



HAL
open science

Higher integrability of the gradient for the Thermal Insulation problem

Camille Labourie, Emmanouil Milakis

► **To cite this version:**

Camille Labourie, Emmanouil Milakis. Higher integrability of the gradient for the Thermal Insulation problem. 2021. hal-03207514v1

HAL Id: hal-03207514

<https://hal.science/hal-03207514v1>

Preprint submitted on 25 Apr 2021 (v1), last revised 11 Oct 2021 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Higher integrability of the gradient for the Thermal Insulation problem

C. Labourie, E. Milakis

Abstract

We prove the higher integrability of the gradient for minimizers of the thermal insulation problem, an analogue of De Giorgi's conjecture for the Mumford-Shah functional. We deduce that the singular part of the free boundary has Hausdorff dimension strictly less than $n - 1$.

AMS Subject Classifications: 35R35, 35J20, 49N60, 49Q20

Keywords: Thermal Insulation, Higher Integrability, Free Boundary Problems.

Contents

1	Introduction	2
2	Minimizers	3
2.1	Definition	3
2.2	General properties	4
2.3	Porosity of the set where K is not regular	7
3	Higher integrability of the gradient	12
4	Dimension of the singular set	18
4.1	Generalities about the density of L^p_{loc} functions	18
4.2	Dimension of the singular set	20
	Appendices	20
A	Generalities about BV functions	20
B	A Robin problem	22
B.1	Statement	22
B.2	Hölder continuity up to the boundary	25
B.3	Gradient estimates	29
C	Extracts from [4]	31

1 Introduction

We fix a bounded connected set $\Omega \subset \mathbf{R}^n$. The thermal insulation problem consists in minimizing the functional

$$\mathcal{I}(A, u) := \int_A |\nabla u|^2 d\mathcal{L}^n + \int_{\partial A} |u^*|^2 d\mathcal{H}^{n-1} + \mathcal{L}^n(A) \quad (1)$$

among all pairs (A, u) where $A \subset \mathbf{R}^n$ is an admissible domain and $u \in W^{1,2}(A)$ is a function such that $u = 1$ for \mathcal{L}^n -a.e. on Ω . Here, u^* is the trace of u on ∂A .

In [3] and [7], Caffarelli and Kriventsov transpose the problem to a slightly different setting in order to apply the direct method of the calculus of variation. The authors represent a pair (A, u) by the function $u\mathbf{1}_A$ and relax the functional on SBV. The new problem consists in minimizing the functional

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} |\nabla u|^2 d\mathcal{L}^n + \int_{J_u} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(\{u > 0\}) \quad (2)$$

among all functions $u \in \text{SBV}(\mathbf{R}^n)$ such that $u = 1$ \mathcal{L}^n -a.e. on Ω . The definition of J_u and \bar{u}, \underline{u} are given in Appendix A. This new setting is more suited to a direct minimization since it enjoys the compactness and closure properties of SBV. In parenthesis, there always exist functions $u \in \text{SBV}(\mathbf{R}^n)$ such that $u = 1$ \mathcal{L}^n -a.e. on Ω and $\mathcal{F}(u) < \infty$. For example, $u = \mathbf{1}_B$ where B is an open ball containing Ω . In [3, Theorem 4.2], Caffarelli and Kriventsov prove that the SBV problem has a solution u . A key point property of solutions is that there exists $0 < \delta < 1$ (depending on n, Ω) such that $\text{spt}(u) \subset B(0, \delta^{-1})$ and

$$u \in \{0\} \cup [\delta, 1] \quad \mathcal{L}^n\text{-a.e. on } \mathbf{R}^n. \quad (3)$$

This property has also been proved in [2].

The main goal of the present article is to prove that there exists $p > 1$ such that $|\nabla u|^2 \in L^p_{\text{loc}}(\mathbf{R}^n \setminus \bar{\Omega})$. A parallel property was conjectured by De Giorgi for minimizers of the Mumford-Shah functional and solved by De Philippis and Figalli in [5]. Our proof is inspired by the technique of [5] and it relies on three key properties: the Ahlfors-regularity of the free boundary, the uniform rectifiability of the free boundary and the ε -regularity theorem. For the thermal insulation problem, the conclusion of the ε -regularity theorem is that the free boundary looks like a pair of graphs rather than just one graph. Moreover, the function u satisfies an elliptic equation with a Robin boundary condition at the boundary rather than a Neumann boundary condition. In our approach we deduce a *porosity* property which means that the singular part Σ of the free boundary has many holes in a quantified way. Finally, once we establish the higher integrability of the gradient, we are also able to conclude that the dimension of Σ is less than $n - 1$.

NOTATIONS. We work in the Euclidean space \mathbf{R}^n with $n > 1$. For $x \in \mathbf{R}^n$ and $r > 0$, $B(x, r)$ is the open ball centered in x and of radius r . Sometimes x is omitted and the open ball is simply denoted by B_r . Given an open ball $B = B(x, r)$, the notation $2B$ means $B(x, 2r)$. Given a set $A \subset \mathbf{R}^n$, the indicator function of A is denoted by $\mathbf{1}_A$. Given two sets $A, B \subset \mathbf{R}^n$, the notation $A \subset\subset B$ means that there exists a compact set $K \subset \mathbf{R}^n$ such that $A \subset K \subset B$. We have gathered some definitions and results from the theory of BV functions in the introduction of Appendix A.

2 Minimizers

Our ambient space is an open set X of \mathbf{R}^n . One can think of X as $\mathbf{R}^n \setminus \bar{\Omega}$. We introduce a few notations for a given $u \in \text{SBV}_{\text{loc}}(X)$. We define the set K_u as the support of the singular part of Du :

$$K_u := \text{spt}(|\bar{u} - \underline{u}| \mathcal{H}^{n-1} \llcorner J_u) \quad (4a)$$

$$:= \text{spt}(\mathcal{H}^{n-1} \llcorner J_u). \quad (4b)$$

When there is no ambiguity, we will write K instead of K_u . For any open ball B such that $\bar{B} \subset X$, we define a *competitor of u in B* as a function $v \in \text{SBV}_{\text{loc}}(X)$ such that $v = u$ \mathcal{L}^n -a.e. on $X \setminus \bar{B}$. For $x \in K_u$ and $r > 0$ such that $\bar{B}(x, r) \subset X$, we define

$$\omega_2(x, r) = r^{-(n-1)} \int_{B(x, r)} |\nabla u|^2 \, d\mathcal{L}^n \quad (5a)$$

$$\beta_2(x, r) = \left(r^{-(n+1)} \inf_V \int_{K \cap B(x, r)} d(y, V)^2 \, d\mathcal{H}^{n-1}(y) \right)^{\frac{1}{2}} \quad (5b)$$

where V runs among $(n-1)$ planes $V \subset \mathbf{R}^n$ passing through x .

2.1 Definition

For the purposes of the present paper, we fix a constant $\delta \in]0, 1[$ considered to be universal.

Definition 2.1. We say that $u \in \text{SBV}_{\text{loc}}(X)$ is a *minimizer* if

1. for \mathcal{L}^n -a.e. $x \in X$, we have $u \in \{0\} \cup]\delta, \delta^{-1}[$;
2. for all open balls B such that $\bar{B} \subset X$ and for all competitors v of u in B ,

$$\begin{aligned} & \int_B |\nabla u|^2 \, d\mathcal{L}^n + \int_{J_u \cap \bar{B}} (\bar{u}^2 + \underline{u}^2) \, d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{u > 0\}) \\ & \leq \int_B |\nabla v|^2 \, d\mathcal{L}^n + \int_{J_v \cap \bar{B}} (\bar{v}^2 + \underline{v}^2) \, d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{v > 0\}). \end{aligned} \quad (6)$$

As a first consequence, we have that $\bar{u}, \underline{u} \in \{0\} \cup [\delta, \delta^{-1}]$ everywhere in X . In particular, $\bar{u} \geq \delta$ everywhere on S_u . For all open balls B such that $\bar{B} \subset X$, we have

$$\int_B |\nabla u|^2 d\mathcal{L}^n + \int_{J_u \cap \bar{B}} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} < \infty. \quad (7)$$

This shows that $|\nabla u|^2 \in L^1_{\text{loc}}(X)$ and that J_u is \mathcal{H}^{n-1} -locally finite in X . One can show that u is harmonic in $X \setminus \bar{S}_u$. We deduce that in each connected component of $X \setminus \bar{S}_u$, we have either $u > \delta$ everywhere or $u = 0$ everywhere.

2.2 General properties

The properties of this section can also be adapted to almost-minimizers ([7, Definition 2.1]). The next result has been proved in [3].

Proposition 2.2 (Ahlfors-regularity). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. There exists $r_0 > 0$ and $C \geq 1$ (both depending on n, δ) such that the following holds true.*

1. For all $x \in X$, for all $0 < r \leq r_0$ such that $\bar{B}(x, r) \subset X$,

$$\int_{B(x, r)} |\nabla u|^2 d\mathcal{L}^n + \mathcal{H}^{n-1}(K \cap B(x, r)) \leq Cr^{n-1}. \quad (8)$$

2. For all $x \in \bar{S}_u$, for all $0 < r \leq r_0$ such that $\bar{B}(x, r) \subset X$,

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \geq C^{-1}r^{n-1}. \quad (9)$$

Corollary 2.3. *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer.*

(i) *We have $K = \bar{S}_u = \bar{J}_u$ and $\mathcal{H}^{n-1}(K \setminus J_u) = 0$.*

(ii) *The set $A_u := \{\bar{u} > 0\} \setminus K$ is open and $\partial A_u = K$.*

Proof. It is straightforward that $K \subset \bar{J}_u \subset \bar{S}_u$. On the other hand, property (9) shows that $\bar{S}_u \subset K$. We justify that $\mathcal{H}^{n-1}(K \setminus J_u) = 0$. The jump set J_u is Borel and \mathcal{H}^{n-1} locally finite in X , so for \mathcal{H}^{n-1} -a.e. $x \in X \setminus J_u$,

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^{n-1}(B(x, r) \cap J_u)}{r^{n-1}} = 0 \quad (10)$$

(see [9, Theorem 6.2]). We draw our claim from the observation that this limit contradicts (9).

We study the set A_u . We recall that the function \bar{u} is continuous in $X \setminus K$ (since it coincides with u outside S_u) and $\bar{u} \in \{0\} \cup [\delta, 1]$ *everywhere* in $X \setminus K$. As a consequence, the sets

$$A_u := \{\bar{u} > 0\} \setminus K, \quad (11)$$

$$B_u := \{\bar{u} = 0\} \setminus K \quad (12)$$

are open subsets of $X \setminus K$ and thus of X . The space X is the disjoint union

$$X = K \cup A_u \cup B_u, \quad (13)$$

where A_u and B_u are open and K is relatively closed, so $\overline{A_u} \subset A_u \cup K$. We show that $S_u \subset \overline{A_u}$. Let us suppose that there exists $x \in S_u$ and $r > 0$ such that $B(x, r) \cap A_u = \emptyset$. Then $B(x, r) \setminus K \subset \{\bar{u} = 0\}$ so we have $u = 0$ \mathcal{L}^n -a.e. on $B(x, r)$ and thus x is a Lebesgue point of u (a contradiction). We conclude that $S_u \subset \overline{A_u}$ and in turn $K \subset \overline{A_u}$ so $\overline{A_u} = A_u \cup K$. \square

We use [4] to justify that K_u is locally contained in a uniformly rectifiable set. For the reader's convenience we have summarised some results of [4] in Appendix C.

Proposition 2.4 (Rectifiability). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. There exists $r_0 > 0$ (depending on n, δ) such that the following holds true. For all $x \in K$ and $0 < r \leq r_0$ such that $\overline{B}(x, 2r) \subset X$, there is a closed, Ahlfors-regular, uniformly rectifiable set E of dimension $(n - 1)$ such that $K \cap B(x, r) \subset E$. The constants for the Ahlfors-regularity and uniform rectifiability depends on n, δ .*

Proof. We want to show that (u, K) satisfy Definition C.1, or rather the alternative Definition given in Remark C.5. Then the Proposition will follow from Theorem C.4. First, it is clear that (u, K) is an admissible pair. Let B be an open ball of radius $r > 0$ such that $\overline{B} \subset X$. Let an admissible pair (v, L) be a competitor of (u, K) in B . As explained in Remark C.2, we can assume without loss of generality that L is \mathcal{H}^{n-1} locally finite. Therefore, $v \in \text{SBV}_{\text{loc}}(X)$ and $\mathcal{H}^{n-1}(J_v \setminus L) = 0$. We have included more details about the construction of SBV functions in Appendix A. We can now apply the minimality inequality. We have

$$\begin{aligned} & \int_B |\nabla u|^2 d\mathcal{L}^n + \int_{J_u \cap \overline{B}} (\bar{u}^2 + \underline{u}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{u > 0\}) \\ & \leq \int_B |\nabla v|^2 d\mathcal{L}^n + \int_{J_v \cap \overline{B}} (\bar{v}^2 + \underline{v}^2) d\mathcal{H}^{n-1} + \mathcal{L}^n(B \cap \{v > 0\}) \end{aligned} \quad (14)$$

so

$$\begin{aligned} & \int_B |\nabla u|^2 d\mathcal{L}^n + \delta^2 \mathcal{H}^{n-1}(J_u \cap \overline{B}) + \mathcal{L}^n(B \cap \{u > 0\}) \\ & \leq \int_B |\nabla v|^2 d\mathcal{L}^n + \delta^{-2} \mathcal{H}^{n-1}(J_v \cap \overline{B}) + \mathcal{L}^n(B \cap \{v > 0\}). \end{aligned} \quad (15)$$

We omit the term $\mathcal{L}^n(B \cap \{u > 0\})$ at the left and we bound the term $\mathcal{L}^n(B \cap \{v > 0\})$ at the right by $\omega_n r^n$ where ω_n is the Lebesgue volume of the unit ball. We can replace J_u by K at the left since $\mathcal{H}^{n-1}(K \setminus J_u) = 0$. We can replace J_v by L at the right since $\mathcal{H}^{n-1}(J_v \setminus L) = 0$. It follows that

$$\mathcal{H}^{n-1}(K \cap \bar{B}) \leq \delta^{-4} \mathcal{H}^{n-1}(L \cap \bar{B}) + \delta^{-2} \Delta E + \delta^{-2} \omega_n r^n \quad (16)$$

where

$$\Delta E = \int_B |\nabla v|^2 - \int_B |\nabla u|^2 d\mathcal{L}^n. \quad (17)$$

□

Next we cite our proper ε -regularity theorem [7, Theorem 14.1]. Contrary to the ε -regularity theorem for the Mumford-Shah problem, it does not require $\omega_2(x, r)$ to be small. It says that when K is very close to a plane, K is given by a pair of smooth graphs. We are going to describe this situation in the next definition. When we are given a point $x_0 \in \mathbf{R}^n$ and a vector $n \in \mathbf{S}^{n-1}$, we define

$$H = \{h \in \mathbf{R}^n \mid h \cdot n = 0\} \quad (18)$$

and we decompose each point $y \in \mathbf{R}^n$ under the form $y = x_0 + (y' + y_n n)$ where $y' \in H$ and $y_n \in \mathbf{R}$.

Definition 2.5. Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. Let $x_0 \in K$ and $R > 0$ be such that $\bar{B}(x_0, R) \subset X$. We say that K is *regular* in $B(x_0, R)$ if it satisfies the three following conditions.

- (i) There exists a vector $n \in \mathbf{S}^{n-1}$ and two C^1 functions

$$f_i: B(0, R) \cap H \rightarrow \mathbf{R} \quad (i = 1, 2) \quad (19)$$

such that $f_i(0) = 0$, $f_2 \leq f_1$ and $K \cap B = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_i = \{y \in B(x_0, R) \mid y_n = f_i(y')\}. \quad (20)$$

- (ii) For all $h \in B(0, R) \cap H$,

$$|\nabla f_i(h)| \leq \frac{1}{4} \quad (21)$$

and there exists $\alpha > 0$ such that for all $h_1, h_2 \in B(0, R) \cap H$

$$|\nabla f_i(h_1) - \nabla f_i(h_2)| \leq \left(\frac{|h_1 - h_2|}{R} \right)^\alpha. \quad (22)$$

- (iii) There are two possible cases. The first case is

$$\begin{cases} u > 0 & \text{in } \{y \in B(x_0, R) \mid y_n > f_1(y') \text{ or } y_n < f_2(y')\} \\ u = 0 & \text{in } \{y \in B(x_0, R) \mid f_1(y') < y_n < f_2(y')\} \end{cases} \quad (23)$$

The second case is $f_1 = f_2$ and

$$\begin{cases} u > 0 & \text{in } \{y \in B(x_0, R) \mid y_n > f_1(y')\} \\ u = 0 & \text{in } \{y \in B(x_0, R) \mid y_n < f_1(y')\} \end{cases} \quad (24)$$

or inversely.

Theorem 2.6 (ε -regularity theorem). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. Fix $x \in K$.*

(i) *For all $\beta > 0$, there exists $\varepsilon > 0$ (depending on n, δ, β) such that the following holds true. For $r > 0$ such that $\overline{B}(x, r) \subset X$ and $\beta_2(x, r) + r \leq \varepsilon$, we have $\omega_2(x, r) \leq \beta$.*

(ii) *There exists $\varepsilon > 0, C \geq 1$ (both depending on n, δ) such that the following holds true. For $r > 0$ such that $\overline{B}(x, r) \subset X$ and $\beta_2(x, r) + r \leq \varepsilon$, the set K is regular in $B(x, C^{-1}R)$.*

2.3 Porosity of the set where K is not regular

The results of this subsection are specific to minimizers rather than almost-minimizers.

Lemma 2.7. *Let $x_0 \in K$ and $R > 0$ be such that $\overline{B}(x_0, R) \subset X$. We assume that K is regular in $B(x_0, R)$ (and we use the notations of Definition 2.5). Then u solves the Robin problem*

$$\begin{cases} \Delta u & = 0 & \text{in } A_1 \\ \partial_\nu u - u_1^* & = 0 & \text{in } \Gamma_1, \end{cases} \quad (25)$$

where $A_1 = \{y \in B \mid y_n > f_1(y')\}$, the vector ν is the inner normal vector to A_1 and the function u_1^* is the trace of $u|_{A_1}$ on Γ_1 .

Proof. We only detail the case (23) since it is the complicated one. We would like to clarify the relationship between traces and upper/lower limits. The easy situation is when a L^1_{loc} function v has a trace on each side of a hyperplane. Say that for some $x \in \mathbf{R}^n$ and $\nu \in \mathbf{S}^{n-1}$, there exists two scalars v_1^*, v_2^* such that

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap H^+} |v(y) - v_1^*| d\mathcal{L}^n(y) = 0 \quad (26a)$$

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap H^-} |v(y) - v_2^*| d\mathcal{L}^n(y) = 0, \quad (26b)$$

where $B_r = B(y, r)$ and

$$H^+ = \{z \in \mathbf{R}^n \mid (y - x) \cdot \nu > 0\} \quad (27a)$$

$$H^- = \{z \in \mathbf{R}^n \mid (y - x) \cdot \nu < 0\}. \quad (27b)$$

Then one has $\bar{v} = \max \{v_1^*, v_2^*\}$ and $\underline{v} = \min \{v_1^*, v_2^*\}$. Thus,

$$\bar{v}^2 + \underline{v}^2 = (v_1^*)^2 + (v_2^*)^2. \quad (28)$$

Our situation is almost the same but we deal with smooth surfaces rather than planes. We include the details below. We partition B in three sets (modulo \mathcal{L}^n)

$$A_1 = \{z \in B \mid y_n > f_1(y')\} \quad (29)$$

$$A_2 = \{z \in B \mid y_n < f_2(y')\} \quad (30)$$

$$A_3 = \{z \in B \mid f_2(y') < x_n < f_1(y')\}. \quad (31)$$

Let $v \in L^\infty(B) \cap W_{\text{loc}}^{1,2}(B \setminus K)$ be such that $v = 0$ in A_3 . For each $i = 1, 2$, there exists $v_i^* \in L^1(\Gamma_i)$ such that for \mathcal{H}^{n-1} -a.e. $x \in \Gamma_i$,

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap A_i} |v(y) - v_i^*(x)| \, d\mathcal{L}^n(y) = 0. \quad (32)$$

We refer to such x as a point where the trace $v_i^*(x)$ exists. We emphasise that, by the regularity of Γ_i at x , there exists $\nu \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap (A_i \Delta H^+)) = 0 \quad (33)$$

so (32) is equivalent to

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap H^+} |v(y) - v_i^*(x)| \, d\mathcal{L}^n(y) = 0. \quad (34)$$

We are going to compute $\bar{v}^2 + \underline{v}^2$ on Γ_1 (a similar reasoning can be done on Γ_2).

Let $x \in \Gamma_1 \setminus \Gamma_2$ be such that (34) holds for $i = 1$. For $r > 0$ small enough, B_r is disjoint from A_2 so

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r \setminus A_1} |v| \, d\mathcal{L}^n = \lim_{r \rightarrow 0} \int_{B_r \cap A_3} |v| \, d\mathcal{L}^n \quad (35)$$

$$= 0 \quad (36)$$

and as before, this is equivalent to

$$\lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap H^-} |v| \, d\mathcal{L}^n = 0. \quad (37)$$

Combining (34) for $i = 1$ and (37), we have

$$\bar{v}^2 + \underline{v}^2 = (v_1^*)^2. \quad (38)$$

Next, let $x \in \Gamma_1 \cap \Gamma_2$ be such that (34) holds for $i = 1$ and $i = 2$. The surfaces Γ_1 and Γ_2 have necessary the same tangent plane at x (and the

inner normal vectors are opposed). Combining (34) for $i = 1$ and $i = 2$, we have

$$\bar{v}^2 + \underline{v}^2 = (v_1^*)^2 + (v_2^*)^2. \quad (39)$$

We come back to our minimizer $u \in \text{SBV}_{\text{loc}}(X)$. We fix $\varphi \in C_c^1(B)$. For $\varepsilon \in \mathbf{R}$, we define $v: X \rightarrow \mathbf{R}$ by

$$v = \begin{cases} u + \varepsilon\varphi & \text{in } A_1 \\ u & \text{in } X \setminus A_1 \end{cases} \quad (40)$$

It is clear that v is C^1 in $X \setminus K$. As K is \mathcal{H}^{n-1} locally finite, we conclude $v \in \text{SBV}_{\text{loc}}(X)$ and $S_v \subset K$.

Remember that $u \geq \delta$ in $A_1 \cup A_2$, and $u = 0$ in A_3 . We take ε small enough so that $|\varepsilon\varphi|_\infty < \delta$. As a consequence $v > 0$ in $A_1 \cup A_2$ and $v = 0$ in A_3 . The sets $B \cap \{u > 0\}$ and $B \cap \{v > 0\}$ are equivalent modulo \mathcal{L}^n .

Let us check the multiplicities on the discontinuity set. As we have seen before, $J_v \cap B \subset \Gamma_1 \cup \Gamma_2$. We observe that for $x \in \Gamma_1$ such that the trace $u_1^*(x)$ exists, we have

$$v_1^*(x) = u_1^*(x) + \varepsilon\varphi(x) \quad (41)$$

and for $x \in \Gamma_2$ such the trace $u_2^*(x)$ exists, we have

$$v_2^*(x) = u_2^*(x). \quad (42)$$

Using the previous discussion, we deduce that for \mathcal{H}^{n-1} -a.e. on $\Gamma_1 \setminus \Gamma_2$,

$$\bar{v}^2 + \underline{v}^2 = (u_1^* + \varepsilon\varphi)^2 \quad (43)$$

$$= (\underline{u}^2 + \bar{u}^2) + 2\varepsilon\varphi u_1^* + \varepsilon^2|\varphi|^2 \quad (44)$$

that for \mathcal{H}^{n-1} -a.e. on $\Gamma_1 \cap \Gamma_2$,

$$\bar{v}^2 + \underline{v}^2 = (u_1^* + \varepsilon\varphi)^2 + (u_2^*)^2 \quad (45)$$

$$= (\underline{u}^2 + \bar{u}^2) + 2\varepsilon\varphi u_1^* + \varepsilon^2|\varphi|^2 \quad (46)$$

and that for \mathcal{H}^{n-1} -a.e. on $\Gamma_2 \setminus \Gamma_1$,

$$\bar{v}^2 + \underline{v}^2 = (u_2^*)^2 \quad (47)$$

$$= \underline{u}^2 + \bar{u}^2. \quad (48)$$

Finally, it is clear that

$$\begin{aligned} \int_B |\nabla v|^2 d\mathcal{L}^n &= \int_B |\nabla u|^2 d\mathcal{L}^n + 2\varepsilon \int_{A_1} \langle \nabla u, \nabla \varphi \rangle d\mathcal{L}^n \\ &\quad + \varepsilon^2 \int_{A_1} |\nabla \varphi|^2 d\mathcal{L}^n. \end{aligned} \quad (49)$$

We plug all these informations in the minimality inequality and we obtain that

$$0 \leq 2\varepsilon \int_{A_1} \langle \nabla u, \nabla \varphi \rangle d\mathcal{L}^n + 2\varepsilon \int_{\Gamma_1} \varphi u_1^* d\mathcal{H}^{n-1} + C(\varphi)\varepsilon^2. \quad (50)$$

As this holds for all small ε (positive or negative), we conclude that

$$\int_{A_1} \langle \nabla u, \nabla \varphi \rangle d\mathcal{L}^n + \int_{\Gamma_1} \varphi u_1^* d\mathcal{H}^{n-1} = 0. \quad (51)$$

□

We now state the *porosity* of the set $\Sigma \subset K$ where K is not regular. It is simpler to obtain than in the Mumford-Shah problem (see for example [10]) because our ε -regularity theorem only requires to control the flatness.

Corollary 2.8 (Porosity). *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. There exists $0 < r_0 \leq 1$ and $C \geq 1$ (both depending on n, δ) for which the following holds true. For all $x \in K$ and all $0 < r \leq r_0$ such that $\overline{B}(x, 2r) \subset X$, there exists $y \in K \cap B(x, r)$ such that K is regular in $B := B(y, C^{-1}r)$. Moreover, we can assume that*

$$\sup_{B \setminus K} |\nabla u|^2 \leq Cr^{-1}. \quad (52)$$

Proof. The letter C is a constant ≥ 1 that depends on n, δ . For $y \in K$ and $t > 0$ such that $\overline{B}(y, t) \subset X$, we define

$$\beta(y, t) = \inf_V \sup_{z \in K \cap \overline{B}(y, t)} t^{-1} d(z, V) \quad (53)$$

where the infimum is taken over the set of all affine hyperplanes V of \mathbf{R}^n . Observe that

$$\beta_2(y, t)^2 \leq r^{-(n-1)} \mathcal{H}^{n-1}(K \cap \overline{B}(y, t)) \beta(y, t)^2 \quad (54)$$

so as soon as t is small enough for the Ahlfors-regularity, we have $\beta_2(y, t) \leq C\beta(y, t)$.

Let r_0 be the minimum between 1, the radius of Proposition 2.2 (Ahlfors-regularity) and the radius of Proposition 2.4 (uniform rectifiability). We fix $x \in K$ and $0 < r \leq r_0$ such that $\overline{B}(x, 2r) \subset X$. According to Proposition 2.4, there exists an Ahlfors-regular and uniformly rectifiable set E such that $K \cap B(x, r) \subset E$. Moreover, the constants for the Ahlfors-regularity and uniform rectifiability depends on n, δ . For $y \in E$ and $t > 0$, we define as before

$$\beta_E(y, t) = \inf_V \sup_{z \in E \cap \overline{B}(y, t)} t^{-1} d(z, \pi) \quad (55)$$

where the infimum is taken on the set of all affine hyperplanes V of \mathbf{R}^n . The key property of β_E is that, according to [4, Theorem 73.11], for all $\varepsilon > 0$, the set

$$\{(y, t) \mid y \in E, 0 < t < \text{diam}(E), \beta_E(y, t) > \varepsilon\} \quad (56)$$

is a Carleson set. This means that for all $\varepsilon > 0$, there exists $C_0(\varepsilon) \geq 1$ (depending on n, δ, ε) such that for all $y \in E$ and all $0 < t < \text{diam}(E)$,

$$\int_0^t \int_{E \cap B(y,t)} \mathbf{1}_{\{\beta_E(z,s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \leq C_0(\varepsilon)t^{n-1}. \quad (57)$$

In particular,

$$\int_0^r \int_{K \cap B(x,r)} \mathbf{1}_{\{\beta(z,s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \leq C_0(\varepsilon)r^{n-1}. \quad (58)$$

We are going to deduce that for all $\varepsilon > 0$, there exists $C(\varepsilon) \geq 1$, a point $y \in K \cap B(x,r)$ and a radius t such that $C(\varepsilon)^{-1}r \leq t \leq r$ and $\beta(y,t) < \varepsilon$. We proceed by contradiction for some $C(\varepsilon)$ to be precised. We get

$$\int_0^r \int_{K \cap B(x,r)} \mathbf{1}_{\{\beta(z,s) > \varepsilon\}}(z) d\mathcal{H}^{n-1}(z) \frac{ds}{s} \quad (59)$$

$$\geq \mathcal{H}^{n-1}(K \cap B(x,r)) \int_{C(\varepsilon)^{-1}r}^r \frac{ds}{s}$$

$$\geq \mathcal{H}^{n-1}(K \cap B(x,r)) \ln(C(\varepsilon)) \quad (60)$$

$$\geq C^{-1}r^{n-1} \ln(C(\varepsilon)) \quad (61)$$

and this contradicts (58) if $C(\varepsilon)$ is too big compared to $C_0(\varepsilon)$.

Now, we assume that we have such a pair (y,t) for a certain ε . In particular, $\beta_2(y,t) \leq C\beta(y,t) \leq C\varepsilon$. We also assume that $r_0 \leq \varepsilon$. According to the second statement of Theorem 2.6, we can choose ε (depending on n, δ) so that K is regular in $B := B(y, C^{-1}t)$. Then $K \cap B$ is given by a pair of graphs $\Gamma_1 \cup \Gamma_2$ and B can be divided in two or three open sets: one where $u = 0$ and one or two where $u > 0$ and satisfies an elliptic equation with Robin boundary conditions on Γ_i . Using Appendix B, and more precisely (166), we deduce that for all z in $(\frac{1}{4}B) \setminus K$,

$$|\nabla u(z)|^2 \leq C \int_B |\nabla u|^2 d\mathcal{L}^n + C|u|_\infty^2. \quad (62)$$

Remember that by Ahlfors-regularity (Proposition 2.2), we have

$$\int_B |\nabla u|^2 d\mathcal{L}^n \leq Ct^{-1}. \quad (63)$$

By definition of minimizers and since $0 < t \leq 1$, we also have trivially $|u|_\infty^2 \leq C \leq Ct^{-1}$. We conclude that for $z \in (\frac{1}{4}B) \setminus K$,

$$|\nabla u(z)|^2 \leq Ct^{-1}. \quad (64)$$

The conclusion of the Lemma holds true for the ball $\frac{1}{4}B$. \square

3 Higher integrability of the gradient

Theorem 3.1. *Let $u \in \text{SBV}(X)$ be minimal. There exists $p > 1$ such that $|\nabla u|^2 \in L^p_{\text{loc}}(X)$.*

Our proof follows the ideas [5]. We present them below. Let B be a (small) open ball of radius R such that $\overline{B} \subset X$. For $\gamma > 1$, we observe that

$$\int_B |\nabla u|^{2\gamma} d\mathcal{L}^n = \int_0^\infty \mathcal{L}^n(B \cap \{|\nabla u|^{2\gamma} > t\}) dt \quad (65)$$

$$= \gamma R^{-\gamma} \int_0^\infty s^{\gamma-1} \mathcal{L}^n(B \cap \{|\nabla u|^2 > sR^{-1}\}) ds \quad (66)$$

and for $M \geq 1$,

$$\gamma R^{-\gamma} \int_1^\infty s^{\gamma-1} \mathcal{L}^n(B \cap \{|\nabla u|^2 > sR^{-1}\}) ds \quad (67)$$

$$\leq \gamma R^{-\gamma} \sum_{h=0}^\infty \int_{M^h}^{M^{h+1}} s^{\gamma-1} \mathcal{L}^n(B \cap \{|\nabla u|^2 > sR^{-1}\}) ds$$

$$\leq \gamma R^{-\gamma} \sum_{h=0}^\infty \left(\int_{M^h}^{M^{h+1}} s^{\gamma-1} ds \right) \mathcal{L}^n(B \cap \{|\nabla u|^2 > M^h R^{-1}\}) \quad (68)$$

$$\leq (M^\gamma - 1) R^{-\gamma} \sum_{h=1}^\infty M^{h\gamma} \mathcal{L}^n(B \cap \{|\nabla u|^2 > M^h R^{-1}\}). \quad (69)$$

We are going to prove that there exists $M \geq 1$, $\alpha > 0$ and $C \geq 1$ such that for all $h \in \mathbf{N}$,

$$\mathcal{L}^n(B \cap \{|\nabla u|^2 > M^h R^{-1}\}) \leq C R^n M^{-h(1+\alpha)} \quad (70)$$

and then take $\gamma = 1 + \frac{1}{2}\alpha$.

We start with a covering Lemma. When we are given a point $x_k \in \mathbf{R}^n$ and a vector $n \in \mathbf{S}^{n-1}$, we define

$$H = \{h \in \mathbf{R}^n \mid h \cdot n = 0\} \quad (71)$$

and we decompose each point $y \in \mathbf{R}^n$ under the form $y = x_k + (y' + y_n n)$ where $y' \in H$ and $y_n \in \mathbf{R}$.

Lemma 3.2 (Covering Lemma). *Let $E \subset \mathbf{R}^n$ be a bounded set. Let (B_k) be a family of open balls such that*

1. for each $k \neq l$, $2B_k \cap B_l = \emptyset$;
2. for each k , the ball B_k is centered at a point $x_k \in E$ and for all $x \in 2B_k$, there exists a vector $n \in \mathbf{S}^{n-1}$ and a $\frac{1}{4}$ -Lipschitz function $f: H \rightarrow \mathbf{R}$ such that $f(0) = 0$ and

$$x \in \{y \in 2B_k \mid y_n = f_i(y')\} \subset E \quad (72)$$

Let $0 < r \leq \inf_k \text{radius}(B_k)$. There exists a sequence of open balls $(D_i)_{i \in I}$ of radius r and centered in $E \setminus \bigcup_k B_k$ such that

$$E \setminus \bigcup_k B_k \subset \bigcup_{i \in I} D_i \quad (73)$$

and the balls $(12^{-1}D_i)_{i \in I}$ are pairwise disjoint and disjoint from $\bigcup_k B_k$.

Proof. Let $0 < r_0 \leq \inf_k \text{radius}(B_k)$. We define

$$F := E \setminus \bigcup_k B_k. \quad (74)$$

The goal is to cover F with a controlled number of balls of radius r_0 . We will in fact work with a radius $0 < r \leq r_0$ which will be precised during the proof. As F is bounded, there exists a maximal sequence of points $(x_i) \in F$ such that $B(x_i, r) \subset \mathbf{R}^n \setminus \bigcup_k B_k$ and $|x_i - x_j| \geq r$. For $i \neq j$, we have $|x_i - x_j| \geq r$ so the balls $(B(x_i, \frac{1}{2}r))_i$ are disjoint.

Next, we show that

$$F \subset \bigcup_i B(x_i, 6r). \quad (75)$$

Let $x \in F$. If $B(x, r) \subset \mathbf{R}^n \setminus \bigcup_k B_k$, then by maximality of (x_i) , there exists i such that $|x - x_i| < r$. We now focus on the case where there exists an index k such that $B(x, r) \cap B_k$. Let us write $B_k = B(x_0, R)$. As $x \in F = E \setminus \bigcup_k B_k$ and $B(x, r) \cap B(x_0, R) \neq \emptyset$, we have $R < |x - x_0| < R + r$. We are going to justify that there exists $y \in K$ such that $R + r < |y - x_0| < R + 3r$ and $|x - y| < 5r$. As $r \leq R$, we have $x \in B(x_0, 2R)$. According to the assumptions of the Lemma, there exists a vector $n \in \mathbf{S}^{n-1}$ and a $\frac{1}{4}$ -Lipschitz function $f: H \rightarrow \mathbf{R}$ such that $f(0) = 0$ and

$$x \in \{y \in B(x_0, 2R) \mid y_n = f(y')\} \subset E. \quad (76)$$

The estimate $R < |x - x_0| < R + r$ can be rewritten

$$R < |x' + f(x')n| < R + r. \quad (77)$$

Then we are looking for a vector $h \in H$ close to x' such that

$$R + r < |h + f(h)n| < R + 3r. \quad (78)$$

As $\lim_{t \rightarrow +\infty} |tx' + f(x')n| = \infty$, there exists $t \geq 1$ such that the vector $h := tx'$ satisfies $|h + f(x')n| = R + 2r$. We estimate how close h is to x' . As $h = tx'$ with $t \geq 1$, we have the identity

$$|h - x'| = \frac{|h|^2 - |x'|^2}{|h| + |x'|} \leq \frac{|h|^2 - |x'|^2}{2|x'|}. \quad (79)$$

We compute

$$|h|^2 - |x'|^2 = \left| |h + f(x')n|^2 - |x' + f(x')n|^2 \right| \quad (80)$$

$$\leq (R + 2r)^2 - R^2 \quad (81)$$

$$\leq 4rR \left(1 + \frac{r}{R} \right). \quad (82)$$

Let ε be the Lipschitz constant of f . We have $|f(x')| \leq \varepsilon|x'|$ so the inequality $|x' + f(x')n| \geq R$ implies $|x'| \geq (1 + \varepsilon^2)^{-\frac{1}{2}}R$. We conclude that

$$|h - x'| \leq 2r(1 + \varepsilon^2)^{\frac{1}{2}} \left(1 + \frac{r}{R} \right). \quad (83)$$

We assume that $r \leq \frac{1}{2}R$ and we use $\varepsilon \leq \frac{1}{4}$ to finally obtain $|h - x'| < 4r$. We have estimated how close h is to x' . We then estimate

$$|(h + f(h)n) - (h + f(x')n)| = |f(h) - f(x')| \quad (84)$$

$$\leq \varepsilon|h - x'| \quad (85)$$

$$< r \quad (86)$$

and since $|h + f(x')n| = R + 2r$, this yields $R + r < |h + f(h)| < R + 3r$. Similarly, we estimate

$$|(h + f(h)n) - (x' + f(x')n)| \leq (1 + \varepsilon)|h - x'| \quad (87)$$

$$< 5r. \quad (88)$$

In conclusion we define $y := x_0 + h + f(h)n$ and we have $|y - x| < 5r$ and $R + r < |y - x_0| < R + 3r$ as promised. We assume $r \leq \frac{1}{3}R$ so that $y \in B(x_0, 2R)$ and thus $y \in E$. We are going to justify that $B(y, r) \subset \mathbf{R}^n \setminus \bigcup_k B_k$. We recall that $B(x_0, 2R)$ is disjoint from all the other balls of the family (B_k) . We observe first that

$$B(y, r) \subset B(x_0, R + 4r) \setminus B(x_0, R) \quad (89)$$

and then we assume $r \leq \frac{1}{4}R$ so that $B(y, r) \subset B(x_0, 2R) \setminus B(x_0, R)$. Our claim follows. By maximality of the family (x_i) , there exists i such that $|y - x_i| < r$ and thus $|x - x_i| < 6r$. We finally choose $r = \frac{1}{6}r_0$. As $r_0 \leq \inf_k \text{radius}(B_k)$, we have all the required bounds on r and $F \subset \bigcup_i B(x_i, 6r) \subset \bigcup_i B(x_i, r_0)$. The balls (D_i) are $D_i = B(x_i, r_0)$. \square

Proof of Theorem 3.1. We define \mathcal{A} as the set of all open balls B centered in K such that $2\bar{B} \subset X$ and K is regular in $2B$. We fix a point $x_0 \in K$. To simplify the notations, we assume $x_0 = 0$. There exists a radius $R > 0$ and constants $C_0, C_1 \geq 1$ such that $\bar{B}(0, 4R) \subset X$ and

(i) for all $x \in B(0, 2R)$, for all $0 < r \leq 2R$,

$$\int_{B(x,r)} |\nabla u|^2 d\mathcal{L}^n + \mathcal{H}^{n-1}(K \cap B(x, r)) \leq C_0 r^{n-1}; \quad (90)$$

(ii) for all $x \in K \cap B(0, 2R)$, for all $0 < r \leq 2R$,

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \geq C_0^{-1} r^{n-1}. \quad (91)$$

(iii) for all $x \in K \cap B(0, 2R)$, for all $0 < r \leq 2R$, there exists $B \in \mathcal{A}$ of radius $C_1^{-1}r$ such that $24B \subset B(x, r)$ and

$$\sup_{2B} |\nabla u|^2 \leq C_1 r^{-1}. \quad (92)$$

Note that C_0, C_1 depends on n, δ . The two first properties are referred to as *Ahlfors-regularity*. The third property is referred to as the *porosity*.

We consider $M = \max \{4C_0, C_0C_1\}$ and we define for $h \in \mathbf{N}$,

$$A_h := \{x \in B(0, \frac{1}{4}R) \setminus K \mid |\nabla u|^2 > M^h R^{-1}\}. \quad (93)$$

Our goal is to show that there exists $C \geq 1$ and $\alpha > 0$ (both depending on n, δ) such that for $h \geq 1$,

$$\mathcal{L}^n(A_h) \leq CR^n M^{-(1+\alpha)h}. \quad (94)$$

The proof is based on the fact that A_h is at distance $\sim M^{-h}R$ from K and has many holes of size $\sim M^{-h}R$ near K . We justify more precisely these observations. Let $x \in A_h$ and assume that $B(x, C_0M^{-h}R)$ is disjoint from K . We use the subharmonicity of $|\nabla u|^2$ and the Ahlfors-regularity to estimate

$$|\nabla u|(x)^2 \leq \int_{B(x, C_0M^{-h}R)} |\nabla u|^2 d\mathcal{L}^n \quad (95)$$

$$\leq M^h R^{-1}. \quad (96)$$

This contradicts the definition of A_h . We deduce that there exists $y \in K$ such that $|x - y| < C_0M^{-h}R$. For our next observation, we consider a point $x \in K \cap B(0, 2R)$ and we apply the porosity property to the ball $B(x, C_1M^{-h}R)$. We obtain an open ball $B \in \mathcal{A}$ of radius $M^{-h}R$ such that $24B \subset B(x, r)$ and

$$\sup_{2B} |\nabla u|^2 \leq M^h R^{-1}. \quad (97)$$

In particular, $2B$ is disjoint from A_h .

We start the proof by defining for $h \geq 1$,

$$r_h = M^{-h}R \quad (98)$$

$$R_h = (\frac{1}{2} + M^{-h+1})R. \quad (99)$$

The sequence $(R_h)_h$ is decreasing, we have $R_1 < 2R$, $\lim_{h \rightarrow \infty} R_h = \frac{1}{2}$ and $R_{h+1} + r_h \leq R_h$. For each $h \geq 1$, we build an index set $I(h)$ and a family of balls $\alpha_h = (B_i)_{i \in I(h)} \in \mathcal{A}$ as follow. First we define $I(1) = \emptyset$ and $\alpha_1 = \emptyset$. Let $h \geq 2$ be such that $\alpha_1, \dots, \alpha_{h-1}$ have been built. We assume that the index sets $I(g)$, where $g = 1, \dots, h-1$, are pairwise disjoint. We also assume that for all $i \in I_g$, the balls B_i have radius $C_1^{-1}r_g$ and that for all indices $i, j \in \bigcup_{g=1}^{h-1} I(g)$ with $i \neq j$, $2B_i \cap B_j = \emptyset$. Then, we introduce the sets

$$K_h := K \cap B(0, R_h) \setminus \bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i \quad (100)$$

$$K_h^* := K \cap B(0, R_{h+1}) \setminus \bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i. \quad (101)$$

According to Lemma 3.2, there exists a sequence of open balls $(D_i)_{i \in I(h)}$ centered in K_h^* of radius r_h such that

$$K_h^* \subset \bigcup_{i \in I(h)} D_i, \quad (102)$$

and such that the balls $(12^{-1}D_i)$ are pairwise disjoint and disjoint from $\bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i$. We can assume that index set $I(h)$ is disjoint from the sets $I(g)$, $g = 1, \dots, h-1$. Since $R_{h+1} + r_h \leq R_h$, we observe that the balls $(12^{-1}D_i)$ are included in

$$B(0, R_h) \setminus \bigcup_{g=1}^{h-1} \bigcup_{i \in I(g)} B_i. \quad (103)$$

Next, we apply the porosity to the balls (D_i) . For each $i \in I(h)$, there exists $B_i \in \mathcal{A}$ of radius $C_1^{-1}r_h$ such that $B_i \subset 24^{-1}D_i$ and

$$\sup_{2B_i} |\nabla u|^2 \leq C_1 r_h^{-1}. \quad (104)$$

We finally define $\alpha_h = (B_i)_{i \in I(h)}$. We should not forget to mention that for all $i \in I(h)$, we have $2B_i \subset 12^{-1}D_i$ so $2B_i$ is disjoint from all the other balls we have built so far.

Now, we estimate $\mathcal{L}^n(A_h)$ for $h \geq 1$. We recall that we have taken $M = \max\{4C_0, C_0C_1\}$. We show first that the points of A_h cannot be too far from K_h^* . Let $x \in A_h$. We have seen earlier that there exists $y \in K$ such that $|x - y| < C_0M^{-h}R$. We are going to show that $y \in K_h^*$. Since $|x| \leq \frac{1}{4}R$, we have

$$|y| \leq \frac{1}{4}R + C_0M^{-h}R \leq \frac{1}{2}R. \quad (105)$$

Let us assume that there exists $g = 1, \dots, h-1$ and $i \in I(g)$ such that $y \in B_i$. The radius of B_i is $C_1^{-1}r_g \geq C_1^{-1}M^{-(h-1)}R$ and since $|x - y| < C_0M^{-h}R$, we have $x \in 2B_i$. Then by construction,

$$\sup_{2B_i} |\nabla u|^2 \leq C_1 r_g^{-1} \leq C_1 M^{h-1} R^{-1} \quad (106)$$

and this contradicts the fact that $x \in A_h$. We have shown that $y \in K_h^*$. As a consequence, there exists $i \in I(h)$ such that $y \in D_i$. As $\text{radius}(D_i) = M^{-h}R$ and $|x - y| < C_0M^{-h}R$, we deduce finally that

$$A_h \subset \bigcup_{i \in I(h)} (1 + C_0)D_i. \quad (107)$$

This allows to estimate $\mathcal{L}^n(A_h)$:

$$\mathcal{L}^n(A_h) \leq \omega_n (1 + C_0)^n |I(h)| r_h^n \quad (108)$$

where ω_n is the Lebesgue measure of the unit ball.

We want to estimate $|I(h)|$. The balls $(12^{-1}D_i)_{i \in I(h)}$ are disjoint and included in the set $B(0, R_h) \setminus \bigcup_{g=2}^{h-1} \bigcup_{i \in I(g)} B_i$ so by Ahlfors-regularity,

$$C_0^{-1} 12^{-(n-1)} r_h^{(n-1)} |I(h)| \leq \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap 12^{-1}D_i) \quad (109)$$

$$\leq \mathcal{H}^{n-1}(K_h). \quad (110)$$

We are going to see that $\mathcal{H}^{n-1}(K_h)$ is bounded from above by a decreasing geometric sequence. We have

$$\mathcal{H}^{n-1}(K_h^*) \leq \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap D_i) \quad (111)$$

$$\leq C_0 \sum_{i \in I(h)} r_h^{n-1} \quad (112)$$

$$\leq C_0 C_1^{n-1} \sum_{i \in I(h)} (C_1 r_h)^{n-1} \quad (113)$$

$$\leq C_0^2 C_1^{n-1} \sum_{i \in I(h)} \mathcal{H}^{n-1}(K \cap B_i) \quad (114)$$

$$\leq C_3 \mathcal{H}^{n-1}(K_h \setminus K_{h+1}). \quad (115)$$

where $C_3 = C_0^2 C_1^{-(n-1)}$. We deduce

$$\mathcal{H}^{n-1}(K_h) \leq C_3 \mathcal{H}^{n-1}(K_h \setminus K_{h+1}) + \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}). \quad (116)$$

We rewrite this inequality as

$$\mathcal{H}^{n-1}(K_{h+1}) \leq \lambda \mathcal{H}^{n-1}(K_h) + C_3^{-1} \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}) \quad (117)$$

where $\lambda = C_3^{-1}(C_3 - 1)$. Then, we multiply both sides of the inequality by λ^{-h+1} :

$$\begin{aligned} \lambda^{-(h+1)}\mathcal{H}^{n-1}(K_{h+1}) \\ \leq \lambda^{-h}\mathcal{H}^{n-1}(K_h) + C_3^{-1}\lambda^{-h}\mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}) \end{aligned} \quad (118)$$

$$\leq \lambda^{-h}\mathcal{H}^{n-1}(K_h) + \mathcal{H}^{n-1}(K \cap B_{R_h} \setminus B_{R_{h+1}}). \quad (119)$$

Summing this telescopic inequality, we obtain that for all $h \geq 1$,

$$\lambda^{-h}\mathcal{H}^{n-1}(K_h) \leq 2\mathcal{H}^{n-1}(K \cap B(0, 2R)) \quad (120)$$

$$\leq 2^n C_0 R^{n-1}. \quad (121)$$

In summary, we have proved that some constant $C \geq 1$ (depending on n, δ) and for $h \geq 1$

$$\mathcal{L}^n(A_h) \leq CR^{n-1}r_h\lambda^h \quad (122)$$

$$\leq CR^n(\lambda M^{-1})^h. \quad (123)$$

As $0 < \lambda < 1$ and $M > 1$, there exists $\alpha > 0$ such that $\lambda = M^{-\alpha}$. \square

4 Dimension of the singular set

The Hausdorff dimension of a set $A \subset \mathbf{R}^n$ is defined by

$$\dim_{\mathcal{H}}(A) = \inf \{ s \geq 0 \mid H^s(A) = 0 \}. \quad (124)$$

We take the convention that for $s < 0$, the term \mathcal{H}^s -almost-everywhere means everywhere and the inequality $\dim_{\mathcal{H}}(A) < 0$ means $A = \emptyset$. As usual, we work in an open set $X \subset \mathbf{R}^n$.

4.1 Generalities about the density of L_{loc}^p functions

Lemma 4.1. *Let μ be a Radon measure in X such that μ is dominated by \mathcal{L}^n and let $s < n$. Then for \mathcal{H}^s -a.e. $x \in X$, we have*

$$\lim_{r \rightarrow 0} r^{-s}\mu(B(x, r)) = 0. \quad (125)$$

Proof. If $s < 0$, the limit is indeed 0 for every $x \in X$. We assume $0 \leq s < n$. We fix a closed ball $\overline{B} \subset X$, a scalar $\lambda > 0$ and a set

$$A = \{ x \in \overline{B} \mid \limsup_{r \rightarrow 0} r^{-s}\mu(B(x, r)) > \lambda \}. \quad (126)$$

According to [1, Theorem 2.56],

$$\mu(A) \geq \lambda H^s(A). \quad (127)$$

As $A \subset \overline{B}$ and μ is a Radon measure, we have $\mu(A) < \infty$. Then (127) gives $H^s(A) < \infty$ and since $s < n$, $\mathcal{L}^n(A) = 0$. The measure μ is dominated by \mathcal{L}^n so $\mu(A) = 0$ and now (127) gives $H^s(A) = 0$. We can take a sequence of scalars $\lambda_k \rightarrow 0$ to deduce

$$\mathcal{H}^s(\{x \in \overline{B} \mid \limsup_{r \rightarrow 0} r^{-s} \mu(B(x, r)) > 0\}) = 0. \quad (128)$$

We can then conclude that

$$\mathcal{H}^s(\{x \in X \mid \limsup_{r \rightarrow 0} r^{-s} \mu(B(x, r)) > 0\}) = 0. \quad (129)$$

by covering X with a sequence of closed balls $\overline{B}_k \subset X$. \square

The previous Lemma can be restated as follow: for $v \in L^1_{\text{loc}}$, for $s < n$ and for \mathcal{H}^s -a.e. $x \in X$,

$$\lim_{r \rightarrow 0} r^{-s} \int_{B(x, r)} v \, d\mathcal{L}^n = 0 \quad (130)$$

We are going to see that we have an improvement when v has a higher integrability.

Corollary 4.2. *Let $v \in L^p_{\text{loc}}(X)$ for some $p \geq 1$ and let $s < n$. Then, for $\mathcal{H}^{n-p(n-s)}$ -a.e. $x \in X$,*

$$\lim_{r \rightarrow 0} r^{-s} \int_{B(x, r)} v \, d\mathcal{L}^n = 0. \quad (131)$$

Proof. Without loss of generality, we assume $v \geq 0$. Let us fix $t < n$. For $x \in X$ and for $r > 0$, the Hölder inequality shows that

$$r^{-(n-\frac{n}{p})} \int_{B(x, r)} v \, d\mathcal{L}^n \leq \left(\int_{B(x, r)} v^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \quad (132)$$

so

$$r^{-(n+\frac{t}{p}-\frac{n}{p})} \int_{B(x, r)} v \, d\mathcal{L}^n \leq \left(r^{-t} \int_{B(x, r)} v^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}. \quad (133)$$

We apply Lemma 4.1 and we see that for \mathcal{H}^t -a.e. $x \in X$,

$$\lim_{r \rightarrow 0} r^{-(n+\frac{t}{p}-\frac{n}{p})} \int_{B(x, r)} v \, d\mathcal{L}^n = 0. \quad (134)$$

The scalar t such that $s = n + \frac{t}{p} - \frac{n}{p}$ is $t = n - p(n - s) < n$. \square

4.2 Dimension of the singular set

Theorem 4.3. *Let $u \in \text{SBV}_{\text{loc}}(X)$ be a minimizer. We define*

$$\Sigma = \{x \in K \mid K \text{ is not regular at } x\}. \quad (135)$$

For $p > 1$ such that $|\nabla u|^2 \in L^p_{\text{loc}}(X)$, we have

$$\dim_{\mathcal{H}}(\Sigma) \leq \max\{n - p, n - 8\} < n - 1. \quad (136)$$

Proof. According to Corollary 4.2, we have for \mathcal{H}^{n-p} -a.e. $x \in X$,

$$\lim_{r \rightarrow 0} \omega_2(x, r) = 0 \quad (137)$$

and according to [3, Theorem 8.2], the set

$$\left\{ x \in X \cap \Sigma \mid \lim_{r \rightarrow 0} \omega_2(x, r) = 0 \right\} \quad (138)$$

has a Hausdorff dimension $\leq n - 8$. \square

Remark 4.4. It is conjectured in [3] that, in the plane, $\Sigma = \emptyset$.

Appendices

A Generalities about BV functions

We recall a few definitions and results from the theory of BV functions ([1]). We work in an open set X of the Euclidean space \mathbf{R}^n ($n > 1$). When a point $x \in X$ is given, we abbreviate the open ball $B(x, r)$ as B_r .

Let $u \in L^1_{\text{loc}}(X)$. The *upper and lower approximate limit* of u at a point $x \in X$ are defined by

$$\bar{u}(x) := \inf \left\{ t \in \mathbf{R} \mid \lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap \{u > t\}} (u - t) \, d\mathcal{L}^n = 0 \right\}, \quad (139)$$

$$\underline{u}(x) := \sup \left\{ t \in \mathbf{R} \mid \lim_{r \rightarrow 0} r^{-n} \int_{B_r \cap \{u < t\}} (t - u) \, d\mathcal{L}^n = 0 \right\}. \quad (140)$$

The functions $\bar{u}, \underline{u}: X \rightarrow \overline{\mathbf{R}}$ are Borel and satisfies $\underline{u} \leq \bar{u}$. We have two examples in mind. We say that x is a *Lebesgue point* if there exists $t \in \mathbf{R}$ such that

$$\lim_{r \rightarrow 0} \int_{B_r} |u - t| \, d\mathcal{L}^n = 0. \quad (141)$$

We then have $\underline{u}(x) = \bar{u}(x) = t$ and we denote t by $\tilde{u}(x)$. The set of non-Lebesgue points $x \in X$ is called the *singular set* S_u . Both the set S_u and the function $X \setminus S_u \rightarrow \mathbf{R}, x \mapsto \tilde{u}(x)$ are Borel ([1, Proposition 3.64]). In

parenthesis, the Lebesgue differentiation theorem states that for \mathcal{L}^n -a.e. $x \in X$, we have $x \in X \setminus S_u$ and $u(x) = \tilde{u}(x)$. We say that x is a *jump point* if there exist two real numbers $s < t$ and a (unique) vector $\nu_u(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} \int_{B_r \cap H^+} |u(y) - s| d\mathcal{L}^n(y) = 0 \quad (142a)$$

$$\lim_{r \rightarrow 0} \int_{B_r \cap H^-} |u(y) - t| d\mathcal{L}^n(y) = 0, \quad (142b)$$

where

$$H^+ = \{y \in \mathbf{R}^n \mid (y - x) \cdot \nu_u(x) > 0\} \quad (143a)$$

$$H^- = \{y \in \mathbf{R}^n \mid (y - x) \cdot \nu_u(x) < 0\}. \quad (143b)$$

We then have $\bar{u}(x) = t$ and $\underline{u}(x) = s$. The set of jump points $x \in X$ is called the *jump set* J_u . Both the set J_u and the function $J_u \rightarrow \mathbf{S}^{n-1}, x \mapsto \nu_u(x)$ are Borel ([1, Proposition 3.69]).

Here we summarize [1, Proposition 3.76, 3.78]). Let $u \in BV(X)$. The singular set S_u is \mathcal{H}^{n-1} rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$. According to the Besicovitch derivation theorem, we can write

$$Du = D^a u + D^s u \quad (144)$$

where $D^a u$ is the *absolutely continuous* part of Du with respect to \mathcal{L}^n and $D^s u$ is the *singular part* of Du with respect to \mathcal{L}^n . As a consequence, there exists a unique vector-valued map $\nabla u \in L^1(X; \mathbf{R}^n)$, the *approximate gradient*, such that $D^a u = \nabla u \mathcal{L}^n$. The measures \mathcal{L}^n and $\|D^s u\|$ are mutually singular which means that there exists a Borel set $S \subset X$ such that

$$\mathcal{L}^n(S) = \|D^s u\|(\mathbf{R}^n \setminus S) = 0. \quad (145)$$

A candidate for S could be S_u but S may not be a $(n-1)$ dimensional set. We can write

$$D^s u = Du^s \llcorner S_u + D^s u \llcorner (X \setminus S_u) \quad (146)$$

where $D^s u \llcorner S_u$ is the *jump part* and $D^s u \llcorner (X \setminus S_u)$ is the *Cantor part*. The jump part has an explicit formula,

$$D^s u \llcorner S_u = (\bar{u} - \underline{u}) \nu_u \mathcal{H}^{n-1} \llcorner J_u, \quad (147)$$

whereas the Cantor part vanishes on \mathcal{H}^{n-1} σ -finite sets $B \subset X$ (and not only $B = S_u$). Remark that Du always vanishes on \mathcal{H}^{n-1} negligible sets. Finally, we define $SBV(X)$ as the subspace of functions $u \in BV(X)$ whose Cantor part is zero, that is

$$\|D^s u\|(X \setminus S_u) = 0. \quad (148)$$

For $u \in \text{BV}(X)$ to be SBV, it suffices that there exists a \mathcal{H}^{n-1} σ -finite set $K \subset X$ such that $\|D^s u\|(X \setminus K) = 0$. Since $\|D^s u\|$ and \mathcal{L}^n are mutually singular, this also amounts to say the measure $\|Du\| \llcorner (X \setminus K)$ is dominated by \mathcal{L}^n . A natural way to build SBV(X) functions is to have a pair (u, K) where $K \subset X$ is relatively closed, \mathcal{H}^{n-1} locally finite and $u \in W^{1,1}(X \setminus K)$.

A set of finite perimeter in X is a Borel set $E \subset X$ such that $\mathbf{1}_E \in \text{BV}(X)$. The singular set of $\mathbf{1}_E$ is called *essential boundary* or *measure-theoretic boundary* and denoted by $\partial_M E$. The jump set of E is denoted by $\partial^* E$. One can see that $\overline{\mathbf{1}_E}, \underline{\mathbf{1}_E} \in \{0, 1\}$ everywhere on X . Thus, if x is a Lebesgue point of $\mathbf{1}_E$, we have either $\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \setminus E) = 0$ or $\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap E) = 0$. The essential boundary $\partial_M E$ can be reformulated as the set of points $x \in X$ such that

$$\limsup_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \setminus E) > 0 \quad (149a)$$

$$\limsup_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap E) > 0. \quad (149b)$$

Similarly, the jump set $\partial^* E$ can be reformulated as the set of points $x \in X$ for which there exists a (unique) vector $n_E(x) \in \mathbf{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} r^{-n} \mathcal{L}^n(B_r \cap (E \Delta H^+)) = 0 \quad (150)$$

where

$$H^+ = \{y \in \mathbf{R}^n \mid (y - x) \cdot n_E(x) > 0\}. \quad (151)$$

The vector $n_E(x)$ is called the *measure-theoretic inner normal* to E at x . We have the inclusions $\partial^* E \subset \partial_M E \subset \partial E$. The measure $D\mathbf{1}_E$ has no absolutely continuous part, neither Cantor part; it is given by the formula

$$D\mathbf{1}_E = n_E \mathcal{H}^{n-1} \llcorner \partial^* E. \quad (152)$$

B A Robin problem

B.1 Statement

We work in the Euclidean space \mathbf{R}^n ($n > 1$) and we denote by (e_1, \dots, e_n) its canonical basis. The space $\{h \in \mathbf{R}^n \mid h \cdot e_n = 0\}$ is denoted by \mathbf{R}^{n-1} . Every $x \in \mathbf{R}^n$ can be written

$$x = x' + x_n e_n \quad (153)$$

where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. We fix an open ball $B := B(0, R) \subset \mathbf{R}^n$. We fix a graph Γ passing through the origin: there exists a continuous function $f: \mathbf{R}^{n-1} \cap B \rightarrow R$ such that $f(0) = 0$ and

$$\Gamma = \{x \in B \mid x_n = f(x')\}. \quad (154)$$

We assume that f is C^1 in $\mathbf{R}^{n-1} \cap B$ so at each point $x \in \Gamma$, there exists a normal vector to Γ going upward

$$\nu(x) := \frac{-\nabla f(x') + e_n}{\sqrt{1 + |\nabla f(x')|^2}}. \quad (155)$$

We assume that there exists $L, L' > 0$ and $\alpha > 0$ such that for all $h \in \mathbf{R}^{n-1} \cap B$,

$$|\nabla f(x)| \leq L \quad (156)$$

and for all $h_1, h_2 \in \mathbf{R}^{n-1} \cap B$,

$$|\nabla f(h_1) - \nabla f(h_2)| \leq L' \left(\frac{|h_1 - h_2|}{R} \right)^\alpha. \quad (157)$$

In practice, we may want to assume L small (depending only on n) so that it is easier to approximate $\nu(x)$ by e_n . We underline that for all $x \in B$,

$$d(x, \Gamma) \leq |x_n - f(x')| \leq (2 + L)d(x, \Gamma). \quad (158)$$

Indeed, we observe first that $|x_n - f(x')| = |x - (x' + f(x')e_n)| \geq d(x, \Gamma)$. Next, we see that for any $y \in \Gamma$,

$$|x - (x' + f(x')e_n)| \leq |x - y| + |(x' + f(x')e_n) - (y' + f(y')e_n)| \quad (159)$$

$$\leq (2 + L)|x - y|. \quad (160)$$

Given a subset $S \subset B$, we define

$$S^+ = \{x \in S \mid x_n > f(x')\} \quad (161)$$

$$S^0 = \{x \in S \mid x_n = f(x')\}. \quad (162)$$

For $u \in \text{BV}(B^+)$, we denote by u^* the trace of u in $L^1(\partial B^+)$. It is characterized by the property that for \mathcal{H}^{n-1} -a.e. $x_0 \in \partial B^+$,

$$\lim_{r \rightarrow 0} r^{-n} \int_{B^+ \cap B_r(x)} |u - u^*(x_0)| d\mathcal{L}^n = 0. \quad (163)$$

We denote by $W_0^{1,2}(B^+ \cup B^0)$ the space of functions $u \in W^{1,2}(B^+)$ such that $u^* = 0$ on $\partial B^+ \setminus B^0$. Our object of study is the functions $u \in W^{1,2}(B^+) \cap L^\infty(B^+)$ such that u is a weak solution of

$$\begin{cases} \Delta u & = 0 & \text{in } B^+ \\ \partial_\nu u - u^* & = 0 & \text{in } B^0, \end{cases} \quad (164)$$

that is, for all $v \in W_0^{1,2}(B^+ \cup B^0)$,

$$\int_{B^+} \langle \nabla u, \nabla v \rangle d\mathcal{L}^n + \int_{B^0} u^* v^* d\mathcal{H}^{n-1} = 0. \quad (165)$$

According to Weyl's lemma, u coincide almost-everywhere in B^+ with an harmonic functions. We replace u by this harmonic representative so that u is pointwise defined and smooth in B^+ . Our goal is to prove that there exists $C \geq 1$ (depending on n, α, L, L') such that for all $x \in (\frac{1}{4}B)^+$,

$$|\nabla u(x)| \leq C \left(\int_B |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} + C|u|_\infty. \quad (166)$$

This will be a consequence of Proposition B.4 and Proposition B.5.

We rely on [8, Theorem 1.2] which establishes that viscosity solutions of such problem are pointwise $C^{1,\alpha}$ up to the boundary. The viscosity approach is based on the maximum principle but it is easy to prove a maximum principle for the weak solutions of our problem. We can thus follow the ideas of [8].

We state below our maximum principle.

Lemma B.1 (Maximum principle). *Let $u \in W^{1,2}(B^+)$ be a weak solution of*

$$\begin{cases} \Delta u & \geq 0 & \text{in } B^+ \\ \partial_\nu u - u^* & \geq 0 & \text{in } B^0, \end{cases} \quad (167)$$

that is, for all non-negative function $v \in W_0^{1,2}(B^+ \cup B^0)$,

$$\int_{B^+} \langle \nabla u, \nabla v \rangle d\mathcal{L}^n + \int_{B^0} u^* v^* d\mathcal{H}^{n-1} \leq 0. \quad (168)$$

If $u^* \leq 0$ on $\partial B^+ \setminus B^0$, then $u \leq 0$ on B^+ .

Proof. Let $u \in W^{1,2}(B)$ and let $u_+ = p(u)$ where $p: \mathbf{R} \rightarrow [0, \infty[$ is the orthogonal projection onto $[0, \infty[$. According to the chain rule, $u_+ \in W^{1,2}(B^+)$ and for \mathcal{L}^n -a.e. $x \in B$,

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) = 0. \end{cases} \quad (169)$$

One can also see that $(u_+)^* = p(u^*)$ using the characterization (163) and the fact that p is Lipschitz. Now, we assume that u is a weak solution of (167) and that $u^* \leq 0$ on $\partial B^+ \setminus B^0$. We have $u_+ \in W_0^{1,2}(B^+)$ so (168) gives

$$\int_{B^+} \langle \nabla u, \nabla u_+ \rangle d\mathcal{L}^n + \int_{B^0} u^* u_+^* d\mathcal{H}^{n-1} \leq 0. \quad (170)$$

As $u^* u_+^* = (u_+^*)^2$ and $\langle \nabla u, \nabla u_+ \rangle = |\nabla u_+|^2$, we conclude that $u_+ = 0$ on B^+ . \square

B.2 Hölder continuity up to the boundary

We aim to prove the following result. We recall that the letter B stands for $B(0, R)$.

Proposition B.2 (Hölder continuity). *Let $u \in W^{1,2}(B^+) \cap L^\infty(B^+)$ be a weak solution of (164). There exists constants $C \geq 1$ (depending on n, α, L, L') and $\beta > 0$ (depending on n) such that for all $x, y \in B^+$,*

$$|u(x) - u(y)| \leq C|u|_\infty \left(\frac{|x - y|}{r} \right)^\beta. \quad (171)$$

here $r = \min \{ d(x, \mathbf{R}^n \setminus B), d(y, \mathbf{R}^n \setminus B) \}$.

We need a weak harnack inequality at the boundary.

Lemma B.3 (Weak Harnack Inequality). *Assume that L is small enough (depending on n). Let $u \in W^{1,2}(B^+)$ be a non-negative weak solution of*

$$\begin{cases} \Delta u & = 0 & \text{in } B^+ \\ \partial_\nu u - u^* & = 0 & \text{in } B^0. \end{cases} \quad (172)$$

Then there exists a constant $C \geq 1$ (depending on n) such that

$$u(\frac{1}{2}Re_n) \leq C \inf \{ u(z) \mid z \in \overline{B}(0, \frac{1}{4}R)^+ \}. \quad (173)$$

Proof. For all $x' \in \mathbf{R}^{n-1}$ such that $|x'_i| \leq \frac{1}{2}R$, we have

$$|f(x')| \leq |f(x') - f(0)| \quad (174)$$

$$\leq \frac{1}{2}LR \quad (175)$$

We define $m = -\frac{1}{2}LR$ so that for all $x' \in \mathbf{R}^{n-1}$ such that $|x'| \leq \frac{1}{2}R$,

$$m \leq f(x') \leq m + LR. \quad (176)$$

We fix some $\delta \in]0, \frac{1}{2}[$ (it will be precised later and depends only on n). We assume that $L \leq \frac{1}{2}\delta \leq 1$. We introduce the cubes

$$Q := \{ x \in \mathbf{R}^n \mid |x'| \leq \frac{1}{2}R, m \leq x_n \leq m + \delta R \} \quad (177)$$

$$Q_0 := \{ x \in \mathbf{R}^n \mid |x'| \leq \frac{1}{2}R, m + \delta R \leq x_n \leq \frac{1}{2}R \}. \quad (178)$$

and the quantity

$$A := \inf \{ u(x) \mid |x'| \leq \frac{1}{2}R, x_n = \delta R \}. \quad (179)$$

We will work in Q_0 to show that $\sup_{Q_0} u \leq C \inf_{Q_0} u$ and then we will work in Q to show that

$$A \leq C \inf \{ u(x) \mid |x'| \leq \frac{1}{4}R, f(x') < x_n \leq \delta R \}. \quad (180)$$

Let us focus of Q_0 . It is clear that $Q_0 \subset B^+$. In addition, we are going to see that for $x \in Q_0$,

$$d(x, \partial B^+) \geq \min \left\{ \frac{1}{6} \delta, \left(1 - \frac{\sqrt{2}}{2}\right) \right\} R. \quad (181)$$

For $x \in Q_0$, we have $|x'| \leq \frac{1}{2}R$ and $|x_n| \leq \frac{1}{2}R$ so $|x| \leq \frac{\sqrt{2}}{2}R$ and then $d(x, \partial B) \geq \left(1 - \frac{\sqrt{2}}{2}\right)R$. We also have by (178) and (176),

$$x_n \geq m + \delta R \quad (182)$$

$$\geq f(x') + \frac{\delta}{2}R \quad (183)$$

so the right hand side of (158) gives

$$d(x, \Gamma) \geq \frac{1}{2(2+L)} \delta R \geq \frac{1}{6} \delta R. \quad (184)$$

The inequality (181) allows us to apply the Harnack inequality for harmonic functions with a controlled constant (remember that δ only depends on n). There exists $C \geq 1$ (depending on n) such that $\sup_{Q_0} u \leq C \inf_{Q_0} u$. Now, we focus on Q . To simplify the notations, we change the origin of the coordinate system so that

$$Q = \{x \in \mathbf{R}^n \mid |x'| \leq \frac{1}{2}R, 0 \leq x_n \leq \delta R\}. \quad (185)$$

We are going to build a paraboloid p such that

- (i) $p \geq 0$ on Q ;
- (ii) $p \geq 1$ on $\{x \in \mathbf{R}^n \mid |x'| = \frac{1}{2}R, 0 \leq x_n \leq \delta R\}$;
- (iii) $p \leq \frac{3}{4}$ on $K := \{x \in \mathbf{R}^n \mid |x'| \leq \frac{1}{4}R, 0 \leq x_n \leq \delta R\}$;
- (iv) $\Delta p \leq -\frac{1}{R^2}$ on Q ;
- (v) $\partial_\nu p \leq -\frac{1}{R}$ on Q^0 .

Then we will apply the maximum principle to $w = u + Ap - A$. Indeed, we check that

$$\begin{cases} \Delta w & \leq 0 & \text{in } Q^+ \\ \partial_\nu w - w & \leq 0 & \text{in } Q^0. \end{cases} \quad (186)$$

As $w \geq 0$ on $\partial Q^+ \setminus Q^0$, the maximum principles implies that $w \geq 0$ on Q^+ and in particular, $u \leq \frac{1}{4}A$ on K^+ . The paraboloid is

$$p(x) = \frac{1}{2} - \frac{1}{4} \left(\frac{x_n}{\delta R}\right) - \frac{1}{4} \left(\frac{x_n}{\delta R}\right)^2 + \left(\frac{2|x'|}{R}\right)^2. \quad (187)$$

Let us check these properties. It is clear that for $x \in Q$,

$$\left(\frac{2|x'|}{R}\right)^2 \leq p \leq \frac{1}{2} + \left(\frac{2|x'|}{R}\right)^2. \quad (188)$$

The first three items follow. We compute

$$\Delta p = -\frac{1}{2(\delta R)^2} + \frac{8(n-1)}{R^2} \quad (189)$$

We take δ small enough (depending on n) so that $\Delta p \leq -\frac{1}{R^2}$ on Q . For $x \in \mathbf{R}^n$, we have

$$\nabla p(x) = -\frac{e_n}{4\delta R} - \frac{x_n e_n}{2(\delta R)^2} + \frac{8x'}{R^2}. \quad (190)$$

so for $x \in Q^0$,

$$\partial_\nu p(x) = \frac{1}{\sqrt{1 + |\nabla f(x')|^2}} \left(-\frac{1}{4\delta R} - \frac{x_n}{2(\delta R)^2} - \frac{8(x' \cdot \nabla f(x'))}{R^2} \right). \quad (191)$$

In addition $x_n \geq 0$ and $|x'| \leq \frac{R}{2}$ so

$$-\frac{1}{4\delta R} - \frac{x_n}{2(\delta R)^2} - \frac{8(x' \cdot \nabla f(x'))}{R^2} \leq -\frac{1}{4\delta R} + \frac{4}{R} |\nabla f(x')| \quad (192)$$

$$\leq -\frac{1}{4\delta R} + \frac{4}{R}. \quad (193)$$

We take δ small enough (depending on n) so that $-\frac{1}{4\delta R} + \frac{4}{R} \leq -\frac{1}{R}$ and we finally conclude that $\partial_\nu p(x) \leq -\frac{1}{R}$. \square

We are ready to prove Proposition B.2.

Proof of Proposition B.2. We introduce

$$\Delta := \{ (x, r) \mid x \in B^+ \cup B^0, \quad B(x, r) \subset B \} \quad (194)$$

and for $(x, r) \in \Delta$,

$$\text{osc}(x, r) := \sup \{ |u(z) - u(y)| \mid y, z \in B(x, r)^+ \}. \quad (195)$$

We are going to show that there exists $0 < \tau < 1$ (depending on n, α, L') and $0 < \lambda < 1$ (depending on n) such that for all $(x, r) \in \Delta$ with $r \leq \tau R$,

$$\text{osc}(x, \frac{1}{16}r) \leq \lambda \text{osc}(x, r) \quad (196)$$

We are going to distinguish three cases: the case $x \in B^0$, the case $B(x, \frac{1}{8}r) \cap \Gamma \neq \emptyset$ and the case $B(x, \frac{1}{8}r) \cap \Gamma = \emptyset$.

Let $(x, r) \in \Delta$ with $x \in \Gamma$. We restrict the system of equation to the ball $B(x, r)$ and we make a translation-rotation to assume that $x = 0$ and $\nabla f(x') = 0$. For all $x' \in \mathbf{R}^{n-1}$ such that $|x'| \leq r$, we have

$$|\nabla f(x')| \leq |\nabla f(x') - \nabla f(0)| \quad (197)$$

$$\leq L' \left(\frac{r}{R} \right)^\alpha. \quad (198)$$

There exists $0 < \tau < 1$ (depending on n, α, L') such that if $r \leq \tau R$, then the Lipschitz of f is small enough in $\mathbf{R}^{n-1} \cap \overline{B}(0, r)$ to apply the previous Lemma. From now on, we assume that $r \leq \tau R$. Let

$$M = \sup \{ u(z) \mid z \in B(0, r)^+ \} \quad (199)$$

$$m = \inf \{ u(z) \mid z \in B(0, r)^+ \} \quad (200)$$

and

$$M' = \sup \{ u(z) \mid z \in B(0, \frac{1}{4}r)^+ \} \quad (201)$$

$$m' = \inf \{ u(z) \mid z \in B(0, \frac{1}{4}r)^+ \}. \quad (202)$$

We apply Lemma B.3 to $M - u$ and $u - m$ in $B(0, r)^+$ and we obtain that there exists $C \geq 1$ (depending on n) such that

$$M - u(q) \leq C(M - M') \quad (203)$$

$$u(q) - m \leq C(m' - m) \quad (204)$$

where $q := \frac{1}{2}re_n$. It follows that $M - m \leq C(M - M' + m' - m)$ and then

$$M' - m' \leq \lambda(M - m) \quad (205)$$

where $\lambda := C^{-1}(C - 1)$. This proves that $\text{osc}(x, \frac{1}{4}r) \leq \lambda \text{osc}(x, r)$.

Let $(x, r) \in \Delta$ be such that $r \leq \tau R$ and such that there exists $x^* \in B(x, \frac{1}{8}r) \cap \Gamma$. We observe that

$$B(x, \frac{1}{16}r) \subset B(x^*, (\frac{1}{8} + \frac{1}{16})r). \quad (206)$$

Then by the previous step

$$\text{osc}(x^*, (\frac{1}{8} + \frac{1}{16})r) \leq \lambda \text{osc}(x^*, 4(\frac{1}{8} + \frac{1}{16})r) \quad (207)$$

$$\leq \lambda \text{osc}(x^*, (\frac{1}{2} + \frac{1}{4})r) \quad (208)$$

and

$$B(x^*, (\frac{1}{2} + \frac{1}{4})r) \subset B(x, (\frac{1}{2} + \frac{1}{4} + \frac{1}{8})r) \quad (209)$$

$$\subset B(x, r). \quad (210)$$

This proves that $\text{osc}(x, \frac{1}{16}r) \leq \lambda \text{osc}(x, r)$.

Let $(x, r) \in \Delta$ be such that $r \leq \tau R$ and $B(x, \frac{1}{8}r) \cap \Gamma = \emptyset$. Then we have

$$\text{osc}(x, \frac{1}{16}r) \leq \lambda \text{osc}(x, \frac{1}{8}r) \quad (211)$$

by reasoning as in the case $x \in \Gamma$. Here we replace the previous Lemma by the usual Harnack inequality for harmonic functions.

We have proved (196) in all cases. We fix $x \in B^+$ and we define the radii $r_0 := d(x, \mathbf{R}^n \setminus B)$ and $r_1 := \tau r_0 \leq \tau R$. For all $0 < r \leq r_1$, we have $\text{osc}(x, \frac{1}{16}r) \leq \lambda \text{osc}(x, r)$ and it is easy to deduce that for all $0 < r \leq r_1$

$$\text{osc}(x, r) \leq \left(\frac{r}{r_1}\right)^\beta \text{osc}(x, r_1) \quad (212)$$

where $\beta = \frac{\ln(\lambda)}{\ln(16)}$. This implies that for all $y \in B(x, r_1)^+$,

$$|u(y) - u(x)| \leq \left(\frac{|y-x|}{r_1}\right)^\beta \text{osc}(x, r_1) \quad (213)$$

$$\leq 2|u|_\infty \left(\frac{|y-x|}{r_1}\right)^\beta. \quad (214)$$

In fact, this inequality is also true for $y \in B^+$ such that $|y-x| \geq r_1$ because we always have $|u(y) - u(x)| \leq 2|u|_\infty$. We conclude that for all $y \in B^+$,

$$|u(y) - u(x)| \leq 2|u|_\infty \tau^{-\beta} \left(\frac{|y-x|}{r_0}\right)^\beta. \quad (215)$$

□

B.3 Gradient estimates

According to [8, Theorem 1.2], the viscosity solutions are pointwise $C^{1,\alpha}$ up to the boundary. Although we use a weak formulation, the proof also applies in our case. The proof relies on the maximum principle (as Lemma B.1), the Hölder continuity (as Proposition B.2) and regularity results for solutions of the Neumann problem in a spherical cap. We extract an estimate that is useful for our paper. We recall that the letter B stands for $B(0, R)$.

Proposition B.4 (Partial Schauder Estimate). *Let $u \in W^{1,2}(B^+) \cap L^\infty(B^+)$ be a weak solution of (164). Then there exists $C \geq 1$ (depending on n, α, L, L') such that for all $x \in (\frac{1}{4}B)^+$,*

$$|\nabla u(x)| \leq \frac{C}{R} \text{osc}(u) + C|u|_\infty. \quad (216)$$

where $\text{osc}(u) = \sup \{ |u(x) - u(y)| \mid x, y \in (\frac{1}{2}B)^+ \}$.

Proof. The letter C is a constant ≥ 1 that depends on n, α, L, L' . We fix any $x_0 \in B^+$. The function $v = u - u(x_0)$ is a weak solution of the Neumann problem

$$\begin{cases} \Delta v = 0 & \text{in } (\frac{1}{2}B)^+ \\ \partial_\nu v = u^* & \text{in } (\frac{1}{2}B)^0. \end{cases} \quad (217)$$

We have restricted the equation to $\frac{1}{2}B$ so that the function u^* is C^β for some $\beta > 0$. More precisely, $|u^*|_\infty \leq |u|_\infty$ and according to Lemma B.2, there exists $\beta > 0$ (depending on n) such that for all $x, y \in (\frac{1}{2}B)^0$,

$$|u^*(x) - u^*(y)| \leq C|u|_\infty \left(\frac{|x - y|}{R} \right)^\beta. \quad (218)$$

Then we apply the scaled version of [8, Theorem 1.2] and we obtain that for $x \in (\frac{1}{4}B)^+$,

$$|\nabla v(x)| \leq \frac{C}{R} (|v|_\infty + R|u|_\infty) \quad (219)$$

□

This last result helps to estimate the oscillations of u .

Proposition B.5 (Local Boundedness). *Let $u \in W^{1,2}(B^+) \cap L^\infty(B^+)$ be a weak solution of (164). Then there exists $C \geq 1$ (depending on n, α, L, L') such that*

$$\text{osc}(u) \leq CR \left(\int_B |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} + CR|u|_\infty. \quad (220)$$

where $\text{osc}(u) = \sup \{ |u(x) - u(y)| \mid x, y \in (\frac{1}{2}B)^+ \}$.

Proof. The letter C is a constant ≥ 1 that depends on n, α, L, L' . Let m be the average value of u on B^+ . The function $v := u - m$ is a weak solution of the Neumann problem

$$\begin{cases} \Delta v = 0 & \text{in } B^+ \\ \partial_\nu v = u^* & \text{in } B^0. \end{cases} \quad (221)$$

We apply a local boundedness estimate for weak solutions of Neumann problems ([6, Theorem 1.6 and Remark 1.12]): for all $x \in (\frac{1}{2}B)^+$

$$|v(x)| \leq C \left(\int_B |v|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} + CR|u|_\infty. \quad (222)$$

The triangular inequality show that

$$\text{osc}(u) \leq 2 \sup \{ |v(x)| \mid x \in (\frac{1}{2}B)^+ \} \quad (223)$$

and the Poincaré inequality gives

$$\left(\int_B |v|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} \leq CR \left(\int_B |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}}. \quad (224)$$

□

C Extracts from [4]

We extract some important results of [4]. We work in an open set X of the Euclidean space \mathbf{R}^n ($n > 1$) and we fix a triple of parameters $\mathcal{P} = (r_0, a, M)$ composed of $r_0 > 0$, $a \geq 0$ and $M \geq 1$. We start by summarizing Definitions 2.1 (*admissible pairs*), 7.2 (*competitors*), 7.21 (*local quasiminimizers*) and 8.24 (*coral pairs*).

Definition C.1. The set of *admissible pairs* \mathcal{A} is the set of all pairs (u, K) where $K \subset X$ is relatively closed in X and $u \in W_{\text{loc}}^{1,2}(X \setminus K)$. Let (u, K) be an admissible pair and let B be an open ball such that $\overline{B} \subset X$. A *competitor* of (u, K) in B is a pair $(v, L) \in \mathcal{A}$ such that $K \setminus \overline{B} = L \setminus \overline{B}$ and $u = v$ \mathcal{L}^n a.e. on $X \setminus (K \cup B)$. In this case, we set

$$E(u) = \int_B |\nabla u|^2 d\mathcal{L}^n, \quad E(v) = \int_B |\nabla v|^2 \quad (225)$$

and

$$\Delta E = \max \{ (E(v) - E(u)), M(E(v) - E(u)) \}. \quad (226)$$

We say that (u, K) is a *local \mathcal{P} -quasiminimizer in X* if for all open balls B of radius $0 < r < r_0$ such that $\overline{B} \subset X$, for all competitors (v, L) of (u, K) in B , we have

$$\mathcal{H}^{n-1}(K \setminus L) \leq M\mathcal{H}^{n-1}(L \setminus K) + \Delta E + ar^{n-1}. \quad (227)$$

In addition, we say that (u, K) is *coral* if $K = \text{spt}(\mathcal{H}^{n-1} \llcorner K)$ in X . This means that for all $x \in K$ and all $r > 0$, $\mathcal{H}^{n-1}(K \cap B(x, r)) > 0$.

Remark C.2. If (u, K) is a quasiminimizer, we can see easily that K is \mathcal{H}^{n-1} locally finite. For all open ball B of radius $r < r_0$ such that $\overline{B} \subset X$, we consider the competitor

$$v = \begin{cases} u & \text{in } X \setminus \overline{B} \\ 0 & \text{in } \overline{B} \end{cases} \quad (228)$$

and $L = (K \setminus \overline{B}) \cup \partial B$. In particular, we have $K \setminus L \subset B$, $L \setminus K \subset \partial B$ and $\Delta E \leq -M \int_B |\nabla u|^2$. This proves that $\mathcal{H}^{n-1}(K \cap B) < \infty$ (and even better).

For all competitors (v, L) of (u, K) , we have either $\mathcal{H}^{n-1}(L \setminus K) = \infty$ and thus (227) says nothing or $\mathcal{H}^{n-1}(K \setminus L) < \infty$ and thus L is \mathcal{H}^{n-1} locally finite. In conclusion, we can always assume that L is \mathcal{H}^{n-1} locally finite.

We are mainly concerned about Ahlfors-regularity (Definition 18.9) and uniform rectifiability (Section 73).

Definition C.3 (Ahlfors-regularity). A closed set $E \subset \mathbf{R}^n$ is Ahlfors-regular of dimension $n - 1$ if there exists a constant $C \geq 1$ such that for all $x \in E$ and for all $0 < r < \text{diam}(E)$

$$C^{-1}r^{n-1}\mathcal{H}^{n-1}(E \cap B(x, r)) \leq Cr^{n-1}. \quad (229)$$

We don't give definitions of uniform rectifiability since there are too many and rich. These definitions are all equivalent for closed, Ahlfors-regular sets. The reader can find a survey of uniform rectifiability in [4, Section 73] and also on Guy David's webpage (Notes-Parkcity.dvi).

Next, we summarize Definition 18.14 (*TRLQ class*), Section 72 and Section 74. It says that quasiminimizers are locally Ahlfors-regular and locally contained in a uniformly rectifiable set. In fact, we have already seen the first item in Remark C.2.

Theorem C.4. *Let $\mathcal{P} = (r_0, a, M)$ be a triple of parameters composed of $r_0 > 0$, $a \geq 0$ and $M \geq 1$. Assume that a is small enough (depending on n, M). Let (u, K) be a coral local \mathcal{P} -quasiminimizer in X .*

1. *For all $x \in X$, for all $0 < r < r_0$ such that $\overline{B}(x, r) \subset X$,*

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \leq Cr^{n-1}. \quad (230)$$

where $C \geq 1$ depends on n, M .

2. *For all $x \in K$, for all $0 < r < r_0$ such that $\overline{B}(x, r) \subset X$,*

$$\mathcal{H}^{n-1}(K \cap B(x, r)) \geq C^{-1}r^{n-1}. \quad (231)$$

where $C \geq 1$ depends on n, M .

3. *For all $x \in K$ and $0 < r < r_0$ such that $\overline{B}(x, 2r) \subset X$, there is a closed, Ahlfors-regular, uniformly rectifiable set E of dimension $(n-1)$ such that $K \cap B(x, r) \subset E$. The constants for the Ahlfors-regularity and uniform rectifiability depends on n, M and a .*

Remark C.5. One can observe that (227) implies

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \leq M\mathcal{H}^{n-1}(L \cap \overline{B}) + \Delta E + ar^{n-1}. \quad (232)$$

This is equivalent when $M = 1$ but strictly weaker when $M > 1$. We claim that (C.4) still holds with (232) in place of (227). The first item is easy (see Remark C.2). The second item works as usual. The most critical point is probably the third item. In Section 74, David builds a suitable competitor (w, G) of (u, K) in a ball B . The set G is of the form $G = (K \setminus B) \cup Z$ where Z a special subset of ∂B containing $K \cap \partial B$. The quasi-minimality condition (C.4) is used only once at line (22) of Section 74. Then David uses the inequalities

$$\mathcal{H}^{n-1}(K \setminus G) \geq \mathcal{H}^{n-1}(K \cap B) \quad (233)$$

$$\mathcal{H}^{n-1}(G \setminus K) \leq \mathcal{H}^{n-1}(Z) \quad (234)$$

but we also have anyway

$$\mathcal{H}^{n-1}(K \cap \overline{B}) \geq \mathcal{H}^{n-1}(K \cap B) \quad (235)$$

$$\mathcal{H}^{n-1}(G \cap \overline{B}) \leq \mathcal{H}^{n-1}(Z). \quad (236)$$

Acknowledgments

This work was co-funded by the European Regional Development Fund and the Republic of Cyprus through the Research and Innovation Foundation (Project: EXCELLENCE/1216/0025).

References

- [1] L. Ambrosio; N. Fusco; D. Pallara Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp. ISBN: 0-19-850245-1
- [2] D. Bucur and S. Luckhaus Monotonicity formula and regularity for general free discontinuity problems. Arch. Ration. Mech. Anal. 211 (2014), no. 2, 489-511.
- [3] L. A. Caffarelli and D. Kriventsov A free boundary problem related to thermal insulation. Comm. Partial Differential Equations 41 (2016), no. 7, 1149-1182.
- [4] G. David Singular sets of minimizers for the Mumford-Shah functional. Progress in Mathematics, 233. Birkhauser Verlag, Basel, 2005. xiv+581 pp. ISBN: 978-3-7643-7182-1; 3-7643-7182-X
- [5] G. De Philippis and A. Figalli Higher integrability for minimizers of the Mumford-Shah functional. Arch. Ration. Mech. Anal. 213 (2014), no. 2, 491-502.
- [6] S. Kim Note on local boundedness for weak solutions of Neumann problem for second-order elliptic equations. J. Korean Soc. Ind. Appl. Math. 19 (2015), no. 2, 189-195.
- [7] D. Kriventsov A free boundary problem related to thermal insulation: flat implies smooth. Calc. Var. Partial Differential Equations 58 (2019), no. 2, Paper No. 78, 83 pp.
- [8] D. Li; K. Zhang Regularity for fully nonlinear elliptic equations with oblique boundary conditions. (English summary) Arch. Ration. Mech. Anal. 228 (2018), no. 3, 923-967.
- [9] P. Mattila Geometry of sets and measures in Euclidean spaces. (English summary) Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. xii+343 pp.

- [10] S. Rigot Big pieces of $C^{1,\alpha}$ -graphs for minimizers of the Mumford-Shah functional. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000), no. 2, 329-349.

Camille Labourie
University of Cyprus
Department of Mathematics & Statistics
P.O. Box 20537
Nicosia, CY- 1678 CYPRUS
labourie.camille@ucy.ac.cy

Emmanouil Milakis
University of Cyprus
Department of Mathematics & Statistics
P.O. Box 20537
Nicosia, CY- 1678 CYPRUS
emilakis@ucy.ac.cy