On the elegance of Ramanujan's series for π.

Chieh-Lei Wong

ON THE ELEGANCE OF RAMANUJAN'S SERIES FOR $\frac{1}{\pi}$

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Abstract. Re presenting the traditional proof of Srinivasa Ramanujan's own favorite series for the reciprocal of π :

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{+\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}} ,$$

as well as several other examples of Ramanujan's infinite series. As a matter of fact, the derivation of such formulae has involved specialized knowledge of identities of classical functions and modular functions.

The Archimedes' constant π appears in many formulae [2] in various areas of mathematics and physics, such as :

$$\text{James Gregory (1671)} \qquad \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} , \tag{0.1}$$

$$\text{Leonhard Euler (1734)} \qquad \sum_{n=0}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} , \tag{0.2}$$

$$\text{Carl Friedrich Gauss (1809)} \qquad \int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi} , \tag{0.3}$$

$$\text{Stephen Hawking (1974)} \qquad T = \frac{1}{8\pi k_B} \frac{\hbar c^3}{GM} . \tag{0.4}$$

The irrationality of π was first proven by Jean-Henri Lambert in 1761. Finally in 1882, Ferdinand von Lindemann established its transcendence, thus laying to rest the problem of « squaring the circle ».

1. Aesthetics in mathematics ?

In 2014, researchers in neurobiology [14] from the University College London (in United Kingdom) used functional MRI to image the brain activity of 15 mathematicians (aged from 22 to 32 years, postgraduate or postdoctoral level, all recruited from colleges in London) when they viewed mathematical formulae. Each subject was given 60 mathematical formulae - including (0.1), (0.2) or (0.3) that correspond successively to $\arctan(1)$, $\zeta(2)$ and $\Gamma\left(\frac{1}{2}\right)$ - to study at leisure and rate as ugly $[-1]$, neutral $[0]$ or beautiful $[+1]$. Note the absence of the nonsimple continued fraction :

$$\text{William Brouncker (1655)} \qquad \frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \ddots}}}}} \tag{1.1}$$

in their list. Results of the study showed that the one most consistently rated as « ugly » was Equation (14) :

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{+\infty} \frac{(4n)!}{(n!)^4} \frac{1103 + 26390n}{396^{4n}} , \tag{1.2}$$

an infinite series due to Ramanujan - with an average rating of $-0,7333$! Truly, beauty is in the eye of the beholder.

Since the starting point of (1.2) lays upon the new foundations of elliptic integrals instilled by the works of both Niels Henrik Abel and Carl Gustav J. Jacobi [9] in the 19^{th} century, we might remember the premonitory words of Felix Klein :

> « *When I was a student, Abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics, and each of us was ambitious to make progress in this field. And now ? The younger generation hardly knows Abelian functions.* »

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Historically, the identity (1.2) appeared in [12]. Afterwards, it fell into near oblivion, until the end of 1985 when it was revived in a modern computational context. Seven decades after its publication, Bill Gosper Jr. used it for computing $17,5.10^6$ decimal digits of π - and briefly held the world record. But a significant issue remained : no mathematical proof existed back then that the series (1.2) actually converges to $\frac{1}{\pi}$. It was somehow a leap of faith, yet an educated one. In fact, he verified beforehand that the sum was correct to 10 million places by comparing this same number of digits of his own calculation to a previous calculation done by Yasumasa Kanada *and al.*

2. Preliminaries

2.1. **Jacobi's elliptic integrals.**

Let $k \in]0,1[$ denote the elliptic modulus, then the quantity $k' = \sqrt{1-k^2}$ is called the complementary modulus. Complete elliptic integrals of the first and second kinds are respectively defined as :

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,k^2\right) \tag{2.1}$$

$$\text{and}\quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}-\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\,k^2\right)\,, \tag{2.2}$$

while their derivatives are given by :

$$\frac{dK}{dk} = \frac{E-k'^2K}{kk'^2} \quad \text{and} \quad \frac{dE}{dk} = \frac{E-K}{k}\,. \tag{2.3}$$

It is also customary to define the complementary integrals K' and E' as :

$$K'(k) = K(k') \quad \text{and} \quad E'(k) = E(k')\,.$$

Finally, these 4 quantities K, K', E and E' are linked by the remarkable Legendre relation :

$$K(k)E'(k) + E(k)K'(k) - K(k)K'(k) = \frac{\pi}{2}\,. \tag{2.4}$$

2.2. **Jacobi's theta functions.**

The theta functions [9], [10] are classically defined as :

$$\theta_2(q) = \sum_{n=-\infty}^{+\infty} q^{\left(n+\frac{1}{2}\right)^2} \quad,\quad \theta_3(q) = \sum_{n=-\infty}^{+\infty} q^{n^2} \quad \text{and} \quad \theta_4(q) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} = \theta_3(-q) \tag{2.5}$$

for $|q|<1$. After rewriting the nome q in terms of the elliptic modulus k :

$$q = \exp\left[-\pi\frac{K'(k)}{K(k)}\right]\,,$$

it is valuable to regard k as a function of q. Thus, we have inversely :

$$k = \frac{\theta_2^2(q)}{\theta_3^2(q)} \quad,\quad k' = \frac{\theta_4^2(q)}{\theta_3^2(q)} \quad \text{and} \quad K(k) = \frac{\pi}{2}\theta_3^2(q)\,. \tag{2.6}$$

2.3. **Ramanujan-Weber's class invariants.**

Let us introduce Ramanujan's class invariants :

$$G = \left(\frac{1}{2kk'}\right)^{1/12} \quad \text{and} \quad g = \left(\frac{k'^2}{2k}\right)^{1/12}\,, \tag{2.7}$$

as well as the Klein's absolute invariant :

$$J = \frac{(4G^{24}-1)^3}{27G^{24}} = \frac{(4g^{24}+1)^3}{27g^{24}} = \frac{4}{27}\frac{\left[1-(kk')^2\right]^3}{(kk')^4}\,. \tag{2.8}$$

In terms of Ramanujan's class invariants, we can explicitly write the elliptic moduli as :

$$k = \frac{1}{2}\left(\sqrt{1+\frac{1}{G^{12}}} - \sqrt{1-\frac{1}{G^{12}}}\right) \quad,\quad k' = \frac{1}{2}\left(\sqrt{1+\frac{1}{G^{12}}} + \sqrt{1-\frac{1}{G^{12}}}\right)\,,$$

$$\text{or}\quad k = g^6\sqrt{g^{12}+\frac{1}{g^{12}}} - g^{12} \quad,\quad k' = \sqrt{2k}\,g^6\,.$$

2.4. **Singular value functions λ^* and α.**

Definition 2.1. *Let $\lambda^*(r) = k(e^{-\pi\sqrt{r}})$ be as in (2.6), then the singular value function of the second kind is defined by :*

$$\alpha(r) = \frac{E'(k)}{K(k)} - \frac{\pi}{4\big[K(k)\big]^2} \tag{2.9}$$

for positive r.

Since $\lim_{r\to+\infty} \lambda^*(r) = 0$, then $\alpha(r)$ converges to $\dfrac{1}{\pi}$ with exponential rate :

$$0 < \alpha(r) - \frac{1}{\pi} \leqslant \sqrt{r}\big[\lambda^*(r)\big]^2 \leqslant 16\sqrt{r}\, e^{-\pi\sqrt{r}}\,.$$

Using the functional equation (2.4) and the fact that $\dfrac{K'\big(\lambda^*(r)\big)}{K\big(\lambda^*(r)\big)} = \sqrt{r}$, we get :

$$\alpha(r) = \frac{\pi}{4\big[K(k)\big]^2} - \sqrt{r}\left[\frac{E(k)}{K(k)} - 1\right]\,.$$

On substituting E with the differential equation (2.3), we may establish that :

$$\alpha(r) = \frac{1}{\pi}\left[\frac{\pi}{2K(k)}\right]^2 - \sqrt{r}\left[kk'^2\frac{1}{K(k)}\frac{dK}{dk} - k^2\right]\,,$$

so that :

$$\frac{1}{\pi} = \sqrt{r}kk'^2\left[\left(\frac{2}{\pi}\right)^2 K(k)\frac{dK}{dk}\right] + \big[\alpha(r) - \sqrt{r}k^2\big]\left[\frac{2}{\pi}K(k)\right]^2 \tag{2.10}$$

where $k = \lambda^*(r)$. Also, observe that $\alpha(r)$ is algebraic for $r \in \mathbb{Q}_+$ (as seen in Tables 1 and 2 in the next section, or in the computation of g_{58}^2 and k_{58} in Subsection 3.2.3). Actually, it is well-known that the quantities $\lambda^*(r)$, G_r, g_r and $\alpha(r)$ are algebraic numbers expressible by surds when r is a positive rational number.

2.5. **Quadratic and cubic transformations of the hypergeometric function ${}_2F_1$.**

Let us recall the definition of the hypergeometric series :

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}\,\middle|\, z\right) = \sum_{n=0}^{+\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\,, \tag{2.11}$$

where parameters a, b and c are arbitrary complex numbers, and $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$ denotes the Pochhammer symbol. However, if and only if the numbers :

$$\pm\,(1-c) \quad , \quad \pm\,(a-b) \quad , \quad \pm\,(a+b-c) \tag{2.12}$$

have the property that one of them equals $\dfrac{1}{2}$ or that two of them are equal, then there exists a so-called quadratic transformation.

Proposition 2.2. *For $k \in \left[0, \dfrac{1}{\sqrt{2}}\right]$, we have :*

$$\frac{2}{\pi}K(k) = {}_2F_1\left(\begin{array}{c} \frac{1}{4}, \frac{1}{4} \\ 1 \end{array}\,\middle|\, (2kk')^2\right) \tag{2.13}$$

$$\textit{and} \quad \left[\frac{2}{\pi}K(k)\right]^2 = {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array}\,\middle|\, (2kk')^2\right)\,. \tag{2.14}$$

Proof. The first identity (2.13) derives from Kummer's identity :

$${}_2F_1\left(\begin{array}{c} 2a, 2b \\ a+b+\frac{1}{2} \end{array}\,\middle|\, z\right) = {}_2F_1\left(\begin{array}{c} a, b \\ a+b+\frac{1}{2} \end{array}\,\middle|\, 4z(1-z)\right) \tag{2.15}$$

and can be verified by showing that both sides satisfy the appropriate hypergeometric differential equation, are analytic and agree at 0. The second identity (2.14) is a special case of Clausen's product identity :

$${}_2F_1\left(\begin{array}{c} \frac{1}{4}+a, \frac{1}{4}+b \\ 1+a+b \end{array}\,\middle|\, z\right)\,{}_2F_1\left(\begin{array}{c} \frac{1}{4}-a, \frac{1}{4}-b \\ 1-a-b \end{array}\,\middle|\, z\right) = {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}+a-b, \frac{1}{2}-a+b \\ 1+a+b, 1-a-b \end{array}\,\middle|\, z\right) \tag{2.16}$$

for hypergeometric functions. ∎

In like fashion, a cubic transformation exists if and only if either two of the numbers in (2.12) are equal to $\frac{1}{3}$ or if :

$$1-c=\pm(a-b)=\pm(a+b-c)\ .$$

Thus, quadratic and cubic transformations of ${}_2F_1$ lead to a variety of alternate hypergeometric expressions for K and K^2.

Proposition 2.3. *We also have :*

$$\frac{2}{\pi}K(k)=\frac{1}{k'}\,{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{1}{4}\\1\end{matrix}\,\middle|\,-\left(\frac{2k}{k'^2}\right)^2\right)\qquad \text{for } k\in[0,\sqrt{2}-1]\ ,$$

$$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{k'}}\,{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{1}{4}\\1\end{matrix}\,\middle|\,-\left(\frac{k^2}{2k'}\right)^2\right)\qquad \text{for } k^2\in[0,2(\sqrt{2}-1)]\ ,$$

$$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{1+k^2}}\,{}_2F_1\left(\begin{matrix}\frac{1}{8},\frac{3}{8}\\1\end{matrix}\,\middle|\,\left(\frac{2}{g^{12}+g^{-12}}\right)^2\right)\qquad \text{for } k\in[0,\sqrt{2}-1]\ ,$$

$$\frac{2}{\pi}K(k)=\frac{1}{\sqrt{k'^2-k^2}}\,{}_2F_1\left(\begin{matrix}\frac{1}{8},\frac{3}{8}\\1\end{matrix}\,\middle|\,-\left(\frac{2}{G^{12}-G^{-12}}\right)^2\right)\qquad \text{for } k\in\left[0,\frac{1-\sqrt{\sqrt{2}-1}}{2^{3/4}}\right]\ ,$$

$$\text{and}\quad \frac{2}{\pi}K(k)=\frac{1}{\left[1-(kk')^2\right]^{1/4}}\,{}_2F_1\left(\begin{matrix}\frac{1}{12},\frac{5}{12}\\1\end{matrix}\,\middle|\,\frac{1}{J}\right)\qquad \text{for } k\in\left[0,\frac{1}{\sqrt{2}}\right]\ .$$

Proof. See e.g. [8] or [1]. ∎

Proposition 2.4. *For k restricted as in Proposition 2.3 :*

$$\left[\frac{2}{\pi}K(k)\right]^2=\frac{1}{k'^2}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\1,1\end{matrix}\,\middle|\,-\left(\frac{2k}{k'^2}\right)^2\right)\ ,$$

$$\left[\frac{2}{\pi}K(k)\right]^2=\frac{1}{k'}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\1,1\end{matrix}\,\middle|\,-\left(\frac{k^2}{2k'}\right)^2\right)\ ,$$

$$\left[\frac{2}{\pi}K(k)\right]^2=\frac{1}{1+k^2}\,{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},\frac{1}{2}\\1,1\end{matrix}\,\middle|\,\left(\frac{2}{g^{12}+g^{-12}}\right)^2\right)\ ,$$

$$\left[\frac{2}{\pi}K(k)\right]^2=\frac{1}{k'^2-k^2}\,{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},\frac{1}{2}\\1,1\end{matrix}\,\middle|\,-\left(\frac{2}{G^{12}-G^{-12}}\right)^2\right)\ ,$$

$$\text{and}\quad \left[\frac{2}{\pi}K(k)\right]^2=\frac{1}{\sqrt{1-(kk')^2}}\,{}_3F_2\left(\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{2}\\1,1\end{matrix}\,\middle|\,\frac{1}{J}\right)\ .$$

Proof. Apply the Clausen's identity (2.16) to Proposition 2.3. ∎

In each case, we have provided series for $\frac{2}{\pi}K$ and $\left(\frac{2}{\pi}K\right)^2$ in terms of the Ramanujan's invariants. Indeed, we have :

$$\left[\frac{2}{\pi}K(k)\right]^2=m(k)F\big(\varphi(k)\big)$$

for algebraic m and φ, while $F(\varphi)$ has a hypergeometric-type power series expansion $\sum_{n=0}^{+\infty}a_n\varphi^n$. Then :

$$\left(\frac{2}{\pi}\right)^2K\frac{dK}{dk}=\frac{1}{2}\left[\frac{dm}{dk}F+m\frac{d\varphi}{dk}\frac{dF}{d\varphi}\right]$$

and substitution in (2.10) lead to :

$$\frac{1}{\pi}=\sum_{n=0}^{+\infty}a_n\left\{\frac{1}{2}\sqrt{r}kk'^2\frac{dm}{dk}+\left[\alpha(r)-\sqrt{r}k^2\right]m+\frac{1}{2}n\sqrt{r}kk'^2\frac{m}{\varphi}\frac{d\varphi}{dk}\right\}\varphi^n\ .\tag{2.17}$$

Thus for rational r, the braced term in (2.17) is of the form $A+nB$ with A and B algebraic.

3. Examples of hypergeometric-like series representations for $\frac{1}{\pi}$

3.1. Deriving Ramanujan's series for $\frac{1}{\pi}$.

By combining Propositions 2.2, 2.3 and 2.4 with the formula (2.17), it is now straightforward to build the next 6 series :

$$\text{(series in } G_N) \qquad \frac{1}{\pi}=\sum_{n=0}^{+\infty}\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3\left[\alpha(N)-\sqrt{N}k_N^2+n\sqrt{N}\left(k_N'^2-k_N^2\right)\right]\left(\frac{1}{G_N^{12}}\right)^{2n} \tag{3.1}$$

$$\text{(series in } g_N) \qquad \frac{1}{\pi}=\sum_{n=0}^{+\infty}(-1)^n\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3\left[\frac{\alpha(N)}{k_N'^2}+n\sqrt{N}\frac{1+k_N^2}{k_N'^2}\right]\left(\frac{1}{g_N^{12}}\right)^{2n} \tag{3.2}$$

$$\text{(series in } g_{4N}=2^{1/4}g_NG_N) \qquad \frac{1}{\pi}=\sum_{n=0}^{+\infty}(-1)^n\left[\frac{1}{n!}\left(\frac{1}{2}\right)_n\right]^3\left\{\left[\alpha(N)-\sqrt{N}\frac{k_N^2}{2}\right]\frac{1}{k_N'}+n\sqrt{N}\left(k_N'+\frac{1}{k_N'}\right)\right\}\left(\frac{1}{g_{4N}^{12}}\right)^{2n} \tag{3.3}$$

On setting $x_N=\dfrac{2}{g_N^{12}+g_N^{-12}}=\dfrac{4k_Nk_N'^2}{(1+k_N^2)^2}$ and $y_N=\dfrac{2}{G_N^{12}-G_N^{-12}}=\dfrac{4k_Nk_N'}{1-(2k_Nk_N')^2}$:

$$\text{(series in } x_N) \qquad \frac{1}{\pi}=\sum_{n=0}^{+\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3}\left[\frac{\alpha(N)}{x_N(1+k_N^2)}-\frac{\sqrt{N}}{4g_N^{12}}+n\sqrt{N}\frac{g_N^{12}-g_N^{-12}}{2}\right]x_N^{2n+1} \tag{3.4}$$

$$\text{(series in } y_N) \qquad \frac{1}{\pi}=\sum_{n=0}^{+\infty}(-1)^n\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3}\left[\frac{\alpha(N)}{y_N(k_N'^2-k_N^2)}+\sqrt{N}\frac{k_N^2G_N^{12}}{2}+n\sqrt{N}\frac{G_N^{12}+G_N^{-12}}{2}\right]y_N^{2n+1} \tag{3.5}$$

And eventually the series in J_N :

$$\frac{1}{\pi}=\frac{1}{3\sqrt{3}}\sum_{n=0}^{+\infty}\frac{\left(\frac{1}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{5}{6}\right)_n}{(n!)^3}\left\{2\left[\alpha(N)-\sqrt{N}k_N^2\right]\left(4G_N^{24}-1\right)+\sqrt{N}\sqrt{1-\frac{1}{G_N^{24}}}+2n\sqrt{N}\left(8G_N^{24}+1\right)\sqrt{1-\frac{1}{G_N^{24}}}\right\}\left(\frac{1}{J_N^{1/2}}\right)^{2n+1} \tag{3.6}$$

that is valid for $N>1$.

3.2. Applications.

Let us first evaluate the Pochhammer symbols. It is well-known that :

$$\frac{1}{n!}\left(\frac{1}{2}\right)_n=\frac{1}{4^n}\binom{2n}{n}$$

in terms of the central binomial coefficient. For the remaining symbols, we may require the following lemma :

Lemma 3.1. *For any $n\in\mathbb{N}$, we have :*

$$\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n=\frac{1}{4^{4n}}\frac{(4n)!}{n!},$$

$$\textit{as well as}\quad \left(\frac{1}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{5}{6}\right)_n=\frac{1}{12^{3n}}\frac{(6n)!}{(3n)!}.$$

Proof. Let $p,q\in\mathbb{N}^*$, observe that :

$$\left(\frac{p}{q}\right)_n=\frac{1}{q^n}\prod_{m=1}^{n}\left[p+(m-1)q\right].$$

Subsequently :

$$\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n=\frac{1}{4^{3n}}\prod_{m=1}^{n}(4m-3)(4m-2)(4m-1)=\frac{1}{4^{4n}}\frac{(4n)!}{n!},$$

$$\text{whereas}\quad \left(\frac{1}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{5}{6}\right)_n=\frac{1}{6^{3n}}\prod_{m=1}^{n}(6m-5)(6m-3)(6m-1)=\frac{1}{12^{3n}}\frac{(6n)!}{(3n)!}.$$

■

Definition 3.2. *Let d be a square-free integer, we consider the real quadratic number field $\mathbb{k}=\mathbb{Q}(\sqrt{d})$. If $\Delta_{\mathbb{k}}$ denotes the discriminant of $\mathbb{k}$ i.e. :*

$$\Delta_{\mathbb{k}}=\begin{cases} d & \text{if } d=1 \pmod 4\\ 4d & \text{if } d=2,3 \pmod 4\end{cases},$$

then the fundamental unit $u_d > 1$ is uniquely characterized as the minimal real number :

$$u_d = \frac{a + b\sqrt{\Delta_{\mathbb{k}}}}{2} \tag{3.7}$$

where (a, b) is the smallest solution to $m^2 - \Delta_{\mathbb{k}} n^2 = \pm 4$ in positive integers. This equation is essentially Pell-Fermat's equation.

Of course, the most challenging part in the formula (2.17) lies in the evaluation of the singular value function α. For positive rational r, many values of $\alpha(r)$ are obtainable. But details would be slightly beyond the scope of this paper, with deep roots in number-theoretic objects and techniques such as modular equations, multipliers, modular forms, the Dedekind's η function, and so on. Alternatively, we shall rely on Weber [13] and Ramanujan [12]. Some of the nicest singular values are collected in the following tables.

N	k_N	$\frac{1}{G_N^{12}}$	$\alpha(N)$	u_N
3	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}-1}{2}$	$2+\sqrt{3}$
5	$\frac{\sqrt{\sqrt{5}-1}-\sqrt{3-\sqrt{5}}}{2}$	$\left(\frac{\sqrt{5}-1}{2}\right)^3$	$\frac{\sqrt{5}-\sqrt{2\sqrt{5}-2}}{2}$	$\frac{1+\sqrt{5}}{2}$
7	$\frac{3-\sqrt{7}}{4\sqrt{2}}$	$\frac{1}{8}$	$\frac{\sqrt{7}-2}{2}$	$8+3\sqrt{7}$
9	$\frac{(\sqrt{2}-3^{1/4})(\sqrt{3}-1)}{2}$	$(2-\sqrt{3})^2$	$\frac{3-3^{3/4}\sqrt{2}(\sqrt{3}-1)}{2}$	–
13	$\frac{\sqrt{10\sqrt{13}-34}-5+\sqrt{13}}{2\sqrt{2}}$	$\left(\frac{\sqrt{13}-3}{2}\right)^3$	$\frac{\sqrt{13}-\sqrt{74\sqrt{13}-258}}{2}$	$\frac{3+\sqrt{13}}{2}$
15	$\frac{(2-\sqrt{3})(3-\sqrt{5})(\sqrt{5}-\sqrt{3})}{8\sqrt{2}}$	$\frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)^4$	$\frac{\sqrt{15}-\sqrt{5}-1}{2}$	$4+\sqrt{15}$
25	$\frac{(\sqrt{5}-2)(3-2\times 5^{1/4})}{\sqrt{2}}$	$\left(\frac{\sqrt{5}-1}{2}\right)^{12}$	$\frac{5\left[1-2\times 5^{1/4}(7-3\sqrt{5})\right]}{2}$	–
37	$\frac{\sqrt{290\sqrt{37}-1762}+29-5\sqrt{37}}{2\sqrt{2}}$	$(\sqrt{37}-6)^3$	$\frac{\sqrt{37}-(171-25\sqrt{37})\sqrt{\sqrt{37}-6}}{2}$	$6+\sqrt{37}$

TABLE 1. Selected singular values, class invariants G_N and fundamental units u_N for N odd.

In Table 1, observe that $G_N^4 = u_N$ for $N = 5$, 13 and 37.

N	k_N	$\frac{1}{g_N^{12}}$	$\alpha(N)$	$u_{N/2}$	u_N
2	$\sqrt{2}-1$	1	$\sqrt{2}-1$	–	$1+\sqrt{2}$
6	$(2-\sqrt{3})(5-2\sqrt{6})^{1/2}$	$(\sqrt{2}-1)^2$	$(\sqrt{2}+1)(2-\sqrt{3})(5-2\sqrt{6})^{1/2}(3-\sqrt{2})$	$2+\sqrt{3}$	$5+2\sqrt{6}$
10	$(\sqrt{2}-1)^2(\sqrt{10}-3)$	$\left(\frac{\sqrt{5}-1}{2}\right)^6$	$\left(\frac{\sqrt{5}+1}{2}\right)^3(\sqrt{2}-1)^2(\sqrt{10}-3)(3\sqrt{5}-4)$	$\frac{1+\sqrt{5}}{2}$	$3+\sqrt{10}$
18	$(7-4\sqrt{3})(5\sqrt{2}-7)$	$(\sqrt{3}-\sqrt{2})^4$	$3(\sqrt{3}+\sqrt{2})^2(7-4\sqrt{3})(5\sqrt{2}-7)(7-2\sqrt{6})$	–	$1+\sqrt{2}$
22	$(10-3\sqrt{11})(197-42\sqrt{22})^{1/2}$	$(\sqrt{2}-1)^6$	$(\sqrt{2}+1)^3(10-3\sqrt{11})(197-42\sqrt{22})^{1/2}(33-17\sqrt{2})$	$10+3\sqrt{11}$	$197+42\sqrt{22}$
58	$(\sqrt{2}-1)^6(13\sqrt{58}-99)$	$\left(\frac{\sqrt{29}-5}{2}\right)^6$	$3\left(\frac{\sqrt{29}+5}{2}\right)^3(\sqrt{2}-1)^6(13\sqrt{58}-99)(33\sqrt{29}-148)$	$\frac{5+\sqrt{29}}{2}$	$99+13\sqrt{58}$

TABLE 2. Selected singular values, class invariants g_N and fundamental units $u_{N/2}$ and u_N for N even.

For $N = 6$, 10, 18, 22 and 58, observe that the values of the function α in Table 2 are all expressed in the form $\alpha(N) = g_N^6 k_N f_N$, where f_N is an element of some quadratic field $\mathbb{Q}(\sqrt{d})$ with $d \mid N$.

Many more singular moduli are given in [4] or [11].

3.2.1. *The case* $N = 7$.

Table 1 provides :

$$G_7^{12} = 8 \quad \text{and} \quad \alpha(7) = \frac{\sqrt{7}}{2} - 1 \,,$$

so that $k_7^2 = \dfrac{8 - 3\sqrt{7}}{16}$. By putting these values in the series (3.1) which is valid for $N > 1$, we obtain :

$$\frac{1}{\pi} = \frac{1}{16} \sum_{n=0}^{+\infty} \frac{\left[(2n)!\right]^3}{(n!)^6} \frac{5 + 7 \times 6n}{64^{2n}} \,. \tag{3.8}$$

This is equivalent to Equation (29) in Ramanujan's original paper [12]. Being composed of fractions whose numerators grow like $\sim 2^{6n}$ and whose denominators are exactly 16×2^{12n}, the above series can be employed to calculate the second block of n binary digits of π without calculating the first n binary digits.

Note that the series (3.5) is valid for $N \geqslant 4$. On using the invariant $y_7 = \dfrac{16}{63}$ in (3.5), we get :

$$\frac{1}{\pi} = \frac{1}{9\sqrt{7}} \sum_{n=0}^{+\infty} (-1)^n \frac{(4n)!}{(n!)^4} \frac{8 + 65n}{63^{2n}} \,, \tag{3.9}$$

while combining $J_7 = \left(\dfrac{85}{4}\right)^3$ with (3.6) shall produce the series :

$$\frac{1}{\pi} = \frac{18}{85} \sqrt{\frac{3}{85}} \sum_{n=0}^{+\infty} \frac{(6n)!}{(3n)!(n!)^3} \frac{8 + 7 \times 19n}{255^{3n}} \,. \tag{3.10}$$

One may recognize Equation (34) of [12] which adds 4 decimal digits a term.

3.2.2. *The case* $N = 37$.

Let us recall that $G_{37}^4 = u_{37} = 6 + \sqrt{37}$. From Table 1, we get :

$$y_{37} = \frac{2}{G_{37}^{12} - G_{37}^{-12}} = \frac{1}{882} \quad , \quad \frac{G_{37}^{12} + G_{37}^{-12}}{2} = 145\sqrt{37} \quad , \quad \alpha(37) = \frac{\sqrt{37} - (171 - 25\sqrt{37}) G_{37}^{-2}}{2} \,,$$

as well as :

$$k_{37}^2 = \frac{1}{2}\left(1 - \frac{1}{G_{37}^6}\sqrt{G_{37}^{12} - \frac{1}{G_{37}^{12}}}\right) = \frac{1}{2}\left(1 - \frac{42}{G_{37}^6}\right) \quad \Longrightarrow \quad \frac{k_{37}^2 G_{37}^{12}}{2} = \frac{G_{37}^6 (G_{37}^6 - 42)}{4} \,.$$

Consequently :

$$\begin{aligned}
\frac{\alpha(37)}{y_{37}(k_{37}'^2 - k_{37}^2)} + \sqrt{37}\,\frac{k_{37}^2 G_{37}^{12}}{2} &= \frac{21}{2}\left[\sqrt{37} - (171 - 25\sqrt{37}) G_{37}^{-2}\right] G_{37}^6 + \sqrt{37}\,\frac{(G_{37}^6 - 42) G_{37}^6}{4} \\
&= \frac{G_{37}^4}{4}\left[-42(171 - 25\sqrt{37}) + \sqrt{37}\, G_{37}^8\right] \\
&= \frac{6 + \sqrt{37}}{4}\left[-42(171 - 25\sqrt{37}) + \sqrt{37}(6 + \sqrt{37})^2\right] \;=\; \frac{1123}{4} \,.
\end{aligned}$$

Putting these numerical values into (3.5) yields :

$$\begin{aligned}
\frac{1}{\pi} &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{4^{4n}} \frac{(4n)!}{(n!)^4} \left[\frac{\alpha(37)}{y_{37}(k_{37}'^2 - k_{37}^2)} + \sqrt{37}\,\frac{k_{37}^2 G_{37}^{12}}{2} + n\sqrt{37}\,\frac{G_{37}^{12} + G_{37}^{-12}}{2}\right] y_{37}^{2n+1} \\
&= \frac{1}{3528} \sum_{n=0}^{+\infty} (-1)^n \frac{(4n)!}{(n!)^4} \frac{1123 + 37 \times 580n}{14112^{2n}}
\end{aligned} \tag{3.11}$$

which can be identified with Equation (39) of [12].

3.2.3. *The case* $N = 58$.

Let $r \in \mathbb{Q}_+^*$, it turns out that :

$$\sum_{m,n=-\infty}^{+\infty}{}' \frac{(-1)^m}{m^2+rn^2} = -\frac{\pi}{\sqrt{r}} \log(2g_r^4) \,. \tag{3.12}$$

Since this zeta sum over a 2-dimensional lattice (with the exception of the origin) can be decomposed into a sum of products of L-series, we have :

$$\sum_{m,n=-\infty}^{+\infty}{}' \frac{(-1)^{m+1}}{m^2+58n^2} = \frac{\pi}{\sqrt{58}} \log 2 + \sum_{d|29} \left[1 - \left(\frac{2}{d}\right)\right] L_{-\frac{232}{d}}(1) L_d(1) = \frac{\pi}{\sqrt{58}} \log 2 + 2L_{-8}(1) L_{29}(1)$$

where $\left(\frac{2}{d}\right)$ denotes the Kronecker symbol of 2 and $d > 0$. Hence :

$$\frac{\pi}{\sqrt{58}} \log(2g_{58}^4) = \frac{\pi}{\sqrt{58}} (\log 2 + 2\log u_{29}) \quad \Longrightarrow \quad g_{58}^2 = u_{29} = \frac{5+\sqrt{29}}{2} \,.$$

From the relation :

$$\sum_{m,n=-\infty}^{+\infty}{}' \frac{(-1)^m}{m^2+2rn^2} - 4 \sum_{m,n=-\infty}^{+\infty}{}' \frac{(-1)^m}{m^2+8rn^2} = -\frac{\pi}{\sqrt{2r}} \log\left(\frac{k_r}{4}\right) \,, \tag{3.13}$$

we may similarly deduce that $k_{58} = \dfrac{1}{u_2^6 u_{58}} = (\sqrt{2}-1)^6(13\sqrt{58}-99)$. So $k_{58} + \dfrac{1}{k_{58}} = 198\sqrt{2}(13\sqrt{29}+70)$.

On inserting now the numerical values :

$$x_{58} = \frac{2}{g_{58}^{12}+g_{58}^{-12}} = \frac{1}{9801} \quad , \quad \frac{g_{58}^{12}-g_{58}^{-12}}{2} = 1820\sqrt{29} \quad , \quad \alpha(58) = 3g_{58}^6 k_{58}(33\sqrt{29}-148) \,,$$

and :

$$\begin{aligned} \frac{\alpha(58)}{x_{58}(1+k_{58}^2)} - \frac{\sqrt{58}}{4g_{58}^{12}} &= \frac{3(33\sqrt{29}-148)}{x_{58}(k_{58}+k_{58}^{-1})} \left(\frac{\sqrt{29}+5}{2}\right)^3 - \frac{1}{2}\sqrt{\frac{29}{2}} \left(\frac{\sqrt{29}-5}{2}\right)^6 \\ &= \frac{1}{2\sqrt{2}} \left[297(33\sqrt{29}-148) - \sqrt{29}(9801-1820\sqrt{29})\right] \;=\; 2\sqrt{2}\times 1103 \end{aligned}$$

into the series (3.4), we find that :

$$\begin{aligned} \frac{1}{\pi} &= \sum_{n=0}^{+\infty} \frac{1}{4^{4n}} \frac{(4n)!}{(n!)^4} \left[\frac{\alpha(58)}{x_{58}(1+k_{58}^2)} - \frac{\sqrt{58}}{4g_{58}^{12}} + n\sqrt{58}\,\frac{g_{58}^{12}-g_{58}^{-12}}{2}\right] x_{58}^{2n+1} \\ &= \frac{2\sqrt{2}}{9801} \sum_{n=0}^{+\infty} \frac{(4n)!}{(n!)^4} \frac{1103+29\times 910n}{396^{4n}} \,. \end{aligned} \tag{3.14}$$

This concludes the proof of Equation (44) in [12]. As observed by Ramanujan himself, the series (1.2) is extremely rapidly convergent by adding 8 decimal digits a term !

As an exercise, the reader is encouraged to determine the other series of [12] with the singular values in Tables 1 and 2. A solution is provided in the companion file https://clwmypage.files.wordpress.com/2021/01/ramanujan-reciprocal-pi.pdf.

4. Conclusion

Srinivasa Ramanujan recorded the bulk of his mathematical results in several notebooks of looseleaf paper and mostly written up without proofs. Hence, his works were often shrouded in a veil of divine magic and mystery. As being a deeply religious Hindu, he credited his substantial capacities to divinity, and stated that formulae were revealed to him by his family goddess, Namagiri Thayar. During the 20$^{\text{th}}$ century, the many results in *Ramanujan's Notebooks* inspired numerous papers by later mathematicians trying to prove what he had previously found.

As demonstrated, the general formula (2.17) produces multiple reciprocal series for π in terms of the function $\alpha(r)$ and related modular quantities. Thus, we showed that the amazing sum (1.2) is a specialization (when $N = 58$) of (2.17) coupled with the invariant $\varphi(k) = \left[\dfrac{4k(1-k^2)}{(1+k^2)^2}\right]^2$.

For the sake of simplicity, we have intentionally skipped here some technical aspects, namely about modular equations of order p (with p prime), modular forms, Eisenstein series, the Dedekind's η function, etc. References [3] and [5] (as well as multiple references therein) are accessible expository papers in connection with Ramanujan's series for $\dfrac{1}{\pi}$. For a deeper insight, material based on the context of elliptic and modular curves can be found e.g. in [4], [6] or [7].

This leads naturally to an other famous instance of Ramanujan-Sato series, to wit :

$$\text{Chudnovsky (1988)} \qquad \frac{1}{\pi} = 12\sum_{n=0}^{+\infty}(-1)^n \frac{(6n)!}{(3n)!(n!)^3}\frac{13591409 + 163\times 3344418n}{640320^{3\left(n+\frac{1}{2}\right)}} \tag{4.1}$$

when $N = 163$. On the quest for digits of π, the series (4.1) was used by Alexander J. Yee and Shigeru Kondo to calculate more than $12,1.10^{12}$ decimal places for a new record-breaking computation in 2013.

It was only recently that Heng Huat Chan and Shaun Cooper [6] discovered a general approach that used the underlying modular congruence subgroup $\Gamma_0(N)$ to generate a set of all-new Ramanujan-Sato series, such as :

$$\text{Chan \& Cooper (2012)} \qquad \frac{1}{\pi} = 2\sqrt{2}\sum_{n=0}^{+\infty}\left[\sum_{m=0}^{n}\frac{(-1)^{n-m}}{64^m}\frac{(4m)!}{(m!)^4}\binom{n+m}{n-m}\right]\left[-24184 + 9801\sqrt{29}\left(n+\frac{1}{2}\right)\right]\left(\frac{\sqrt{29}-5}{2}\right)^{12\left(n+\frac{1}{2}\right)}$$

which can be considered as a counterpart of (1.2).

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