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# Robust Nonlinear Predictive Control through qLPV embedding and Zonotope Uncertainty Propagation<sup>\*</sup>

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**Abstract:** This paper presents a novel robust Model Predictive Control (MPC) algorithm for nonlinear systems represented through quasi-Linear Parameter Varying (qLPV) models. The nominal MPC predictions are made considering a frozen scheduling parameter guess, which is computationally cheaper than nonlinear predictions, while zonotopes bound the disturbance propagation along the prediction. These sets are computed with respect to the bounds of the variation of scheduling parameters, offering reduced conservatism of the closed-loop dynamics and ensuring input-to-state stability and recursive feasibility properties. A DC-DC converter benchmark example is used to illustrate the advantages of the proposed method.

*Keywords:* Robust Model Predictive Control, Quasi-Linear Parameter Varying Systems, Zonotopes, Set-based Control, Constraint Tightening.

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## 1. INTRODUCTION

Model Predictive Control (MPC) is well established and widely used for the optimal regulation of constrained dynamic systems (Camacho and Bordons, 2013). Yet with great practical value, the standard MPC design was originally attached to the idea of a nominal linear time-invariant (LTI) prediction model. Albeit nominal LTI MPC inherently offering a certain degree of robustness, it lacks guarantees of recursive feasibility or closed-loop stability at the presence of disturbances.

Since the 00's, literature has provided robust MPC tools, such as terminal ingredients (Mayne et al., 2000) and integral quadratic constraints (Cisneros and Werner, 2018), which enable input-to-state stability and constraint satisfaction for bounded disturbances. Nonetheless, the application of robust nonlinear MPC (NMPC) is not trivial and comes with increased numerical burden, which becomes an impediment for real-time applications (Allgöwer and Zheng, 2012).

In parallel to the theoretical establishment of robust MPC, the Linear Parameter Varying (LPV) toolkit has been brought to focus (Mohammadpour and Scherer, 2012; Sename et al., 2013). LPV models represent nonlinear dynamics with the use of known, bounded scheduling parameters  $\rho$ . Recent advances on NMPC algorithms conceived

through quasi-LPV (qLPV) embedding have been presented, e.g. Morato et al. (2020b) and references therein, enabling fast operation. The elegance of the qLPV embedding is that the "full-blown" nonlinear predictions can be replaced by sequential linear predictions (Abbas et al., 2018; González Cisneros and Werner, 2020).

With respect to this context, we investigate the problem of robust NMPC based on nominal predictions obtained through quasi-Linear Parameter Varying (qLPV) embedding, as done in (Morato et al., 2019; Cisneros and Werner, 2017). Based on the one-step-ahead disturbance propagation, we apply tightened constraints over the nominal qLPV predictions, enforcing the satisfactions of the performance requirements by the real system trajectories (Santos et al., 2019; Köhler et al., 2018). The main novelty of the proposed approach is that it does not require sequential QPs or estimation of the future scheduling variables, as in the prior, but binds the prediction error within zonotopes.

In the nonlinear setting, the majority of uncertainty propagation methods, such as Lipschitz constants and  $\mathcal{K}$ -functions, result in exceeding conservatism, limiting performances and reducing the closed-loop domain of attraction (Marruedo et al., 2002). Nonetheless, the use of zonotopes comes as an alternative, offering representation simplicity and flexibility, as well as low implementation cost. These sets are symmetrical, convex, and compact polyhedrons, originally used for state estimation and fault detection (Pourasghar et al., 2019).

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In this paper, we provide a new method for the uncertainty propagation considering nonlinear systems with nominal qLPV predictions, applying the mean value zonotope extension framework from Alamo et al. (2005); Cunha and Santos (2020). Additionally, we "robustify" the parameter-dependent terminal ingredients from González Cisneros and Werner (2020), which now ensure input-to-state stability and recursive feasibility properties even in the case of (bounded) additive disturbances and model prediction uncertainties. The main restricting hypothesis of the procedure is the necessity of bounds upon the variation of the scheduling variables between sampling instants.

The rest of this paper is organised as follows. Sec. 2 provides preliminary discussions, the problem setup, and the qLPV terminal ingredients. Sec. 3 details the use of zonotopes to bound the qLPV nominal prediction errors. Sec. 4 illustrates a DC-DC converter case study application of the method, together with comparisons to other NMPC algorithms from the literature. General conclusions are drawn in Section 5.

**Notation.** The index set  $\mathbb{N}_{[a,b]}$  represents  $\{i \in \mathbb{N} \mid a \leq i \leq b\}$ , with  $0 \leq a \leq b$ . The identity matrix of size  $j$  is denoted as  $I_j$ ;  $I_{\{j\}}$  denotes the  $j$ -th row of  $I_j$ ;  $\text{col}\{\cdot\}$  denotes the vectorization of the entries and  $\text{diag}\{v\}$  denotes the diagonal matrix generated with the line vector  $v$ .  $\mathbf{1}_{n \times m}$  stands for the  $n \times m$  vector of unit entries; The value of a given variable  $v(k)$  at time instant  $k+i$ , computed based on the information available at instant  $k$ , is denoted as  $v(k+i|k)$ .  $\mathcal{K}$  refers to the class of positive and strictly increasing scalar functions that pass through the origin. A  $\mathcal{C}^1$  function  $f$  is such that it is differentiable with continuous derivatives. In this case,  $\nabla^T f : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$  denotes its Jacobian matrix. Consider sets  $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$ ,  $\mathcal{C} \subset \mathbb{R}^m$  and a matrix  $R \in \mathbb{R}^{n \times m}$ . The Minkowski set addition is defined by  $A \oplus B := \{a + b \mid a \in A, b \in B\}$ , while the Pontryagin set difference is defined by  $A \ominus B := \{a \mid a \oplus B \subseteq A\}$ . A linear mapping is  $R\mathcal{A} = \{y \in \mathbb{R}^n : y = Ra, a \in \mathcal{A} \subseteq \mathbb{R}^m, R \in \mathbb{R}^{n \times m}\}$ , while the cartesian product holds as  $\mathcal{A} \times \mathcal{C} = \{z \in \mathbb{R}^{n+m} : z = (a^T c^T)^T, a \in \mathcal{A}, c \in \mathcal{C}\}$ . The unitary  $m$ -dimensional box is denoted  $\mathcal{B}_\infty^m = \{\xi \in \mathbb{R}^m : \|\xi\|_\infty \leq 1\}$ . The set of real compact intervals is given by  $\mathbb{I} = \{[a, b], a, b \in \mathbb{R}, a \leq b\}$ . An interval matrix  $\mathbf{J} \in \mathbb{I}^{n \times m}$  has  $\text{mid}(\mathbf{J})$  and  $\text{rad}(\mathbf{J})$  denoting its middle point and radius, respectively.  $\|\nu\|$  denotes the 2-norm.

## 2. ROBUST NMPC THROUGH QLPV EMBEDDING

### 2.1 The Nonlinear System

We consider the following discrete-time nonlinear system:

$$x(k+1) = f(x(k), u(k)) + Ew(k), \quad (1)$$

being  $k \in \mathbb{N}$  the sampling instant,  $x \in \mathbb{R}^{n_x}$  the vector of states,  $u \in \mathbb{R}^{n_u}$  the vector of control inputs, and  $w \in \mathbb{R}^{n_w}$  an unknown additive disturbance, assumed bounded to a compact set  $\mathcal{W} \subset \mathbb{R}^{n_w}$  is such way that  $\|w(k)\| \leq \bar{w}, \forall k$ . The states  $x$  are measurable at all sampling instants.

The system in Eq. (1) is subject to state and input constraints, which define admissible operation, as follows:  $x \in \mathcal{X} := \{x_j \in \mathbb{R} : \|x_j\| \leq \bar{x}_j, \forall j \in \mathbb{N}_{[1, n_x]}\}$  and  $u \in \mathcal{U} := \{u_i \in \mathbb{R} : \|u_i\| \leq \bar{u}_i, \forall i \in \mathbb{N}_{[1, n_u]}\}$ .

Without loss of generality, the origin is considered an equilibrium point of Eq. (1), in such way that  $f(0, 0) = 0$ . Moreover, the nonlinearity  $f(x, u)$  is of class  $\mathcal{C}^1$  over  $\mathcal{Z} := \mathcal{X} \times \mathcal{U}$ . The constraint set are also expressed as:

$$\begin{aligned} \mathcal{X} &:= \{x \in \mathbb{R}^{n_x} : H_x x \leq g_x\}, \\ \mathcal{U} &:= \{u \in \mathbb{R}^{n_u} : H_u u \leq g_u\}. \end{aligned}$$

### 2.2 qLPV Embedding

The nonlinear system satisfies the Linear Differential Inclusion property (Shamma, 2012), which means that it can be embedded under a qLPV formalism. Suppose that  $\exists H(x, u) \subseteq \mathbb{R}^{n_x \times n_x}, \forall (x, u) \in \mathcal{Z}$  s.t.  $f(x, u) := H(x, u)[x \ u]^T$ . Then, Eq. (1), for all  $(x, u) \in \mathcal{Z}$ , equivalent to:

$$\begin{aligned} x(k+1) &= A(\rho(k))x(k) + B(\rho(k))u(k) + Ew(k), \quad (2) \\ \rho(k) &= f_\rho(x(k)) \in \mathcal{P}. \quad (3) \end{aligned}$$

The model in Eq. (2) is qLPV, with an endogenous nonlinear function  $f_\rho(\cdot)$  that results in the scheduling parameters  $\rho(k)$ , which are bounded and known at each instant  $k$ . We note, nonetheless, that they are generally unknown for any future instant  $k+j, \forall j \in \mathbb{N}_{[1, \infty]}$ . We consider the following set:  $\mathcal{P} := \{\rho_j \in \mathbb{R} \mid \underline{\rho}_j \leq \rho_j \leq \bar{\rho}_j, \forall j \in \mathbb{Z}_{[1, n_\rho]}\}$ .

*Assumption 1.* The scheduling variables  $\rho(k)$  evolve along the horizon with bounded rates of variation. This is:  $\delta\rho(k+1) = (\rho(k+1) - \rho(k)) \in \delta\mathcal{P}$ , with:

$$\delta\mathcal{P} := \{\delta\rho_j \in \mathbb{R} : \underline{\delta\rho}_j \leq \delta\rho_j \leq \bar{\delta\rho}_j, \forall j \in \mathbb{N}_{[1, n_\rho]}\}. \quad (4)$$

*Remark 1.* Assuming bounds upon the variation of scheduling parameters between sampling instants is quite reasonable in many real LPV applications (Jungers et al., 2011). Nonetheless, if the qLPV-embedded nonlinear dynamics are s.t.  $\delta\rho$  is unbounded, the proposed method can be deployed considering uncertainty propagation as based on the whole scheduling set  $\mathcal{P}$ , which leads to further conservatism, but with no loss of generality.

### 2.3 Closed-loop Paradigm

We consider that the qLPV system is regulated according to the following LPV state-feedback control law:

$$u(k) = \pi(x(k), v(k)) = K(\rho(k))x(k) + v(k), \quad (5)$$

where the virtual input  $v(k)$  is used to ensure constraints satisfaction, and the parameter-dependent feedback gain  $K(\rho(k))$  attenuates the propagation of disturbances. The closed-loop dynamics are given by:

$$\begin{aligned} x(k+1) &= (A(\rho) + B(\rho)K(\rho))x(k) + B(\rho)v(k) + Ew(k), \\ &= A_\pi(\rho)x(k) + B(\rho)v(k) + Ew(k). \quad (6) \end{aligned}$$

We note that the process constraints  $[x^T(k) \ u^T(k)] \in \mathcal{Z}$  can be expressed in terms of the closed-loop dynamics and virtual entry as  $\begin{pmatrix} x(k) \\ v(k) \end{pmatrix} \in \mathcal{Z}_\pi$ , with:

$$\mathcal{Z}_\pi := \left\{ z \in \mathbb{R}^{n_x+n_u} : \begin{pmatrix} H_x & 0 \\ H_u K(\rho) & H_u \end{pmatrix} z \leq \begin{pmatrix} g_x \\ g_u \end{pmatrix} \right\}. \quad (7)$$

The control law in Eq. (5) will be defined by an MPC algorithm based on a frozen prediction model, which gives the sequence of future virtual inputs  $\mathbf{v}_{[k,k+N_p-1]} = \{v(k|k) \dots v(k+N_p-1|k)\}$  as output, considering a prediction horizon of  $N_p$  steps. The MPC recursive feasibility property ensured through terminal ingredients (Theorem 1) is not violated by the time-varying nature of  $\mathcal{Z}_\pi$ , since the energy-dissipation arguments hold for the wider set  $\mathcal{Z}$ .

We proceed by detailing disturbance propagation, the predictive control design, and the used terminal ingredients.

#### 2.4 Disturbance Propagation

Given an initial condition  $x(k) \in \mathbb{R}^n$ , the future trajectory of the closed-loop system in Eq. (6) evolves, for  $j \geq 0$ , as:

$$\begin{aligned} x(k+j) &= \phi_\pi(j, x(k), \mathbf{v}_{[k,k+j-1]}, \mathbf{w}_{[k,k+j-1]}, \rho_{[k,k+j-1]}) \\ &:= A_j(\rho_{[k,k+j-1]})x(k) + B_j(\rho_{[k,k+j-1]})\mathbf{v}_{[k,k+j-1]} \\ &\quad + E_j\mathbf{w}_{[k,k+j-1]}, \end{aligned} \quad (8)$$

where  $\mathbf{w}_{[k,k+j]} = \text{col}\{w(k+j|k)\}$ ,  $\forall j \in \mathbb{N}_{[0,j]}$  gives the future disturbances. It is worth noting that  $\rho_{[k,k+j]} = \text{col}\{\rho(k+j|k)\}$ ,  $\forall j \in \mathbb{N}_{[0,j]}$  gives the future values of the scheduling parameters, which are **unknown** (function of the future states). Matrices  $A_j$ ,  $B_j$  and  $E_j$  are analytically given in (González Cisneros and Werner, 2020).

Since both  $\rho_{[k,k+j-1]}$  and  $\mathbf{w}_{[k,k+j-1]}$  are unknown from the viewpoint of sampling instant  $k$ , we pursue the development of an MPC algorithm based on nominal prediction with a "frozen" qLPV scheduling guess, as if  $\rho(k)$  would remain constant along the future horizon, and as if the disturbances were null. These nominal predictions are:

$$\begin{aligned} x(k+j|k) &= \phi_\pi(j, x(k), \mathbf{v}_{[k,k+j-1]}, \mathbf{0}, \rho(k)\mathbf{1}_{n_p \times j}) \\ &= A_j(I_\rho(k))x(k) + B_j(I_\rho(k))\mathbf{v}_{[k,k+j-1]}, \end{aligned} \quad (9)$$

where  $I_\rho(k) := \rho(k)I_{n_p \times j}$  denotes the frozen scheduling guess that replaces the real future scheduling values.

There is a clear mismatch between the nominal predictions (9) and real qLPV trajectories (8). Therefore, in order to guarantee recursive feasibility of the MPC, *one-step ahead* disturbance propagation sets  $\mathcal{S}(j) \subseteq \mathbb{R}^n$ ,  $j = 0 \dots N_p$  are developed (Santos et al., 2019).

*Condition 1.* The one-step-ahead disturbance propagation sets  $\mathcal{S}(j)$ ,  $j \in \mathbb{N}_{[0,N_p]}$ , must be compact and satisfy the following conditions:

- (1)  $\mathcal{S}(0)$  contains  $EW$ .
- (2) For all  $x_a, x_b \in \mathbb{R}^{n_x}$ ,  $v \in \mathbb{R}^{n_u}$ ,  $\rho_a, \rho_b \in \mathcal{P}$  and  $j \in \mathbb{N}_{[1,N_p]}$ , with  $(x_a, v) \in \mathcal{Z}_\pi \ominus (\mathcal{S}(j-1) \times \{0\})$ ,  $(x_b - x_a) \in \mathcal{S}(j-1)$ , and  $(\rho_b - \rho_a) \in \delta\mathcal{P}$ , the set  $\mathcal{S}(j)$  must be such that  $(A_\pi(\rho_b)x_b + B(\rho_b)v) - (A_\pi(\rho_a)x_a + B(\rho_a)v) \in \mathcal{S}(j)$ .

With  $x_a = x(k+1|k) = A(\rho_a)x + B(\rho_a)v$ ,  $x_b = x(k+1) = A(\rho_b)x + B(\rho_b)v$ ,  $\rho_a = \rho(k-1)$  and  $\rho_b = \rho(k)$ , we have  $x_b - x_a \in EW$  and  $(\rho_b - \rho_a) \in \delta\mathcal{P}$ . It follows by induction from Condition 1 that  $x(k+j|k+1) \in x(k+j|k) \oplus \mathcal{S}(j -$

$1)$ ,  $j = 1 \dots N_p + 1$  for any admissible sequence  $\mathbf{v}_{[k,k+j-1]}$ . The sets  $\mathcal{S}(j)$  thus bound the one-step-ahead prediction error between the nominal MPC predictions and the real trajectories.

We note that, in the linear setting, the smallest sets that satisfy Condition 1 can be directly computed through linear expansions of the closed-loop dynamics (Ferramosca et al., 2012). Anyhow, for generic nonlinear systems, there are no efficient algorithms for the exact computation of these sets (Köhler et al., 2018), which means that more conservative strategies, taking into account upper bounds on the uncertainty propagation through the nonlinear system, must be sought.

One simple alternative for the computation of these sets in the nonlinear case is to bound the disturbance propagation with Lipschitz constants, e.g. Theorem 1 of (Morato et al., 2020a). In this case, for nonlinear closed-loop dynamics  $x(k+1) = f_\pi(x(k), v(k)) + w(k)$ , Lipschitz constant  $\Gamma \in \mathbb{R}$  satisfying inequality  $\|f_\pi(x_a, v) - f_\pi(x_b, v)\| \leq \Gamma\|x_a - x_b\|$  for any  $(x_a^T, v^T)^T, (x_b^T, v^T)^T \in \mathbb{Z}_\pi$  and  $\mathcal{S}(0) := \{x \in \mathbb{R}^{n_x} : \|x\| \leq \bar{w}\} \supseteq EW$ . Then, the Lipschitz sets  $\mathcal{S}(j), \forall j \in \mathbb{N}_{[0,N_p]}$  that satisfy the *one-step-ahead* disturbance propagation condition are:

$$\mathcal{S}(j) := \{x \in \mathbb{R}^{n_x} : \|x\| \leq \Gamma^j \bar{w}\}, \forall j \in \mathbb{N}_{[0,N_p]}. \quad (10)$$

Lipschitz disturbance propagation sets, in the form of Eq. (10), are often quite conservative, since they inherently consider the worst-case disturbance propagation in all directions. Less conservative sets can be obtained by zonotope-based disturbance propagation methods, as described in (Cunha and Santos, 2020).

Furthermore, predictions that rely directly on nonlinear models result in nonlinear and non-convex optimisation problems, which are computationally costly and do not guarantee global optimality. Herein, we provide an extension of the zonotopic disturbance propagation method considering the qLPV setting, developing zonotopes  $\mathcal{S}(j)$  that satisfy Condition 1.

*Remark 2.* The qLPV embedding in Eq. (2) stands for a realisation of the nonlinear dynamics with only state-dependent scheduling parameter. This framework holds for many applications. Nevertheless, in some cases, the scheduling parameters are state- and input-dependent, i.e.  $\rho = f_\rho(x, u)$ . Thereby, alternative formulations to Condition 1 should be sought, since the uncertainty will propagate along the horizon depending on the future control inputs, which may belong to a wide set, thus leading to poor performances. One option is to represent the system as nonlinear parameter varying (NLPV), with an explicit input-dependent nonlinearity that can be handled with Lipschitz propagation, as done in (Morato et al., 2020a).

#### 2.5 Controller Design

This process must be regulated in such way that the state trajectories are steered to the origin, in an admissible manner, despite disturbances and the discrepancy between the nominal prediction model and the real qLPV model. As discussed, we apply a robust MPC scheme, for which contracted constraints are used. Considering an initial

constraint set  $\mathcal{Z}_\pi(0) = \mathcal{Z}_\pi$ , the following sets for  $j \in \mathbb{N}_{[1, N_p]}$  are iteratively taken as:

$$\mathcal{Z}_\pi(j+1) = \mathcal{Z}_\pi(j) \ominus (S(j) \times \{0\}). \quad (11)$$

Therefore, at each sampling instant  $k$ , we measure the state  $x(k)$ , compute the scheduling parameter  $\rho(k)$ , and solve the following optimization problem, which embeds the performance objectives of the system, as well as the operational constraints:

$$\begin{aligned} \min_{\mathbf{v}_{[k, k+N_p-1]}} \quad & V(x(k+N_p|k)) \\ & + \sum_{j=0}^{N_p-1} \ell(x(k+j|k), v(k+j|k)) \end{aligned} \quad (12)$$

$$\begin{aligned} \text{s.t.} \quad & x(k+j+1|k) = A_\pi(\rho(k))x(k+j|k) \\ & + B(\rho(k))v(k+j|k), j \in \mathbb{N}_{[0, N_p-1]} \\ & (x^T(k+j|k) v^T(k+j|k))^T \in \mathcal{Z}_\pi(j), j \in \mathbb{N}_{[0, N_p-1]} \\ & x(k+N_p|k) \in \mathbf{X}_f, \end{aligned} \quad (13)$$

where  $\ell(x, v) = \|x\|_Q^2 + \|v\|_R^2$  is a quadratic stage cost,  $V(\cdot)$  is a terminal cost and  $\mathbf{X}_f$  is a terminal set. We denote  $\mathbf{v}_{[k, k+N_p-1]}^*$  as the optimal solution of this problem, of which the first entry  $v^*(k|k)$  is applied to the process according to Eq. (5).

## 2.6 Terminal Ingredients

We proceed by detailing the terminal ingredients and the feedback gain such that input-to-state stability and recursive feasibility properties are ensured.

In order to ensure these properties, first of all, the cost functions  $\ell$  and  $V$  must be  $\mathcal{K}$ -class bounded, as details Mayne et al. (2000):

$$\ell(x, v) \geq \beta_\ell(\|x\|), \forall x \in \mathcal{X}, \beta_\ell(\cdot) \in \mathcal{K}\text{-class}, \quad (15)$$

$$0 \leq V(x) \leq \beta_V(\|x\|), \forall x \in \mathcal{X}, \beta_V(\cdot) \in \mathcal{K}\text{-class}. \quad (16)$$

In this paper, we choose parameter-dependent quadratic terminal ingredients, being  $X_f := \{x \mid x^T P(\rho)x \leq 1\}$  an ellipsoidal terminal set and a sub-level Lyapunov terminal cost:  $V(x) = x^T P(\rho)x$ . Likewise, we consider a terminal qLPV feedback  $K(\rho)$ , which is given in terms of the positive-definite matrix  $P(\rho)$ .

The terminal set  $\mathbf{X}_f$  must be robust positively invariant regarding the closed-loop dynamics of Eq. (6), as provide the following Theorems.

*Theorem 1.* Mayne et al. (2000)

Suppose  $\exists u = K(\rho)x$ . Consider that the MPC is given by Eq. (12), with a terminal state set given by  $\mathbf{X}_f(\rho)$  and a terminal cost  $V(x, \rho)$ . Then, input-to-state stability is ensured if the following conditions hold  $\forall \rho \in \mathcal{P}$ :

- (C1) The origin lies in the interior of  $\mathbf{X}_f(\rho)$ ;
- (C2) Any consecutive state to  $x$ , in closed-loop given by  $(A(\rho) + B(\rho)K(\rho))x$  lies within  $\mathbf{X}_f(\rho)$ ;
- (C3) The discrete Lyapunov equation is verified within this invariant set, this is,  $\forall x \in \mathbf{X}_f(\rho)$  and  $\forall \rho \in \mathcal{P}$  and  $\forall \delta\rho \in \delta\mathcal{P}$ :  $V((A(\rho) + B(\rho)K(\rho))x, \rho + \delta\rho) - V(x, \rho) \leq -x^T Q x - x^T (K(\rho))^T R K(\rho)x$ .

(C4) The image of the nominal feedback lies within the admissible control domain:  $K(\rho)x \in \mathcal{U}, \forall \rho \in \mathcal{P}$ .

(C5) The terminal set  $\mathbf{X}_f(\rho)$  is a subset of  $\mathcal{X}$ .

Assuming that the initial solution of the MPC problem  $\mathbf{v}_{[0, N_p-1]}^*$  is feasible, then, the MPC is recursively feasible, stabilizing the state origin.

*Proof 1.* Refer to (Mayne et al., 2000).  $\square$

*Theorem 2.* Robust Positively Invariant Terminal set  
Assume that there exists an ellipsoidal terminal set  $\mathbf{X}_f$ .  $\mathbf{X}_f$  is a robust positively invariant set iff, for any  $x \in \mathbf{X}_f$  and  $\rho \in \mathcal{P}$ , i.e.  $x^T P(\rho)x \leq 1$ , it follows that:

$$x^T A_\pi^T(\rho)P(\rho + \delta\rho)A_\pi(\rho)x \leq 1. \quad (17)$$

*Proof 2.* It is trivial to verify Eq. (17) by using  $x(k), x(k+1) \in \mathbf{X}_f, \forall k \in \mathbb{N}$ .  $\square$

*Theorem 3.* Terminal Ingredients

The conditions (C1)-(C5) of Theorem 1 and the inequality of Theorem 2 are satisfied if there exist a symmetric parameter-dependent positive definite matrix  $P(\rho) : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_x \times n_x}$ , a parameter-dependent rectangular matrix  $W(\rho) : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{n_u \times n_x}$ , and a scalar  $0 < \lambda \in \mathbb{R}$  such that  $Y(\rho) = (P(\rho))^{-1} > 0$ ,  $W(\rho) = K(\rho)Y(\rho)$  such that LMIs (19)-(21) and the BMI (18) hold for  $w^\dagger$  given as the vertices of  $\mathcal{S}(N_p)$ , and for all  $\rho \in \mathcal{P}$  and  $\delta\rho \in \delta\mathcal{P}$ , under the minimization of  $\log \det\{Y(\rho)\}$ .

$$\begin{aligned} & \left[ \begin{array}{c|c} \lambda Y(\rho) & \star \\ \hline 0 & (1-\lambda) \end{array} \middle| \begin{array}{c} \star \\ \star \end{array} \right] > \alpha(18) \\ & \left[ \begin{array}{c|c} Y(\rho) & \star \\ \hline (A(\rho)Y(\rho) + B(\rho)W(\rho)) & Y(\rho + \delta\rho) \end{array} \middle| \begin{array}{c} \star \\ \star \end{array} \right] \geq \alpha(19) \\ & \left[ \begin{array}{c|c} Y(\rho) & 0 \\ \hline W(\rho) & 0 \end{array} \middle| \begin{array}{c} Q^{-1} \\ \star \\ 0 \\ R^{-1} \end{array} \right] \geq \alpha(19) \end{aligned}$$

$$\left[ \begin{array}{c|c} \bar{u}_i^2 & I_{\{i\}}W(\rho) \\ \hline \star & Y(\rho) \end{array} \right] \geq 0, i \in \mathbb{N}_{[1, n_u]}, \quad (20)$$

$$\left[ \begin{array}{c|c} \bar{x}_j^2 & I_{\{j\}}Y(\rho) \\ \hline I_{\{j\}}^T & Y^T(\rho) \end{array} \middle| \begin{array}{c} Y(\rho) \end{array} \right] \geq 0, j \in \mathbb{N}_{[1, n_x]}. \quad (21)$$

The proof of Theorem 3 is provided in the Appendix. This Theorem ensures a positive definite parameter dependent matrix  $P(\rho)$  which is used to compute the MPC terminal ingredients  $V(\cdot)$  and  $\mathbf{X}_f$  such that input-to-state stability of the closed-loop is guaranteed. Furthermore, when the MPC is designed with these terminal ingredients, for any initial condition  $x(0) \in \mathbf{X}_f$ , the MPC is recursively feasible for all  $k > 0$ .

*Remark 3.* The BMI in Theorem 3 can be solved through simple bisection search over the optimization plane since  $0 < \lambda \leq 1$ , by construction, as argues Yang et al. (2016).

*Remark 4.* LMIs (20)-(21) in Theorem 3 require box-type constraints on each  $u_i$  and  $x_j$ . Nonetheless, these can be converted into more generic polyhedral constraints, if necessary. We chose box-type constraints for the simplicity of the proofs.

*Remark 5.* Theorem 3 provides infinite-dimensional inequalities, which must hold  $\forall \rho \in \mathcal{P}$  and  $\forall \delta\rho \in \delta\mathcal{P}$ . In practice, the solution can be found by enforcing the inequalities over a sufficiently dense grid of points  $(\rho, \delta\rho)$  along the  $\mathcal{P} \times \delta\mathcal{P}$  plane. Then, the solution can be verified

over a denser grid. The parameter-dependency of  $P$  may be dropped if the system is quadratically stabilizable, but this may result in quite conservative performances.

*Remark 6.* The solution of Theorem 3 is a parameter-dependent  $Y(\rho) = \sum_{j=1}^{n_p} \rho_j Y_j$ . The online procedure, nonetheless, depends on an inversion of  $Y(\rho)$  to compute  $V$ ,  $K(\rho)$  and  $\mathbf{X}_f$ , at each sampling instant.

### 3. UNCERTAINTY PROPAGATION VIA ZONOTOPES

Zonotopes are a particular class of convex, compact, and symmetrical polyhedrons (Pourasghar et al., 2019), which can be described by a Minkowsky sum of line segments or by the affine image of a unitary box  $\mathcal{B}_\infty^{n_g}$ :

$$Z = \{G, c\} = c \oplus G\mathcal{B}_\infty^{n_g}, \quad (22)$$

where  $c \in \mathbb{R}^n$  is the center of the zonotope and its shape is given by the rows of  $G \in \mathbb{R}^{n \times n_g}$  (generators). The number of generators  $n_g \geq n$  defines the complexity of the zonotope;  $n = n_g$  defines a parallelotope.

The use of zonotopes for set-based state estimation and disturbance propagation is tied to their simplicity and to the numerical efficiency of linear transformations and Minkowsky sums of zonotopes (Alamo et al., 2005), as well as the Pontryagin difference of a polyhedron by zonotope (Alvarado, 2007). Given  $Z_1 = \{G_1, c_1\}$ ,  $Z_2 = \{G_2, c_2\} \subseteq \mathbb{R}^n$ ,  $R \in \mathbb{R}^{m \times n}$ , it follows that:

$$\begin{cases} RZ_1 &= \{RG_1, Rc_1\}, \\ Z_1 \oplus Z_2 &= \{(G_1 \ G_2), c_1 + c_2\}. \end{cases} \quad (23)$$

We aim to compute zonotopic reachable sets  $\mathcal{S}(j), \forall j \in \mathbb{N}_{[0, N_p]}$  that satisfy Condition 1. For such, we develop Theorem 4, based on Lemma 1.

*Lemma 1.* Consider a centered zonotope  $X = M\mathcal{B}_\infty^{n_g} \subseteq \mathbb{R}^m$ , an interval matrix  $\mathbf{J} \in \mathbb{I}^{n \times m}$ , and a zonotope family  $\mathbf{Z} = \mathbf{J}\mathbf{X}$ . A zonotopic inclusion is defined by:

$$\diamond(\mathbf{Z}) := \text{mid}(\mathbf{J})\mathbf{X} \oplus P\mathcal{B}_\infty^{n_g}, \quad (24)$$

where  $P$  is a diagonal matrix such that:

$$P_{ii} = \sum_{j=1}^{n_g} \sum_{k=1}^m \text{rad}(\mathbf{J})_{ik} |M_{kj}|, \forall i \in \mathbb{N}_{[1, n]}. \quad (25)$$

It holds that  $\mathbf{Z} \subseteq \diamond(\mathbf{Z})$ .

*Proof 3.* Refer to (Rego et al., 2020).  $\square$

*Theorem 4.* Consider the qLPV system described in Eq. (6), with constraints  $(x(k), v(k)) \in \mathcal{Z}_\pi$ , disturbances  $w(k) \in \mathcal{W}$  and scheduling parameters  $\rho(k) \in \mathcal{P}$ ,  $\delta\rho(k) \in \delta\mathcal{P}$ . Let  $Z \subseteq \mathbb{R}^{n_x + n_u}$ ,  $\mathcal{S}_0 \subseteq \mathbb{R}^{n_x}$  be zonotopes and  $\mathbf{A}, \mathbf{\Delta}_A \in \mathbb{I}^{n_x \times n_x}$ ,  $\mathbf{\Delta}_B \in \mathbb{I}^{n_x \times n_u}$  interval matrices satisfying  $E\mathcal{W} \subseteq \mathcal{S}_0$ ,  $\mathcal{Z}_\pi \subseteq Z$ ,  $A_\pi(\rho_a) \in \mathbf{A}$ ,  $A_\pi(\rho_b) - A_\pi(\rho_a) \in \mathbf{\Delta}_A$  and  $B(\rho_b) - B(\rho_a) \in \mathbf{\Delta}_B$  for all  $\rho_a, \rho_b \in \mathcal{P}$ ,  $\rho_b - \rho_a \in \delta\mathcal{P}$ . The sets  $\mathcal{S}(j)$ ,  $j \in \mathbb{N}_{[0, N_p]}$  defined iteratively by  $\mathcal{S}(0) = \mathcal{S}_0$  and

$$\mathcal{S}(j) = \mathcal{V} \oplus \diamond(\mathbf{A}\mathcal{S}(j-1)), \quad j \in \mathbb{N}_{[1, N_p]}, \quad (26)$$

where  $\mathcal{V} = \diamond((\mathbf{\Delta}_A \ \mathbf{\Delta}_B) Z)$ , satisfy Condition 1.

*Proof 4.* The first condition is satisfied by design. Considering  $x_a, x_b \in \mathbb{R}^{n_x}$ ,  $v \in \mathbb{R}^{n_u}$ ,  $\rho_a, \rho_b \in \mathbb{R}^{n_\rho}$ , as given in the second condition for some  $j \in \mathbb{N}_{[1, N_p]}$ , and  $\Delta_j = (A_\pi(\rho_b)x_b + B(\rho_b)v) - (A_\pi(\rho_a)x_a + B(\rho_a)v)$ , we have:

$$\begin{aligned} \Delta_j &= (A_\pi(\rho_b) - A_\pi(\rho_a))x_a + A_\pi(\rho_b)(x_b - x_a) \\ &\quad + (B_\pi(\rho_b) - B_\pi(\rho_a))v \\ &\in (\mathbf{\Delta}_A \ \mathbf{\Delta}_B) \mathcal{Z}_\pi \oplus \mathbf{A}\mathcal{S}(j-1) \\ &\subseteq \mathcal{V} \oplus \diamond(\mathbf{A}\mathcal{S}(j-1)) = \mathcal{S}(j). \end{aligned}$$

Therefore, the sets  $\mathcal{S}(j)$  satisfy Condition 1.  $\square$

*Remark 7.* In the case of qLPV systems with  $A_\pi(\rho)$  and  $B(\rho)$  affine on  $\rho$ , it follows that  $A_\pi(\rho(k+1)) - A_\pi(\rho(k)) = \bar{A}_\pi(\delta\rho(k+1))$  and  $B(\rho(k+1)) - B(\rho(k)) = \bar{B}(\delta\rho(k+1))$ , with  $A_\pi(\cdot)$  and  $\bar{B}(\cdot)$  being linear mappings. Then, the interval matrices  $\mathbf{\Delta}_A$  and  $\mathbf{\Delta}_B$  can be computed directly from  $\delta\mathcal{P}$ . In the case of non-affine models, a simple alternative to convert them into affine by augmenting the number of scheduling parameters, which may result in slightly more conservatism.

*Remark 8.* Bounds on scheduling parameters' variations  $\delta\mathcal{P}$  can be obtained based on their dependence on the states  $\rho(k) = f_\rho(x(k))$ . Specifically, this can be done by placing bounds on  $\delta\rho = f_\rho(f(x, u) + Ew) - f_\rho(x)$  for  $(x, u), (f(x, u) + Ew, u) \in \mathcal{Z}$  and  $w \in \mathcal{W}$ , either by interval arithmetics or optimisation.

*Remark 9.* Due to the zonotope inclusion and Minkowsky sum, the number of generators of the zonotopes  $\mathcal{S}(j)$  increase by  $2n_x$  for each iteration. Methods for complexity reduction can be used to restrict the number of generators of each  $\mathcal{S}(j)$  to a predefined value (Scott et al., 2016).

*Remark 10.* We stress that the qLPV model in Eq. (2) requires a linear nominal prediction model, as in Eq. (9), which changes at each sampling instant  $k$ , since it is based on a frozen prediction for the future scheduling parameters. Therefore, the propagation of disturbances along the horizon is crucial, but may yield conservative sets  $\mathcal{S}(j)$ . This is nonetheless expected, since we are using a linear time-invariant model to make predictions on a nonlinear system. Nonlinear predictions can result in less conservative sets  $\mathcal{S}(j)$ , but also require more online computational cost, as discussed in Section 2.4.

### 4. APPLICATION EXAMPLE

For illustration purposes of the proposed method, we consider a nonlinear model from (Lazar et al., 2008), which represents a DC-DC Buck-Boost converter system. The nonlinear dynamics are:

$$f(x, u) = \begin{pmatrix} x_1 + \alpha_1 x_2 + (\beta_1 - \gamma_2 x_2) u \\ -\alpha_2 x_1 + \alpha_3 x_2 + (\beta_2 + \gamma_1 x_1) u \end{pmatrix}, \quad (27)$$

which can be embedded into the qLPV form of Eq. (2) with:

$$A(\rho) = \begin{pmatrix} 1 & \alpha_1 \\ -\alpha_2 & \alpha_3 \end{pmatrix}, \quad B(\rho) = \begin{pmatrix} \beta_1 - \rho_2 \\ \beta_2 + \rho_1 \end{pmatrix}, \quad (28)$$

$$f_\rho(x) = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} x, \quad E = I_{n_x}. \quad (29)$$

The states  $x_1$  and  $x_2$  represent, respectively, the inductor current and the output tension, while  $u$  is a duty-cycle input signal. The model parameters are:  $\alpha_1 = 0.0541\Omega^{-1}$ ,

$\alpha_2 = 0.1033\Omega$ ,  $\alpha_3 = 0.9909$ ,  $\beta_1 = 2.619A$ ,  $\beta_2 = 0.2400V$ ,  $\gamma_1 = 0.2273\Omega$  and  $\gamma_2 = 0.1190\Omega^{-1}$ .

The state and input constraints are:  $x(k) \in \mathcal{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq \bar{x} = 1\}$  and  $u(k) \in \mathcal{U} = \{u \in \mathbb{R} : |u| \leq \bar{u} = 0.1\}$ . The scheduling parameter constraints are:

$$\rho \in \mathcal{P} = \{\rho \in \mathbb{R}^2 : |\rho_1| \leq \gamma_1 x_{max}, |\rho_2| \leq \gamma_2 x_{max}\}. \quad (30)$$

In order to compute the bounds on  $\delta\rho$ , one can use either constrained optimisation or interval arithmetic methods as described in Remark 8. The tightest bounds were obtained through constrained optimisation, which gives:

$$\delta\mathcal{P} := \left\{ \delta\rho \in \mathbb{R}^2 : \begin{pmatrix} -0.0859 \\ -0.0249 \end{pmatrix} \leq \delta\rho \leq \begin{pmatrix} 0.0805 \\ 0.0195 \end{pmatrix} \right\}. \quad (31)$$

The system operates subject to additive load disturbance bounded to the box  $w(k) \in \mathcal{W} := \{w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.02\}$ . The following results are obtained with the aid of Matlab, yalmip, SDPT3, fmincon and Gurobi.

We apply our proposed zonotopic MPC with stage cost weights  $Q = I_{n_x}$  and  $R = I_{n_u}$ . The chosen prediction horizon is  $N_p = 4$ . The parameter-dependent matrices  $W(\rho)$  and  $Y(\rho)$ , used to compute the terminal cost  $V(x) = x^T Y^{-1}(\rho)x$  and feedback gain  $K(\rho) = W(\rho)Y^{-1}(\rho)$ , found by the bisection LMI solution of Theorem 3 with  $\lambda = 0.3$ , are:

$$Y(\rho) = \rho_1 \begin{bmatrix} -0.3014 & -0.0304 \\ \star & -0.2216 \end{bmatrix} + \rho_2 \begin{bmatrix} -1.0018 & -0.0559 \\ \star & -0.8395 \end{bmatrix},$$

$$W(\rho) = \rho_1 [0.1130 \ 0.0221] + \rho_2 [0.3814 \ 0.0661].$$

For illustration purposes, we compare the performances obtained with our method against those obtained with "full-blown" NMPC method with Lipschitz disturbance propagation (Santos et al., 2019). We note that this NMPC approach requires the solution of a nonlinear program at each sampling period, which is numerically much more costly than the online program of the proposed method, which benefits from the linear predictions of the qLPV realisation, as gave Eq. (9).

First and foremost, we show the disturbance propagation reachable sets  $\mathcal{S}(j), \forall j \in \mathbb{N}_{[1, N_p]}$  with Zonotope disturbance propagation. These sets are computed according to Theorem 4. In Fig. 1, we show the collection of sets  $\mathcal{S}(j)$  over the  $x_1 \times x_2$  plane (Condition 1).

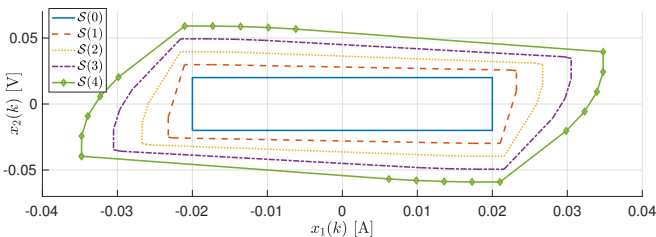


Fig. 1. Zonotopic sets  $\mathcal{S}(j)$ .

Then, considering the same uniformly randomly generated disturbance vector  $w_{[0,30]}$  with unitary seed and entries inside  $\mathcal{W}$ , we evaluate the closed-loop performances with these two robust MPC algorithms, considering the initial condition  $x(0) = (0.5 \ -0.5)^T$ . In Fig. 2, we show the

state feasibility set  $\mathcal{X}$ , the obtained closed-loop trajectories for 20 discrete-time samples, and the terminal set  $\mathbf{X}_f$  (parameter-dependent ellipsoid for the proposed method and polyhedron for the Lipschitz robust MPC, as detailed in (Santos et al., 2019)). The results were obtained in a 2.4 GHz, 8 GB RAM Macintosh computer. We note both approaches are able to stabilise the system at the origin, as well as to ensure input feasibility  $u \in \mathcal{U}$ .

In order to stress the advantages of the proposed method, we assess the obtained closed-loop trajectories in terms of performances indexes. In Tab. 1, we provide the RMS values for each state and for the stage cost  $\ell(x, u)$ , along the whole simulation. With these indexes, we can conclude that the proposed robust Zonotopic qLPV MPC method can obtain good performances, with a reasonable increase of between 13% and 25% on the RMS values compared to the "full-blown" NMPC. This performance deterioration derives from the model-process mismatches by using a "frozen qLPV scheduling guess". Anyhow, we must point out that the average online computational stress ( $t_c$  index) with the proposed solution is much smaller than the NMPC one (less than 2.8 times smaller), since the nominal predictions are linear at each sampling instant, resulting in simpler, convex, optimisation problems. This indicates the flexibility of the proposed method for embedded applications, as well as its robustness against bounded additive uncertainties and the inherent model-process mismatches derived using the frozen qLPV scheduling predictions.

As a final comment, we indicate that the use of scheduling prediction guesses, as done in (Morato et al., 2020b) would certainly refine the closed-loop performances, since the mismatches between the nominal linear prediction and the real qLPV trajectories would decrease. Moreover, the uncertainty propagation w.r.t. these mismatches would also lead to even smaller zonotopes. These predictions guesses could be formulated by iterating the MPC QP (González Cisneros and Werner, 2020) or through recursive mechanisms (Morato et al., 2019). We note that using such scheduling prediction estimates would not compromise the real-time capabilities of the proposed method, since the model predictions  $\phi_\pi(\cdot)$  would still be linear and resulting optimization would still be convex. We also stress that the case of stochastic disturbances can be directly treated by the zonotopic inclusions of the proposed method, similarly to what is done in (Santos et al., 2019).

Table 1. Performance Results

Method	RMS $\{x_1\}$	RMS $\{x_2\}$	RMS $\{\ell(\cdot)\}$	$t_c$
Lipschitz	0.3358	0.2626	0.3386	50 ms
Zonotope	0.3813	0.3253	0.4233	17.5 ms

## 5. CONCLUSIONS

In this paper, we provided a novel MPC algorithm for nonlinear systems represented through qLPV models. The method uses a frozen scheduling parameter guess as a prediction model, and bounds the propagation of model-process mismatches and additive load disturbances with zonotopes. We offer LMI-solvable remedies to compute a stabilising LPV feedback gain and parameter-dependent terminal conditions, used to ensure recursive feasibility of the optimisation procedure. The terminal set is a robust

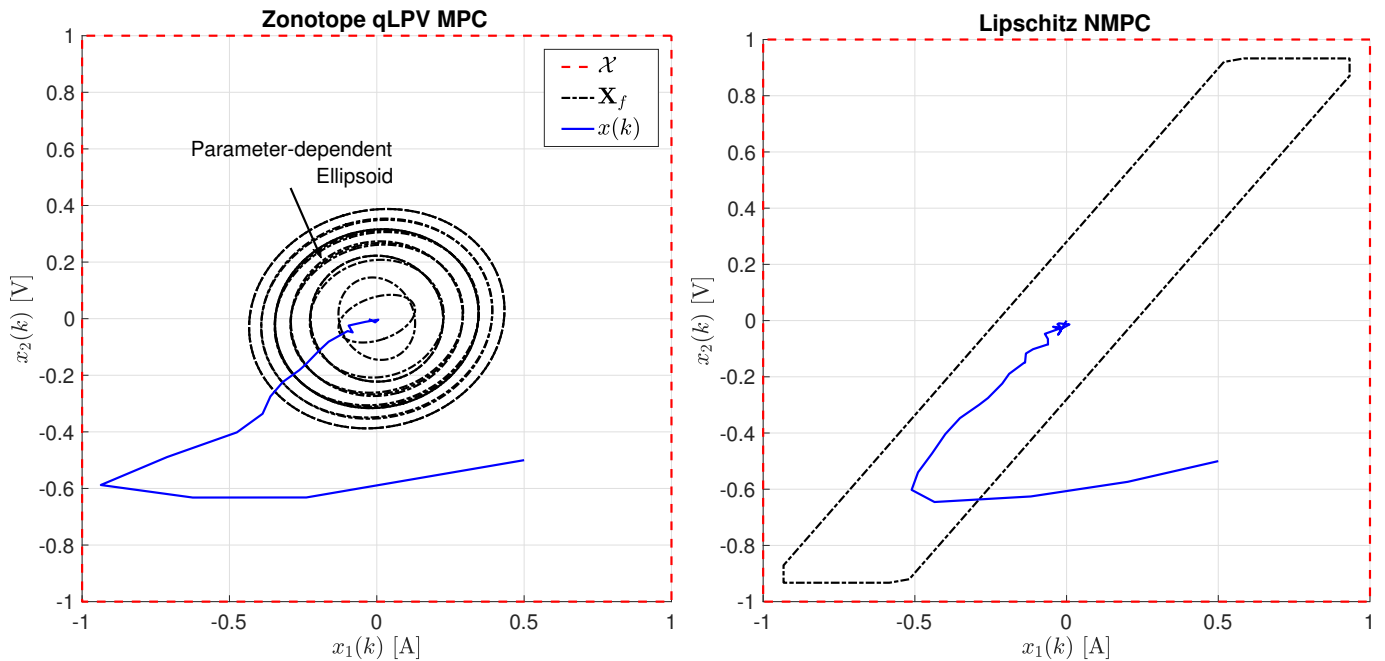


Fig. 2. Closed-Loop trajectories and terminal set  $\mathbf{X}_f$ : Zonotope (left) and Lipschitz (right).

positively invariant ellipsoid. We compare our method against a NMPC with Lipschitz disturbance propagation. As a result, we find similar performances with smaller computational stress with the proposed scheme, which benefits from the linear predictions of the qLPV realisation. While the method is ready for embedded applications (the online stress is similar to that of a constrained QP), we still plan on investigating conservatism reduction of the solution as well as its adaptation for tracking purposes.

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## APPENDIX

### PROOF OF THEOREM 3

We begin by showing the positive invariance of the ellipsoid. Applying the  $\mathcal{S}$ -procedure, with  $\lambda > 0$  to (17) and ( $x^T P(\rho)x \leq 1$ ), we get:

$$x^T A_\pi^T(\rho)P(\rho + \delta\rho)A_\pi(\rho)x + \lambda(1 - x^T P(\rho)x) < 0,$$

which can be rewritten as:

$$\left[ x^T P^T \quad I \right] \overbrace{\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \star & \Pi_{22} \end{bmatrix}}^{\Pi} \begin{bmatrix} Px \\ I \end{bmatrix} < 0. \quad (32)$$

$$\Pi_{11} = Y(\rho)A_\pi^T(\rho)P(\rho + \delta\rho)A_\pi(\rho)Y(\rho) - \lambda Y(\rho), \quad (33)$$

$$\Pi_{12} = Y A_\pi^T(\rho)P(\rho + \delta\rho)w^\dagger, \quad (34)$$

$$\Pi_{22} = (w^\dagger)^T P(\rho + \delta\rho)w^\dagger + (\lambda - 1). \quad (35)$$

Applying a Schur complement over  $P(\rho + \delta)$  for each entry of  $\Pi$  leads to (18), as detail Limón et al. (2008). This ensures Theorem 2.

Complementary, we proceed by demonstrating that the resulting  $P(\rho)$  satisfies all five conditions of Theorem 1. (C1) trivially holds due to the ellipsoidal form of  $\mathbf{X}_f$ . (C2) is verified due to the fact that  $\mathbf{X}_f$  is a sub-level set of the terminal cost  $V(\cdot)$ . Therefore, if condition (C3) is verified, (C2) is consequently ensured.

The discrete Ricatti condition (C3) is verified through the solution of LMI (19). Since  $Q^{-1} > 0$ ,  $R^{-1} > 0$  and  $Y(\rho + \delta\rho) > 0$ , we can take  $W(\rho) = K(\rho)Y(\rho)$  and apply two consecutive Schur, complements. This procedures leads to:

$$\begin{aligned} & \left( Y(\rho) (A(\rho) + B(\rho)K(\rho))^T \right) (Y(\rho + \delta\rho))^{-1} \\ & \quad (A(\rho) + B(\rho)K(\rho))Y(\rho) - Y(\rho) \leq \\ & \quad -Y(\rho)QY(\rho) - Y(\rho)(K(\rho))^T R K(\rho)Y(\rho) \quad . \end{aligned}$$

This condition can be pre and post-multiplied by  $x^T P(\rho)$  and  $P(\rho)x$ , respectively, which leads to:

$$\begin{aligned} & x^T (A(\rho) + B(\rho)K(\rho))^T P(\rho + \delta\rho) (A(\rho) + B(\rho)K(\rho)) x \\ & \quad - x^T P(\rho)x \leq -x^T Qx - x^T (K(\rho))^T R K(\rho)x \quad . \end{aligned}$$

This inequality is a sufficient condition for (C3) with  $V(\cdot)$  as a sub-level of  $\mathbf{X}_f$ .

The fourth and fifth conditions (C4-C5) are verified by the direct application of the Schur complement to Eq. (20) and Eq. (21), respectively, using  $W(\rho) = K(\rho)Y(\rho)$ . They lead, respectively, to:

$$\begin{aligned} & (I_{\{i\}}K(\rho)) (Y(\rho)) (I_{\{i\}}K(\rho))^T \leq \bar{u}_i^2 \quad . \\ & \quad I_{\{j\}}^T (Y(\rho)) I_{\{j\}} \leq \bar{x}_j^2 \quad . \end{aligned}$$

Since the maximum normed  $Fx$  of an  $x$  that belongs to some ellipsoid  $x^T P x \leq 1$  is given by  $\sqrt{F^T (P^{-1}) F}$ , it holds that the first inequality implies that the projection  $I_{\{i\}}K(\rho)x$  (i.e.  $i$ -th control signal) is upper-bounded, in norm, by  $\bar{u}_i$ , which satisfies (C4). Analogously, the second inequality ensures that the projection  $I_{\{j\}}x$  (i.e.  $j$ -th state) is norm-bounded by  $\bar{x}_j$ , which satisfies condition (C5). This concludes the proof.  $\square$