbf{n}) \right) \bar{\theta}^2 \\ 466 & = \int_{\Omega} \theta_{\varepsilon} \bar{\mathbf{u}} \cdot \nabla \bar{\theta} - C \int_{\Omega} (k(\alpha) - k(\alpha_{\varepsilon})) \nabla \theta \cdot \nabla \bar{\theta} \\ & \quad - \int_{\Gamma_{\text{out}}} [((\bar{\mathbf{u}} \cdot \mathbf{n}) + \beta (\text{neg}(\mathbf{u} \cdot \mathbf{n}) - \text{neg}_{\varepsilon}(\mathbf{u}_{\varepsilon} \cdot \mathbf{n})))] \theta_{\varepsilon} \bar{\theta}. \end{aligned}$$

467 As shown in Proposition 2.2,  $\int_{\Gamma_{\text{out}}} \left( \frac{1}{2} (\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}(\mathbf{u} \cdot \mathbf{n}) \right) \bar{\theta}^2$  is positive. There-  
468 fore, using (2.17), (2.18), Proposition 1.2 and Young's inequality, one proves that  
469 there exists  $C_3 > 0, C_4 > 0$ , such that:

$$470 \frac{d}{dt} \|\bar{\theta}\|_{L^2(\Omega)}^2 + C_3 \|\nabla \bar{\theta}\|_{L^2(\Omega)}^2 \lesssim \|\theta_{\varepsilon}\|_{L^2(\Omega)}^2 \|\nabla \theta_{\varepsilon}\|_{L^2(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + C_4 \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \\ + \left( C \int_{\Omega} (k(\alpha) - k(\alpha_{\varepsilon}))^2 |\nabla \theta|^2 \right) + g_1^{\theta} \|\bar{\theta}\|_{L^2(\Omega)}^2 + g_2^{\theta},$$



471 where  $g_1^\theta = 1 + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla\theta_\varepsilon\|_{L^2(\Omega)}^2$ ,  $g_2^\theta = C_\varepsilon^2 \|\theta_\varepsilon\|_{L^2(\Omega)} \|\nabla\theta_\varepsilon\|_{L^2(\Omega)}$ .

472 Summing (2.19) and (2.20) and choosing  $C_4$  small enough, there exists  $C^* > 0$   
 473 such that:

$$\begin{aligned}
 & \frac{d}{dt} (\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\bar{\theta}\|_{L^2(\Omega)}^2) + C^* (\|\nabla\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\nabla\bar{\theta}\|_{L^2(\Omega)}^2) \lesssim g_2^u + g_2^\theta \\
 474 \quad (2.21) \quad & + (1 + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla\theta_\varepsilon\|_{L^2(\Omega)}^2 + g_1^u) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + (g_1^\theta + 1) \|\bar{\theta}\|_{L^2(\Omega)}^2 \\
 & + \int_\Omega (k(\alpha) - k(\alpha_\varepsilon))^2 |\nabla\theta|^2 + \int_\Omega |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2.
 \end{aligned}$$

475 We now introduce the following functions

$$\begin{aligned}
 476 \quad a_\varepsilon^u &= (1 + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla\theta_\varepsilon\|_{L^2(\Omega)}^2 + g_1^u), & b_\varepsilon^u &= \int_\Omega |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2 + g_2^u, \\
 477 \quad a_\varepsilon^\theta &= (1 + g_1^\theta), & b_\varepsilon^\theta &= \int_\Omega (k(\alpha) - k(\alpha_\varepsilon))^2 |\nabla\theta|^2 + g_2^\theta.
 \end{aligned}$$

478 Since  $\mathbf{u}$  and  $\mathbf{u}_\varepsilon$  both belong to  $L^2(0, T; H^1(\Omega)^2) \cap L^\infty(0, T; L^2(\Omega)^2)$  (the same holds  
 479 for  $\theta$  and  $\theta_\varepsilon$ ),  $a_\varepsilon^u$ ,  $b_\varepsilon^u$ ,  $a_\varepsilon^\theta$  and  $b_\varepsilon^\theta$  are integrable, and so are  $a_\varepsilon = \max(a_\varepsilon^u, a_\varepsilon^\theta)$  and  $b_\varepsilon =$   
 480  $b_\varepsilon^u + b_\varepsilon^\theta$ . Grönwall's lemma proves that for all  $t \in [0, T]$ ,  $\|\bar{\mathbf{u}}(t)\|_{L^2(\Omega)}^2 + \|\bar{\theta}(t)\|_{L^2(\Omega)}^2 \leq$   
 481  $\left(\int_0^t b_\varepsilon(s) ds\right) \exp\left(\int_0^t a_\varepsilon(s) ds\right)$ . Since  $a_\varepsilon \geq 0$  and  $b_\varepsilon \geq 0$ ,  $t \mapsto \left(\int_0^t b_\varepsilon(s) ds\right)$  and  $t \mapsto$   
 482  $\exp\left(\int_0^t a_\varepsilon(s) ds\right)$  are non-decreasing and we have

$$483 \quad (2.22) \quad \sup_{t \in [0, T]} (\|\bar{\mathbf{u}}(t)\|_{L^2(\Omega)} + \|\bar{\theta}(t)\|_{L^2(\Omega)}) \leq \left(\int_0^T b_\varepsilon(s) ds\right)^{\frac{1}{2}} \exp\left(\frac{1}{2} \int_0^T a_\varepsilon(s) ds\right).$$

484 Since, on one hand,  $\alpha_\varepsilon \rightarrow \alpha$  in  $L^1(\Omega)$  and  $\alpha_\varepsilon$  is independent of time, and on the other  
 485 hand,  $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$  strongly in  $L^2(0, T; L^2(\Omega))$ , Lebesgue's dominated convergence gives a  
 486 subsequence  $(\varepsilon_k)$  such that:

$$487 \quad (2.23) \quad \int_0^T \int_\Omega |h(\alpha) - h(\alpha_{\varepsilon_k})|^2 |\mathbf{u}_\varepsilon|^2 \xrightarrow{k \rightarrow +\infty} 0, \quad \int_0^T \int_\Omega |k(\alpha) - k(\alpha_{\varepsilon_k})|^2 |\nabla\theta|^2 \xrightarrow{k \rightarrow +\infty} 0.$$

488 Notice that, owing to the convergence of  $\mathbf{u}_\varepsilon$  and  $\theta_\varepsilon$ ,  $\|\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}\|_{L^2(\Omega)} \|\nabla(\mathbf{u}_\varepsilon -$   
 489  $\mathbf{u}^{\text{ref}})\|_{L^2(\Omega)}$  and  $\|\theta_\varepsilon\|_{L^2(\Omega)} \|\nabla\theta_\varepsilon\|_{L^2(\Omega)}$  are bounded w.r.t  $\varepsilon$  in  $L^1([0, T])$ . Therefore,  
 490 since  $C_\varepsilon \xrightarrow{\varepsilon \rightarrow +\infty} 0$ , it proves that  $\int_0^T (g_2^u + g_2^\theta) \xrightarrow{\varepsilon_k \rightarrow +\infty} 0$ . Gathering the previous  
 491 convergence results then prove that  $\int_0^T b_{\varepsilon_k}(s) ds \xrightarrow{k \rightarrow +\infty} 0$ . In addition, thanks to the  
 492 convergence proved in [Theorem 2.5](#), we show that  $\int_0^T a_\varepsilon(s) ds$  is bounded w.r.t.  $\varepsilon$ .  
 493 Therefore, it proves that  $\|\mathbf{u} - \mathbf{u}_{\varepsilon_k}\|_{L^\infty(0, T; L^2(\Omega))} + \|\theta - \theta_{\varepsilon_k}\|_{L^\infty(0, T; L^2(\Omega))} \xrightarrow{k \rightarrow +\infty} 0$ .

494 We now move back to (2.21). We integrate each side of the inequality:

$$\begin{aligned}
 495 \quad & \int_0^T \|\nabla\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\nabla\bar{\theta}\|_{L^2(\Omega)}^2 \lesssim F_\varepsilon^{u, \theta} + \int_0^T (g_1^u + \|\theta_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla\theta_\varepsilon\|_{L^2(\Omega)}^2 + 1) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \\
 & + \int_0^T (g_1^\theta + 1) \|\bar{\theta}\|_{L^2(\Omega)}^2,
 \end{aligned}$$

496 with  $F_\varepsilon^{u, \theta} = \|\mathbf{u}_0(\alpha_\varepsilon) - \mathbf{u}_0(\alpha)\|_{L^2(\Omega)}^2 + \|\theta_0(\alpha_\varepsilon) - \theta_0(\alpha)\|_{L^2(\Omega)}^2 + \int_0^T (g_2^u + g_2^\theta)$   
 497  $+ \int_0^T \int_\Omega |k(\alpha) - k(\alpha_\varepsilon)|^2 |\nabla\theta|^2 + \int_0^T \int_\Omega |h(\alpha) - h(\alpha_\varepsilon)|^2 |\mathbf{u}_\varepsilon|^2$ . From [Assumptions 2.1](#),

498 the initial conditions are continuous with respect to  $\alpha$  and thus the two first terms  
499 above goes to 0 as  $\varepsilon \rightarrow +\infty$ . The third, fourth and fifth terms have been already  
500 treated (see (2.23)). Concerning the two last terms, notice that due to the con-  
501 vergence of  $\mathbf{u}_\varepsilon$ , one has  $\|\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}\|_{L^2(\Omega)} \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\|_{L^2(\Omega)}$  bounded w.r.t  $\varepsilon$  in  
502  $L^1([0, T])$ . The main problem concerns the term  $\int_0^T \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$ .  
503 Since  $\mathbf{u}_\varepsilon \in L^2(0, T; L^2(\Omega)^2)$ , we only need to deal with  $\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$ . How-  
504 ever, as proved in Theorem 2.5, up to a subsequence,  $\nabla \mathbf{u}_\varepsilon$  weakly converges to  $\nabla \mathbf{u}$   
505 in  $L^2(0, T, L^2(\Omega))$  and  $\|\mathbf{u} - \mathbf{u}_\varepsilon\|_{L^2(\Omega)}^2 \rightarrow 0$  in  $L^\infty([0, T])$ . Concerning the other terms  
506 in  $g_1^u$ , they are all independent of  $\varepsilon$ , and we mainly use the fact that  $\|\bar{\mathbf{u}}\|_{L^2(\Omega)} \rightarrow 0$   
507 in  $L^2([0, T])$ . We may do the same proof concerning  $\int_0^T \|\theta_\varepsilon\|_{L^2(\Omega)} \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$   
508 and  $\int_0^T \|\theta_\varepsilon\|_{L^2(\Omega)} \|\nabla \theta_\varepsilon\|_{L^2(\Omega)}^2 \|\bar{\theta}\|_{L^2(\Omega)}^2$ .  
509 Therefore,  $\int_0^T (1 + \|\theta_{\varepsilon_k}\|_{L^2(\Omega)} \|\nabla \theta_{\varepsilon_k}\|_{L^2(\Omega)}^2 + g_1) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 \xrightarrow{\varepsilon_k \rightarrow +\infty} 0$  and  $\int_0^T (g_1^\theta +$   
510  $1) \|\bar{\theta}\|_{L^2(\Omega)}^2 \xrightarrow{\varepsilon_k \rightarrow +\infty} 0$ . It eventually proves that  $\|\nabla(\mathbf{u} - \mathbf{u}_{\varepsilon_k})\|_{L^2(0, T; L^2(\Omega))} + \|\nabla(\theta -$   
511  $\theta_{\varepsilon_k})\|_{L^2(0, T; L^2(\Omega))} \xrightarrow{k \rightarrow +\infty} 0$ .

512 Concerning the pressure, we use once again the inf-sup condition (2.8) on  $\bar{p}$ , which  
513 proves that:

$$\begin{aligned}
\|\bar{p}\|_{L^2(\Omega)} &\lesssim \|\partial_t \bar{\mathbf{u}}\|_{V'} + \|\bar{\mathbf{u}}\|_{L^2(\Omega)} + \|h\| \|\bar{\mathbf{u}}\|_{L^2(\Omega)} + \|(h(\alpha) - h(\alpha_\varepsilon))\mathbf{u}_\varepsilon\|_{V'} + \\
514 \quad (2.24) \quad &\|\mathcal{B}(\mathbf{u}, \mathbf{u}) - \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)\|_{V'} + \|\mathcal{T}\bar{\theta}\|_{V'} + \\
&\frac{1}{2} \|\mathcal{N}(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\|_{V'}.
\end{aligned}$$

515 Most of the terms in the right hand side were proved to converge strongly to 0 in  
516  $L^2([0, T])$ .

- 517 • In the same pattern of proof as in (2.23), one proves easily that  $(h(\alpha) -$   
518  $h(\alpha_{\varepsilon_k}))\mathbf{u}_{\varepsilon_k}$  converges strongly to 0 in  $L^2(0, T; V')$
- 519 • Using (2.12), one proves that  $\mathcal{B}(\mathbf{u}, \mathbf{u}) - \mathcal{B}(\mathbf{u}_{\varepsilon_k}, \mathbf{u}_{\varepsilon_k})$  strongly converges to 0  
520 in  $L^2(0, T; V')$ .
- 521 • Summing (2.13) and (2.15) prove that:  $\|\mathcal{N}(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\|_{V'}$   
522  $\lesssim \|\bar{\mathbf{u}}\|_{H^1(\Omega)} + C_\varepsilon$ . Since  $\mathbf{u}_{\varepsilon_k} \rightarrow \mathbf{u}$  strongly in  $L^2(0, T; H^1(\Omega))$  and  $C_\varepsilon \rightarrow 0$ ,  
523 this proves that  $\mathcal{N}(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_{\varepsilon_k}, \mathbf{u}_{\varepsilon_k} - \mathbf{u}^{\text{ref}})$  converges strongly to 0  
524 in  $L^2(0, T; V')$ .
- 525 • Finally, in a similar fashion as in [10, Lemma V.1.6], the above bounds prove  
526 that  $\partial_t \bar{\mathbf{u}} \rightarrow 0$  strongly in  $L^2(0, T; V')$ .

527 Therefore, all the convergence results above prove that, up to a subsequence,  $p_\varepsilon \rightarrow p$   
528 strongly in  $L^2(0, T; L^2(\Omega))$ . Owing to Urysohn's subsequence principle and the  
529 uniqueness of the solution to (WF), we actually obtain that the whole sequence  
530  $(\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$  strongly converges toward  $(\mathbf{u}, \theta, p)$ .  $\square$

531 **3. Optimal control and necessary conditions.** We now begin the analysis of  
532 the optimal control problems (OPT) and (OPTe). Let us detail first some assumptions  
533 made on the objective functional:

- 534 ASSUMPTIONS 3.1. • For  $d = 2$ ,  $\mathcal{J}$  is lower semi-continuous with respect to  
535 the (weak-\*, strong, strong, strong) topology of  $\mathcal{U}_{\text{ad}} \times L^2(0, T; V^u) \times L^2(0, T;$   
536  $V^\theta) \times L^2(0, T; L^2(\Omega))$ .  
537 • In dimension 3,  $\mathcal{J}$  is either lower semi-continuous with respect to the (weak-  
538 \*, strong, strong, weak) topology of  $\mathcal{U}_{\text{ad}} \times L^2(0, T; H^u) \times L^2(0, T; H^\theta) \times$

539  $L^2(0, T; L^2(\Omega))$ , or lower semi-continuous with respect to the (weak-\*, weak,  
540 weak, weak) topology of  $\mathcal{U}_{ad} \times L^2(0, T; V^u) \times L^2(0, T; V^\theta) \times L^2(0, T; L^2(\Omega))$ .

541 The existence of solutions to (OPTe) and (OPT) is rather classical and we refer  
542 for instance to [20, 31, 33]. We state a first result that let us see that a solution of  
543 (OPT) can be approximated by (OPTe).

544 **THEOREM 3.2.** *Assume Assumptions 3.1 is verified. Let  $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$  be a glob-*  
545 *ally optimal solution of (OPTe). Then  $(\alpha_\varepsilon^*) \subset \mathcal{U}_{ad}$  is a bounded sequence. Further-*  
546 *more, there exists  $(\alpha^*, \mathbf{u}^*, \theta^*, p^*) \in \mathcal{U}_{ad} \times L^2(0, T; V^u) \times L^2(0, T; V^\theta) \times L^2(0, T; L^2(\Omega))$*   
547 *such that a subsequence of  $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$  converges to  $(\alpha^*, \mathbf{u}^*, \theta^*, p^*)$  in the topol-*  
548 *ogy of Assumptions 3.1, and for all  $(\alpha, \mathbf{u}, \theta, p)$  in  $\mathcal{U}_{ad} \times L^2(0, T; V^u) \times L^2(0, T; V^\theta) \times$*   
549  *$L^2(0, T; L^2(\Omega))$ :  $\mathcal{J}(\alpha^*, \mathbf{u}^*, \theta^*, p^*) \leq \mathcal{J}(\alpha, \mathbf{u}, \theta, p)$ . Hence, any accumulation point of*  
550  *$(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$  is a globally optimal solution of (OPT).*

551 *Proof.* The proof can be adapted from [20, Theorem 15] or [31, Theorem 3].  $\square$

552 However, the fact that this only concerns global solutions may appear restrictive.  
553 Under an additional assumption, we can state a slightly stronger result.

554 **COROLLARY 3.3.** *Assume Assumptions 3.1 hold. Let  $\alpha^*$  be a local strict solution*  
555 *of (OPT), meaning that there exists  $\rho > 0$  such that  $\mathcal{J}(\alpha^*, \mathbf{u}^*, \theta^*, p^*) < \mathcal{J}(\alpha, \mathbf{u}, \theta, p)$*   
556 *for all  $\alpha$  such that  $\|\alpha^* - \alpha\|_{BV} < \rho$ . Then, there exists a family of local solution  $(\alpha_\varepsilon^*)$*   
557 *of (OPTe) such that  $(\alpha_\varepsilon^*)$  converges weak-\* to  $\alpha^*$ .*

558 *Proof.* Similar to [35, Theorem 3.14].  $\square$

559 **3.1. First order necessary conditions for (OPTe).** From now on, we set  
560  $d = 2$ , in order to have uniqueness of solution of (WFe). We make the following  
561 assumption on the cost function:

562 **ASSUMPTIONS 3.4.** *Assume  $d = 2$  and  $\mathcal{J}$  is Fréchet-differentiable.*

563 We define the sets  $W^u(0, T) = \{\mathbf{u} \in L^2(0, T; (V^u)); \partial_t \mathbf{u} \in L^2(0, T; (V^u)')\}$ , and  
564  $W^\theta(0, T) = \{\theta \in L^2(0, T; (V^\theta)); \partial_t \theta \in L^2(0, T; (V^\theta)')\}$ . Write, in  $(V^u)' \times (V^\theta)' \times$   
565  $L^2(0, T; L^2(\Omega))$ , the equation (WFe) as  $e(\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon, \alpha_\varepsilon) = 0$ , where  $e : W^u(0, T) \times$   
566  $W^\theta(0, T) \times L^2(0, T; L^2(\Omega)) \times \mathcal{U}_{ad} \rightarrow L^2(0, T; (V^u)') \times L^2(0, T; L^2(\Omega)) \times L^2(0, T; (V^\theta)')$   
567  $\times H^u \times H^\theta$  is defined as:

$$568 \quad e(\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon, \alpha_\varepsilon) = \begin{pmatrix} \partial_t \mathbf{u}_\varepsilon + \mathcal{A} \mathbf{u}_\varepsilon + \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + h(\alpha_\varepsilon) \mathbf{u}_\varepsilon + \mathcal{P} p \\ \quad \quad \quad + \frac{1}{2} \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) - f - \sigma^{\text{ref}} \\ \mathcal{P}^* \mathbf{u}_\varepsilon \\ \partial_t \theta_\varepsilon - \mathcal{D}(\mathbf{u}_\varepsilon, \theta_\varepsilon) + \mathcal{C}(\alpha_\varepsilon) \theta_\varepsilon + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon) - \phi \\ \quad \quad \quad \mathbf{u}_\varepsilon(0, \cdot) - \mathbf{u}_0(\alpha_\varepsilon) \\ \quad \quad \quad \theta_\varepsilon(0, \cdot) - \theta_0(\alpha_\varepsilon) \end{pmatrix}.$$

569 The operators  $\mathcal{N}_\varepsilon$  and  $\mathcal{M}_\varepsilon$  are Fréchet differentiable with the same smoothness as  
570 the approximation  $\text{neg}_\varepsilon$ . Their derivatives with respect to  $\mathbf{u}_\varepsilon$  are denoted by  $d_u \mathcal{N}_\varepsilon :$   
571  $W^u(0, T)^2 \rightarrow \mathcal{L}(W(0, T), L^2(0, T; (V^u)'))$ ,  $d_u \mathcal{M}_\varepsilon : W^u(0, T) \times W^\theta(0, T) \rightarrow$   
572  $\mathcal{L}(W^u(0, T), L^2(0, T; (V^\theta)'))$ , defined by:

$$573 \quad d_u \mathcal{N}_\varepsilon(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) \mathbf{v} = \mathcal{N}_\varepsilon(\mathbf{u}, \mathbf{v}) + \mathcal{N}'(\mathbf{u}, \mathbf{u} - \mathbf{u}^{\text{ref}}) \mathbf{v},$$

574

$$575 \quad \langle d_u \mathcal{M}_\varepsilon(\mathbf{u}, \theta) \mathbf{v}, \varphi \rangle_{(V^\theta)', V^\theta} = \int_{\Gamma_{\text{out}}} (1 + \beta \text{neg}'_\varepsilon(\mathbf{u} \cdot \mathbf{n})) (\mathbf{v} \cdot \mathbf{n}) \theta \varphi,$$

576 where  $\mathcal{N}'(\mathbf{u}, \mathbf{w})$  is defined by:

$$577 \quad \langle \mathcal{N}'(\mathbf{u}, \mathbf{w})\mathbf{v}, \mathbf{\Psi} \rangle_{(V^u)', V^u} = \int_{\Gamma_{\text{out}}} \text{neg}'_\varepsilon(\mathbf{u} \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n}) \mathbf{w} \cdot \mathbf{\Psi}.$$

578 Furthermore, these operators are bounded, as proved in the following lemma:

579 LEMMA 3.5. *Given  $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$  solution of (WFe):*

$$580 \quad \|d_u \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\mathbf{v}\|_{L^2(0,T;(V^u)')} \leq c(\|\mathbf{u}_\varepsilon\|_{L^2(0,T;V^u)})\|\mathbf{v}\|_{L^2(0,T;V^u)},$$

$$581 \quad \|d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon)\mathbf{v}\|_{L^2(0,T;(V^\theta)')} \leq c(\|\theta_\varepsilon\|_{L^2(0,T;V^\theta)})\|\mathbf{v}\|_{L^2(0,T;V^u)},$$

583 where  $c(\cdot)$  is a Lipschitz function.

584 *Proof.* The proof is similar to the proof of Theorem 2.5 from which we infer,  
 585 for  $\mathbf{\Psi} \in V$ , that:  $\langle \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{v}), \mathbf{\Psi} \rangle_{V', V} \leq \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}\|\mathbf{\Psi}\|_{H^1(\Omega)}$ . Due to (A3),  
 586 one has straight away the existence of a constant  $C > 0$  such that:  $\langle \mathcal{N}'(\mathbf{u}, \mathbf{u} -$   
 587  $\mathbf{u}^{\text{ref}})\mathbf{v}, \mathbf{\Psi} \rangle_{V', V} \leq (\|\mathbf{u}_\varepsilon\|_{H^1(\Omega)} + C)\|\mathbf{v}\|_{H^1(\Omega)}\|\mathbf{\Psi}\|_{H^1(\Omega)}$ . Adding the two inequalities  
 588 and dividing by  $\|\mathbf{\Psi}\|_{H^1(\Omega)}$  concludes the proof. The proof of the second inequality  
 589 being similar is thus omitted.  $\square$

590 Using the results of [33, Section 1.8.2], one shows easily that  $e$  is Fréchet differ-  
 591 entiable w.r.t.  $(\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon)$ , with derivative given by:

$$592 \quad e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon}(\alpha_\varepsilon) \begin{pmatrix} \mathbf{v} \\ \ell \\ q \end{pmatrix} = \begin{pmatrix} \partial_t \mathbf{v} + \mathbf{A}\mathbf{v} + \mathbf{B}(\mathbf{v}, \mathbf{u}_\varepsilon) + \mathbf{B}(\mathbf{u}_\varepsilon, \mathbf{v}) + h(\alpha_\varepsilon)\mathbf{v} + \mathcal{P}q + \mathcal{T}\ell \\ + \frac{1}{2}d_u \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}})\mathbf{v} \\ \mathcal{P}^*\mathbf{v} \\ \partial_t \ell - \mathcal{D}(\mathbf{u}_\varepsilon, \ell) - \mathcal{D}(\mathbf{v}, \theta_\varepsilon) + \mathcal{C}(\alpha_\varepsilon)\ell + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \ell) \\ + d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon)\mathbf{v} \\ \mathbf{v}(0, \cdot) \\ \ell(0, \cdot) \end{pmatrix}.$$

593 For defining first order conditions (see [33]), a question of interests is to determine  
 594 if, for all  $g = (g^u, g^p, g^\theta, \mathbf{v}_0, \ell_0) \in L^2(0, T; (V^u)') \times L^2(0, T; (V^\theta)') \times L^2(0, T; L^2(\Omega)) \times$   
 595  $H^u \times H^\theta$ , the following linearized equation

$$596 \quad (3.1) \quad e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon}(\alpha_\varepsilon) \begin{pmatrix} \mathbf{v} \\ \ell \\ q \end{pmatrix} = g$$

597 admits a solution  $(\mathbf{v}, \ell, q) \in W^u(0, T) \times W^\theta(0, T) \times L^2(0, T; L^2(\Omega))$ .

598 THEOREM 3.6. *For all  $\alpha_\varepsilon \in \mathcal{U}_{ad}$ , Eq. (3.1) admits a unique solution. Therefore,*  
 599  $e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon}(\alpha_\varepsilon)$  *is invertible.*

600 *Sketch of proof.* First, notice that using [28, Corollary 2.4], there exists  $\mathbf{v}^{\text{ref}} \in$   
 601  $L^2(0, T; H_0^1(\Omega))$  such that  $g^p = \nabla \cdot \mathbf{v}^{\text{ref}}$ . Thus, defining  $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{v}^{\text{ref}}$ , we bring the  
 602 system back in the framework of a solenoidal vector function  $\bar{\mathbf{v}}$ . Using Lemma 3.5,  
 603 the proof becomes a simple adaptation of the proof of Theorem 2.5 and [32, Appendix  
 604 A2]. Uniqueness is proved as for Proposition 2.6 (see also [32, Appendix A2]).  $\square$

605 A consequence of Theorem 3.6 is that for all  $G = (g_1, g_2, g_3) \in W^u(0, T)' \times$   
 606  $W^\theta(0, T)' \times L^2(0, T; L^2(\Omega))$ , the following adjoint equation admits a unique solution  
 607  $\Lambda_\varepsilon = (\lambda_\varepsilon^u, \lambda_\varepsilon^\theta, \lambda_\varepsilon^p, \lambda_\varepsilon^{\mathbf{u}_0}, \lambda_\varepsilon^{\ell_0}) \in L^2(0, T; V^u) \times L^2(0, T; V^\theta) \times L^2(0, T; L^2(\Omega)) \times H^u \times H^\theta$ :

$$608 \quad (3.2) \quad (e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon}(\alpha_\varepsilon))^* \Lambda_\varepsilon = G,$$

609 where  $(e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon}(\alpha_\varepsilon))^*$  denotes the adjoint operator of  $e'_{\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon}(\alpha_\varepsilon)$ .

610 After some calculations, equation (3.2) is equivalent to solve, for all  $(\mathbf{v}, \ell, q) \in$   
 611  $W^u(0, T) \times W^\theta(0, T) \times L^2(0, T; L^2(\Omega))$ , the following variational problem:

$$\begin{aligned}
 & \langle -\partial_t \lambda_\varepsilon^{\mathbf{u}} + \mathcal{A} \lambda_\varepsilon^{\mathbf{u}} + (\nabla \mathbf{u}_\varepsilon)^\top \lambda_\varepsilon^{\mathbf{u}} + \mathcal{B}(\mathbf{u}_\varepsilon, \lambda_\varepsilon^{\mathbf{u}}) + h(\alpha_\varepsilon) \lambda_\varepsilon^{\mathbf{u}} + \mathcal{P} \lambda_\varepsilon^p - \mathcal{D}_1(\theta_\varepsilon) \lambda_\varepsilon^\theta \\
 & \quad + \mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \lambda_\varepsilon^{\mathbf{u}}) + (\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}))^* \lambda_\varepsilon^{\mathbf{u}} \\
 & \quad + \mathcal{N}'_2(\mathbf{u}_\varepsilon, \lambda_\varepsilon^{\mathbf{u}}) + (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda_\varepsilon^\theta, \mathbf{v} \rangle_{W(0, T)', W(0, T)} \\
 & \quad + \langle \mathbf{v}(0, \cdot), \lambda_\varepsilon^{u_0} \rangle_H \\
 612 \quad (3.3) \quad & = \langle g_1, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)}, \\
 & \langle \mathcal{P}^* \lambda_\varepsilon^{\mathbf{u}}, q \rangle_{L^2(0, T; L^2(\Omega))} = \langle g_2, q \rangle_{L^2(0, T; L^2(\Omega))} \\
 & \langle -\partial_t \lambda_\varepsilon^\theta + \mathcal{T}^* \lambda_\varepsilon^{\mathbf{u}} + \mathcal{C}(\alpha_\varepsilon) \lambda_\varepsilon^\theta - \mathcal{D}_2(\mathbf{u}_\varepsilon) \lambda_\varepsilon^\theta + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon)^* \lambda_\varepsilon^\theta, \ell \rangle_{W^\theta(0, T)', W^\theta(0, T)} \\
 & = \langle g_3, \ell \rangle_{W^\theta(0, T)', W^\theta(0, T)}
 \end{aligned}$$

613 where  $\langle \mathcal{D}(\theta, \mathbf{u}), \varphi \rangle = \langle \mathcal{D}_1(\theta) \varphi, \mathbf{u} \rangle = \langle \mathcal{D}_2(\mathbf{u}) \varphi, \theta \rangle$ ,  $\langle \mathcal{M}_\varepsilon(\mathbf{u}) \theta, \varphi \rangle = \langle \mathcal{M}_\varepsilon(\mathbf{u}) \varphi, \theta \rangle$   
 614  $= \int_{\Gamma_{\text{out}}} ((\mathbf{u} \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u} \cdot \mathbf{n})) \theta \varphi$ , and  $\langle \mathcal{N}'_2(\mathbf{u}_\varepsilon, \lambda_\varepsilon^{\mathbf{u}}), \mathbf{v} \rangle = \int_{\Gamma_{\text{out}}} (\mathbf{u}_\varepsilon \cdot \mathbf{n}) \lambda_\varepsilon^{\mathbf{u}} \cdot \mathbf{v}$ . This  
 615 equation, in turn, is the weak formulation of:

$$\begin{aligned}
 & -\partial_t \lambda_\varepsilon^{\mathbf{u}} - \Delta \lambda_\varepsilon^{\mathbf{u}} + h(\alpha_\varepsilon) \lambda_\varepsilon^{\mathbf{u}} + \nabla \lambda_\varepsilon^p + (\nabla \mathbf{u}_\varepsilon)^\top \lambda_\varepsilon^{\mathbf{u}} - (\mathbf{u}_\varepsilon \cdot \nabla) \lambda_\varepsilon^{\mathbf{u}} - \theta_\varepsilon \nabla \lambda_\varepsilon^\theta = g_1 \\
 & \nabla \cdot \lambda_\varepsilon^{\mathbf{u}} = g_2 \\
 & -\partial_t \lambda_\varepsilon^\theta + B \lambda_\varepsilon^{\mathbf{u}} \cdot e_y - \nabla \cdot (Ck(\alpha_\varepsilon) \nabla \lambda_\varepsilon^\theta) - \nabla \cdot (\mathbf{u}_\varepsilon \lambda_\varepsilon^\theta) = g_3 \\
 & \lambda_\varepsilon^{\mathbf{u}}|_{\Gamma_w \cup \Gamma_{\text{in}}} = 0, \\
 & \lambda_\varepsilon^\theta|_{\Gamma_{\text{in}}} = 0, \\
 616 \quad (3.4a) \quad & \partial_n \lambda_\varepsilon^\theta|_{\Gamma_w} = 0, \\
 & A \partial_n \lambda_\varepsilon^{\mathbf{u}} - \mathbf{n} \lambda_\varepsilon^p|_{\Gamma_{\text{out}}} = \left( \frac{1}{2} \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) + (\mathbf{u}_\varepsilon \cdot \mathbf{n}) \right) \lambda_\varepsilon^{\mathbf{u}} + (1 + \beta \mu_\varepsilon) \theta_\varepsilon \lambda_\varepsilon^\theta \mathbf{n} \\
 & \quad + \frac{1}{2} \mu_\varepsilon ((\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \lambda_\varepsilon^{\mathbf{u}}) \mathbf{n}, \\
 & Ck(\alpha_\varepsilon) \partial_n \lambda_\varepsilon^\theta + \beta \lambda_\varepsilon^\theta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})|_{\Gamma_{\text{out}}} = 0 \\
 & \lambda_\varepsilon^{\mathbf{u}}(T) = 0, \lambda_\varepsilon^\theta(T) = 0,
 \end{aligned}$$

617

$$618 \quad (3.4b) \quad \mu_\varepsilon = \text{neg}'_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})$$

619 and, as shown in a similar fashion in [32],  $\lambda_\varepsilon^{u_0} = \lambda_\varepsilon^{\mathbf{u}}(0, \cdot)$ ,  $\lambda_\varepsilon^{\theta_0} = \lambda_\varepsilon^\theta(0, \cdot)$ .

620 An other consequence of Theorem 3.6 is that we can apply [33, Corollary 1.3]  
 621 which states that at any local solution  $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*)$  of (OPTe), the following opti-  
 622 mality conditions hold:

623 **THEOREM 3.7.** *Let  $\alpha_\varepsilon^*$  be an optimal solution of (OPTe) with associated states*  
 624  *$(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*)$ . Then there exists adjoint states  $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta, \lambda_\varepsilon^p) \in L^2(0, T; V^u) \times L^2(0, T; V^\theta)$*   
 625  *$\times L^2(0, T; L^2(\Omega))$  such that, denoting  $(\lambda_\varepsilon^{u_0}, \lambda_\varepsilon^{\theta_0}) = (\lambda_\varepsilon^{\mathbf{u}}(0, \cdot), \lambda_\varepsilon^\theta(0, \cdot))$  and  $\Lambda_\varepsilon = (\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta,$   
 626  $\lambda_\varepsilon^p, \lambda_\varepsilon^{u_0}, \lambda_\varepsilon^{\theta_0})$ :*

$$\begin{aligned}
 & e(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*) = 0, \\
 & \mathcal{J}'_{\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*}(\alpha_\varepsilon^*) + (e_{\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*}(\alpha_\varepsilon^*))' \Lambda_\varepsilon = 0, \\
 627 \quad (3.5) \quad & \left\langle \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*))' \Lambda_\varepsilon, \alpha - \alpha_\varepsilon^* \right\rangle_{\mathcal{U}'_{ad}, \mathcal{U}_{ad}} \geq 0, \quad \forall \alpha \in \mathcal{U}_{ad}, \\
 & \alpha_\varepsilon \in \mathcal{U}_{ad}.
 \end{aligned}$$

628 REMARK 3.8. As stated in [33, Eq. (1.89)], since  $e$  and  $\mathcal{J}$  are Fréchet differ-  
629 entiable, the mapping  $\alpha_\varepsilon \mapsto \tilde{\mathcal{J}}(\alpha_\varepsilon) = \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon, p_\varepsilon)$  is Fréchet differentiable, and  
630  $\tilde{\mathcal{J}}'(\alpha_\varepsilon) = \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*))^* \Lambda_\varepsilon$ , which reads as:

$$631 \quad (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*))^* \Lambda_\varepsilon = \int_0^T (h'(\alpha_\varepsilon) \mathbf{u}_\varepsilon \cdot \lambda_\varepsilon^{\mathbf{u}} + Ck'(\alpha_\varepsilon) \nabla \theta_\varepsilon \cdot \nabla \lambda_\varepsilon^\theta) \\ + \mathbf{u}'_0(\alpha_\varepsilon) \cdot \lambda_\varepsilon^{u_0} + \theta'_0(\alpha_\varepsilon) \lambda_\varepsilon^{\theta_0}.$$

632 **3.2. Limit adjoint system.** To conclude this paper, we will now study the con-  
633 vergence of the adjoint states  $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta, \lambda_\varepsilon^p)$  to functions  $(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p)$ . The only trouble  
634 concerns the multiplier  $\mu_\varepsilon$  defined in (3.4b). We will prove that at the limit,  $\mu$  is  
635 defined thanks to the convex-hull of the Heaviside function  $H : \mathbb{R} \rightarrow [0, 1]$ , defined as:

$$636 \quad (3.6) \quad H(u) = \begin{cases} \{0\} & \text{if } u < 0, \\ \{1\} & \text{if } u > 0, \\ [0, 1] & \text{if } u = 0. \end{cases}$$

637 As we will prove in this section, these limit adjoint states  $(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p)$  let us define  
638 necessary conditions of optimality for the unrelaxed problem (OPT).

639 LEMMA 3.9. Let  $(\alpha_\varepsilon) \subset \mathcal{U}_{ad}$  and  $\alpha \in \mathcal{U}_{ad}$  such that  $\alpha_\varepsilon \xrightarrow{*} \alpha$ . Define by  $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta, \lambda_\varepsilon^p)$   
640 a weak solution of (3.4) parametrized by  $\alpha_\varepsilon$ . Then, there exists  $(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p) \in L^\infty(0, T;$   
641  $H^u) \cap L^2(0, T; V^u) \times L^\infty(0, T, H^\theta) \cap L^2(0, T; V^\theta) \times L^\infty(0, T; L^2(\Omega))$  such that, up to  
642 a subsequence:

- 643 •  $\lambda_\varepsilon^{\mathbf{u}} \rightarrow \lambda^{\mathbf{u}}$  in  $L^\infty(0, T; (L^2(\Omega))^2)$  and  $\lambda_\varepsilon^\theta \rightarrow \lambda^\theta$  in  $L^\infty(0, T; L^2(\Omega))$ ,
- 644 •  $\lambda_\varepsilon^{\mathbf{u}} \xrightarrow{\varepsilon \rightarrow +\infty} \lambda^{\mathbf{u}}$  in  $L^2(0, T; (H^1(\Omega))^2)$  and  $\lambda_\varepsilon^\theta \xrightarrow{\varepsilon \rightarrow +\infty} \lambda^\theta$  in  $L^2(0, T; (H^1(\Omega)))$ ,
- 645 •  $\lambda_\varepsilon^{\mathbf{u}} \xrightarrow{\varepsilon \rightarrow +\infty} \lambda^{\mathbf{u}}$  in  $L^2(0, T; (L^2(\Gamma))^2)$  and  $\lambda_\varepsilon^\theta \xrightarrow{\varepsilon \rightarrow +\infty} \lambda^\theta$  in  $L^2(0, T; (L^2(\Gamma)))$ ,
- 646 •  $\lambda_\varepsilon^p \rightarrow \lambda^p$  in  $L^2(0, T; L^2(\Omega))$ .

647 Furthermore, there exists  $\mu \in L^\infty(0, T; L^\infty(\Gamma_{out}))$  defined by  $-\mu(x) \in H(-\mathbf{u}(x) \cdot$   
648  $\mathbf{n}(x))$  a.e. in  $\Gamma_{out}$  such that  $(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p)$  is a weak solution to (3.4a) parametrized by  
649  $\alpha$  and  $\mu$ .

650 *Proof.* The proof is very similar to the ones presented in section 2.

- 651 • In a similar manner as for Proposition 2.2 and Proposition 2.3, one shows  
652 that, for all  $\sigma \in [0, \frac{1}{6}]$ , there exist constants  $c_\lambda^\theta(\sigma)$  and  $c_\lambda^u(\sigma)$ , independent of  
653  $\varepsilon$ , such that:

$$654 \quad \sup_{[0, T]} \|\lambda_\varepsilon^{\mathbf{u}}\|_{L^2(\Omega)} + \int_0^T \|\nabla \lambda_\varepsilon^{\mathbf{u}}\|_{L^2(\Omega)} + \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F} \left( \widetilde{\lambda_\varepsilon^{\mathbf{u}}} \right) \right\|_{L^2(\Omega)} d\tau \leq c_\lambda^u(\sigma),$$

$$655 \quad \sup_{[0, T]} \|\lambda_\varepsilon^\theta\|_{L^2(\Omega)} + \int_0^T \|\nabla \lambda_\varepsilon^\theta\|_{L^2(\Omega)} + \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathcal{F} \left( \widetilde{\lambda_\varepsilon^\theta} \right) \right\|_{L^2(\Omega)} d\tau \leq c_\lambda^\theta(\sigma).$$

- 657 • These bounds prove a weaker set of convergence in the same manner as in  
658 Theorem 2.5. Since once again, we set  $d = 2$ , one proves the strong conver-  
659 gence stated above as in Corollary 2.7.

660 We only need to prove that  $(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p)$  is a weak solution to (3.4a). The terms  
661  $\langle (\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}))^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v} \rangle$  and  $\langle (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda_\varepsilon^\theta, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)}$  need a more thor-  
662 ough examination. We start with the first term for which we have

$$663 \quad \langle (\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}))^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v} \rangle_{W^u} = \int_0^T \int_{\Gamma_{out}} \text{neg}'_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) ((\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \lambda_\varepsilon^{\mathbf{u}}) \mathbf{n} \cdot \mathbf{v}.$$

664 Thanks to assumptions **(A3)** and **(A4)** and  $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$  pointwise a.e. in  $\Gamma$  (due to  
665 strong convergence in  $L^2([0, T] \times \Gamma)$ ), there is a subsequence (not relabeled) such that  
666  $\text{neg}'_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) \xrightarrow{*} \mu$  in  $L^\infty(0, T; L^\infty(\Gamma_{\text{out}}))$ , and such that  $0 \leq \mu \leq 1$  a.e. in  $\Gamma_{\text{out}}$  and

667  $\mu = 1$  a.e. in  $\{x \in \Gamma_{\text{out}}, \mathbf{u}(x) \cdot \mathbf{n}(x) > 0\}$ ,  $\mu = 0$  a.e. in  $\{x \in \Gamma_{\text{out}}, \mathbf{u}(x) \cdot \mathbf{n}(x) < 0\}$ .

668 Furthermore, due to the convergence presented above,  $((\mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}) \cdot \lambda_\varepsilon^{\mathbf{u}}) \rightarrow ((\mathbf{u} - \mathbf{u}^{\text{ref}}) \cdot \lambda^{\mathbf{u}})$   
669  $\lambda^{\mathbf{u}}$  in  $L^1(0, T; L^1(\Gamma_{\text{out}}))$ . Therefore, it proves that:

$$670 \quad \langle (\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{u}^{\text{ref}}))^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)} \rightarrow \int_0^T \int_{\Gamma_{\text{out}}} \mu ((\mathbf{u} - \mathbf{u}^{\text{ref}}) \cdot \lambda^{\mathbf{u}}) \mathbf{n} \cdot \mathbf{v}.$$

671 Similarly, one proves that:

$$672 \quad \langle (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda_\varepsilon^\theta, \mathbf{v} \rangle_{W^u(0, T)', W^u(0, T)} \rightarrow \int_0^T \int_{\Gamma_{\text{out}}} (1 + \beta\mu) (\mathbf{v} \cdot \mathbf{n}) \theta \lambda^\theta.$$

673 All other terms in **(3.3)** are easily proved to converge in the same manner as in  
674 **Theorem 2.5**. Therefore,  $(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p)$  is a weak solution to **(3.4a)** parametrized by  $\alpha$   
675 and  $\mu$ .  $\square$

676 We may now prove the final result of this paper ; namely the necessary optimality  
677 conditions of **(OPT)**.

678 **THEOREM 3.10.** *Let  $\alpha^*$  be an optimal solution of **(OPT)** with associated state*  
679  *$\mathbf{u}^*, \theta^*, p^*$ . Then there exist a multiplier  $\mu \in L^\infty(0, T; L^\infty(\Gamma_{\text{out}}))$  and adjoint states*  
680  *$(\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p) \in L^2(0, T; V^u) \times L^2(0, T; V^\theta) \times L^2(0, T; L^2(\Omega))$  solution of **(3.4a)** such*  
681 *that, denoting  $(\lambda^{\mathbf{u}_0}, \lambda^{\theta_0}) = (\lambda^{\mathbf{u}}(0, \cdot), \lambda^\theta(0, \cdot))$  and  $\Lambda = (\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^p, \lambda^{\mathbf{u}_0}, \lambda^{\theta_0})$ :*

$$682 \quad \langle \mathcal{J}'_{\alpha^*}(\mathbf{u}^*, \theta^*, p^*) + (e_{\alpha^*}(\mathbf{u}^*, \theta^*, p^*))^* \Lambda, \alpha - \alpha^* \rangle_{\mathcal{U}'_{ad}, \mathcal{U}_{ad}} \geq 0, \quad \forall \alpha \in \mathcal{U}_{ad}.$$

683 *Proof.* The proof follows the lines of [17, Theorem 4.4]. Denote by  $S_\varepsilon$  the solution  
684 operator which to  $\alpha$  associates the solution of the relaxed equations **(WFe)** and by  $S$   
685 the solution operator which to  $\alpha$  associates the solution of **(WF)**. For some  $\rho > 0$ ,  
686 consider the auxiliary optimal control problem:

$$687 \quad (3.7) \quad \begin{aligned} \min F_\varepsilon(\alpha_\varepsilon) &= \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon) + \frac{1}{2} \|\alpha^* - \alpha_\varepsilon\|_{L^2(\Omega)}^2 \\ \text{s.t.} \quad &\begin{cases} (\mathbf{u}_\varepsilon, \theta_\varepsilon, p_\varepsilon) = S_\varepsilon(\alpha_\varepsilon), \\ \alpha_\varepsilon \in \mathcal{U}_{ad}, \\ \|\alpha_\varepsilon - \alpha^*\|_{L^2(\Omega)} \leq \rho. \end{cases} \end{aligned}$$

688 Since  $\alpha_\varepsilon$  and  $\alpha^*$  are both in  $\mathcal{U}_{ad}$ , they are both bounded in  $L^\infty(\Omega)$  and therefore,  
689  $\|\alpha^* - \alpha_\varepsilon\|_{L^2(\Omega)}$  is well defined. It is classical to show that **(3.7)** admits a global  
690 minimizer  $\alpha_\varepsilon^* \in \mathcal{U}_{ad}$ .

691 Using **(2.22)** and **(2.24)** (but with  $\alpha_\varepsilon \equiv \alpha$ ), one proves that (in the norm of the  
692 topology from **Assumptions 3.1** with  $d = 2$ ):

$$693 \quad (3.8) \quad \|S(\alpha) - S_\varepsilon(\alpha)\| \lesssim C_\varepsilon, \quad \forall \alpha \in \mathcal{U}_{ad},$$

694 where  $C_\varepsilon$  has been defined in **(2.14)**.

695 Note that due to the Fréchet-differentiability of  $\mathcal{J}$  supposed in **Assumptions 3.4**  
696 and **(3.8)**, it holds, for  $\varepsilon$  large enough:

$$697 \quad |\mathcal{J}(\alpha, S(\alpha)) - \mathcal{J}(\alpha, S_\varepsilon(\alpha))| \lesssim C_\varepsilon, \quad \forall \alpha \in \mathcal{U}_{ad}, \quad \|\alpha - \alpha^*\| \leq \rho.$$



698 We obtain as a consequence that  $F_\varepsilon(\alpha^*) \lesssim C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*))$ , and:

$$699 \quad F_\varepsilon(\alpha) \gtrsim -C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*)) + \frac{1}{2}\|\alpha - \alpha^*\|_{L^2(\Omega)}^2, \quad \forall \alpha \in \mathcal{U}_{\text{ad}}, \quad \|\alpha - \alpha^*\|_{L^2(\Omega)} \leq \rho.$$

700 Therefore, for all  $\alpha \in \mathcal{U}_{\text{ad}}$  such that  $\|\alpha - \alpha^*\|_{L^2(\Omega)} \leq \rho$ :

$$701 \quad F_\varepsilon(\alpha^*) \lesssim C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*)) \lesssim C_\varepsilon + \mathcal{J}(\alpha, S(\alpha)) \lesssim 2C_\varepsilon + F_\varepsilon(\alpha).$$

702 Hence, for some constant  $C'$ , and denoting  $C'_\varepsilon = C' C_\varepsilon$ , one has the implication:

$$703 \quad \forall \alpha \in \mathcal{U}_{\text{ad}}, \quad 2C'_\varepsilon < \frac{1}{2}\|\alpha - \alpha^*\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\rho^2 \implies F_\varepsilon(\alpha^*) < F_\varepsilon(\alpha).$$

704 One has therefore the following necessary condition of optimality:

$$705 \quad (3.9) \quad \|\alpha_\varepsilon^* - \alpha^*\|_{L^2(\Omega)} \leq \sqrt{4C'_\varepsilon}.$$

706 Hence, for  $\varepsilon$  large enough,  $\alpha_\varepsilon^*$  is in the  $\rho$ -ball around  $\alpha^*$ ; therefore,  $\alpha_\varepsilon^*$  is a local  
707 solution of (OPTe). Using [Theorem 3.7](#), one then proves that there exists adjoint  
708 states  $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta, \lambda_\varepsilon^p)$  solution of (3.4a) such that, for all  $\alpha \in \mathcal{U}_{\text{ad}}$ :

$$709 \quad (3.10) \quad \left\langle \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*, p_\varepsilon^*))' \Lambda_\varepsilon, \alpha - \alpha_\varepsilon^* \right\rangle_{\mathcal{U}'_{\text{ad}}, \mathcal{U}_{\text{ad}}} + \langle \alpha_\varepsilon^* - \alpha^*, \alpha - \alpha_\varepsilon^* \rangle_{L^2(\Omega)} \geq 0.$$

710 From (3.9), one has  $\alpha_\varepsilon^* \rightarrow \alpha^*$  strongly in  $L^2(\Omega)$ , and therefore, in  $L^1(\Omega)$ . Since  
711  $(\alpha_\varepsilon^* - \alpha^*)_\varepsilon \subset \mathcal{U}_{\text{ad}}$ , one has also  $(\alpha_\varepsilon^* - \alpha^*)_\varepsilon$  bounded in  $BV(\Omega)$ . Hence,  $\alpha_\varepsilon^* \xrightarrow{*} \alpha^*$  in  
712  $\mathcal{U}_{\text{ad}}$ . Using then [Corollary 2.7](#), [Assumptions 3.1](#) and [Lemma 3.9](#), we can pass to the  
713 limit in (3.10), which concludes this proof.  $\square$

#### 714 REFERENCES

- 715 [1] J. Alexandersen and C. S. Andreasen. A review of topology optimisation for fluid-based prob-  
716 lems. *Fluids*, 5(1):29, 2020.
- 717 [2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity*  
718 *problems*, volume 254. Clarendon Press Oxford, 2000.
- 719 [3] S. Amstutz. The topological asymptotic for the Navier–Stokes equations. *ESAIM: Control,*  
720 *Optimisation and Calculus of Variations*, 11(3):401–425, 2005.
- 721 [4] P. Angot, C.-H. Bruneau, and P. Fabrie. A penalization method to take into account obstacles  
722 in incompressible viscous flows. *Numerische Mathematik*, 81(4):497–520, 1999.
- 723 [5] D. Arndt, M. Braack, and G. Lube. Finite elements for the Navier-Stokes problem with outflow  
724 condition. In *Numerical Mathematics and Advanced Applications ENUMATH 2015*, pages  
725 95–103. Springer, 2016.
- 726 [6] R. Arndt, A. N. Ceretani, and C. Rautenberg. On existence and uniqueness of solutions to a  
727 Boussinesq system with nonlinear and mixed boundary conditions. *Journal of Mathematical*  
728 *Analysis and Applications*, 490(1):124201, 2020.
- 729 [7] J. Boland and W. Layton. Error analysis for finite element methods for steady natural convec-  
730 tion problems. *Numerical Functional Analysis and Optimization*, 11(5-6):449–483, 1990.
- 731 [8] T. Borrvall and J. Petersson. Topology optimization of fluids in Stokes flow. *International*  
732 *Journal for Numerical Methods in Fluids*, 41(1):77–107, 2003.
- 733 [9] F. Boyer and P. Fabrie. Outflow boundary conditions for the incompressible non-homogeneous  
734 Navier–Stokes equations. *Discrete and Continuous Dynamical Systems-Series B*, 7(2):pp-  
735 219, 2007.
- 736 [10] F. Boyer and P. Fabrie. *Mathematical Tools for the Study of the Incompressible Navier–Stokes*  
737 *Equations and Related Models*, volume 183. Springer Science & Business Media, 2012.
- 738 [11] M. Braack and P. B. Mucha. Directional do-nothing condition for the Navier–Stokes equations.  
739 *Journal of Computational Mathematics*, pages 507–521, 2014.

- 740 [12] B. Brangeon, P. Joubert, and A. Bastide. Influence of the dynamic boundary conditions on  
741 natural convection in an asymmetrically heated channel. International Journal of Thermal  
742 Sciences, 95:64–72, 2015.
- 743 [13] C-H Bruneau and P Fabrie. New efficient boundary conditions for incompressible Navier–  
744 Stokes equations: a well-posedness result. ESAIM: Mathematical Modelling and Numerical  
745 Analysis, 30(7):815–840, 1996.
- 746 [14] F. Caubet, C. Conca Rosende, and M. Godoy. On the detection of several obstacles in 2D  
747 Stokes flow: topological sensitivity and combination with shape derivatives. IPI : Inverse  
748 Problems and Imaging, 2016.
- 749 [15] A. Ceretani and C. Rautenberg. The Boussinesq system with mixed non-smooth boundary  
750 conditions and do-nothing boundary flow. Zeitschrift für angewandte Mathematik und  
751 Physik, 70(1):1–24, 2019.
- 752 [16] N. Chami and A. Zoughaib. Modeling natural convection in a pitched thermosyphon system in  
753 building roofs and experimental validation using particle image velocimetry. Energy and  
754 buildings, 42(8):1267–1274, 2010.
- 755 [17] C. Christof, C. Meyer, S. Walther, and C. Clason. Optimal control of a non-smooth semilinear  
756 elliptic equation. Mathematical Control & Related Fields, 8(1):247, 2018.
- 757 [18] C. Clason and K. Kunisch. A duality-based approach to elliptic control problems in non-reflexive  
758 banach spaces. ESAIM: Control, Optimisation and Calculus of Variations, 17(1):243–266,  
759 2011.
- 760 [19] C. Clason, K. Kunisch, and A. Rund. Optimal control of partial differential equations with  
761 nonsmooth cost functionals. In Proceedings of the XXIV Congress on Differential Equations  
762 and Applications / XIV Congress on Applied Mathematics, Cádiz, 2015.
- 763 [20] P.-H. Cocquet, S. Riffo, and J. Salomon. Optimization of bathymetry for long waves with small  
764 amplitude. arXiv preprint arXiv:2003.09135, 2020.
- 765 [21] E. Colmenares, G. N. Gatica, and R. Oyarzúa. A posteriori error analysis of an augmented  
766 fully-mixed formulation for the stationary Boussinesq model. Computers & Mathematics  
767 with Applications, 77(3):693–714, 2019.
- 768 [22] T. Dbouk. A review about the engineering design of optimal heat transfer systems using  
769 topology optimization. Applied Thermal Engineering, 112:841–854, 2017.
- 770 [23] G. Desrayaud, E. Chénier, A. Joulin, A. Bastide, B. Brangeon, J.P. Caltagirone, Y. Cherif,  
771 R. Eymard, C. Garnier, S. Giroux-Julien, et al. Benchmark solutions for natural convection  
772 flows in vertical channels submitted to different open boundary conditions. International  
773 journal of thermal sciences, 72:18–33, 2013.
- 774 [24] F. Feppon, G. Allaire, F. Bordeu, J. Cortial, and C. Dapogny. Shape optimization of a coupled  
775 thermal fluid–structure problem in a level set mesh evolution framework. SeMA Journal,  
776 76(3):413–458, 2019.
- 777 [25] L. Formaggia, J.-F. Gerbeau, F. Nobile, and A. Quarteroni. Numerical treatment of defec-  
778 tive boundary conditions for the Navier–Stokes equations. SIAM Journal on Numerical  
779 Analysis, 40(1):376–401, 2002.
- 780 [26] G. Galdi. An introduction to the mathematical theory of the Navier–Stokes equations:  
781 Steady-state problems. Springer Science & Business Media, 2011.
- 782 [27] H. Garcke, M. Hinze, C. Kahle, and K. F. Lam. A phase field approach to shape optimiza-  
783 tion in Navier–Stokes flow with integral state constraints. Advances in Computational  
784 Mathematics, 44(5):1345–1383, 2018.
- 785 [28] V. Girault and P.-A. Raviart. Finite element methods for Navier–Stokes equations: theory and  
786 algorithms, volume 5. Springer Science & Business Media, 2012.
- 787 [29] T. Goudon, S. Krell, and G. Lissoni. Ddfv method for Navier–Stokes problem with outflow  
788 boundary conditions. Numerische Mathematik, 142(1):55–102, 2019.
- 789 [30] F. Harder and G. Wachsmuth. Comparison of optimality systems for the optimal control of  
790 the obstacle problem. GAMM-Mitteilungen, 40(4):312–338, 2018.
- 791 [31] Jaroslav Haslinger and Raino AE Mäkinen. On a topology optimization problem governed  
792 by two-dimensional Helmholtz equation. Computational Optimization and Applications,  
793 62(2):517–544, 2015.
- 794 [32] M. Hinze. Optimal and instantaneous control of the instationary Navier–Stokes equations.  
795 Habilitationsschrift, Fachbereich Mathematik, Technische Universität, Berlin, 2002.
- 796 [33] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. Optimization with PDE constraints, vol-  
797 ume 23. Springer Science & Business Media, 2008.
- 798 [34] S. Kračmar and J. Neustupa. Modeling of the unsteady flow through a channel with an artificial  
799 outflow condition by the Navier–Stokes variational inequality. Mathematische Nachrichten,  
800 291(11-12):1801–1814, 2018.
- 801 [35] K. Kunisch and D. Wachsmuth. Sufficient optimality conditions and semi-smooth Newton meth-

- 802 ods for optimal control of stationary variational inequalities. ESAIM: Control, Optimisation  
803 and Calculus of Variations, 18(2):520–547, 2012.
- 804 [36] J.L. Lions. Quelques méthodes de résolution des problèmes aux limites non linéaires. Les Cours  
805 de référence. Dunod, 2002.
- 806 [37] C. Meyer and L. Susu. Optimal control of nonsmooth, semilinear parabolic equations. SIAM  
807 Journal on Control and Optimization, 55(4):2206–2234, 2017.
- 808 [38] B. Mohammadi and O. Pironneau. Shape optimization in fluid mechanics. Annual Review of  
809 Fluid Mechanics, 36:255–279, 2004.
- 810 [39] B. Mohammadi and O. Pironneau. Applied shape optimization for fluids. Oxford University  
811 Press, 2010.
- 812 [40] A. A. Novotny, J. Sokółowski, and A. Żochowski. Topological derivatives of shape function-  
813 als. Part II: first-order method and applications. Journal of Optimization Theory and  
814 Applications, 180(3):683–710, 2019.
- 815 [41] C. Popa, D. Ospir, S. Fohanno, and C. Chereches. Numerical simulation of dynamical aspects  
816 of natural convection flow in a double-skin façade. Energy and Buildings, 50:229–233, 2012.
- 817 [42] D. Ramalingom, P.-H. Cocquet, and A. Bastide. Numerical study of natural convection in  
818 asymmetrically heated channel considering thermal stratification and surface radiation.  
819 Numerical Heat Transfer, Part A: Applications, 72(9):681–696, 2017.
- 820 [43] D. Ramalingom, P.-H. Cocquet, R. Maleck, and A. Bastide. A multi-objective optimization  
821 problem in mixed and natural convection for a vertical channel asymmetrically heated.  
822 Structural and Multidisciplinary Optimization, 60(5):2001–2020, 2019.
- 823 [44] Delphine Ramalingom, Pierre-Henri Cocquet, and Alain Bastide. A new interpolation tech-  
824 nique to deal with fluid-porous media interfaces for topology optimization of heat transfer.  
825 Computers & Fluids, 168:144–158, 2018.
- 826 [45] A. Schiela and D. Wachsmuth. Convergence analysis of smoothing methods for optimal control  
827 of stationary variational inequalities with control constraints. ESAIM: Mathematical  
828 Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique,  
829 47(3):771–787, 2013.
- 830 [46] C. Suárez, P. Joubert, J. Molina, and F. Sánchez. Heat transfer and mass flow correlations for  
831 ventilated facades. Energy and Buildings, 43(12):3696–3703, 2011.
- 832 [47] R. Temam. Navier–Stokes equations: theory and numerical analysis, volume 343. American  
833 Mathematical Soc., 2001.
- 834 [48] Alexandre Vieira, Alain Bastide, and Pierre-Henri Cocquet. Topology optimization for  
835 steady-state anisothermal flow targeting solid with piecewise constant thermal diffusivity.  
836 hal-02569142, version 2, 2020.
- 837 [49] I. Yousept. Optimal control of non-smooth hyperbolic evolution Maxwell equations in type-II  
838 superconductivity. SIAM Journal on Control and Optimization, 55(4):2305–2332, 2017.