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Higher integrability of the gradient for the Thermal Insulation problem

C. Labourie, E. Milakis

Abstract

We prove the higher integrability of the gradient for minimizers of the thermal insulation problem, an analogue of De Giorgi's conjecture for the Mumford-Shah functional. We deduce that the singular part of the free boundary has Hausdorff dimension strictly less than $n - 1$.

AMS Subject Classifications: 35R35, 35J20, 49N60, 49Q20

Keywords: Thermal Insulation, Higher Integrability, Free Boundary Problems.

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1 Introduction

We fix a bounded connected set $\Omega \subset \mathbf{R}^n$. The thermal insulation problem consists in minimizing the functional

$$\mathcal{I}(A, u) := \int_A |\nabla u|^2 d\mathcal{L}^n + \int_{\partial A} |u^*|^2 d\mathcal{H}^{n-1} + \mathcal{L}^n(A) \quad (1)$$

among all pairs (A, u) where $A \subset \mathbf{R}^n$ is an admissible domain and $u \in W^{1,2}(A)$ is a function such that $u = 1$ for \mathcal{L}^n -a.e. on Ω . Here, u^* is the trace of u on ∂A .

The problem has been studied by Caffarelli–Kriventsov in [4], [8] and Bucur–Giacomini–Luckhaus in [2], [3]. The authors transpose the problem to a slightly different setting in order to apply the direct method of the calculus of variation. The authors represent a pair (A, u) by the function $u\mathbf{1}_A$ and relax the functional on SBV. The new problem consists in minimizing the functional

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} |\nabla u|^2 d\mathcal{L}^n + \int_{J_u} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(\{u > 0\}) \quad (2)$$

among all functions $u \in \text{SBV}(\mathbf{R}^n)$ such that $u = 1$ \mathcal{L}^n -a.e. on Ω . The definition of J_u and \bar{u}, \underline{u} are given in Appendix A. This new setting is more suited to a direct minimization since it enjoys the compactness and closure properties of SBV. In parenthesis, there always exist functions $u \in \text{SBV}(\mathbf{R}^n)$ such that $u = 1$ \mathcal{L}^n -a.e. on Ω and $\mathcal{F}(u) < \infty$. For example, $u = \mathbf{1}_B$ where B is an open ball containing Ω . In [4, Theorem 4.2], Caffarelli and Kriventsov prove that the SBV problem has a solution u . A key point property of solutions is that there exists $0 < \delta < 1$ (depending on n, Ω) such that $\text{spt}(u) \subset B(0, \delta^{-1})$ and

$$u \in \{0\} \cup [\delta, 1] \quad \mathcal{L}^n\text{-a.e. on } \mathbf{R}^n. \quad (3)$$

This property has also been proved in [2]

The main goal of the present article is to prove that there exists $p > 1$ such that $|\nabla u|^2 \in L^p_{\text{loc}}(\mathbf{R}^n \setminus \bar{\Omega})$. A parallel property was conjectured by De Giorgi for minimizers of the Mumford-Shah functional and solved by De Philippis and Figalli in [6]. Our proof is inspired by the technique of [6] and it relies on three key properties: the Ahlfors-regularity of the free boundary, the uniform rectifiability of the free boundary and the ε -regularity theorem. For the thermal insulation problem, the conclusion of the ε -regularity theorem is that the free boundary looks like a pair of graphs rather than just one graph. Moreover, the function u satisfies an elliptic equation with a Robin boundary condition at the boundary rather than a Neumann boundary condition. In our approach we deduce a *porosity* property which means that the singular part Σ of the free boundary has many holes in a quantified way. Finally,

once we establish the higher integrability of the gradient, we are also able to conclude that the dimension of Σ is less than $n - 1$.

NOTATIONS. We work in the Euclidean space \mathbf{R}^n with $n > 1$. For $x \in \mathbf{R}^n$ and $r > 0$, $B(x, r)$ or $B_r(x)$ is the open ball centered in x and of radius r . Sometimes x is omitted and it is simply denoted by B_r . Given an open ball $B = B(x, r)$, the notation $2B$ means $B(x, 2r)$. Given a set $A \subset \mathbf{R}^n$, the indicator function of A is denoted by $\mathbf{1}_A$. Given two sets $A, B \subset \mathbf{R}^n$, the notation $A \subset\subset B$ means that there exists a compact set $K \subset \mathbf{R}^n$ such that $A \subset K \subset B$. We have gathered some definitions and results from the theory of BV functions in the introduction of Appendix A.

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2 Generalities about minimizers

2.1 Definition

Our ambient space is an open set X of \mathbf{R}^n . One can think of X as $\mathbf{R}^n \setminus \overline{\Omega}$. We introduce a few notations. Given $u \in \text{SBV}_{\text{loc}}(X)$, we denote by K the support of the singular part of Du :

$$K := \text{spt}(|\bar{u} - \underline{u}| \mathcal{H}^{n-1} \llcorner J_u) \quad (4a)$$

$$:= \text{spt}(\mathcal{H}^{n-1} \llcorner J_u). \quad (4b)$$

For $x \in K$ and $r > 0$ such that $\overline{B}(x, r) \subset X$, we define

$$\omega_2(x, r) = r^{-(n-1)} \int_{B(x, r)} |\nabla u|^2 d\mathcal{L}^n, \quad (5a)$$

$$\beta_2(x, r) = \left(r^{-(n+1)} \inf_V \int_{K \cap \overline{B}(x, r)} d(y, V)^2 d\mathcal{H}^{n-1}(y) \right)^{\frac{1}{2}}, \quad (5b)$$

where V runs among $(n - 1)$ planes $V \subset \mathbf{R}^n$ passing through x . When there is an ambiguity, we will write $\beta_{K,2}$ instead of β_2 . For any open ball B such that $\overline{B} \subset X$, we define a *competitor of u in B* as a function $v \in \text{SBV}_{\text{loc}}(X)$ such that $v = u$ \mathcal{L}^n -a.e. on $X \setminus \overline{B}$. Finally, we fix a constant $\delta \in]0, 1[$ for all the paper.

Definition 2.1. We say that $u \in \text{SBV}_{\text{loc}}(X)$ is a *minimizer* if

1. for \mathcal{L}^n -a.e. $x \in X$, we have $u \in \{0\} \cup [\delta, \delta^{-1}]$;
2. for all open balls B such that $\overline{B} \subset X$, for all competitors v of u in B ,

$$\begin{aligned} & \int_B |\nabla u|^2 d\mathcal{L}^n + \int_{J_u \cap \overline{B}} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{u > 0\}) \\ & \leq \int_B |\nabla v|^2 d\mathcal{L}^n + \int_{J_v \cap \overline{B}} (\bar{v}^2 + \underline{v}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{v > 0\}). \end{aligned} \quad (6)$$

As a first consequence, we have that $\bar{u}, \underline{u} \in \{0\} \cup [\delta, \delta^{-1}]$ everywhere in X . In particular, $\bar{u} \geq \delta$ everywhere on S_u . For all open balls B such that $\bar{B} \subset X$, we have

$$\int_B |\nabla u|^2 d\mathcal{L}^n + \int_{J_u \cap \bar{B}} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} < \infty. \quad (7)$$

This shows that $|\nabla u|^2 \in L^1_{\text{loc}}(X)$ and that S_u is \mathcal{H}^{n-1} -locally finite in X . In $X \setminus \bar{S}_u$, the function u is $W^{1,2}_{\text{loc}}$ and locally minimizes its Dirichlet energy. Therefore, u is harmonic (and thus continuous) in $X \setminus \bar{S}_u$. We conclude that in each connected component of $X \setminus \bar{S}_u$, we have either $u > \delta$ everywhere or $u = 0$ everywhere.

2.2 Properties

The next results (Ahlfors-regularity, uniform rectifiability and ε -regularity theorem) also hold for almost-minimizers ([8, Definition 2.1]). We are going to cite [4, Corollary 3.3 and Theorem 5.1].

Proposition 2.2 (Ahlfors-regularity). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. There exists $0 < r_0 \leq 1$ and $C \geq 1$ (both depending on n, δ) such that the following holds.*

1. For all $x \in X$, for all $0 < r \leq r_0$ such that $\bar{B}(x, r) \subset X$,

$$\int_{B(x, r)} |\nabla u|^2 d\mathcal{L}^n + \mathcal{H}^{n-1}(K \cap \bar{B}(x, r)) \leq Cr^{n-1}. \quad (8)$$

2. For all $x \in \bar{S}_u$, for all $0 < r \leq r_0$ such that $\bar{B}(x, r) \subset X$,

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \geq C^{-1}r^{n-1}. \quad (9)$$

Corollary 2.3. *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer.*

(i) *We have $K = \bar{S}_u = \bar{J}_u$ and $\mathcal{H}^{n-1}(K \setminus J_u) = 0$.*

(ii) *The set $A_u := \{\bar{u} > 0\} \setminus K$ is open and $\partial A_u = K$.*

Proof. It is straightforward by definition that $K \subset \bar{J}_u \subset \bar{S}_u$. On the other hand, property (9) shows that $\bar{S}_u \subset K$. We justify that $\mathcal{H}^{n-1}(K \setminus J_u) = 0$. The jump set J_u is Borel and \mathcal{H}^{n-1} locally finite in X , so for \mathcal{H}^{n-1} -a.e. $x \in X \setminus J_u$,

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^{n-1}(B(x, r) \cap J_u)}{r^{n-1}} = 0 \quad (10)$$

(see [10, Theorem 6.2]). We draw our claim from the observation that this limit contradicts (9).

We study the set A_u . We recall that the function \bar{u} is continuous in $X \setminus K$ (since it coincides with u outside S_u) and $\bar{u} \in \{0\} \cup [\delta, 1]$ everywhere in $X \setminus K$. As a consequence, the sets

$$A_u := \{\bar{u} > 0\} \setminus K, \quad (11)$$

$$B_u := \{\bar{u} = 0\} \setminus K \quad (12)$$

are open subsets of $X \setminus K$ and thus of X . The space X is the disjoint union

$$X = K \cup A_u \cup B_u, \quad (13)$$

where A_u and B_u are open and K is relatively closed, so $\overline{A_u} \subset A_u \cup K$. We show that $S_u \subset \overline{A_u}$. Let us suppose that there exists $x \in S_u$ and $r > 0$ such that $B(x, r) \cap A_u = \emptyset$. Then $B(x, r) \setminus K \subset \{\bar{u} = 0\}$ so we have $u = 0$ \mathcal{L}^n -a.e. on $B(x, r)$ and thus x is a Lebesgue point of u (a contradiction). We conclude that $S_u \subset \overline{A_u}$ and in turn $K \subset \overline{A_u}$ so $\overline{A_u} = A_u \cup K$. \square

We are going to apply [5] to justify that K is locally contained in a uniformly rectifiable set. For the reader's convenience we have summarised the relevant results of [5] in Appendix C.

Proposition 2.4 (Uniform Rectifiability). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. There exists $0 < r_0 \leq 1$ (depending on n, δ) such that the following holds. For all $x \in K$ and $0 < r \leq r_0$ such that $\overline{B}(x, 2r) \subset X$, there is a closed, Ahlfors-regular, uniformly rectifiable set E of dimension $(n-1)$ such that $K \cap B(x, r) \subset E$. The constants for the Ahlfors-regularity and uniform rectifiability depends on n, δ .*

Proof. We want to show that (u, K) satisfy Definition C.1, or rather the alternative Definition given in Remark C.5. Then the Proposition will follow from Theorem C.4. First, it is clear that (u, K) is an admissible pair. Let B be an open ball of radius $r > 0$ such that $\overline{B} \subset X$. Let an admissible pair (v, L) be a competitor of (u, K) in B . As explained in Remark C.2, we can assume without loss of generality that L is \mathcal{H}^{n-1} locally finite. Therefore, $v \in \text{SBV}_{\text{loc}}(X)$ and $\mathcal{H}^{n-1}(J_v \setminus L) = 0$. We have included more details about the construction of SBV functions in Appendix A. We can now apply the minimality inequality. We have

$$\begin{aligned} & \int_B |\nabla u|^2 d\mathcal{L}^n + \int_{J_u \cap \overline{B}} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{u > 0\}) \\ & \leq \int_B |\nabla v|^2 d\mathcal{L}^n + \int_{J_v \cap \overline{B}} (\bar{v}^2 + \underline{v}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{v > 0\}) \end{aligned} \quad (14)$$

so

$$\begin{aligned} & \int_B |\nabla u|^2 d\mathcal{L}^n + \delta^2 \mathcal{H}^{n-1}(J_u \cap \overline{B}) + \mathcal{L}^n(B \cap \{u > 0\}) \\ & \leq \int_B |\nabla v|^2 d\mathcal{L}^n + \delta^{-2} \mathcal{H}^{n-1}(J_v \cap \overline{B}) + \mathcal{L}^n(B \cap \{v > 0\}). \end{aligned} \quad (15)$$

We omit the term $\mathcal{L}^n(B \cap \{u > 0\})$ at the left and we bound the term $\mathcal{L}^n(B \cap \{v > 0\})$ at the right by $\omega_n r^n$ where ω_n is the Lebesgue volume of the unit ball. We can replace J_u by K at the left since $\mathcal{H}^{n-1}(K \setminus J_u) = 0$. We can replace J_v by L at the right since $\mathcal{H}^{n-1}(J_v \setminus L) = 0$. It follows that

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \leq \delta^{-4} \mathcal{H}^{n-1}(L \cap \overline{B}) + \delta^{-2} \Delta E + \delta^{-2} \omega_n r^n \quad (16)$$

where

$$\Delta E = \int_B |\nabla v|^2 - \int_B |\nabla u|^2 \, d\mathcal{L}^n. \quad (17)$$

□

We are going to cite the ε -regularity theorem for our problem [8, Theorem 14.1]. Contrary to the ε -regularity theorem for the Mumford-Shah problem, it does not require $\omega_2(x, r)$ to be small. It says that when K is very close to a plane, K is given by a pair of smooth graphs. We to describe this situation in the next definition.

Given a point $x \in \mathbf{R}^n$, a vector $e_n \in \mathbf{S}^{n-1}$, we can decompose each point $y \in \mathbf{R}^n$ under the form $y = x + (y' + y_n e_n)$ where $y' \in e_n^\perp$ and $y_n \in \mathbf{R}$. Then for all function $f: e_n^\perp \rightarrow \mathbf{R}$, we define the graph of f in the coordinate system (x, n) as

$$\Gamma_{(x,n)}(f) := \{y \in \mathbf{R}^n \mid y' = f(y_n)\}. \quad (18)$$

Definition 2.5. Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. Let $x \in K$ and $R > 0$ be such that $\overline{B}(x, R) \subset X$. Let $0 < \alpha < 1$. We say that K is $C^{1,\alpha}$ -regular in $B = B(x, R)$ if it satisfies the three following conditions.

- (i) There exists a vector $n \in \mathbf{S}^{n-1}$ and two functions $f_i: e_n^\perp \rightarrow \mathbf{R}$ ($i = 1, 2$) such that $f_1 \leq f_2$ and

$$K \cap B = \bigcup_{i=1,2} \Gamma_{(x,n)}(f_i) \cap B. \quad (19)$$

The functions f_1, f_2 are $C^{1,\alpha}$ and

$$R^{-1} |f_i|_{C^0} + |\nabla f_i|_{C^0} + R^\alpha |\nabla f_i|_{C^{0,\alpha}} \leq \frac{1}{4}. \quad (20)$$

- (ii) There are two possible cases. The first case is

$$\begin{cases} u > 0 & \text{in } \{y \in B \mid y_n < f_1(y') \text{ or } y_n > f_2(y')\} \\ u = 0 & \text{in } \{y \in B \mid f_1(y') < y_n < f_2(y')\} \end{cases} \quad (21)$$

The second case is $f_1 = f_2$ and

$$\begin{cases} u > 0 & \text{in } \{y \in B \mid y_n > f_1(y')\} \\ u = 0 & \text{in } \{y \in B \mid y_n < f_1(y')\} \end{cases} \quad (22)$$

or inversely.

Theorem 2.6 (ε -regularity theorem). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer and let $x \in K$.*

- (i) *For all $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ (depending on n, δ, β) such that the following holds. For $r > 0$ such that $\overline{B}(x, r) \subset X$ and $\beta_2(x, r) + r \leq \varepsilon_1$, we have $\omega_2(x, r) \leq \varepsilon$.*
- (ii) *There exists $\varepsilon > 0, C \geq 1$ and $0 < \alpha < 1$ (both depending on n, δ) such that the following holds. For $r > 0$ such that $\overline{B}(x, r) \subset X$ and $\beta_2(x, r) + r \leq \varepsilon$, the set K is $C^{1, \alpha}$ -regular in $B(x, C^{-1}R)$.*

This following last result is specific to minimizers rather than almost-minimizers.

Proposition 2.7. *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. Let $x \in K$ and $R > 0$ be such that $B(x, R) \subset X$. We assume that K is regular in $B = B(x, R)$ (Definition 2.5) and we denote by Γ_i the graph of f_i in B ($i = 1, 2$). Then for each $i = 1, 2$, $u|_{A_i}$ solves the Robin problem*

$$\begin{cases} \Delta u & = 0 & \text{in } A_i \\ \partial_{\nu_i} u - u_i & = 0 & \text{in } \Gamma_i, \end{cases} \quad (23)$$

where

$$A_1 = \{y \in B \mid y_n < f_1(y')\} \quad (24)$$

$$A_2 = \{y \in B \mid y_n > f_2(y')\} \quad (25)$$

and ν_i is the inner normal vector to A_i .

Proof. We only detail the case (21) of Definition 2.5 and we prove the Proposition for $i = 2$. We partition B in three sets (modulo \mathcal{L}^n)

$$A_1 = \{z \in B \mid y_n < f_1(y')\} \quad (26)$$

$$A_2 = \{z \in B \mid y_n > f_2(y')\} \quad (27)$$

$$A_3 = \{z \in B \mid f_1(y') < x_n < f_2(y')\}. \quad (28)$$

The first paragraph is devoted to detail a few generalities about traces and upper/lower limits. We consider a general $v \in L^\infty(B) \cap W_{\text{loc}}^{1,2}(B \setminus K)$ such that $v = 0$ in A_3 . For each $i = 1, 2$, there exists $v_i^* \in L^1(\Gamma_i)$ such that for \mathcal{H}^{n-1} -a.e. $x \in \Gamma_i$,

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r(x) \cap A_i} |v(y) - v_i^*(x)| \, d\mathcal{L}^n(y) = 0. \quad (29)$$

The boundary Γ_i is C^1 so for all $x \in \Gamma_i$, there a vector $\nu_i(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r(x) \cap (A_i \Delta H_i^+(x))) = 0 \quad (30)$$

where

$$H_i^+(x) = \{y \in \mathbf{R}^n \mid (y-x) \cdot \nu_i(x) > 0\}. \quad (31)$$

Therefore, (29) is equivalent to

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r(x) \cap H_i^+(x)} |v(y) - v_i^*(x)| d\mathcal{L}^n(y) = 0. \quad (32)$$

For \mathcal{H}^{n-1} -a.e. $x \in \Gamma_2$, we detail the relationship between $\bar{v}(x)^2 + \underline{v}(x)^2$ and $v_i^*(x)$. We fix $x \in \Gamma_2 \setminus \Gamma_1$ such that (32) is satisfied for $i = 2$. We have $v = 0$ on A_3 and $B_r(x)$ is disjoint from A_1 for small $r > 0$ so

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r(x) \setminus A_2} |v| d\mathcal{L}^n = 0 \quad (33)$$

which is equivalent to

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r(x) \cap H_2^-(x)} |v| d\mathcal{L}^n = 0. \quad (34)$$

Combining (32) for $i = 2$ and (34), we deduce

$$\bar{v}(x)^2 + \underline{v}(x)^2 = v_2^*(x)^2. \quad (35)$$

Next, we fix $x \in \Gamma_1 \cap \Gamma_2$ such that (32) holds for $i = 1$ and $i = 2$. The surfaces Γ_1 and Γ_2 have necessary the same tangent plane at x and the vectors ν_i are opposed. Combining (32) for $i = 1$ and $i = 2$, we deduce

$$\bar{v}^2 + \underline{v}^2 = (v_1^*)^2 + (v_2^*)^2. \quad (36)$$

We come back to our minimizer $u \in \text{SBV}_{\text{loc}}(X)$. We fix $\varphi \in C_c^1(B)$. For $\varepsilon \in \mathbf{R}$, we define $v: X \rightarrow \mathbf{R}$ by

$$v = \begin{cases} u + \varepsilon\varphi & \text{in } A_2 \\ u & \text{in } X \setminus A_2 \end{cases} \quad (37)$$

It is clear that $\{v \neq u\} \subset\subset B$ and that v is C^1 in $X \setminus K$. As K is \mathcal{H}^{n-1} locally finite in X , we have $v \in \text{SBV}_{\text{loc}}(X)$ and $S_v \subset K$. Remember that $u \geq \delta$ in $A_1 \cup A_2$, and $u = 0$ in A_3 . We take ε small enough so that $\varepsilon|\varphi|_\infty < \delta$. As a consequence $v > 0$ in $A_1 \cup A_2$ and $v = 0$ in A_3 . Let us check the multiplicities on the discontinuity set. As we have seen before, $J_v \cap B \subset \Gamma_1 \cup \Gamma_2$. We observe that for $x \in \Gamma_2$ such that the trace $u_2^*(x)$ exists, we have

$$v_2^*(x) = u_2^*(x) + \varepsilon\varphi(x) \quad (38)$$

and for $x \in \Gamma_1$ such the trace $u_1^*(x)$ exists, we have

$$v_1^*(x) = u_1^*(x). \quad (39)$$

Using the previous discussion, we deduce that for \mathcal{H}^{n-1} -a.e. on $\Gamma_2 \setminus \Gamma_1$,

$$\bar{v}^2 + \underline{v}^2 = (u_2^* + \varepsilon\varphi)^2 \quad (40)$$

$$= (\underline{u}^2 + \bar{u}^2) + 2\varepsilon\varphi u_2^* + \varepsilon^2|\varphi|^2 \quad (41)$$

that for \mathcal{H}^{n-1} -a.e. on $\Gamma_2 \cap \Gamma_1$,

$$\bar{v}^2 + \underline{v}^2 = (u_2^* + \varepsilon\varphi)^2 + (u_1^*)^2 \quad (42)$$

$$= (\underline{u}^2 + \bar{u}^2) + 2\varepsilon\varphi u_2^* + \varepsilon^2|\varphi|^2 \quad (43)$$

and that for \mathcal{H}^{n-1} -a.e. on $\Gamma_1 \setminus \Gamma_2$,

$$\bar{v}^2 + \underline{v}^2 = (u_1^*)^2 \quad (44)$$

$$= \underline{u}^2 + \bar{u}^2. \quad (45)$$

Finally, it is clear that

$$\begin{aligned} \int_B |\nabla v|^2 d\mathcal{L}^n &= \int_B |\nabla u|^2 d\mathcal{L}^n + 2\varepsilon \int_{A_2} \langle \nabla u, \nabla \varphi \rangle d\mathcal{L}^n \\ &\quad + \varepsilon^2 \int_{A_2} |\nabla \varphi|^2 d\mathcal{L}^n. \end{aligned} \quad (46)$$

We plug all these informations in the minimality inequality and we obtain that

$$0 \leq 2\varepsilon \int_{A_2} \langle \nabla u, \nabla \varphi \rangle d\mathcal{L}^n + 2\varepsilon \int_{\Gamma_2} \varphi u_2^* d\mathcal{H}^{n-1} + C(\varphi)\varepsilon^2. \quad (47)$$

As this holds for all small ε (positive or negative), we conclude that

$$\int_{A_2} \langle \nabla u, \nabla \varphi \rangle d\mathcal{L}^n + \int_{\Gamma_2} \varphi u_2^* d\mathcal{H}^{n-1} = 0. \quad (48)$$

□

3 Porosity of the subset where K is not regular

The following results says that the set $\Sigma \subset K$ where K is not regular has many holes in a quantified way. It is simpler to obtain than in the Mumford-Shah problem (see [11]) because the ε -regularity theorem of [8] only requires to control the flatness.

Proposition 3.1 (Porosity). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. There exists $0 < r_0 \leq 1$, $C \geq 1$ and $0 < \alpha < 1$ (all depending on n, δ) for which the following holds. For all $x \in K$ and all $0 < r \leq r_0$ such that $\bar{B}(x, 2r) \subset X$, there exists $y \in K \cap B(x, r)$ such that K is $C^{1,\alpha}$ -regular in $B := B(y, C^{-1}r)$ and moreover,*

$$\sup_{B \setminus K} |\nabla u|^2 \leq Cr^{-1}. \quad (49)$$

Proof. The letter C is a constant ≥ 1 that depends on n, δ . The letter α is the constant of Theorem 2.6. For $y \in K$ and $t > 0$ such that $\overline{B}(y, t) \subset X$, we define the flatness

$$\beta_K(y, t) = \inf_V \sup_{z \in K \cap \overline{B}(y, t)} t^{-1} d(z, V), \quad (50)$$

where the infimum is taken over the affine hyperplanes V of \mathbf{R}^n passing through y . In [5, (41.2)], the infimum is taken over all affine hyperplanes V of \mathbf{R}^n (not necessarily passing through y); this would decrease our β number but no more than a factor $\frac{1}{2}$. Indeed, if V is an hyperplane of \mathbf{R}^n and p is the orthogonal projection of y onto V , then $V - (p - y)$ is passing through y and we have

$$\beta_K(y, t) \leq \sup_{z \in K \cap \overline{B}(y, t)} d(z, V - (p - y)) \quad (51)$$

$$\leq |p - y| + \sup_{z \in K \cap \overline{B}(y, t)} d(z, V) \quad (52)$$

$$\leq 2 \sup_{z \in K \cap \overline{B}(y, t)} d(z, V). \quad (53)$$

We also observe that

$$\beta_{K,2}(y, t)^2 \leq r^{-(n-1)} \mathcal{H}^{n-1}(K \cap \overline{B}(y, t)) \beta_K(y, t)^2 \quad (54)$$

so as soon as t is small enough for the Ahlfors-regularity to hold, we have $\beta_{K,2}(y, t) \leq C \beta_K(y, t)$.

Let r_0 be the minimum between the radius of Proposition 2.2 (Ahlfors-regularity) and the radius of Proposition 2.4 (uniform rectifiability). We fix $x \in K$ and $0 < r \leq r_0$ such that $\overline{B}(x, 2r) \subset X$. According to Proposition 2.4, there exists an Ahlfors-regular and uniformly rectifiable set E such that $K \cap B(x, r) \subset E$. Moreover, the constants for the Ahlfors-regularity and uniform rectifiability depends on n, δ . For $y \in E$ and $t > 0$, we define as before

$$\beta_E(y, t) = \inf_V \sup_{z \in E \cap \overline{B}(y, t)} t^{-1} d(z, V) \quad (55)$$

where the infimum is taken on the set of all affine hyperplanes V of \mathbf{R}^n passing through x . As E is Ahlfors-regular and uniformly rectifiable, the Weak Geometric Lemma [5, (73.13)] states that for all $\varepsilon > 0$, the set

$$\{(y, t) \mid y \in E, 0 < t < \text{diam}(E), \beta_E(y, t) > \varepsilon\} \quad (56)$$

is a Carleson set. This means that for all $\varepsilon > 0$, there exists $C_0(\varepsilon) \geq 1$ (depending on n, δ, ε) such that for all $y \in E$ and all $0 < t < \text{diam}(E)$,

$$\int_0^t \int_{E \cap B(y, t)} \mathbf{1}_{\{\beta_E(z, s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \leq C_0(\varepsilon) t^{n-1}. \quad (57)$$

For all $z \in K \cap B(x, \frac{1}{2}r)$ and for all $0 < s < \frac{1}{2}r$, we have $K \cap \overline{B}(z, s) \subset E \cap \overline{B}(z, s)$ and thus $\beta_K(z, s) \leq \beta_E(z, s)$. Then for all $t > 0$ such that $t \leq \frac{1}{2}r$ and $t < \text{diam}(K \cap B(x, r))$, we have

$$\int_0^t \int_{K \cap B(x, t)} \mathbf{1}_{\{\beta_K(z, s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \leq C_0(\varepsilon)t^{n-1}. \quad (58)$$

We only apply this property with $\rho = \frac{1}{4}\text{diam}(K \cap B(x, r))$;

$$\int_0^\rho \int_{K \cap B(x, \rho)} \mathbf{1}_{\{\beta_K(z, s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \leq C_0(\varepsilon)\rho^{n-1}. \quad (59)$$

In parenthesis, we have $\rho \geq C^{-1}r$ by Ahlfors-regularity of K . We are going to deduce from (59) that for all $\varepsilon > 0$, there exists $C(\varepsilon) \geq 1$, a point $y \in K \cap B(x, \rho)$ and a radius t such that $C(\varepsilon)^{-1}\rho \leq t \leq \rho$ and $\beta_K(y, t) < \varepsilon$. We proceed by contradiction for some $C(\varepsilon)$ to be precised. We have therefore

$$\int_0^\rho \int_{K \cap B(x, \rho)} \mathbf{1}_{\{\beta_K(z, s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \quad (60)$$

$$\geq \mathcal{H}^{n-1}(K \cap B(x, \rho)) \int_{C(\varepsilon)^{-1}\rho}^\rho \frac{ds}{s} \quad (61)$$

$$\geq C^{-1}\rho^{n-1} \ln(C(\varepsilon)) \quad (62)$$

This contradicts (59) if $C(\varepsilon)$ is too big compared to $C_0(\varepsilon)$.

We fix $\varepsilon > 0$ (to be precised soon) and we assume that we have a corresponding pair (y, t) as above. In particular, $\beta_{K,2}(y, t) \leq C\beta_K(y, t) \leq C\varepsilon$ and we also assume that $r_0 \leq \varepsilon$. According to the second statement of Theorem 2.6, we can fix ε (depending on n, δ) so that K is $C^{1,\alpha}$ -regular in $B := B(y, C^{-1}t)$. Thus $K \cap B$ is given by a pair of $C^{1,\alpha}$ graphs Γ_1, Γ_2 and B can be divided in two or three open sets: one where $u = 0$ and one or two where $u > 0$ and satisfies an elliptic equation with a Robin boundary conditions on Γ_i . Using Appendix B, and more precisely (158), we deduce that for all z in $(\frac{1}{2}B) \setminus K$,

$$|\nabla u(z)|^2 \leq C \int_B |\nabla u|^2 d\mathcal{L}^n + C|u|_\infty^2. \quad (63)$$

Remember that by Ahlfors-regularity (Proposition 2.2), we have

$$\int_B |\nabla u|^2 d\mathcal{L}^n \leq Ct^{-1}. \quad (64)$$

By definition of minimizers and since $0 < t \leq r_0 \leq 1$, we also have trivially $|u|_\infty^2 \leq C \leq Ct^{-1}$. We conclude that for $z \in (\frac{1}{2}B) \setminus K$,

$$|\nabla u(z)|^2 \leq Ct^{-1}. \quad (65)$$

□

4 Higher integrability of the gradient

Theorem 4.1. *Let $u \in \text{SBV}(X)$ be minimal. There exists $p > 1$ such that $|\nabla u|^2 \in L^p_{\text{loc}}(X)$.*

We follow [6] but we are going to simplify the proof by singling out a covering lemma and we will see that [6, Lemma 3.2] is not needed. Here is a summary. Let B be a (small) open ball of radius $R > 0$ such that $\bar{B} \subset X$. For $p > 1$, we observe that

$$\int_B |\nabla u|^{2p} d\mathcal{L}^n = \int_0^\infty \mathcal{L}^n(B \cap \{|\nabla u|^{2p} > t\}) dt \quad (66)$$

$$= pR^{-p} \int_0^\infty s^{p-1} \mathcal{L}^n(B \cap \{|\nabla u|^2 > sR^{-1}\}) ds \quad (67)$$

and for $M \geq 1$,

$$pR^{-p} \int_1^\infty s^{p-1} \mathcal{L}^n(B \cap \{|\nabla u|^2 > sR^{-1}\}) ds \quad (68)$$

$$\leq pR^{-p} \sum_{h=0}^\infty \int_{M^h}^{M^{h+1}} s^{p-1} \mathcal{L}^n(B \cap \{|\nabla u|^2 > sR^{-1}\}) ds$$

$$\leq pR^{-p} \sum_{h=0}^\infty \left(\int_{M^h}^{M^{h+1}} s^{p-1} ds \right) \mathcal{L}^n(B \cap \{|\nabla u|^2 > M^h R^{-1}\}) \quad (69)$$

$$\leq (M^p - 1)R^{-p} \sum_{h=1}^\infty M^{hp} \mathcal{L}^n(B \cap \{|\nabla u|^2 > M^h R^{-1}\}). \quad (70)$$

We will prove that there exists $M \geq 1$, $\gamma > 1$ and $C \geq 1$ such that for all $h \in \mathbf{N}$,

$$\mathcal{L}^n(B \cap \{|\nabla u|^2 > M^h R^{-1}\}) \leq CR^n M^{-\gamma h} \quad (71)$$

and then take $p = \frac{1}{2}(1 + \gamma)$.

In the following Lemma, we use the notation $\Gamma_{(x,n)}$ that was defined at line (18).

Lemma 4.2 (Covering Lemma). *Let $E \subset \mathbf{R}^n$ be a bounded set. Let (B_k) be a family of open balls of center $x_k \in \mathbf{R}^n$ and radius $R_k > 0$. We assume that*

1. for all $k \neq l$, $2B_k \cap B_l = \emptyset$;
2. for all k , for all $x \in 2B_k \cap E$, there exists a vector $e_n \in \mathbf{S}^{n-1}$ and a $\frac{1}{2}$ -Lipschitz function $f: e_n^\perp \rightarrow \mathbf{R}$ such that $|f| \leq \frac{1}{2}R_k$ and

$$x \in 2B_k \cap \Gamma_{(x_k,n)}(f) \subset E. \quad (72)$$

Let $0 < r \leq \inf_k R_k$. There exists a sequence of open balls $(D_i)_{i \in I}$ of radius r and centered in $E \setminus \bigcup_k B_k$ such that

$$E \setminus \bigcup_k B_k \subset \bigcup_{i \in I} D_i \quad (73)$$

and the balls $(20^{-1}D_i)_{i \in I}$ are pairwise disjoint and disjoint from $\bigcup_k B_k$.

Proof. Let $0 < r_0 \leq \inf_k R_k$. We define

$$F := E \setminus \bigcup_k B_k. \quad (74)$$

The goal is to cover F with a controlled number of balls of radius r_0 . Let r be a radius $0 < r \leq r_0$ which will be precised during the proof. As F is bounded, there exists a maximal sequence of points $(x_i) \in F$ such that $B(x_i, r) \subset \mathbf{R}^n \setminus \bigcup_k B_k$ and $|x_i - x_j| \geq r$. For $i \neq j$, we have $|x_i - x_j| \geq r$ so the balls $(B(x_i, \frac{1}{2}r))_i$ are disjoint. Next, we show that

$$F \subset \bigcup_i B(x_i, 10r). \quad (75)$$

Let $x \in F$. If $B(x, r) \subset \mathbf{R}^n \setminus \bigcup_k B_k$, then by maximality of (x_i) , there exists i such that $x \in B(x_i, r) \subset B(x_i, 10r)$. Now we focus on the case where there exists an index k_0 such that $B(x, r) \cap B_{k_0} \neq \emptyset$. We assume without loss of generality that the center of B_{k_0} is 0 and we denote by R its radius. As $x \in F = E \setminus \bigcup_k B_k$ and $B(x, r) \cap B(0, R) \neq \emptyset$, we have $R < |x| < R + r$. We are going to build a point $y \in E$ such that $R + r < |y| < R + 7r$ and $|x - y| < 9r$.

Since $r \leq R$, we observe that $x \in B(0, 2R)$. According to the assumptions of the Lemma, there exists two scalars $0 < \varepsilon, L \leq \frac{1}{2}$, a vector $e_n \in \mathbf{S}^{n-1}$ and a L -Lipschitz $f: e_n^\perp \rightarrow \mathbf{R}$ such that $|f| \leq \varepsilon R$ and

$$x \in \{y \in B(0, 2R) \mid y_n = f(y')\} \subset E. \quad (76)$$

Here, we have decomposed each point $y \in \mathbf{R}^n$ under the form $y = y' + y_n e_n$ where $y' \in n^\perp$ and $y_n \in \mathbf{R}$. The estimate $R < |x| < R + r$ can be rewritten

$$R < |x' + f(x')e_n| < R + r. \quad (77)$$

We consider $t \geq 1$ such that $|tx' + f(x')e_n| = R + 4r$ and we estimate how close tx' is to x' . We have

$$|x'| \geq \sqrt{R^2 - |f(x')|^2} \quad (78)$$

$$|tx'| \leq \sqrt{(R + 4r)^2 - |f(x')|^2} \quad (79)$$

so

$$|tx' - x'| \leq \sqrt{(R + 4r)^2 - |f(x')|^2} - \sqrt{R^2 - |f(x')|^2} \quad (80)$$

$$\leq \frac{4Rr + 8r^2}{\sqrt{R^2 - |f(x')|^2}}. \quad (81)$$

We assume $r \leq \frac{1}{8}R$ and we recall that $|f(x')| \leq \varepsilon R$ with $\varepsilon \leq \frac{1}{2}$ so this simplifies to

$$|tx' - x'| \leq \frac{5r}{\sqrt{1 - \varepsilon^2}} < 6r. \quad (82)$$

Next, we define $y := tx' + f(tx')e_n$ and we recall that f is L -Lipschitz with $L \leq \frac{1}{2}$ to estimate

$$|y - [tx' + f(x')e_n]| = |f(tx') - f(x')| \quad (83)$$

$$< 3r. \quad (84)$$

Since $|tx' + f(x')e_n| = R + 4r$, this yields

$$R + r < |y| < R + 7r. \quad (85)$$

We also estimate

$$|y - x| \leq |tx' - x'| + |f(tx') - f(x')| \quad (86)$$

$$< 9r. \quad (87)$$

As $r \leq \frac{1}{8}R$, the inequalities (85) imply $y \in B(0, 2R)$ and thus $y \in E$. We are going to justify that $B(y, r) \subset \mathbf{R}^n \setminus \bigcup_k B_k$. We recall that $B(0, 2R)$ is disjoint from all the other balls of the family (B_k) . By (85), we observe that

$$B(y, r) \subset B(0, R + 8r) \setminus B(0, R) \quad (88)$$

$$\subset B(0, 2R) \setminus B(0, R) \quad (89)$$

and our claim follows. By maximality of the family (x_i) , there exists i such that $|y - x_i| < r$ and in turn by (87), $|x - x_i| < 10r$. We finally choose $r = \frac{1}{10}r_0$. The balls (D_i) are given by $D_i = B(x_i, 10r) = B(x_i, r_0)$. \square

Proof of Theorem 4.1. The letter α is the constant of Theorem 2.6 and Proposition 3.1. We define \mathcal{A} as the set of all open balls B centered in K such that $2\bar{B} \subset X$ and K is $C^{1,\alpha}$ -regular in $2B$. We fix a point $x_0 \in K$ and to simplify the notations, we assume $x_0 = 0$. There exists a radius $R > 0$ and constants $C_0, C_1 \geq 1$ such that $\bar{B}(0, 4R) \subset X$ and

(i) for all $x \in B(0, 2R)$, for all $0 < r \leq 2R$,

$$\int_{B(x,r)} |\nabla u|^2 d\mathcal{L}^n + \mathcal{H}^{n-1}(K \cap B(x, r)) \leq C_0 r^{n-1}; \quad (90)$$

(ii) for all $x \in K \cap B(0, 2R)$, for all $0 < r \leq 2R$,

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \geq C_0^{-1} r^{n-1}. \quad (91)$$

(iii) for all $x \in K \cap B(0, 2R)$, for all $0 < r \leq 2R$, there exists $B \in \mathcal{A}$ of radius $C_1^{-1}r$ such that $40B \subset B(x, r)$ and

$$\sup_{2B} |\nabla u|^2 \leq C_1 r^{-1}. \quad (92)$$

Note that C_0, C_1 depends on n, δ . The two first properties are referred to as *Ahlfors-regularity*. The third property is referred to as the *porosity*.

We consider $M = \max\{4C_0, C_0C_1\}$ and we define for $h \in \mathbf{N}$,

$$A_h := \{x \in B(0, \frac{1}{4}R) \setminus K \mid |\nabla u|^2 > M^h R^{-1}\}. \quad (93)$$

Our goal is to show that there exists $C \geq 1$ and $\gamma > 1$ (both depending on n, δ) such that for $h \geq 1$,

$$\mathcal{L}^n(A_h) \leq CR^n M^{-\gamma h}. \quad (94)$$

The proof is based on the fact that A_h is at distance $\sim M^{-h}R$ from K and has many holes of size $\sim M^{-h}R$ near K . We justify more precisely these observations. Let $x \in A_h$ and assume that $B(x, C_0M^{-h}R)$ is disjoint from K . We use the subharmonicity of $|\nabla u|^2$ and the Ahlfors-regularity to estimate

$$|\nabla u|(x)^2 \leq \int_{B(x, C_0M^{-h}R)} |\nabla u|^2 d\mathcal{L}^n \quad (95)$$

$$\leq M^h R^{-1}. \quad (96)$$

This contradicts the definition of A_h . We deduce that there exists $y \in K$ such that $|x - y| < C_0M^{-h}R$. For our next observation, we consider a point $x \in K \cap B(0, 2R)$ and we apply the porosity property to the ball $B(x, C_1M^{-h}R)$. We obtain an open ball $B \in \mathcal{A}$ of radius $M^{-h}R$ such that $40B \subset B(x, r)$ and

$$\sup_{2B} |\nabla u|^2 \leq M^h R^{-1}. \quad (97)$$

In particular, $2B$ is disjoint from A_h .

We start the proof by defining for $h \geq 1$,

$$r_h = M^{-h}R \quad (98)$$

$$R_h = (\frac{1}{2} + M^{-h+1})R. \quad (99)$$

The sequence $(R_h)_h$ is decreasing, we have $R_1 < 2R$, $\lim_{h \rightarrow \infty} R_h = \frac{1}{2}$ and $R_{h+1} + r_h \leq R_h$. For each $h \geq 1$, we build an index set $I(h)$ and a family

of balls $\alpha_h = (B_i)_{i \in I(h)} \in \mathcal{A}$ as follow. First we define $I(1) = \emptyset$ and $\alpha_1 = \emptyset$. Let $h \geq 2$ be such that $\alpha_1, \dots, \alpha_{h-1}$ have been built. We assume that the index sets $I(g)$, where $g = 1, \dots, h-1$, are pairwise disjoint. We also assume that for all $i \in I_g$, the balls B_i have radius $C_1^{-1}r_g$ and that for all indices $i, j \in \bigcup_{g=1}^{h-1} I(g)$ with $i \neq j$, $2B_i \cap B_j = \emptyset$. Then, we introduce the sets

$$K_h := K \cap B(0, R_h) \setminus \bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i \quad (100)$$

$$K_h^* := K \cap B(0, R_{h+1}) \setminus \bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i. \quad (101)$$

According to Lemma 4.2, there exists a sequence of open balls $(D_i)_{i \in I(h)}$ centered in K_h^* of radius r_h such that

$$K_h^* \subset \bigcup_{i \in I(h)} D_i, \quad (102)$$

and such that the balls $(20^{-1}D_i)$ are pairwise disjoint and disjoint from $\bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i$. We can assume that index set $I(h)$ is disjoint from the sets $I(g)$, $g = 1, \dots, h-1$. Since $R_{h+1} + r_h \leq R_h$, we observe that the balls $(20^{-1}D_i)$ are included in

$$B(0, R_h) \setminus \bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i. \quad (103)$$

Next, we apply the porosity to the balls (D_i) . For each $i \in I(h)$, there exists $B_i \in \mathcal{A}$ of radius $C_1^{-1}r_h$ such that $B_i \subset 40^{-1}D_i$ and

$$\sup_{2B_i} |\nabla u|^2 \leq C_1 r_h^{-1}. \quad (104)$$

We finally define $\alpha_h = (B_i)_{i \in I(h)}$. We should not forget to mention that for all $i \in I(h)$, we have $2B_i \subset 20^{-1}D_i$ so $2B_i$ is disjoint from all the other balls we have built so far.

Now, we estimate $\mathcal{L}^n(A_h)$ for $h \geq 1$. We recall that we have taken $M = \max\{4C_0, C_0C_1\}$. We show first that the points of A_h cannot be too far from K_h^* . Let $x \in A_h$. We have seen earlier that there exists $y \in K$ such that $|x - y| < C_0M^{-h}R$. We are going to show that $y \in K_h^*$. Since $|x| \leq \frac{1}{4}R$, we have

$$|y| \leq \frac{1}{4}R + C_0M^{-h}R \leq \frac{1}{2}R. \quad (105)$$

Let us assume that there exists $g = 1, \dots, h-1$ and $i \in I(g)$ such that $y \in B_i$. The radius of B_i is $C_1^{-1}r_g \geq C_1^{-1}M^{-(h-1)}R$ and since $|x - y| < C_0M^{-h}R$, we have $x \in 2B_i$. Then by construction,

$$\sup_{2B_i} |\nabla u|^2 \leq C_1 r_g^{-1} \leq C_1 M^{h-1} R^{-1} \quad (106)$$

and this contradicts the fact that $x \in A_h$. We have shown that $y \in K_h^*$. As a consequence, there exists $i \in I(h)$ such that $y \in D_i$. As $\text{radius}(D_i) = M^{-h}R$ and $|x - y| < C_0 M^{-h}R$, we deduce finally that

$$A_h \subset \bigcup_{i \in I(h)} (1 + C_0)D_i. \quad (107)$$

This allows to estimate $\mathcal{L}^n(A_h)$:

$$\mathcal{L}^n(A_h) \leq \omega_n (1 + C_0)^n |I(h)| r_h^n \quad (108)$$

where ω_n is the Lebesgue measure of the unit ball.

We want to estimate $|I(h)|$. The balls $(20^{-1}D_i)_{i \in I(h)}$ are disjoint and included in the set $B(0, R_h) \setminus \bigcup_{g=2}^{h-1} \bigcup_{i \in I(g)} B_i$ so by Ahlfors-regularity,

$$C_0^{-1} 20^{-(n-1)} r_h^{(n-1)} |I(h)| \leq \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap 12^{-1}D_i) \quad (109)$$

$$\leq \mathcal{H}^{n-1}(K_h). \quad (110)$$

We are going to see that $\mathcal{H}^{n-1}(K_h)$ is bounded from above by a decreasing geometric sequence. We have

$$\mathcal{H}^{n-1}(K_h^*) \leq \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap D_i) \quad (111)$$

$$\leq C_0 \sum_{i \in I(h)} r_h^{n-1} \quad (112)$$

$$\leq C_0 C_1^{n-1} \sum_{i \in I(h)} (C_1 r_h)^{n-1} \quad (113)$$

$$\leq C_0^2 C_1^{n-1} \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap B_i) \quad (114)$$

$$\leq C_3 \mathcal{H}^{n-1}(K_h \setminus K_{h+1}). \quad (115)$$

where $C_3 = C_0^2 C_1^{-(n-1)}$. We deduce

$$\mathcal{H}^{n-1}(K_h) \leq C_3 \mathcal{H}^{n-1}(K_h \setminus K_{h+1}) + \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}). \quad (116)$$

We rewrite this inequality as

$$\mathcal{H}^{n-1}(K_{h+1}) \leq \lambda \mathcal{H}^{n-1}(K_h) + C_3^{-1} \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}) \quad (117)$$

where $\lambda = C_3^{-1}(C_3 - 1)$. Then, we multiply both sides of the inequality by λ^{-h+1} :

$$\begin{aligned} \lambda^{-(h+1)} \mathcal{H}^{n-1}(K_{h+1}) & \\ & \leq \lambda^{-h} \mathcal{H}^{n-1}(K_h) + C_3^{-1} \lambda^{-h} \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}) \end{aligned} \quad (118)$$

$$\leq \lambda^{-h} \mathcal{H}^{n-1}(K_h) + \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}). \quad (119)$$

Summing this telescopic inequality, we obtain that for all $h \geq 1$,

$$\lambda^{-h} \mathcal{H}^{n-1}(K_h) \leq 2\mathcal{H}^{n-1}(K \cap B(0, 2R)) \quad (120)$$

$$\leq 2^n C_0 R^{n-1}. \quad (121)$$

In summary, we have proved that some constant $C \geq 1$ (depending on n, δ) and for $h \geq 1$

$$\mathcal{L}^n(A_h) \leq CR^{n-1} r_h \lambda^h \quad (122)$$

$$\leq CR^n (\lambda M^{-1})^h. \quad (123)$$

As $0 < \lambda < 1$ and $M > 1$, there exists $\gamma > 1$ such that $\lambda = M^{1-\gamma}$. \square

5 Dimension of the singular set

The Hausdorff dimension of a set $A \subset \mathbf{R}^n$ is defined by

$$\dim_{\mathcal{H}}(A) = \inf \{ s \geq 0 \mid H^s(A) = 0 \}. \quad (124)$$

We take the convention that for $s < 0$, the term \mathcal{H}^s -almost-everywhere means everywhere and the inequality $\dim_{\mathcal{H}}(A) < 0$ means $A = \emptyset$. As usual, we work in an open set $X \subset \mathbf{R}^n$.

5.1 Generalities about the density of L_{loc}^p functions

Lemma 5.1. *Let μ be a Radon measure in X such that μ is dominated by \mathcal{L}^n and let $s < n$. Then for \mathcal{H}^s -a.e. $x \in X$, we have*

$$\lim_{r \rightarrow 0} r^{-s} \mu(B(x, r)) = 0. \quad (125)$$

Proof. If $s < 0$, the limit is indeed 0 for every $x \in X$. We assume $0 \leq s < n$. We fix a closed ball $\overline{B} \subset X$, a scalar $\lambda > 0$ and a set

$$A = \{ x \in \overline{B} \mid \limsup_{r \rightarrow 0} r^{-s} \mu(B(x, r)) > \lambda \}. \quad (126)$$

According to [1, Theorem 2.56],

$$\mu(A) \geq \lambda H^s(A). \quad (127)$$

As $A \subset \overline{B}$ and μ is a Radon measure, we have $\mu(A) < \infty$. Then (127) gives $H^s(A) < \infty$ and since $s < n$, $\mathcal{L}^n(A) = 0$. The measure μ is dominated by \mathcal{L}^n so $\mu(A) = 0$ and now (127) gives $H^s(A) = 0$. We can take a sequence of scalars $\lambda_k \rightarrow 0$ to deduce

$$\mathcal{H}^s(\{ x \in \overline{B} \mid \limsup_{r \rightarrow 0} r^{-s} \mu(B(x, r)) > 0 \}) = 0. \quad (128)$$

We can then conclude that

$$\mathcal{H}^s(\{x \in X \mid \limsup_{r \rightarrow 0} r^{-s} \mu(B(x, r)) > 0\}) = 0. \quad (129)$$

by covering X with a sequence of closed balls $\overline{B_k} \subset X$. \square

The previous Lemma can be restated as follow: for $v \in L^1_{\text{loc}}$, for $s < n$ and for \mathcal{H}^s -a.e. $x \in X$,

$$\lim_{r \rightarrow 0} r^{-s} \int_{B(x, r)} v \, d\mathcal{L}^n = 0 \quad (130)$$

We are going to see that we have an improvement when v has a higher integrability.

Corollary 5.2. *Let $v \in L^p_{\text{loc}}(X)$ for some $p \geq 1$ and let $s < n$. Then, for $\mathcal{H}^{n-p(n-s)}$ -a.e. $x \in X$,*

$$\lim_{r \rightarrow 0} r^{-s} \int_{B(x, r)} v \, d\mathcal{L}^n = 0. \quad (131)$$

Proof. Without loss of generality, we assume $v \geq 0$. Let us fix $t < n$. For $x \in X$ and for $r > 0$, the Hölder inequality shows that

$$r^{-(n-\frac{n}{p})} \int_{B(x, r)} v \, d\mathcal{L}^n \leq \left(\int_{B(x, r)} v^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \quad (132)$$

so

$$r^{-(n+\frac{t}{p}-\frac{n}{p})} \int_{B(x, r)} v \, d\mathcal{L}^n \leq \left(r^{-t} \int_{B(x, r)} v^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}. \quad (133)$$

We apply Lemma 5.1 and we see that for \mathcal{H}^t -a.e. $x \in X$,

$$\lim_{r \rightarrow 0} r^{-(n+\frac{t}{p}-\frac{n}{p})} \int_{B(x, r)} v \, d\mathcal{L}^n = 0. \quad (134)$$

The scalar t such that $s = n + \frac{t}{p} - \frac{n}{p}$ is $t = n - p(n - s) < n$. \square

5.2 Dimension of the singular set

Theorem 5.3. *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. We define*

$$\Sigma = \{x \in K \mid K \text{ is not regular at } x\}. \quad (135)$$

For $p > 1$ such that $|\nabla u|^2 \in L^p_{\text{loc}}(X)$, we have

$$\dim_{\mathcal{H}}(\Sigma) \leq \max\{n - p, n - 8\} < n - 1. \quad (136)$$

Proof. According to Corollary 5.2, we have for \mathcal{H}^{n-p} -a.e. $x \in X$,

$$\lim_{r \rightarrow 0} \omega_2(x, r) = 0 \quad (137)$$

and according to [4, Theorem 8.2], the set

$$\left\{ x \in X \cap \Sigma \mid \lim_{r \rightarrow 0} \omega_2(x, r) = 0 \right\} \quad (138)$$

has a Hausdorff dimension $\leq n - 8$. \square

Appendices

A Generalities about BV functions

We recall a few definitions and results from the theory of BV functions ([1]). We work in an open set X of the Euclidean space \mathbf{R}^n ($n > 1$). When a point $x \in X$ is given, we abbreviate the open ball $B(x, r)$ as B_r .

Let $u \in L^1_{loc}(X)$. The *upper and lower approximate limit* of u at a point $x \in X$ are defined by

$$\bar{u}(x) := \inf \left\{ t \in \mathbf{R} \mid \lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap \{u > t\}} (u - t) d\mathcal{L}^n = 0 \right\}, \quad (139)$$

$$\underline{u}(x) := \sup \left\{ t \in \mathbf{R} \mid \lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap \{u < t\}} (t - u) d\mathcal{L}^n = 0 \right\}. \quad (140)$$

The functions $\bar{u}, \underline{u}: X \rightarrow \bar{\mathbf{R}}$ are Borel and satisfies $\underline{u} \leq \bar{u}$. We have two examples in mind. We say that x is a *Lebesgue point* if there exists $t \in \mathbf{R}$ such that

$$\lim_{r \rightarrow 0} \int_{B_r} |u - t| d\mathcal{L}^n = 0. \quad (141)$$

We then have $\underline{u}(x) = \bar{u}(x) = t$ and we denote t by $\tilde{u}(x)$. The set of non-Lebesgue points $x \in X$ is called the *singular set* S_u . Both the set S_u and the function $X \setminus S_u \rightarrow \mathbf{R}, x \mapsto \tilde{u}(x)$ are Borel ([1, Proposition 3.64]). The Lebesgue differentiation theorem states that for \mathcal{L}^n -a.e. $x \in X$, we have $x \in X \setminus S_u$ and $u(x) = \tilde{u}(x)$. We say that x is a *jump point* if there exist two real numbers $s < t$ and a (unique) vector $\nu_u(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} \int_{B_r \cap H^+} |u(y) - s| d\mathcal{L}^n(y) = 0 \quad (142a)$$

$$\lim_{r \rightarrow 0} \int_{B_r \cap H^-} |u(y) - t| d\mathcal{L}^n(y) = 0, \quad (142b)$$

where

$$H^+ = \{y \in \mathbf{R}^n \mid (y - x) \cdot \nu_u(x) > 0\} \quad (143a)$$

$$H^- = \{y \in \mathbf{R}^n \mid (y - x) \cdot \nu_u(x) < 0\}. \quad (143b)$$

We then have $\bar{u}(x) = t$ and $\underline{u}(x) = s$. The set of jump points $x \in X$ is called the *jump set* J_u . Both the set J_u and the function $J_u \rightarrow \mathbf{S}^{n-1}, x \mapsto \nu_u(x)$ are Borel ([1, Proposition 3.69]).

Here we summarize [1, Proposition 3.76, 3.78]). Let $u \in \text{BV}(X)$. The singular set S_u is \mathcal{H}^{n-1} rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. According to the Besicovitch derivation theorem, we can write

$$Du = D^a u + D^s u \quad (144)$$

where $D^a u$ is the *absolutely continuous* part of Du with respect to \mathcal{L}^n and $D^s u$ is the *singular part* of Du with respect to \mathcal{L}^n . As a consequence, there exists a unique vector-valued map $\nabla u \in L^1(X; \mathbf{R}^n)$, the *approximate gradient*, such that $D^a u = \nabla u \mathcal{L}^n$. The measures \mathcal{L}^n and $\|D^s u\|$ are mutually singular which means that there exists a Borel set $S \subset X$ such that

$$\mathcal{L}^n(S) = \|D^s u\|(\mathbf{R}^n \setminus S) = 0. \quad (145)$$

A candidate for S could be S_u but S may not be a $(n - 1)$ dimensional set. We can write

$$D^s u = Du^s \llcorner S_u + D^s u \llcorner (X \setminus S_u) \quad (146)$$

where $D^s u \llcorner S_u$ is the *jump part* and $D^s u \llcorner (X \setminus S_u)$ is the *Cantor part*. The jump part has an explicit formula,

$$D^s u \llcorner S_u = (\bar{u} - \underline{u}) \nu_u \mathcal{H}^{n-1} \llcorner J_u, \quad (147)$$

whereas the Cantor part vanishes on \mathcal{H}^{n-1} σ -finite sets $B \subset X$ (and not only $B = S_u$). Remark that Du always vanishes on \mathcal{H}^{n-1} negligible sets. Finally, we define $\text{SBV}(X)$ as the subspace of functions $u \in \text{BV}(X)$ whose Cantor part is zero, that is

$$\|D^s u\|(X \setminus S_u) = 0. \quad (148)$$

For $u \in \text{BV}(X)$ to be SBV , it suffices that there exists a \mathcal{H}^{n-1} σ -finite set $K \subset X$ such that $\|D^s u\|(X \setminus K) = 0$. Since $\|D^s u\|$ and \mathcal{L}^n are mutually singular, this also amounts to say the measure $\|Du\| \llcorner (X \setminus K)$ is dominated by \mathcal{L}^n . A natural way to build $\text{SBV}(X)$ functions is to have a pair (u, K) where $K \subset X$ is relatively closed, \mathcal{H}^{n-1} locally finite and $u \in W^{1,1}(X \setminus K)$.

A set of finite perimeter in X is a Borel set $E \subset X$ such that $\mathbf{1}_E \in \text{BV}(X)$. The singular set of $\mathbf{1}_E$ is called *essential boundary* or *measure-theoretic boundary* and denoted by $\partial_M E$. The jump set of E is denoted by $\partial^* E$. One can see that $\overline{\mathbf{1}_E}, \underline{\mathbf{1}_E} \in \{0, 1\}$ everywhere on X . Thus, if x

is a Lebesgue point of $\mathbf{1}_E$, we have either $\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \setminus E) = 0$ or $\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap E) = 0$. The essential boundary $\partial_M E$ can be reformulated as the set of points $x \in X$ such that

$$\limsup_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \setminus E) > 0 \quad (149a)$$

$$\limsup_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap E) > 0. \quad (149b)$$

Similarly, the jump set $\partial^* E$ can be reformulated as the set of points $x \in X$ for which there exists a (unique) vector $n_E(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap (E \Delta H^+)) = 0 \quad (150)$$

where

$$H^+ = \{y \in \mathbf{R}^n \mid (y - x) \cdot n_E(x) > 0\}. \quad (151)$$

The vector $n_E(x)$ is called the *measure-theoretic inner normal* to E at x . We have the inclusions $\partial^* E \subset \partial_M E \subset \partial E$. The measure $D\mathbf{1}_E$ has no absolutely continuous part, neither Cantor part; it is given by the formula

$$D\mathbf{1}_E = n_E \mathcal{H}^{n-1} \llcorner \partial^* E. \quad (152)$$

B A Robin problem

B.1 Statement

We fix a connected bounded open set $V \subset \mathbf{R}^n$ with Lipschitz boundary. We assume that $0 \in \overline{V}$ and for $0 < R \leq 1$, we define

$$V_R = V \cap B(0, R) \quad (153)$$

$$\Gamma_R = \partial V \cap B(0, R) \quad (154)$$

Also for $0 < t \leq 1$, we write tV_R for V_{tR} and $t\Gamma_R$ for Γ_{tR} . For $u \in \text{BV}(V_R)$, we denote by u^* the trace of u in $L^1(\partial V_R)$. It is characterized by the property that for \mathcal{H}^{n-1} -a.e. $x \in \partial V_R$,

$$\lim_{r \rightarrow 0} r^{-n} \int_{V_R \cap B_r(x)} |u - u^*(x)| \, d\mathcal{L}^n = 0. \quad (155)$$

We denote by $W_0^{1,2}(V_R \cup \Gamma_R)$ the space of functions $u \in W^{1,2}(V_R)$ such that $u^* = 0$ on $\partial V_R \setminus \Gamma_R$. Our object of study are the functions $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ such that u is a weak solution of

$$\begin{cases} \Delta u & = 0 & \text{in } V_R \\ \partial_\nu u - u & = 0 & \text{in } \Gamma_R, \end{cases} \quad (156)$$

that is, for all $v \in W_0^{1,2}(V_R \cup \Gamma_R)$,

$$\int_{V_R} \langle \nabla u, \nabla v \rangle d\mathcal{L}^n + \int_{\Gamma_R} u^* v^* d\mathcal{H}^{n-1} = 0. \quad (157)$$

According to Weyl's lemma, u coincide almost-everywhere in V_R with a harmonic functions. We replace u by this harmonic representative so that u is pointwise defined and smooth in V_R . Our goal is to prove the following estimate. If V_R has $C^{1,\alpha}$ boundary ($0 < \alpha < 1$), then there exists $C \geq 1$ (depending on n, V but not R) such that for all $x \in \frac{1}{2}V_R$,

$$|\nabla u(x)| \leq C \left(\int_{V_R} |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} + C|u|_\infty. \quad (158)$$

This will be a consequence of Proposition B.4 and Proposition B.5.

According to [9], the viscosity solutions of such a problem are pointwise $C^{1,\alpha}$ up to the boundary. The viscosity approach is based on the maximum principle but it is easy to prove a maximum principle for the weak solutions of our problem and thus to follow the ideas of [9]. Once we prove that u is C^α up to the boundary, the Robin boundary condition can be written as a Neumann boundary condition with a C^α right-hand side and we will be able to apply the usual results for this kind of problem.

Lemma B.1 (Maximum principle). *Let $u \in W^{1,2}(V_R)$ be a weak solution of*

$$\begin{cases} \Delta u & \geq 0 & \text{in } V_R \\ \partial_\nu u - u & \geq 0 & \text{in } \Gamma_R, \end{cases} \quad (159)$$

that is, for all non-negative function $v \in W_0^{1,2}(V_R \cup \Gamma_R)$,

$$\int_{V_R} \langle \nabla u, \nabla v \rangle d\mathcal{L}^n + \int_{\Gamma_R} u^* v^* d\mathcal{H}^{n-1} \leq 0. \quad (160)$$

If $u^* \leq 0$ on $\partial V_R \setminus \Gamma_R$, then $u \leq 0$ on V_R .

Proof. Let $u \in W^{1,2}(V_R)$ and let $u_+ = p(u)$ where $p: \mathbf{R} \rightarrow [0, \infty[$ is the orthogonal projection onto $[0, \infty[$. According to the chain rule, $u_+ \in W^{1,2}(V_R)$ and for \mathcal{L}^n -a.e. $x \in V_R$,

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) = 0. \end{cases} \quad (161)$$

One can also see that $(u_+)^* = p(u^*)$ using the characterization (155) and the fact that p is Lipschitz. Now, we assume in addition that u is a weak solution of (159) and that $u^* \leq 0$ on $\partial V_R \setminus \Gamma_R$. We have $u_+ \in W_0^{1,2}(V_R)$ so (160) gives

$$\int_{V_R} \langle \nabla u, \nabla u_+ \rangle d\mathcal{L}^n + \int_{\Gamma_R} u^* u_+^* d\mathcal{H}^{n-1} \leq 0. \quad (162)$$

As $u^* u_+^* = (u_+^*)^2$ and $\langle \nabla u, \nabla u_+ \rangle = |\nabla u_+|^2$, we conclude that $u_+ = 0$ on V_R . \square

B.2 Hölder continuity up to the boundary

Lemma B.2 (Hölder continuity). *Assume that V_R has a C^1 boundary. Let $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ be a weak solution of (156). There exists constants $C \geq 1$ (depending on n, V) and $l > 0$ (depending on n) such that for all $x, y \in V_R$,*

$$|u(x) - u(y)| \leq C|u|_\infty \left(\frac{|x - y|}{r} \right)^\ell. \quad (163)$$

where $r = \max \{ d(x, \mathbf{R}^n \setminus B(0, R)), d(y, \mathbf{R}^n \setminus B(0, R)) \}$.

The Hölder continuity relies on a weak Harnack inequality at the boundary. We temporarily redefine the notation V_R, Γ_R in the next Lemma because is more convenient to work with cylinders rather than balls.

Lemma B.3 (Weak Harnack Inequality). *We fix a radius $0 < R \leq 1$. We fix a vector $e_n \in \mathbf{S}^{n-1}$ and we decompose each point $x \in \mathbf{R}^n$ as $x = x' + x_n e_n$ where $x' \in e_n^\perp$ and $x_n \in \mathbf{R}$. We fix a 1-Lipschitz function $f: e_n^\perp \rightarrow \mathbf{R}$ and we assume that $0 \leq f \leq \delta R$ for a certain $0 < \delta \leq \frac{1}{2}$ small enough (depending on n). Finally, we define*

$$V_R = \{ x \in \mathbf{R}^n \mid |x'| < R, f(x') < x_n < 2R \} \quad (164)$$

$$\Gamma_R = \{ x \in \mathbf{R}^n \mid |x'| < R, x_n = f(x') \}. \quad (165)$$

Let $u \in W^{1,2}(V_R)$ be a non-negative weak solution of

$$\begin{cases} \Delta u & = 0 & \text{in } V_R \\ \partial_\nu u - u & = 0 & \text{in } \Gamma_R. \end{cases} \quad (166)$$

Then there exists a constant $C \geq 1$ (depending on n) such that

$$\begin{aligned} & \sup \{ u(x) \mid |x'| \leq \frac{1}{2}R, 2\delta R \leq x_n \leq \frac{3}{2}R \} \\ & \leq C \inf \{ u(x) \mid |x'| \leq \frac{1}{2}R, f(x') < x_n \leq \frac{3}{2}R \} \end{aligned} \quad (167)$$

Proof. The letter C is a constant ≥ 1 that depends on n and whose value might change from one line to the other. We introduce the closed cubes

$$Q := \{ x \in \mathbf{R}^n \mid |x'| \leq \frac{3}{4}R, 0 \leq x_n \leq 2\delta R \} \quad (168)$$

$$Q_i := \{ x \in \mathbf{R}^n \mid |x'| \leq \frac{3}{4}R, 2\delta R \leq x_n \leq \frac{3}{2}R \}. \quad (169)$$

It suffices to prove that $\sup_{Q_i} u \leq C \inf_{Q_i} u$ and that $A \leq CB$ where

$$A = \inf \{ u(x) \mid |x'| \leq \frac{3}{4}R, x_n = 2\delta R \} \quad (170)$$

$$B = \inf \{ u(x) \mid |x'| \leq \frac{1}{2}R, f(x') < x_n \leq 2\delta R \}. \quad (171)$$

We show that $\sup_{Q_i} u \leq C \inf_{Q_i} u$ by applying the interior Harnack inequality for harmonic functions and by observing that

$$d(Q_i, \mathbf{R}^n \setminus V_R) \geq \min \{ \frac{1}{4}R, \delta R \}. \quad (172)$$

Here, remember that δ will be chosen in function of n only. Now, we focus on Q . We define two subsets of Q ,

$$W_R = \{x \in \mathbf{R}^n \mid |x'| \leq \frac{3}{4}R, f(x') < x_n < 2\delta R\} \quad (173)$$

$$\Sigma_R = \{x \in \mathbf{R}^n \mid |x'| \leq \frac{3}{4}R, x_n = f(x')\} \quad (174)$$

and we recall that for \mathcal{H}^{n-1} -a.e. $x \in \Sigma_R$,

$$\nu(x) = \frac{-\nabla f(x') + e_n}{\sqrt{1 + |\nabla f(x')|^2}}. \quad (175)$$

We are going to build a paraboloid p such that

- (i) $p \geq 0$ on Q ;
- (ii) $p \geq 1$ on $\{x \in \mathbf{R}^n \mid |x'| = \frac{3}{4}R, 0 \leq x_n \leq 2\delta R\}$;
- (iii) $p \leq \frac{99}{100}$ on $\{x \in \mathbf{R}^n \mid |x'| \leq \frac{1}{2}R, 0 \leq x_n \leq 2\delta R\}$;
- (iv) $\Delta p \leq 0$ on Q ;
- (v) $\partial_\nu p \leq -1$ on Σ_R .

Once p is built, we apply the maximum principle to $w = u + Ap - A$. More precisely, one can check that $\Delta w \leq 0$ on W_R and $\partial_\nu w \leq u - A \leq w$ on Σ_R . One can also check that $w \geq 0$ on $\partial W_R \setminus \Sigma_R$. The maximum principles implies that $w \geq 0$ on W_R and in particular, $u \geq \frac{1}{100}A$ on $\{x \in \mathbf{R}^n \mid |x'| \leq \frac{1}{2}R, f(x') < x_n \leq 2\delta R\}$. The paraboloid is

$$p(x) = \frac{1}{2} - \left(\frac{x_n}{8\delta R}\right) - \left(\frac{x_n}{4\delta R}\right)^2 + \left(\frac{4|x'|}{3R}\right)^2. \quad (176)$$

Let us check these properties. For $x \in Q$, we have $0 \leq x_n \leq 2\delta R$ so

$$\left(\frac{4|x'|}{3}\right)^2 \leq p \leq \frac{1}{2} + \left(\frac{4|x'|}{3R}\right)^2. \quad (177)$$

The first three items follow. We compute

$$\Delta p = -\frac{1}{8\delta^2 R^2} + \frac{32(n-1)}{9R^2} \quad (178)$$

and we take δ small enough (depending on n) so that $\Delta p \leq 0$ on Q . For $x \in \mathbf{R}^n$, we have

$$\nabla p(x) = -\frac{e_n}{8\delta R} - \frac{x_n e_n}{8\delta^2 R^2} + \frac{32x'}{9R^2}. \quad (179)$$

so for \mathcal{H}^{n-1} -a.e. $x \in \Sigma_R$,

$$\partial_\nu p(x) = \frac{1}{\sqrt{1 + |\nabla f(x')|^2}} \left(-\frac{1}{8\delta R} - \frac{x_n}{8\delta^2 R^2} - \frac{32(x' \cdot \nabla f(x'))}{9R^2} \right). \quad (180)$$

In addition $x_n \geq 0$, $|x'| \leq \frac{3}{4}R$ and f is 1-Lipschitz so

$$-\frac{1}{8\delta R} - \frac{x_n}{8\delta^2 R^2} - \frac{32(x' \cdot \nabla f(x'))}{9R^2} \leq -\frac{1}{8\delta R} + \frac{8}{3R}. \quad (181)$$

We take δ small enough so that $-\frac{1}{8\delta R} + \frac{8}{3R} \leq -\frac{\sqrt{2}}{R}$ and thus $\partial_\nu p \leq -\frac{1}{R} \leq -1$ (because $R \leq 1$). \square

We are ready to prove Lemma B.2 (we come back to the notations V_R , Γ_R of the introduction).

Proof of Lemma B.2. We start with a preliminary observation. There exists a radius $0 < r_0 \leq 1$ (depending on n, V) such that the following holds. For all $x \in \partial V$, there exists a vector $e_n \in \mathbf{S}^{n-1}$ inducing a coordinate system $y = x + (y' + y_n e_n)$ and a $\frac{1}{2}\delta$ -Lipschitz function $f: e_n^\perp \rightarrow \mathbf{R}$ (where $0 < \delta \leq \frac{1}{2}$ is given by Lemma B.3) such that $f(0) = 0$ and

$$\begin{aligned} V \cap \{y \in \mathbf{R}^n \mid |y'| < r_0, y_n < 2r_0\} \\ = \{y \in \mathbf{R}^n \mid |y'| < r_0, f(y') < y_n < 2r_0\} \end{aligned} \quad (182)$$

and

$$\begin{aligned} \partial V \cap \{y \in \mathbf{R}^n \mid |y'| < r_0, y_n < 2r_0\} \\ = \{y \in \mathbf{R}^n \mid |y'| < r_0, y_n = f(y')\}. \end{aligned} \quad (183)$$

Let us fix $x \in \partial V$ and $0 < r \leq r_0$. Using the previous notations, we have in particular $|f| \leq \frac{1}{2}\delta r$ in $e_n^\perp \cap B(0, r)$. We consider the translated coordinate system whose origin is $x - \frac{1}{2}\delta r e_n$,

$$y = x - \frac{1}{2}\delta r e_n + (z' + z_n e_n). \quad (184)$$

Equivalently, the new coordinates are $z' = y'$ and $z_n = y_n + \frac{1}{2}\delta r$. Setting $g = f + \frac{1}{2}\delta r$, we have $0 \leq g \leq \delta r$ and the description (182) and (183) imply

$$\begin{aligned} V \cap \{y \in \mathbf{R}^n \mid |z'| < r, z_n < 2r\} \\ = \{y \in \mathbf{R}^n \mid |z'| < r, g(z') < z_n < 2r\} \end{aligned} \quad (185)$$

and

$$\begin{aligned} \partial V \cap \{y \in \mathbf{R}^n \mid |z'| < r, z_n < 2r\} \\ = \{y \in \mathbf{R}^n \mid |z'| < r, z_n = g(z')\}. \end{aligned} \quad (186)$$

Therefore, we can apply Lemma B.3 in the new coordinates. Whenever a non-negative function $v \in W^{1,2}(V \cap B(x, 4r))$ is a weak solution of

$$\begin{cases} \Delta v & = 0 & \text{in } V \cap B(x, 4r) \\ \partial_\nu v - v & = 0 & \text{in } \partial V \cap B(x, 4r), \end{cases} \quad (187)$$

we can draw the conclusion that there exists $C_0 > 1$ (depending on n) such that

$$v(x + re_n) \leq C_0 \inf \{ v(y) \mid y \in B(x, \frac{1}{2}r) \cap V \}. \quad (188)$$

Now we define

$$\Delta := \{ (x, r) \mid x \in \bar{V}, 0 < r \leq r_0, B(x, r) \subset B(0, R) \} \quad (189)$$

and for $(x, r) \in \Delta$,

$$\text{osc}(x, r) := \sup \{ |u(z) - u(y)| \mid y, z \in V \cap B(x, r) \}. \quad (190)$$

We are going to show that there exists $L > 1$ (close to 1, depending on n) such that for all $(x, r) \in \Delta$,

$$\text{osc}(x, \frac{1}{32}r) \leq L^{-1} \text{osc}(x, r). \quad (191)$$

We are going to distinguish three cases: the case $x \in \partial V$, the case $B(x, \frac{1}{12}r) \cap \partial V \neq \emptyset$ and the case $B(x, \frac{1}{12}r) \subset V$.

Let $(x, r) \in \Delta$ with $x \in \partial V$. We define

$$M = \sup \{ u(z) \mid z \in B(x, r) \cap V \} \quad (192)$$

$$m = \inf \{ u(z) \mid z \in B(x, r) \cap V \} \quad (193)$$

and

$$M' = \sup \{ u(z) \mid z \in B(x, \frac{1}{8}r) \cap V \} \quad (194)$$

$$m' = \inf \{ u(z) \mid z \in B(x, \frac{1}{8}r) \cap V \}. \quad (195)$$

We apply the preliminary paragraph to $M - u$ and $u - m$ as functions of $W^{1,2}(V \cap B(x, r))$ and we obtain that there exists $C_0 > 1$ (depending on n) such that

$$M - u(x + \frac{1}{4}re_n) \leq C_0(M - M') \quad (196)$$

$$u(x + \frac{1}{4}re_n) - m \leq C_0(m' - m). \quad (197)$$

It follows that $M - m \leq C_0(M - M' + m' - m)$ and then

$$M' - m' \leq L^{-1}(M - m) \quad (198)$$

where $L := C_0(C_0 - 1)^{-1} > 1$. This proves that $\text{osc}(x, \frac{1}{8}r) \leq L^{-1} \text{osc}(x, r)$.

Let $(x, r) \in \Delta$ be such that there exists $x^* \in \partial V$ with $|x - x^*| < \frac{1}{12}r$. We observe that

$$B(x, \frac{1}{32}r) \subset B(x^*, (\frac{1}{32} + \frac{1}{12})r), \quad (199)$$

that by the previous step

$$\text{osc}(x^*, (\frac{1}{32} + \frac{1}{12})r) \leq L^{-1} \text{osc}(x^*, (\frac{1}{4} + \frac{2}{3})r) \quad (200)$$

and finally that

$$B(x^*, (\frac{1}{4} + \frac{2}{3})r) \subset B(x, r). \quad (201)$$

This proves that $\text{osc}(x, \frac{1}{32}r) \leq L^{-1}\text{osc}(x, r)$.

Let $(x, r) \in \Delta$ be such that $B(x, \frac{1}{12}r) \subset V$. We can proceed as in the case $x \in \partial V$ (but replacing Lemma B.3 by the interior Harnack inequality for harmonic functions) to get

$$\text{osc}(x, \frac{1}{32}r) \leq L^{-1}\text{osc}(x, \frac{1}{16}r). \quad (202)$$

We have proved (191) in all cases. We fix $x \in V_R$ and we define the radii $r_1 = d(x, \mathbf{R}^n \setminus B(0, R))$ and $\rho = \min\{r_0, r_1\}$ so that $(x, \rho) \in \Delta$. For all $0 < r \leq \rho$, we have $\text{osc}(x, \frac{1}{32}r) \leq L^{-1}\text{osc}(x, r)$ and it is easy to deduce that for all $0 < r \leq \rho$,

$$\text{osc}(x, r) \leq L \left(\frac{r}{\rho}\right)^\ell \text{osc}(x, \rho), \quad (203)$$

where $\ell = \frac{\ln(L)}{\ln(32)} > 0$. This implies that for all $y \in V \cap B(x, \rho)$,

$$|u(y) - u(x)| \leq L \left(\frac{|y-x|}{\rho}\right)^\ell \text{osc}(x, \rho) \quad (204)$$

$$\leq 2L|u|_\infty \left(\frac{|y-x|}{\rho}\right)^\ell \quad (205)$$

In fact, this inequality is also true for $y \in V_R$ such that $|y-x| \geq \rho$ because we always have $|u(y) - u(x)| \leq 2|u|_\infty$. We finally underline $\rho \geq r_0 r_1$ because $r_0 \leq 1$ and $r_1 \leq R \leq 1$. The Lemma holds therefore with $C = 2Lr_0^{-\ell}$ (which depends on n and r_0). \square

B.3 Gradient estimates

According to [9, Theorem 1.2], the viscosity solutions are pointwise $C^{1,\alpha}$ up to the boundary. Although we use a weak formulation, the proof also applies in our case because it relies on the maximum principle (Lemma B.1), the Hölder continuity (Lemma B.2) and regularity results for solutions of the Neumann problem in a spherical cap. We only extract the Schauder estimate that is useful for our paper.

Proposition B.4 (Partial Schauder Estimate). *Let $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ be a weak solution of (156). Then there exists $C \geq 1$ (depending on n, V) such that for all $x \in \frac{1}{2}V_R$,*

$$|\nabla u(x)| \leq \frac{C}{R} \text{osc}(u) + C|u|_\infty. \quad (206)$$

where $\text{osc}(u) = \sup\{|u(x) - u(y)| \mid x, y \in \frac{3}{4}V_R\}$.

Proof. The letter C is a constant ≥ 1 that depends on n, V and whose value might change from one line to the other. We fix any $x_0 \in \frac{3}{4}V_R$. The function $v = u - u(x_0)$ is a weak solution of the Neumann problem

$$\begin{cases} \Delta v &= 0 & \text{in } \frac{3}{4}V_R \\ \partial_\nu v &= u & \text{in } \frac{3}{4}\Gamma_R. \end{cases} \quad (207)$$

We have restricted the domain of the equation so that the trace u^* is C^ℓ on $\frac{3}{4}\Gamma_R$ for some $\ell > 0$. More precisely, $|u^*|_\infty \leq |u|_\infty$ and according to Lemma B.2, there exists $\ell > 0$ (depending on n, V) such that for all $x, y \in \frac{3}{4}\Gamma_R$,

$$|u^*(x) - u^*(y)| \leq C|u|_\infty \left(\frac{|x - y|}{R} \right)^\ell. \quad (208)$$

Then we apply the scaled version of [9, Theorem 1.2] and we obtain that for all $x \in \frac{1}{2}V_R$,

$$|\nabla v(x)| \leq \frac{C}{R} (|v|_\infty + R|u|_\infty) \quad (209)$$

□

The last result allows to estimate the oscillations of u .

Proposition B.5 (Local Boundedness). *Let $u \in W^{1,2}(V_R) \cap L^\infty(V_R)$ be a weak solution of (156). Then there exists $C \geq 1$ (depending on n, V) such that*

$$\text{osc}(u) \leq CR \left(\int_{V_R} |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} + CR|u|_\infty. \quad (210)$$

where $\text{osc}(u) = \sup \{ |u(x) - u(y)| \mid x, y \in \frac{3}{4}V_R \}$.

Proof. The letter C is a constant ≥ 1 that depends on n, V and whose value might change from one line to the other. Let m be the average value of u on V_R . The function $v := u - m$ is a weak solution of the Neumann problem

$$\begin{cases} \Delta v &= 0 & \text{in } V_R \\ \partial_\nu v &= u & \text{in } \Gamma_R. \end{cases} \quad (211)$$

We apply a local boundedness estimate for weak solutions of Neumann problems ([7, Theorem 1.6 and Remark 1.12]): for all $x \in \frac{3}{4}V_R$

$$|v(x)| \leq C \left(\int_{V_R} |v|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} + CR|u|_\infty. \quad (212)$$

The triangular inequality show that

$$\text{osc}(u) \leq 2 \sup \{ |v(x)| \mid x \in \frac{3}{4}V_R \} \quad (213)$$

and the Poincaré inequality gives

$$\left(\int_{V_R} |v|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} \leq CR \left(\int_{V_R} |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}}. \quad (214)$$

□

C Extracts from [5]

We extract some important results of [5]. We work in an open set X of the Euclidean space \mathbf{R}^n ($n > 1$) and we fix a triple of parameters $\mathcal{P} = (r_0, a, M)$ composed of $r_0 > 0$, $a \geq 0$ and $M \geq 1$. We start by summarizing Definitions 2.1 (*admissible pairs*), 7.2 (*competitors*), 7.21 (*local quasiminimizers*) and 8.24 (*coral pairs*).

Definition C.1. The set of *admissible pairs* \mathcal{A} is the set of all pairs (u, K) where $K \subset X$ is relatively closed in X and $u \in W_{\text{loc}}^{1,2}(X \setminus K)$. Let (u, K) be an admissible pair and let B be an open ball such that $\overline{B} \subset X$. A *competitor* of (u, K) in B is a pair $(v, L) \in \mathcal{A}$ such that $K \setminus \overline{B} = L \setminus \overline{B}$ and $u = v$ \mathcal{L}^n a.e. on $X \setminus (K \cup \overline{B})$. In this case, we set

$$E(u) = \int_B |\nabla u|^2 d\mathcal{L}^n, \quad E(v) = \int_B |\nabla v|^2 \quad (215)$$

and

$$\Delta E = \max \{ (E(v) - E(u)), M(E(v) - E(u)) \}. \quad (216)$$

We say that (u, K) is a *local \mathcal{P} -quasiminimizer in X* if for all open balls B of radius $0 < r \leq r_0$ such that $\overline{B} \subset X$, for all competitors (v, L) of (u, K) in B , we have

$$\mathcal{H}^{n-1}(K \setminus L) \leq M\mathcal{H}^{n-1}(L \setminus K) + \Delta E + ar^{n-1}. \quad (217)$$

In addition, we say that (u, K) is *coral* if $K = \text{spt}(\mathcal{H}^{n-1} \llcorner K)$ in X . This means that for all $x \in K$ and all $r > 0$, $\mathcal{H}^{n-1}(K \cap B(x, r)) > 0$.

Remark C.2. If (u, K) is a quasiminimizer, we can see that K is \mathcal{H}^{n-1} locally finite. For all open ball B of radius $r \leq r_0$ such that $\overline{B} \subset X$, we consider the competitor

$$v = \begin{cases} u & \text{in } X \setminus \overline{B} \\ 0 & \text{in } \overline{B} \end{cases} \quad (218)$$

and $L = (K \setminus \overline{B}) \cup \partial B$. In particular, we have $K \setminus L = K \cap B$ and $L \setminus K \subset \partial B$ and $\Delta E \leq 0$. This proves that $\mathcal{H}^{n-1}(K \cap \overline{B}) \leq ((M+1)n\omega_n + a)r^{n-1}$.

For all competitors (v, L) of (u, K) , we have either $\mathcal{H}^{n-1}(L \setminus K) = \infty$ and thus (217) says nothing or $\mathcal{H}^{n-1}(K \setminus L) < \infty$ and thus L is \mathcal{H}^{n-1} locally finite in X . This justifies that we can assume that L is \mathcal{H}^{n-1} locally finite without loss of generality.

We are mainly concerned about Ahlfors-regularity (Definition 18.9) and uniform rectifiability (Section 73).

Definition C.3 (Ahlfors-regularity). A closed set $E \subset \mathbf{R}^n$ is Ahlfors-regular of dimension $n - 1$ if there exists a constant $C \geq 1$ such that for all $x \in E$ and for all $0 < r < \text{diam}(E)$

$$C^{-1}r^{n-1} \leq \mathcal{H}^{n-1}(E \cap B(x, r)) \leq Cr^{n-1}. \quad (219)$$

We don't give definitions of uniform rectifiability since there are too many. They are equivalent for closed, Ahlfors-regular sets. The reader can find a survey of uniform rectifiability in [5, Section 73] and also on Guy David's webpage (Notes-Parkcity.dvi).

Next, we summarize Definition 18.14 (*TRLQ class*), Section 72 and Section 74. It says that quasiminimizers are locally Ahlfors-regular and locally contained in a uniformly rectifiable set. In fact, we have already seen the first item in Remark C.2.

Theorem C.4. *Let $\mathcal{P} = (r_0, a, M)$ be a triple of parameters composed of $r_0 > 0$, $a \geq 0$ and $M \geq 1$. Assume that a is small enough (depending on n, M). Let (u, K) be a coral and local \mathcal{P} -quasiminimizer in X .*

1. *For all $x \in X$, for all $0 < r \leq r_0$ such that $\overline{B}(x, r) \subset X$,*

$$\mathcal{H}^{n-1}(K \cap \overline{B}(x, r)) \leq Cr^{n-1}. \quad (220)$$

where $C \geq 1$ depends on n, M .

2. *For all $x \in K$, for all $0 < r \leq r_0$ such that $\overline{B}(x, r) \subset X$,*

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \geq C^{-1}r^{n-1}. \quad (221)$$

where $C \geq 1$ depends on n, M .

3. *For all $x \in K$ and $0 < r < r_0$ such that $\overline{B}(x, 2r) \subset X$, there is a closed, Ahlfors-regular, uniformly rectifiable set E of dimension $(n-1)$ such that $K \cap B(x, r) \subset E$. The constants for the Ahlfors-regularity and uniform rectifiability depends on n, M and a .*

Remark C.5. One can observe that (217) implies

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \leq M\mathcal{H}^{n-1}(L \cap \overline{B}) + \Delta E + ar^{n-1}. \quad (222)$$

This is equivalent when $M = 1$ but strictly weaker when $M > 1$. We claim that Theorem C.4 still holds with (222) in place of (217). The first item is easy (see Remark C.2). The second item works as usual. The most critical point is the third item. In Section 74, David builds a suitable competitor (w, G) of (u, K) in a ball B . The set G is of the form $G = (K \setminus B) \cup Z$ where Z a special subset of ∂B containing $K \cap \partial B$. The quasi-minimality condition (217) is used only once at line (22) of Section 74. Then David uses the inequalities

$$\mathcal{H}^{n-1}(K \setminus G) \geq \mathcal{H}^{n-1}(K \cap B) \quad (223)$$

$$\mathcal{H}^{n-1}(G \setminus K) \leq \mathcal{H}^{n-1}(Z) \quad (224)$$

but we also have anyway

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \geq \mathcal{H}^{n-1}(K \cap B) \quad (225)$$

$$\mathcal{H}^{n-1}(G \cap \overline{B}) \leq \mathcal{H}^{n-1}(Z). \quad (226)$$

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References

- [1] L. Ambrosio; N. Fusco; D. Pallara Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp. ISBN: 0-19-850245-1
- [2] D. Bucur and S. Luckhaus Monotonicity formula and regularity for general free discontinuity problems. Arch. Ration. Mech. Anal. 211 (2014), no. 2, 489-511.
- [3] D. Bucur and A. Giacomini Shape optimization problems with Robin conditions on the free boundary. Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016), no. 6, 1539–1568.
- [4] L. A. Caffarelli and D. Kriventsov A free boundary problem related to thermal insulation. Comm. Partial Differential Equations 41 (2016), no. 7, 1149-1182.
- [5] G. David Singular sets of minimizers for the Mumford-Shah functional. Progress in Mathematics, 233. Birkhauser Verlag, Basel, 2005. xiv+581 pp. ISBN: 978-3-7643-7182-1; 3-7643-7182-X
- [6] G. De Philippis and A. Figalli Higher integrability for minimizers of the Mumford-Shah functional. Arch. Ration. Mech. Anal. 213 (2014), no. 2, 491-502.
- [7] S. Kim Note on local boundedness for weak solutions of Neumann problem for second-order elliptic equations. J. Korean Soc. Ind. Appl. Math. 19 (2015), no. 2, 189-195.
- [8] D. Kriventsov A free boundary problem related to thermal insulation: flat implies smooth. Calc. Var. Partial Differential Equations 58 (2019), no. 2, Paper No. 78, 83 pp.
- [9] D. Li; K. Zhang Regularity for fully nonlinear elliptic equations with oblique boundary conditions. (English summary) Arch. Ration. Mech. Anal. 228 (2018), no. 3, 923-967.

- [10] P. Mattila Geometry of sets and measures in Euclidean spaces. (English summary) Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. xii+343 pp.
- [11] S. Rigot Big pieces of $C^{1,\alpha}$ -graphs for minimizers of the Mumford-Shah functional. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000), no. 2, 329-349.

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