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AN INTEGRABILITY CRITERION FOR A PROJECTIVE LIMIT OF BANACH DISTRIBUTIONS

FERNAND PELLETIER

ABSTRACT. We give an integrability criterion for a projective limit of Banach distributions on a Fréchet manifold which is a projective limit of Banach manifolds. This leads to a result of integrability of projective limit of involutive bundles on a projective sequence of Banach manifolds. This can be seen as a version of Frobenius Theorem in Fréchet setting. As consequence, we obtain a version of the third Lie theorem for a Fréchet-Lie group which is a submersive projective limit of Banach Lie groups. We also give an application to a sequence of prolongations of a Banach Lie algebroid.

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1. INTRODUCTION

In classical differential geometry, a *distribution* on a smooth manifold M , is an assignment $\Delta : x \mapsto \Delta_x \subset T_x M$ on M , where Δ_x is a subspace of $T_x M$. This distribution is *integrable* if, for any $x \in M$, there exists an immersed submanifold $f : L \rightarrow M$ such that $x \in f(L)$ and for any $z \in L$, we have $Tf(T_z L) = \Delta_{f(z)}$. On the other hand, Δ is called *involutive* if, for any vector fields X and Y on M tangent to Δ , their Lie bracket $[X, Y]$ is also tangent to Δ .

On a finite dimensional manifold, when Δ is a subbundle of TM , the classical Frobenius Theorem gives an equivalence between integrability and involutivity. In the other cases, the distribution is *singular* and, even under assumptions of smoothness on Δ , in general, the involutivity is not a sufficient condition for integrability (one needs some more additional local conditions). These problems were clarified and resolved essentially in [20] and [19].

In the context of Banach manifolds, the Frobenius Theorem is again true for distributions which are complemented subbundles in the tangent bundle (cf. [16]). For singular Banach distributions closed and complemented (i.e. Δ_x is a complemented Banach subspace of $T_x M$) we also have the integrability property under some natural geometrical conditions (see [5] for instance). In a more general way, weak Banach distributions Δ (i.e. Δ_x which is a Banach subspace of $T_x M$ not necessary complemented), the integrability property is again true under some additional geometrical assumptions (see [18] or [4] for more details).

The proof of this last results is essentially based on the existence of the flow of a local vector field. In a more general infinite dimensional context as distributions on convenient manifolds or on locally convex manifolds, in general the local flow for a vector field does not exist. Analog results exists in such previous settings: [12], [13], [8], [21] [2] for instance. But essentially, all these integrability criteria are proved under strong assumptions which, either implies the existence of a family of vector fields which are tangent and generate locally a distribution and each one of these vector fields have a local flow, or implies the

existence of an implicit function theorem in such a setting.

The purpose of this paper is to give an integrability criterion for projective limit of Banach distributions on a Fréchet manifold which is a projective limit of Banach manifolds. The precise assumptions on the distribution are presented in assumptions (*) in Definition 1) and this criterion is formulated in a local way in Theorem 2 and in a global way in Theorem 3. These results are obtained under conditions which permits to used a Theorem of existence and unicity of solution of ODE in a Fréchet space proved in [17], and which can be reformulated in the context of projective limit of Banach spaces (cf. Appendix C). Using such a Theorem, this proof needs, in the one hand, an adaptation of some arguments used in the proof of Theorem 1 in [18] for closed distributions on Banach manifolds, and on the other hand, some properties of the Banach Lie group of "uniformly bounded" automorphisms of a Fréchet space (cf. Appendix B). As application, this criterion permits to obtain a kind of *projective limit of "Banach Frobenius theorem"* for submersive projective limit of involutive bundles on a submersive projective sequence of Banach manifolds (cf. Theorem 4). By the way, as consequence, a submersive projective limit of complemented Banach Lie subalgebras of a submersive projective limit of Banach Lie group algebras is the Lie algebra of a Fréchet Lie group (cf. Theorem 5). This result can be considered as *a version of the third Lie Theorem for a Fréchet-Lie group* which is a submersive projective limit of Banach Lie groups. We also give an application to a sequence of prolongations of a Banach Lie algebroid (cf. Theorem 7). We complete these results, by an example of integrable Fréchet distribution which is a projective limit of non integrable distributions but which satisfies the assumptions (*).

This paper is organized as follows. The first paragraph of the next section describes the context and the assumptions of this criterion of integrability used in Theorem 2 and Theorem 3. In order to present these Theorems in a more accessible way for a quick reading, the useful definitions and results take place in Appendix A, and we formulate the assumptions with precise references to this Appendix. The Theorem 3 permits to show that, under some natural conditions, the projective limit E of a submersive projective sequence of involutive subbundles E_i of the tangent bundle TM_i of a submersive projective sequence of Banach manifold M_i , is a Fréchet involutive and integrable subbundle of the tangent bundle TM of the Fréchet manifold $M = \varprojlim M_i$.

A first application of these results, is the existence of a Fréchet Lie group whose Lie algebra is a submersive projective limit of complemented Banach Lie subalgebra of the a submersive projective limit of Banach Lie groups Lie algebras (cf. Theorem 5 in § 3.1). In § 3.2, we give an application of Theorem 2 to a sequence of prolongations of a Banach-Lie algebroid (see [3]) and we end this paragraph by the announced contre-exemple of Theorem 2 and Theorem 3.

The proof of the basic Theorem 2 is located in §4.

All properties concerning the set of uniformly bounded endomorphisms of a Fréchet space are developed in Appendix B. Also in a series of other Appendices, we expose all the definitions and results needed in the statements of Theorems and in the proof of Theorem 2.

2. AN INTEGRABILITY CRITERION FOR A SUBMERSIVE PROJECTIVE LIMIT OF BANACH DISTRIBUTIONS

2.1. The criterion and its corollaries. *The context needed in this section is detailed in Appendix A.*

Let $(M_i, \delta_i^j)_{j \geq i}$ be a sequence of projective Banach manifolds with projective limit $M = \varprojlim M_i$ ¹. In order to give a criterion of integrability for projective limits of local Banach

¹cf. section A.4

bundles on M under some additional assumptions, we need to introduce some notations. Let ν (resp. μ) be a norm on a Banach space \mathbb{E} (resp. \mathbb{M}). We denote by $\|\cdot\|^{\text{op}}$ the associated norm on the linear space of continuous linear mappings $\mathcal{L}(\mathbb{E}, \mathbb{M})$.

Then we have:

Definition 1. Let $(M_i, \delta_i^j)_{j \geq i}$ be a projective sequence of Banach manifolds where the maps δ_i^j are submersions and $M = \varprojlim M_i$ its projective limit.

A closed distribution Δ on M will be called a submersive projective limit of local anchored bundles if the following property is satisfied:

- (*) For any $x = \varprojlim x_i \in M$, there exists an open neighbourhood $U = \varprojlim U_i$ of x , a submersive projective sequence of anchored Banach bundles $(E_i, \pi_i, U_i, \rho_i)$ ^{2 3 4} fulfilling the following properties for any $z = \varprojlim z_i \in U$:
1. $\varprojlim \rho_i((E_i)_{z_i}) = \Delta_z$, for any $z \in U$.
 2. The kernel of $(\rho_i)_{z_i}$ is complemented in $(E_i)_{z_i}$ and the range of $(\rho_i)_{z_i}$ is closed, for all $i \in \mathbb{N}$.
 3. There exists a constant $C > 0$ and a Finsler norm $\|\cdot\|^{E_i}$ (resp. $\|\cdot\|^{M_i}$) on $(E_i)_{U_i}$ (resp. $TM_i|_{U_i}$) such that:

$$\forall i \in \mathbb{N}, \|(\rho_i)_{z_i}\|_i^{\text{op}} \leq C, \forall z_i \in U_i.$$
⁵

We have the following criterion of integrability:

Theorem 2. Let M be a projective limit of a submersive projective sequence $(M_i, \delta_i^j)_{j \geq i}$ of Banach manifolds and Δ be a local projective limit of local Banach bundles on M . Assume that, under the property (*), there exists a Lie bracket $[\cdot, \cdot]_i$ on $(E_i, \pi_i, U_i, \rho_i)$ such that $(E_i, \pi_i, U_i, \rho_i, [\cdot, \cdot]_i)$ is a submersive projective sequence of Banach-Lie algebroids⁶. Then the distribution Δ is integrable and the maximal integral manifold N through $x = \varprojlim x_i$ is a closed Fréchet submanifold of M which is a submersive projective limit of the set of maximal leaves N_i of $\rho_i(E_i)$ through x_i in M_i .

The proof of Theorem takes place in section 4.

Now, we have the following consequences of Theorem 2:

Theorem 3. Let $(E_i, \pi_i, M_i, \rho_i, [\cdot, \cdot]_i)$ be a submersive projective sequence of split Lie algebroids⁷. Then we have:

1. $(E := \varprojlim E_i, \pi := \varprojlim \pi_i, M := \varprojlim M_i, \rho = \varprojlim \rho_i)$ is Fréchet anchored bundle and $\Delta = \rho(E)$ is a closed distribution on M
2. If (ρ_i) satisfies the condition (3) in Definition 1, then Δ is integrable and each leaf L of Δ is a projective limit of leaves L_i of Δ_i .

Proof The property (1) is a consequence of Proposition 29. From this property, it follows that locally Δ satisfies assumption (1) and (2) of Definition 1 so if the assumption (3) is satisfied, the result is a direct consequence of Theorem 2. □

Theorem 4. Let $(M_i, \delta_i^j)_{j \geq i}$ be a submersive projective sequence of Banach manifold and (E_i, π_i, M_i) an involutive subbundle of TM_i where π_i is the restriction of the natural projection $p_{M_i} : TM_i \rightarrow M_i$. Assume that the restriction $T\delta_i^j : E_j \rightarrow E_i$ is a surjective

² see: Definition 25 and Notations 24 for a submersive sequence of projective Banach bundles

³ see: Definition 27 for an anchored Banach bundle

⁴ a sequence of projective anchored bundle $(E_i, \pi_i, M_i, \rho_i)$ is a projective sequence of Banach bundles which satisfies assumption **(PSBLA 2)** in Definition 28

⁵ More precisely, $\|(\rho_i)_{z_i}\|_i^{\text{op}} = \sup\{\|(\rho_i)_{z_i}(u)\|^{M_i}, \|u\|^{E_i} \leq 1\}$

⁶ cf. Definition 28

⁷ cf. Definition 28

bundle morphism for all $i \in \mathbb{N}$ and $j \geq i$. Then (E_i, π_i, M_i) is a submersive projective sequence of Banach bundles, and $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ is an integrable Fréchet subbundle of TM whose each leaf L of E in M is a projective limit of leaves L_i of E_i in M_i

Proof Since $\delta_i^j : M_j \rightarrow M_i$ is a surjective submersion, so is $T\delta_i^j : TM_j \rightarrow TM_i$. If $T\delta_i^j : E_j \rightarrow E_i$ is a surjective morphism, this implies that $T\delta_i^j$ is a submersion onto E_i and so (E_i, π_i, M_i) is a submersive projective sequence of Banach bundles. Let $\iota_i : E_i \rightarrow TM_i$ the natural inclusion and $[\cdot, \cdot]_i$ the restriction of Lie bracket of vector fields to (local) sections of E_i . Then $(E_i, M_i, \iota_i, [\cdot, \cdot]_i)$ is a Banach Lie algebroid and since $T\delta_i^j \circ \iota_j = \iota_i \circ T\delta_i^j$ it follows that $(E_i, M_i, \iota_i, [\cdot, \cdot]_i)$ is a submersive projective sequence of Banach-Lie algebroids. Fix some $x = \varprojlim x_i \in M = \varprojlim M_i$. According to Theorem 3 we have only to show that the condition (3) of Definition 1 is satisfied by ι_i .

Given any norm $\|\cdot\|^{M_i}$ on $T_{x_i}M_i$ we denote by $\|\cdot\|^{E_i}$ the induced norm on the fiber $\{E_i\}_{x_i}$, then, for the associated norm operator we have $\|\{\iota_i\}_{x_i}\|_i^{\text{op}} = 1$. So all the assumption of Theorem 2 are satisfied which ends the proof. \square

3. SOME APPLICATIONS AND CONTRE-EXAMPLE

3.1. Application to submersive projective sequence of Banach Lie groups. Let $(G_i, \delta_i^j)_{j \geq i}$ be a submersive a projective sequence of Banach-Lie groups where G_i is modelled on \mathbb{G}_i (cf. Definition 19). We denote by $\mathbf{L}(G_i)$ the Lie algebra of G_i . Then $\mathbf{L}(G_i) \equiv T_e G_i$ is isomorphic to \mathbb{G}_i . If we set $\bar{\delta}_i^j := T_e \delta_i^j$, then each $\bar{\delta}_i^j$ is a surjective linear map from $\mathbf{L}(G_j)$ to $\mathbf{L}(G_i)$ whose kernel is complemented.

Consider a sequence \mathfrak{h}_i of complemented sub-Lie algebra of $\mathbf{L}(G_i)$ such that the restriction of $\bar{\delta}_i^j$ to \mathfrak{h}_j is a continuous surjective map. Then $(\mathfrak{h}_i, \bar{\delta}_i^j)_{j \geq i}$ is a submersive projective sequence of Banach Lie algebra and so $\mathfrak{h} = \varprojlim \mathfrak{h}_i$ is a Fréchet Lie algebra (cf. [4] chapter 4). Now from classic result on Banach Lie groups (cf. [16]), by left translation each \mathfrak{h}_i gives rise to a complemented involutive subbundle of \mathcal{H}_i of TG_i and the leaf H_i through the neutral e_i in G_i has a structure of connected Banach Lie group so that the inclusion $\iota_i : H_i \rightarrow G_i$ is a Banach Lie morphism. Note that $\hat{\iota}_i := T_{e_i} \iota_i$ is nothing but else that the inclusion of \mathfrak{h}_i in $\mathbf{L}(G_i)$ and which induces the natural inclusion $\hat{\iota}_i$ of \mathcal{H}_i in TG_i .

Moreover, since $(\mathfrak{h}_i, \bar{\delta}_i^j)_{j \geq i}$ is a submersive projective sequence of Banach Lie algebra and $(G_i, \delta_i^j)_{j \geq i}$ be a submersive a projective sequence of Banach-Lie groups, it follows that $(\mathcal{H}_i, G_i, \hat{\iota}_i, [\cdot, \cdot]_i)$ ⁸ is a submersive projective sequence of split Banach Lie algebroids.

On the other hand, from Theorem 21 the Lie algebra $\mathbf{L}(G)$ of $G = \varprojlim G_i$ is $\varprojlim \mathbf{L}G_i$ and so $\mathfrak{h} = \varprojlim \mathfrak{h}_i$ is a closed complemented Lie subalgebra of $\mathbf{L}(G)$. As in the context of Banach setting, by left translation, \mathfrak{h} gives rise to an involutive Fréchet subbundle \mathcal{H} of TG which is clearly the projective limit of the submersive sequence $(\mathcal{H}_i, G_i, \hat{\iota}_i, [\cdot, \cdot]_i)$. So from Theorem 4 by same arguments as in Banach Lie groups, we obtain:

Theorem 5. *Let $G = \varprojlim G_i$ be a the projective limit of a submersive projective sequence of Banach-Lie groups $(G_i, \delta_i^j)_{j \geq i}$ and for each $i \in \mathbb{N}$, consider a closed complemented Banach Lie subalgebra \mathfrak{h}_i of $\mathbf{L}(G_i)$. Assume that the restriction of $\bar{\delta}_i^j$ to \mathfrak{h}_j is a continuous surjective map. Then there exists a Fréchet Lie group H in G such that $\mathbf{L}(H)$ is isomorphic to \mathfrak{h} and H is the projective limit of $(H_i, \delta_i^j|_{H_j})_{j \geq i}$.*

Remark 6. *The reader will also find an application of Theorem 2 in the proof of Theorem 8.23 on submersive projective limit of a projective sequence of Banach groupoids in*

⁸here $[\cdot, \cdot]_i$ denote again the restriction to sections of \mathcal{H}_i of the Lie bracket of vector fields on G_i

[4] which is a kind of generalization of Theorem 5 to Lie groupoids setting .

3.2. Application to sequences of prolongations of a Banach-Lie algebroid over a Banach manifold. Consider an anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ with typical fiber \mathbb{A} . Let $V\mathcal{A} \subset T\mathcal{A}$ be the vertical subbundle of $p_{\mathcal{A}} : T\mathcal{A} \rightarrow \mathcal{A}$. If $\mathcal{A}_x := \pi^{-1}(x)$ is the fiber over $x \in M$, according to [3], the prolongation $\mathbf{T}\mathcal{A}$ of the anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ over \mathcal{A} is the set $\{(x, a, b, c), (x, b) \in \mathcal{A}_x, (x, a, c) \in V_{(x,a)}\mathcal{A}\}$. It is a Banach vector bundle $\hat{\mathbf{p}} : \mathbf{T}\mathcal{A} \rightarrow \mathcal{A}$ with typical fiber $\mathbb{A} \times \mathbb{A}$ and we have an anchor $\hat{\rho} : \mathbf{T}\mathcal{A} \rightarrow T\mathcal{A}$ given by

$$\hat{\rho}(x, a, b, c) = (x, a, \rho_x(b), c) \in T_{(x,a)}\mathcal{A}.$$

From now on, we fix a Banach Lie algebroid $(\mathcal{A}, \pi, M, \rho, [\cdot, \cdot]_{\mathcal{A}})$ such that the typical fiber \mathbb{A} of \mathcal{A} is finite dimensional. By the way, we have a Banach Lie algebroid structure $(\mathbf{T}\mathcal{A}, \hat{\mathbf{p}}, \mathcal{A}, \hat{\rho}, [\cdot, \cdot]_{\mathbf{T}\mathcal{A}})$ (cf. [3] Corollary 44)

We denote $(\mathcal{A}_1, \pi_1, \mathcal{A}_0, \rho_1, [\cdot, \cdot]_1)$ the Banach-Lie algebroid $(\mathcal{A}, \pi, M, \rho, [\cdot, \cdot]_{\mathcal{A}})$ over a Banach manifold $\mathcal{A}^0 = M$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\rho_1} & T\mathcal{A}_0 \\ & \searrow \pi_1 & \swarrow p_{\mathcal{A}_0} \\ & & \mathcal{A}_0 \end{array} \quad (1)$$

According to the notations of Theorem 43 and Corollary 44 in [3], we set $\mathcal{A}_2 = \mathbf{T}\mathcal{A}_1$, $\rho_2 = \hat{\rho}$, $[\cdot, \cdot]_2 = [\cdot, \cdot]_{\mathbf{T}\mathcal{A}_1}$ and $\pi_2 = \hat{\mathbf{p}}$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A}_2 & \xrightarrow{\rho_2} & T\mathcal{A}_1 & & \\ & \searrow \pi_2 & \swarrow p_{\mathcal{A}_1} & \searrow T\pi_1 & \\ & & \mathcal{A}_1 & \xrightarrow{\rho_1} & T\mathcal{A}_0 \\ & & \searrow \pi_1 & \swarrow p_{\mathcal{A}_0} & \\ & & & & \mathcal{A}_0 \end{array} \quad (2)$$

Fix some $x \in \mathcal{A}_0$ and a norm $\|\cdot\|_0$ (resp. $\|\cdot\|_1$) on the fibre $T_x\mathcal{A}_0 \equiv \mathbb{A}_0$ (resp. $\mathcal{A}_x = \pi_1^{-1}(x) \equiv \mathbb{A}_1$). Since the fiber $\mathbf{T}_{(x,a)}\mathcal{A}_1$ (resp. $T_{(x,a)}\mathcal{A}_1$) is isomorphic to $\mathbb{A}_1 \times \mathbb{A}_1$ (resp. $\mathbb{A}_0 \times \mathbb{A}_1$), it follows that $\sup\{\|\cdot\|_1, \|\cdot\|_1\}$ (resp. $\sup\{\|\cdot\|_0, \|\cdot\|_1\}$) gives rise to a norm on $\mathbf{T}_{(x,a)}\mathcal{A}_1$ (resp. $T_{(x,a)}\mathcal{A}_1$). Then for the associated operator norm $\|\cdot\|^{\text{op}}$ we have

$$\|(\rho_2)_{(x,a)}\|^{\text{op}} \leq \sup(\|(\rho_1)_x\|^{\text{op}}, 1) \quad (3)$$

By induction, for $i \geq 1$, again according to notations of Theorem 43 and Corollary 44 in [3], we set

$\mathcal{A}^{i+1} = \mathbf{T}^{\mathcal{A}_i}\mathcal{A}_i$, $\rho_{i+1} = \hat{\rho}_i$, $[\cdot, \cdot]_{i+1} = [\cdot, \cdot]_{\mathbf{T}^{\mathcal{A}_i}\mathcal{A}_i}$ and $\pi_{i+1} = \hat{\mathbf{p}}$ and, as before, we have the following commutative diagrams:

$$\begin{array}{ccccc} \mathcal{A}_{i+1} & \xrightarrow{\rho_{i+1}} & T\mathcal{A}_i & & \\ & \searrow \pi_{i+1} & \swarrow p_{\mathcal{A}_i} & \searrow T\pi_i & \\ & & \mathcal{A}_i & \xrightarrow{\rho_i} & T\mathcal{A}_{i-1} \\ & & \searrow \pi_i & \swarrow p_{\mathcal{A}_{i-1}} & \\ & & & & \mathcal{A}_{i-1} \end{array} \quad (4)$$

Also, by same arguments as for (3) we obtain:

$$\|(\rho_{i+1})_{(x, a_1, \dots, a_{i+1})}\|^{\text{op}} \leq \sup(\|(\rho_i)_{(x, a_1, \dots, a_i)}\|^{\text{op}}, 1) \quad (5)$$

It follows that we have a submersive projective sequence of Banach-Lie algebroids $(\mathcal{A}_i, \mathcal{A}_{i-1}, \rho_i, [\cdot, \cdot]_i)$ over a submersive projective sequence of Banach manifolds $(\mathcal{A}_i)_{i \in \mathbb{N}}$ which satisfies the assumptions of Theorem 2. Thus, we obtain:

Theorem 7. *Under the previous context,*

$$(\mathcal{A} = \varprojlim_{i \geq 1} \mathcal{A}_i, \mathcal{M} = \varprojlim_{j \geq 0} \mathcal{A}_j, \rho = \varprojlim_{i \geq 1} \rho_i, [\cdot, \cdot] = \varprojlim_{i \geq 1} [\cdot, \cdot]_i)$$

is a Fréchet Lie algebroid on the Fréchet manifold \mathcal{M} and the distribution $\rho(\mathcal{A})$ is integrable. Each leaf L is a projective limit of a projective sequence of leaves of type (L_i) defined by induction in the following way:

L_0 is a leaf of $\rho_1(\mathcal{A}_1)$ and if L_i is a leaf of $\rho_i(\mathcal{A}_i)$ then $L_{i+1} = (\mathcal{A}_i)_{|L_i}$.

3.3. A contre-example. In this subsection we give a Example of an integrable distribution on a Fréchet bundle over a finite dimensional manifold which satisfies the assumptions (*) in Definition 1 but is a projective limit of a submersive sequence of Banach **not integrable** distributions.

Let $E = M \times \mathbb{R}^m$ the trivial bundle over a manifold M of dimension n . The set $J^k(E)$ of the k -jets of section of E over M is a finite dimensional manifold which is a vector bundle $\pi^k : J^k(E) \rightarrow M$ and whose typical fiber is is the space $\prod_{j=0}^k \mathcal{L}_{\text{sym}}^j(\mathbb{R}^n, \mathbb{R}^m)$ where $\mathcal{L}_{\text{sym}}^j(\mathbb{R}^n, \mathbb{R}^m)$ is the space of continuous j -linear symmetric mappings $\mathbb{R}^n \rightarrow \mathbb{R}^m$. Then each projection $\pi_k^l : J^l(E) \rightarrow J^k(E)$ defined, for $l \geq k$, by

$$\pi_k^l \left[j^l(s)(x) \right] = j^k(s)(x)$$

is a smooth surjection.

Proposition 8. ([4]) $(J^k(E), \pi_k^l)$ is a submersive projective sequence of Banach vector bundles and the projective limit

$$J^\infty(E) = \varprojlim J^k(E)$$

can be endowed with a structure of Fréchet vector bundle whose fibre is isomorphic to the Fréchet space $\prod_{j=0}^{\infty} \mathcal{L}_{\text{sym}}^j(\mathbb{R}^n, \mathbb{R}^m)$.

Let s be a section of π on a neighbourhood U of $x \in M$. For $\xi = j^k(s)(x) \in J^k(E)$, the n -dimensional subspace $R(s, x)$ of $T_\xi J^k(E)$ equals to the tangent space at ξ to the submanifold $j^k(s)(U) \subset J^k(\pi)$ is called an *R-plane*.

The Cartan subspace $\mathcal{C}^k(E)$ of $T_\xi J^k(E)$ is the linear subspace spanned by all *R*-planes $R(s', x)$ such that $j^k(s')(x) = \xi$. So it is the hull of the union of $(j^k(s))_*(x)(T_x M)$ where s is any local section of π around x .

The Cartan subspaces form a smooth distribution on $J^k(\pi)$ called *Cartan distribution* and denoted \mathcal{C}^k . Then \mathcal{C}^k is a regular distribution which a contact distribution and so is not integrable (cf. [14]). We have a submersive projective limit of bundle $(\mathcal{C}^k, T\pi_k^l, j^k(E))$ whose projective limit $\mathcal{C} = \varprojlim \mathcal{C}^k$ is called the Cartan distribution on $J^\infty(E)$. In fact \mathcal{C} is integrable (cf. [14]). Note that since \mathcal{C}^k is a subbundle of $TJ^k(E)$, from the proof of Theorem 4, the condition (3) of Definition 1 is satisfied.

4. PROOF OF THEOREM 2

Fix some $x \in M$. According to the property (*) and the Definition 1, we can choose a submersive projective sequence of charts $(U_i, \delta_i^j|_{U_j})_{j \geq i}$ and a submersive projective sequence of Banach bundles $(E_i, \lambda_i^j)_{j \geq i}$, such that:

- $U = \varprojlim U_i$ is an open neighbourhood of x in M ;
- If $x = \varprojlim(x_i)$, each U_i is the contractile domain of a chart (U_i, ϕ_i) around x_i in M_i and $(U = \varprojlim(U_i), \phi = \varprojlim(\phi_i))$ is a projective limit chart in M around x .
- the projective sequence of Banach bundles $(E_i, \lambda_i^j)_{j \geq i}$ satisfies the assumption (*) on U .

Step 1: The kernel of ρ_x is supplemented.

There exists a trivialization $\tau_i : E_i \rightarrow U_i \times \mathbb{E}_i$ which satisfies the compatibility condition:

$$(\delta_i^j \times \overline{\lambda_i^j}) \circ \tau_j = \tau_i \circ \lambda_i^j \quad (6)$$

where $(\mathbb{E}_i, \overline{\lambda_i^j})_{j \geq i}$ is the projective sequence of Banach spaces on which $(E_i, \lambda_i^j)_{j \geq i}$ is modeled.

Under these conditions, without loss of generality we may assume that, for each $i \in \mathbb{N}$, we have

- $x_i \equiv 0 \in \mathbb{M}_i$;
- U_i is an open subset of \mathbb{M}_i and so $TU_i = U_i \times \mathbb{M}_i$;
- $E_i = U_i \times \mathbb{E}_i$.

The projection δ_i^j at point $x_j \equiv 0$ (resp. λ_i^j in restriction to the fibre of E_j over $x_j \equiv 0$) is denoted d_i^j (resp. ℓ_i^j). The morphism ρ_i in restriction to the fibre E_i over $x_i \equiv 0$ is denoted r_i and ρ_x is denoted r , so that $r = \varprojlim r_i$. Now, according to the context of Assumption 1 in Definition 1, we have the following result:

Lemma 9. *There exists a decomposition $\mathbb{E} = \ker r \oplus \mathbb{F}'$ with the following property: if (ν'_n) (resp. (μ_n)) is the graduation on \mathbb{F}' (resp. (μ_n) on \mathbb{M}) induced by the norm $\|\cdot\|_{\mathbb{E}_i}$ (resp. $\|\cdot\|_{\mathbb{M}_i}$) on $(E_i)_{x_i}$ (resp. $T_{x_i}M_i$), then the restriction of r to \mathbb{F}' is a closed uniformly bounded operator according to these graduations ⁹.*

Proof of Lemma 9

At first, in such a context we have $T_{x_j}\delta_i^j \equiv d_i^j$ and the following compatibility condition:

$$d_i^j \circ r_j = r_i \circ \ell_i^j. \quad (7)$$

We set $\mathbb{F}_i = r_i(\mathbb{E}_i)$ for all $i \in \mathbb{N}$. From Definition 1 assumption 1, \mathbb{F}_i is a Banach subspace of \mathbb{E}_i and there exists a decomposition $\mathbb{E}_i = \ker r_i \oplus \mathbb{F}'_i$. Thus the restriction r'_i of r_i to \mathbb{F}'_i is an isomorphism onto \mathbb{F}_i . Now, from (7), we have

$$\forall (i, j) \in \mathbb{N}^2 : j > i, d_i^j \circ r'_j = r'_i \circ \ell_i^j.$$

But since each r'_i is an isomorphism, the restriction $(\ell_i^j)'$ of ℓ_i^j to \mathbb{F}'_j takes values in \mathbb{F}'_i for all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$. Moreover, as δ_i^j is surjective, according to (7) again, this implies that $d_i^j(\mathbb{F}_j) = \mathbb{F}_i$ and so $\ell_i^j(\mathbb{F}'_j) = \mathbb{F}'_i$. Since $(\mathbb{E}_i, \ell_i^j)_{j \geq i}$ is a projective sequence, this implies that $(\mathbb{F}'_i, (\ell_i^j)')_{j \geq i}$ is a surjective projective system. The vector space $\mathbb{F}' = \varprojlim \mathbb{F}'_i$ is then a Fréchet subspace of \mathbb{E} .

On the other hand, let $(\ell_i^j)''$ be the restriction of ℓ_i^j to $\ker r_j$. Always from (7), we have

$$\forall (i, j) \in \mathbb{N}^2 : j > i, (\ell_i^j)''(\ker r_j) \subset \ker r_i.$$

⁹cf. Appendix B

By same argument, it implies that $(\ker r_i, (\ell_i^j)'')$ $_{j \geq i}$ is a projective sequence and $\ker r = \varprojlim \ker r_i$. Moreover, since $\mathbb{E}_i = \ker r_i \oplus \mathbb{F}'_i$, it follows that $\mathbb{E} = \ker r \oplus \mathbb{F}'$ and also the restriction r' of r to \mathbb{F}' is obtained as $r' = \varprojlim r'_i$ and r' is an injective continuous operator $r' : \mathbb{F}' \rightarrow \mathbb{M}$ whose range is the closed subspace $\mathbb{F} = \varprojlim \mathbb{F}_i$. It remains to show that r' is uniformly bounded.

According to the context of assumption 2 of Definition 1, there exists a constant $C > 0$ and, for each $i \in \mathbb{N}$, we have a norm $\|\cdot\|_{\mathbb{E}_i}$ on \mathbb{E}_i and a norm $\|\cdot\|_{\mathbb{M}_i}$ on \mathbb{M}_i such that $\|r_i\|_i^{\text{op}} \leq C$. As \mathbb{F}'_i is a closed Banach subspace of \mathbb{E}_i , it follows that for the induced norm on \mathbb{F}'_i we have

$$\forall i \in \mathbb{N}, \|r'_i\|_i^{\text{op}} \leq C. \quad (8)$$

Set $\ell_i = \varprojlim \ell_i^j$ and $d_i = \varprojlim d_i^j$. By construction we have $\ell_i(\mathbb{F}') = \mathbb{F}'_i$ and $d_i(\mathbb{M}) = \mathbb{M}_i$. The norm $\|\cdot\|_{\mathbb{E}_i}$ on \mathbb{E}_i induces a norm $\|\cdot\|_{\mathbb{F}'_i}$ on the Banach subspace \mathbb{F}'_i and we get a natural graduation (ν'_i) on \mathbb{F}' given by $\nu'_i(u) = \|\ell_i(u)\|_{\mathbb{F}'_i}$ (cf. Appendix B (29)). In the same way, the norm $\|\cdot\|_{\mathbb{M}_i}$ induces a graduation (μ_i) on \mathbb{M} given by $\mu_i(v) = \|d_i(v)\|_{\mathbb{M}_i}$. Now (8) implies that r' is uniformly bounded and $r'(\mathbb{F}') = r(\mathbb{E}) = \Delta_x$ is closed by assumption. Therefore, the proof of Lemma 9 is complete. \square

Step 2: There exists a neighbourhood $V \subset U$ of x such that the map $\rho' = \rho|_{U \times \mathbb{F}'}$ takes values in $\mathcal{H}_b(\mathbb{F}', \mathbb{M})^{10}$ and is K -Lipschitz on V for some $K > 0$.

Since $E_i = U_i \times \mathbb{E}_i$, it follows that $E = U \times \mathbb{E}$ and so $\rho : E \rightarrow TU$ can be seen as a smooth map from U into $\mathcal{H}(\mathbb{E}, \mathbb{M})$. Let ρ' be the restriction of ρ to $U \times \mathbb{F}'$ and so consider ρ' as a smooth map from U to $\mathcal{H}(\mathbb{F}', \mathbb{M})$. From the definition of a Finsler norm, Assumption 2 in Definition 1 and Lemma 9, the map $x \mapsto \rho'_x$ takes value in $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$.

Lemma 10. *There exists a neighbourhood $V_1 \subset V$ of 0 such that the map $\rho' : V_1 \rightarrow \mathcal{H}_b(\mathbb{F}', \mathbb{M})$ is Lipschitz, that is:*

there exists $K > 0$ such that

$$\hat{\mu}_i^{\text{op}}(\rho'_z - \rho'_x) \leq K \hat{\mu}_i(z - z'), \quad \forall (z, z') \in V_1^2$$

¹¹

Proof of Lemma 10 Let $(\hat{\nu}'_i)_{i \in \mathbb{N}}$ (resp. $(\hat{\mu}_i)_{i \in \mathbb{N}}$) be the canonical increasing graduation associated to the graduation $(\nu'_i)_{i \in \mathbb{N}}$ on \mathbb{M} (resp. $(\mu_i)_{i \in \mathbb{N}}$) (cf. Appendix (B 29)). Since the map $x \mapsto \rho'_x$ is a smooth map from U to $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ it follows that for each x the differential map $d_x \rho'$ is a continuous linear map from \mathbb{M} to the Banach space $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ and so there exists $i_0 \in \mathbb{N}$ and a constant $A_x > 0$ such that

$$\|d_x \rho'(u)\|_\infty \leq A_x \hat{\nu}'_{i_0}(u) \quad (9)$$

for all $u \in \mathbb{F}'$ and so

$$\|d_x \rho'(u)\|_\infty \leq A_x \hat{\nu}'_i(u) \quad (10)$$

for all $u \in \mathbb{F}'$ and and $i \geq i_0$ and according to Remark 32 we set

$$\|d_x \rho'\|_i^{\text{op}} := \sup\{\hat{\mu}_i(d_x \rho'(u)) : \hat{\nu}'_i(u) \leq 1\} \leq A_x, \quad \forall i \geq i_0 \quad (11)$$

On the other hand, for $1 \leq i < i_0$ we set $C_x^i = \|d_x \rho'\|_i^{\text{op}}$. Then $d_x \rho'$ belongs to $\mathcal{H}_b(\mathbb{M}, \mathcal{H}_b(\mathbb{F}', \mathbb{M}))$ for all $x \in U$ since we have

$$\|d_x \rho'\|_\infty := \sup_{i \in \mathbb{N}} \|d_x \rho'\|_i^{\text{op}} \leq \sup\{A_x, C_x^1, \dots, C_x^{i_0-1}\}.$$

¹⁰cf. Appendix B

¹¹cf. Remark 32

We set $C = \|d_0\rho'\|_\infty$. By continuity, there exists an open neighbourhood V_1 of 0 such that

$$\|d_x\rho'\|_\infty \leq 2C \quad (12)$$

By choosing $K = 2C$, from the definition of $\|d_x\rho'\|_\infty$ it implies the announced result. \square

Step 3: Local flow of the vector field $X_u = \rho'(u)$.

Consider a neighbourhood V as announced in step 2. As δ_i is surjective, $V_i = \delta_i(V)$ is an open set of U_i and so we have $V = \varprojlim V_i$. For each $u \in \mathbb{F}'$, let $X_u = \rho'(u)$ be the vector field on V . If $u = \varprojlim u_i$ with $u_i \in \mathbb{F}'_i$, then $X_{u_i} = \rho'_i(u_i)$ is a vector field on V_i and $X_u = \varprojlim X_{u_i}$. Now since ρ' takes values in $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})$, from our assumption, there exists a constant $C > 0$ such that $\|(\rho'_i)_{z_i}\|_i^{\text{op}} \leq C$ for all $z_i \in V_i$. Therefore, if u belongs to \mathbb{F}' and $u_i = \lambda(u)$, then $\|X_{u_i}\|^{\mathbb{M}_i} \leq C\|u\|_i^{\mathbb{E}_i}$ and, from Lemma 10 and the definition of $\hat{\mu}_i^{\text{op}}$ we have

$$\forall (x_i, x'_i) \in (\delta_i(V))^2, \|X_{u_i}(x_i) - X_{u_i}(x'_i)\|^{\mathbb{M}_i} \leq K\|u_i\|_i^{\mathbb{E}_i}\|x_i - x'_i\|^{\mathbb{M}_i}. \quad (13)$$

Now, recall that we have provided \mathbb{F}' and \mathbb{M} with seminorms $(\hat{\nu}'_n)$ and $(\hat{\mu}_n)$ respectively defined by

$$\nu_n(u) = \|\lambda_i(u)\|_i^{\mathbb{E}_i} \quad \text{and} \quad \mu_n(x) = \|\delta_i(x)\|^{\mathbb{M}_i}.$$

Since $\delta_i(X_u)(x) = X_{u_i}(\delta_i(x))$ and in this way, from (13), for all $n \in \mathbb{N}$, we have

$$\forall (x, x') \in V^2, \mu_n(X_u(x) - X_u(x')) \leq K\nu_n(u)\mu_n(x - x'). \quad (14)$$

Therefore, X_u satisfies the assumption of Corollary 37. Let $\epsilon > 0$ such that the pseudo-ball

$$B_{\mathbb{M}}(0, 2\epsilon) = \{x \in \mathbb{M}, : \hat{\mu}_{n_i}(x) < 2\epsilon, 1 \leq i \leq k\}$$

is contained in V and set

$$C_1 := \max_{1 \leq i \leq k} \{K\nu_{n_i}(u)\} = K \max_{1 \leq i \leq k} \nu_{n_i}(u);$$

$$C_2 := \sup_{x \in B_{\mathbb{M}}(0, \epsilon)} \left\{ \max_{1 \leq i \leq k} \mu_{n_i}(X(x)) \right\} \leq C \max_{1 \leq i \leq k} \nu_{n_i}(u).$$

By application of Corollary 37, for any u such that $\max_{1 \leq i \leq k} \nu_{n_i}(u) \leq 1$, there exists $\alpha > 0$ such that

$$\alpha e^{2\alpha K} \leq \frac{\epsilon}{2C},$$

such that the local flow Fl_t^u is defined on the pseudo-ball $B_{\mathbb{M}}(0, \epsilon)$ for all $t \in [-\alpha, \alpha]$ for all u which satisfied the previous inequality

Note that for any $s \in \mathbb{R}$ we have $X_{su} = sX_u$. Therefore, from the classical properties of a flow of a vector field,

there exists $\eta > 0$ such that the local flow Fl_t^u is defined on $[-1, 1]$, for all u in the open pseudo-ball

$$B_{\mathbb{F}'}(0, \eta) := \{u \in \mathbb{F}', : \nu'_{n_i}(u) \leq \eta, 1 \leq i \leq k\}.$$

(cf. for instance proof of Corollary 4.2 in [5]).

We set $B_{\mathbb{M}_i}(0, \epsilon) = \delta_i(B_{\mathbb{M}}(0, \epsilon))$. Then X_{u_i} is a vector field on $V_i = \delta_i(V)$ and $\text{Fl}_t^{u_i} := \delta_i \circ (\text{Fl}_t^u) \circ \delta_i$ is the local flow of X_{u_i} which is defined on $B_{\mathbb{M}_i}(0, \epsilon)$ for all $t \in [-1, 1]$ and $\text{Fl}_t^{u_i}(x_i)$ belongs to V_i for all $x_i \in B_{\mathbb{M}_i}(0, \epsilon)$ and $t \in [-1, 1]$ and, from (15), we have

$$\text{Fl}_t^u = \varprojlim \text{Fl}_t^{u_i}. \quad (15)$$

Step 4: Existence of an integral manifold.

Since $(E_i, \pi_i, U_i, \rho_i, [\cdot, \cdot]_i)$ is a Banach-Lie algebroid, the Lie bracket $[X_{u_i}, X_{u'_i}]$ is tangent to Δ_i and so by Definition 3.2 and Lemma 3.6 in [18] we have

$$\forall t \in [-1, 1], (T\text{Fl}_t^{u_i})(\Delta_i)_{x_i} = (\Delta_i)_{\text{Fl}_t^{u_i}(x_i)}.$$

Therefore, according to the notations at the end of step 3, for each $i \in \mathbb{N}$, we set:

$$\forall u_i \in B_{\mathbb{F}'_i}(0, \eta) = \lambda_i(B_{\mathbb{F}'}(0, \eta)), \quad \Phi_i(u_i) = (\text{Fl}_1^{u_i})(0);$$

$$\forall u \in B_{\mathbb{F}'}(0, \eta), \quad \Phi(u) = (\text{Fl}_1^u)(0);$$

Lemma 11. $\Phi = \varprojlim \Phi_i$ is smooth and there exists $0 < \eta' \leq \eta$ such that the restriction of Φ to $B_{\mathbb{F}'}(0, \eta')$ is injective and $T_u \Phi$ belongs to $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ for all $u \in B_{\mathbb{F}'}(0, \eta')$.

Recall that, for each $i \in \mathbb{N}$, $B_{\mathbb{F}'_i}(0, \eta')$ is an open ball in \mathbb{F}'_i and $\delta_i(V)$ is open neighbourhood of $0 \in \mathbb{M}_i$. From Lemma 11, since Φ is injective and each differential $T_u \Phi$ is injective, it follows that the same is true for each $\Phi_i : B_{\mathbb{F}'_i}(0, \eta') \rightarrow \delta_i(V)$. Thus we can apply the proof of Theorem 1 in [18] for Φ_i . By the way, $\mathcal{U}_i = \Phi_i(B_{\mathbb{F}'_i}(0, \eta'))$ is an integral manifold of $\rho_i(\mathbb{F}'_i)$ and $(\mathcal{U}_i, \Phi_i^{-1})$ is a (global) chart for this integral manifold modeled on \mathbb{F}'_i . As $\Phi = \varprojlim \Phi_i$, $B_{\mathbb{F}'}(0, \eta') = \varprojlim B_{\mathbb{F}'_i}(0, \eta')$, $\Phi_i \circ \lambda_i^j = \delta_i^j \circ \Phi_j$ for all $j \geq i$, it follows that $(\mathcal{U}_i, \delta_i^j)_{j \geq i}$ is a surjective projective sequence and so $\mathcal{U} = \varprojlim \mathcal{U}_i$ is a Fréchet manifold modeled on \mathbb{F}' . This last result clearly **ends the proof of Theorem 2**.

Proof of Lemma 11

According to step 3, Φ_i is well defined on $B_{\mathbb{F}'_i}(0, \eta)$ and $\Phi = \varprojlim \Phi_i$.

Now, for every $x \in V$, ρ'_x belongs to $\mathcal{I}\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ and so is injective; after shrinking V , if necessary, there exists some $M > 0$ such that

$$\forall x \in V, \quad \|\rho'_x\|_\infty \leq M. \quad (16)$$

Now, by construction, we have $T_0 \Phi(u) = \rho'_0(u)$ and so $T_0 \Phi$ is injective.

Claim 12. *The map $u \mapsto T_u \Phi$ is a smooth map from $B'(0, \eta)$ to $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$.*

Proof of Claim 12

We will use some argument of the proof of Lemma 2.12 of [18]. We fix the index i and, for any $y \in B_{\mathbb{F}'_i}(0, \epsilon)$, $v \in \mathbb{F}'_i$ we set

$$\begin{aligned} &: X_v(y) = \rho'_i(y, v) \\ &: \varphi(t, v) = \text{Fl}_t^{X_v}(0) \\ &: A(t) = \partial_1 \rho'_i(\varphi(t, v), v) \text{ (partial derivative relative to the first variable)} \\ &: B(t) = \rho'_i(\varphi(t, v), \cdot) \end{aligned}$$

Note that A and B are smooth fields on $[0, 1]$ of operators in $\mathcal{L}(\mathbb{M}_i, \mathbb{M}_i)$ and $\mathcal{L}(\mathbb{F}'_i, \mathbb{M}_i)$ respectively. Therefore the differential equation

$$\dot{S} = A \circ S + B$$

has a unique solution S_t with initial condition $S_0 = 0$ given by

$$S_t = \int_0^t G_{t-s} \circ B(s) ds \quad (17)$$

where G_t is the unique solution of

$$\dot{G} = A \circ G$$

with initial condition $G_0 = \text{Id}_{\mathbb{M}}$. Given by

$$G_t = \text{Id}_{\mathbb{M}} + \int_0^t A \circ G_s ds. \quad (18)$$

Under these notations, from [6], Chapter X § 7, we have $\partial_2 \varphi(t, v)(\cdot) = S_t$.

On the one hand, from the choice of ϵ , $\varphi(t, v)$ belongs to V_i and so from (12), we have $\|A(t)\|_i^{\text{op}} \leq K$ for any $t \in [-1, 1]$ and from (16) we have $\|B(t)\|_i^{\text{op}} \leq M$. Thus from (18), using Gronwal equality, we obtain

$$\|G_t\|_i^{\text{op}} \leq e^K,$$

And so from (17) we obtain

$$\|S_i\|_i^{\text{op}} \leq Me^K$$

This implies that

$$\|\partial_2 \varphi(t, v)\|_i^{\text{op}} \leq Me^K.$$

We set $M_1 = Me^K$. Since $\Phi_i(u_i) = \varphi(1, u_i)$ it follows that:

$$\|T_{u_i} \Phi(v_i)\|^{M_i} \leq M_1 \|v_i\|^{E_i} \quad (19)$$

from (19) we obtain:

$$\mu_i(T_u \Phi(v)) \leq M_1 \nu_i(v). \quad (20)$$

But, since $T_u \Phi(\mathbb{F}') = \Delta_{\Phi(u)} \subset \{\Phi(u)\} \times \mathbb{M}$, it follows that the map $u \mapsto T_u \Phi$ can be considered as a continuous linear map from \mathbb{F}' to \mathbb{M} which takes values in $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$. Now as each Φ_i is a smooth map from $B_{\mathbb{F}'_i}(0, \eta')$ to $\delta_i(V)$ and $\Phi = \varprojlim \Phi_i$, this imply that Φ is a smooth map on $B_{\mathbb{F}'}(0, \eta') = \varprojlim B_{\mathbb{F}'_i}(0, \eta')$ to $V = \varprojlim \delta_i(V)$ which ends the proof of the Claim. \square

End of the proof of Lemma 10.

At first, from Claim 12, the map $u \mapsto T_u \Phi$ takes values in the Banach space $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$, as in step 2 for ρ , we can show that this map is Lipschitz on $B_{\mathbb{F}'}(0, \eta)$ for η small enough. As $T_0 \Phi = \rho_0$, from Proposition 34, it follows that, again for η small enough, $T\Phi$ is injective on $B_{\mathbb{F}'}(0, \eta')$, and we have (cf. (19))

$$\forall u \in B_{\mathbb{F}'}(0, \eta'), \forall v \in \mathbb{F}' \mu_i(T_u \Phi(v)) \leq M_1 \nu_i(v) \quad (21)$$

using the fact that the range of $T_u \Phi$ is always closed for $u \in B_{\mathbb{F}'}(0, \eta')$. Moreover, for $u \in B_{\mathbb{F}'}(0, \eta')$, as for the relation (33) in the proof of Proposition 34, we obtain:

$$\frac{1}{\ell_u} \nu_i(v) \leq \mu_i(T_u \Phi(v)) \leq \ell_u \cdot \nu_n(v) \quad (22)$$

for all $i \in \mathbb{N}$, where $\ell_u = \|T_u \Phi\|_\infty \leq M_1$.

Finally we obtain:

$$\forall i \in \mathbb{N}, \frac{1}{M_1} \nu_i(v) \leq \mu_i(T_u \Phi(v)) \leq M_1 \cdot \nu_i(u) \quad (23)$$

Suppose that, for any $0 \leq \eta' \leq \eta$, the restriction of each Φ_i to $B_{\mathbb{F}'_i}(0, \eta)$ is not injective. Consider any pair $(u, v) \in [B_{\mathbb{F}'}(0, \eta)]^2$ such that $u \neq v$ but $\Phi(u) = \Phi(v)$, we set $h = v - u$. For any $\alpha \in \mathbb{M}^*$, we consider the smooth curve $c_\alpha : [0, 1] \rightarrow \mathbb{R}$ defined by:

$$c_\alpha(t) = \langle \alpha, (\Phi(u + th) - \Phi(u)) \rangle.$$

Of course, we have $\dot{c}_\alpha(t) = \langle \alpha, T_{u+th} \Phi(h) \rangle$.

Denote by $]u, v[$ the set of points $\{w = u + th, t \in]0, 1[\}$. As we have $c_\alpha(0) = c_\alpha(1) = 0$, from Rolle's Theorem, there exists $u_\alpha \in]u, v[$ such that

$$\langle \alpha, \delta_t(T_{u_\alpha} \Phi(h)) \rangle = 0 \quad (24)$$

Note that, for any $t \in \mathbb{R}$, this relation is also true for any th . From our assumption, it follows that, for each $k \in \mathbb{N} \setminus \{0\}$, there exists u_k and v_k in $B_{\mathbb{F}'_i}(0, \frac{\eta}{k})$ so that $u_k \neq v_k$ but with $\Phi(u_k) = \Phi(v_k)$. So from the previous argument, for any $\alpha \in \mathbb{M}^*$, we have

$$\langle \alpha, (T_{u_{\alpha,k}} \Phi(h_k)) \rangle = 0 \quad (25)$$

for some $u_{\alpha,k} \in]u_k, v_k[$ and $h_k = v_k - u_k$. From (25), for any $i \in \mathbb{N}$, any $t \in \mathbb{R}$ and any $\alpha_i \in \mathbb{M}_i^*$, if $\alpha = \delta_i^*(\alpha_i)$, we have:

$$| \langle \alpha, T_0 \Phi(th_k) \rangle | = | \langle \alpha_i, \delta_i([T_0 \Phi - T_{u_{\alpha}} \Phi](th_k)) \rangle | = | \langle \alpha_i, [T_0 \Phi_i - T_{\delta_i(u_{\alpha})} \Phi_i](\lambda_i(th_k)) \rangle |$$

Denote by $\|\cdot\|_{\mathbb{M}_i^*}$ the canonical norm on \mathbb{M}_i^* associated to $\|\cdot\|_{\mathbb{M}_i}$. Then from (25) and since $u \mapsto T_u \Phi$ is K_1 -Lipschitz on $B'(0, \eta)$ (for some constant K_1) we obtain:

$$\begin{aligned} | \langle \alpha, T_0 \Phi(th_k) \rangle | &\leq (\|\alpha_i\|_{\mathbb{M}_i^*} \cdot K_1 \cdot \|\delta_i(u_{\alpha})\|_{\mathbb{M}_i} \cdot \|\lambda_i(th_k)\|_{\mathbb{E}_i}) \\ &\leq (\|\alpha_i\|_{\mathbb{M}_i^*} \cdot K_1 \cdot \|\lambda_i(th_k)\|_{\mathbb{E}_i} \frac{\eta}{k}). \end{aligned} \quad (26)$$

Since $u_k \neq v_k$, there must exist at least one integer $i \in \mathbb{N}$ such that $\lambda_i(h_k) \neq 0$. Thus, by taking $t = \frac{u_k - v_k}{\nu_i((u_k - v_k))}$ in (26), we may assume $t = 1$ and $\|\lambda_i(h_k)\|_{\mathbb{E}_i} = 1$. In this way, for this choice of h_k , we have a 1-form $\bar{\beta}_{k,i}$ on the linear space generated by $\lambda_i(h_k)$ in \mathbb{F}_i' such that $\langle \bar{\beta}_{k,i}, \lambda_i(h_k) \rangle = 1$ and with $(\|\bar{\beta}_{k,i}\|_{\mathbb{E}_i})^* = 1$. From the Hahn-Banach Theorem, we can extend this linear form to a form $\beta_{k,i} \in \mathbb{M}_i^*$ such that $\langle \beta_{k,i}, \lambda_i(h_k) \rangle = 1$ and $(\|\beta_{k,i}\|_{\mathbb{E}_i})^* = 1$. But since each $T_0 \Phi$ is injective, this implies that $T_0 \Phi_i$ is injective and so the adjoint $T_0^* \Phi_i$ is surjective. This implies that there exists $\alpha_{k,i} \in \mathbb{M}_i^*$ such that $T_0^* \Phi_i(\alpha_{k,i}) = \beta_{k,i}$. Thus from (26) we obtain

$$\begin{aligned} 1 = | \langle \beta_{k,i}, \lambda_i(h_k) \rangle | &= | \langle \alpha_{k,i}, T_0 \Phi_i(\lambda_i(h_k)) \rangle | \\ &= | \langle \delta_i^* \alpha_{k,i}, T_0 \Phi(h_k) \rangle | \leq \|\alpha_{k,i}\|_{\mathbb{M}_i^*} K_1 \frac{\eta}{k} \end{aligned} \quad (27)$$

But, on the other hand since the operation "adjoint" is an isometry, from (23) we have

$$\frac{1}{M_1} \|\alpha_{k,i}\|_{\mathbb{M}_i^*} \leq \|T_0^* \Phi \alpha_{k,i}\|_{\mathbb{M}_i^*} = 1 \quad (28)$$

which gives a contradiction with (27) for k large enough. \square

APPENDIX A. PROJECTIVE LIMITS

A.1. Projective limits of topological spaces.

Definition 13. A projective sequence of topological spaces is a sequence

$((X_i, \delta_i^j))_{(i,j) \in \mathbb{N}^2, j \geq i}$ where

(PSTS 1): For all $i \in \mathbb{N}$, X_i is a topological space;

(PSTS 2): For all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, $\delta_i^j : X_j \rightarrow X_i$ is a continuous map;

(PSTS 3): For all $i \in \mathbb{N}$, $\delta_i^i = Id_{X_i}$;

(PSTS 4): For all $(i, j, k) \in \mathbb{N}^3$ such that $k \geq j \geq i$, $\delta_i^j \circ \delta_j^k = \delta_i^k$.

Notation 14. For the sake of simplicity, the projective sequence $((X_i, \delta_i^j))_{(i,j) \in \mathbb{N}^2, j \geq i}$ will be denoted $(X_i, \delta_i^j)_{j \geq i}$.

An element $(x_i)_{i \in \mathbb{N}}$ of the product $\prod_{i \in \mathbb{N}} X_i$ is called a *thread* if, for all $j \geq i$, $\delta_i^j(x_j) = x_i$.

Definition 15. The set $X = \varprojlim X_i$ of all threads, endowed with the finest topology for which all the projections $\delta_i : X \rightarrow X_i$ are continuous, is called the projective limit of the sequence $(X_i, \delta_i^j)_{j \geq i}$.

A basis of the topology of X is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where U_i is an open subset of X_i (and so δ_i is open whenever δ_i is surjective).

Definition 16. Let $(X_i, \delta_i^j)_{j \geq i}$ and $(Y_i, \gamma_i^j)_{j \geq i}$ be two projective sequences whose respective projective limits are X and Y .

A sequence $(f_i)_{i \in \mathbb{N}}$ of continuous mappings $f_i : X_i \rightarrow Y_i$, satisfying, for all $(i, j) \in \mathbb{N}^2$, $j \geq i$, the coherence condition

$$\gamma_i^j \circ f_j = f_i \circ \delta_i^j$$

is called a projective sequence of mappings.

The projective limit of this sequence is the mapping

$$\begin{aligned} f : X &\rightarrow Y \\ (x_i)_{i \in \mathbb{N}} &\mapsto (f_i(x_i))_{i \in \mathbb{N}} \end{aligned}$$

The mapping f is continuous if all the f_i are continuous (cf. [1]).

A.2. Projective limits of Banach spaces. Consider a projective sequence $(\mathbb{E}_i, \delta_i^j)_{j \geq i}$ of Banach spaces.

Remark 17. Since we have a countable sequence of Banach spaces, according to the properties of bonding maps, the sequence $(\delta_i^j)_{(i,j) \in \mathbb{N}^2, j \geq i}$ is well defined by the sequence of bonding maps $(\delta_i^{i+1})_{i \in \mathbb{N}}$.

A.3. Projective limits of differential maps. The following proposition (cf. [9], Lemma 1.2 and [4], Chapter 4) is essential

Proposition 18. Let $(\mathbb{E}_i, \delta_i^j)_{j \geq i}$ be a projective sequence of Banach spaces whose projective limit is the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}_i$ and $(f_i : \mathbb{E}_i \rightarrow \mathbb{E}_i)_{i \in \mathbb{N}}$ a projective sequence of differential maps whose projective limit is $f = \varprojlim f_i$. Then the following conditions hold:

- (1) f is smooth in the convenient sense (cf. [15])
- (2) For all $x = (x_i)_{i \in \mathbb{N}}$, $df_x = \varprojlim (df_i)_{x_i}$.
- (3) $df = \varprojlim df_i$.

A.4. Projective limits of Banach manifolds and Banach Lie groups.

Definition 19. [9] The projective sequence $(M_i, \delta_i^j)_{j \geq i}$ is called projective sequence of Banach manifolds if

(PSBM 1): M_i is a manifold modelled on the Banach space \mathbb{M}_i ;

(PSBM 2): $(\mathbb{M}_i, \delta_i^j)_{j \geq i}$ is a projective sequence of Banach spaces;

(PSBM 3): For all $x = (x_i) \in M = \varprojlim M_i$, there exists a projective sequence of local charts $(U_i, \xi_i)_{i \in \mathbb{N}}$ such that $x_i \in U_i$ where one has the relation

$$\xi_i \circ \delta_i^j = \delta_i^j \circ \varphi_j;$$

(PSBM 4): $U = \varprojlim U_i$ is a non empty open set in M .

Under the assumptions **(PSBM 1)** and **(PSBM 2)** in Definition 19, the assumptions **(PSBM 3)** and **(PSBM 4)** around $x \in M$ is called the projective limit chart property around $x \in M$ and $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ is called a projective limit chart.

The projective limit $M = \varprojlim M_i$ has a structure of Fréchet manifold modelled on the Fréchet space $\mathbb{M} = \varprojlim \mathbb{M}_i$ and is called a PLB-manifold. The differentiable structure is defined via the charts (U, φ) where $\varphi = \varprojlim \xi_i : U \rightarrow (\xi_i(U_i))_{i \in \mathbb{N}}$.

φ is a homeomorphism (projective limit of homeomorphisms) and the charts changings $(\psi \circ \varphi^{-1})|_{\varphi(U)} = \varprojlim ((\psi_i \circ \xi_i^{-1})|_{\xi_i(U_i)})$ between open sets of Fréchet spaces are smooth in the sense of convenient spaces.

Definition 20. [9] $(G_i, \delta_i^j)_{j \geq i}$ is a projective sequence of Banach-Lie groups where G_i is modelled on \mathbb{G}_i if, for all $i \in \mathbb{N}$, there exists a chart (U_i, φ_i) centered at the unity $e_i \in G_i$ such that:

(PLBLG 1): $\forall (i, j) \in \mathbb{N}^2 : j \geq i, \delta_i^j(U_j) \subset U_i$;

(PLBLG 2): $\forall (i, j) \in \mathbb{N}^2 : j \geq i, \delta_i^j \circ \varphi_j = \varphi_i \circ \delta_i^j$;

(PLBLG 3): $\varprojlim \varphi_i(U_i)$ is a non empty open set of \mathbb{G} and $\varprojlim U_i$ is open in G according to the projective limit topology.

A projective sequence of Banach-Lie groups $(G_i, \delta_i^j)_{j \geq i}$ is *submersive* if each δ_i^j is a surjective submersion.

Theorem 21. [9] Let $G = \varprojlim G_i$ be a the projective limit of a projective sequence of Banach-Lie groups $(G_i, \delta_i^j)_{j \geq i}$. Then we have the following properties:

- (1) G is a Fréchet-Lie group.
- (2) If $\mathbf{L}(G_i)$ is the Lie algebra of G_i then $\mathbf{L}(G) = \varprojlim \mathbf{L}G_i$.
- (3) If \exp_{G_i} is the exponential map for G_i , then $\exp_G = \varprojlim \exp_{G_i}$ is the exponential map of the Fréchet-Lie group G .

A.5. Projective limits of Banach vector bundles. Let $(M_i, \delta_i^j)_{j \geq i}$ be a projective sequence of Banach manifolds where each manifold M_i is modeled on the Banach space \mathbb{M}_i .

For any integer i , let (E_i, π_i, M_i) be the Banach vector bundle whose type fibre is the Banach vector space \mathbb{E}_i where $(\mathbb{E}_i, \lambda_i^j)_{j \geq i}$ is a projective sequence of Banach spaces.

Definition 22. $((E_i, \pi_i, M_i), (\xi_i^j, \delta_i^j))_{j \geq i}$, where $\xi_i^j : E_j \rightarrow E_i$ is a morphism of vector bundles, is called a projective sequence of Banach vector bundles on the projective sequence of manifolds $(M_i, \delta_i^j)_{j \geq i}$ if, for all (x_i) , there exists a projective sequence of trivializations (U_i, τ_i) of (E_i, π_i, M_i) , where $\tau_i : (\pi_i)^{-1}(U_i) \rightarrow U_i \times \mathbb{E}_i$ are local diffeomorphisms, such that $x_i \in U_i$ (open in M_i) and where $U = \varprojlim U_i$ is a non empty open set in M where, for all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, we have the compatibility condition

(PLBVB): $(\delta_i^j \times \lambda_i^j) \circ \tau_j = \tau_i \circ \xi_i^j$.

With the previous notations, $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ is called a *projective bundle chart limit*. The triple of projective limit $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ is called a *projective limit of Banach bundles* or **PLB-bundle** for short.

The following proposition generalizes the result of [10] about the projective limit of tangent bundles to Banach manifolds (cf. [7] and [4]).

Proposition 23. Let $((E_i, \pi_i, M_i), (\xi_i^j, \delta_i^j))_{j \geq i}$ be a projective sequence of Banach vector bundles.

Then $(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i)$ is a Fréchet vector bundle.

Notation 24. For the sake of simplicity, the projective sequence $((E_i, \pi_i, M_i), (\xi_i^j, \delta_i^j))_{j \geq i}$ will be denoted (E_i, π_i, M_i) .

Definition 25. A sequence (E_i, π_i, M_i) is called a *submersive projective sequence of Banach vector bundles* if $(M_i, \delta_i^j)_{j \geq i}$ is a submersive projective sequence of Banach manifolds and if around each $x \in M = \varprojlim M_i$, there exists a projective limit chart bundle $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ such that for all $i \in \mathbb{N}$, we have a decomposition $\mathbb{E}_{i+1} = \ker \bar{\lambda}_i^{i+1} \oplus \mathbb{E}'_i$ such that the condition **(PLBVB)** is true.

The projective limit (E, π, M) of a projective sequence of Banach vector bundles (E_i, π_i, M_i) is called a *submersive projective limit of Banach bundles* or *submersive PLB-bundle* for short.

Now, we have the following result:

Proposition 26. *Let (E_i, π_i, M_i) be a submersive projective sequence of Banach bundles. Then, for each $i \in \mathbb{N}$, the maps $\delta_i : M \rightarrow M_i$ and $\lambda_i : E \rightarrow E_i$ are submersions.*

A.6. Projective limit of Banach Lie algebroids.

Definition 27. *Let $\pi : E \rightarrow M$ be a Banach bundle.*

- (1) *an anchor is a vector bundle morphism $\rho : E \rightarrow TM$ and (E, ρ) is called an anchored bundle*
- (2) *An almost Lie bracket $[\cdot, \cdot]_E$ on an anchored bundle E is a sheaf of antisymmetric bilinear maps*

$$[\cdot, \cdot]_{E_U} : \Gamma(E_U) \times \Gamma(E_U) \rightarrow \Gamma(E_U)$$

for any open set $U \subseteq M$ and which satisfies the following properties

(AL 1) *the Leibniz identity:*

$$\forall (\mathbf{a}_1, \mathbf{a}_2) \in \Gamma(E_U)^2, \forall f \in C^\infty(M), [\mathbf{a}_1, f\mathbf{a}_2]_{E_U} = f \cdot [\mathbf{a}_1, \mathbf{a}_2]_{E_U} + df(\rho(\mathbf{a}_1)) \cdot \mathbf{a}_2.$$

(AL 2) *For any open set $U \subseteq M$ and any $(\mathbf{a}_1, \mathbf{a}_2) \in \Gamma(E_U)^2$, the map*

$$(\mathbf{a}_1, \mathbf{a}_2) \mapsto [\mathbf{a}_1, \mathbf{a}_2]_{E_U}$$

only depends on the 1-jets of \mathbf{a}_1 and \mathbf{a}_2 .

- (3) *An anchored bundle (E, ρ) provided with an almost Lie bracket $[\cdot, \cdot]_E$ which satisfies the Jacobi identity*

$$[[\mathbf{a}_1, \mathbf{a}_2]_E, \mathbf{a}_3]_E + [[\mathbf{a}_2, \mathbf{a}_3]_E, \mathbf{a}_1]_E + [[\mathbf{a}_3, \mathbf{a}_1]_E, \mathbf{a}_2]_E = 0$$

$$\forall (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \in \Gamma(E_U)^3$$

is called a Lie algebroid.

Definition 28. *$(E_i, \pi_i, M_i, \rho_i, [\cdot, \cdot]_i)$ is called a submersive projective sequence of split Lie algebroids if*

(PSBLA 1): *$(E_i, \xi_i^j)_{j \geq i}$ is a submersive projective sequence of Banach vector bundles $(\pi_i : E_i \rightarrow M_i)_{i \in \mathbb{N}}$ over the projective sequence of manifolds $(M_i, \delta_i^j)_{j \geq i}$;*

(PSBLA 2): *For all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, one has*

$$\rho_i \circ \xi_i^j = T\delta_i^j \circ \rho_j$$

(PSBLA 3): *For all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, one has*

$$\xi_i^j([\cdot, \cdot]_j) = [\xi_i^j(\cdot), \xi_i^j(\cdot)]_i$$

(PSBLA 4): *For all $i \in \mathbb{N}$ and $x_i \in M$ the kernel $\ker(\rho_i)_{x_i}$ is complemented in the fiber E_{x_i} .*

Proposition 29. *([4]) Let $(E_i, \pi_i, M_i, \rho_i, [\cdot, \cdot]_i)$ be a submersive projective sequence of split Lie algebroids. Then $(E := \varprojlim E_i, \pi := \varprojlim \pi_i, M := \varprojlim M_i, \rho = \varprojlim \rho_i)$ is Fréchet anchored bundle and $\Delta = \rho(E)$ is a closed distribution on E*

Remark 30. *Under the assumptions of Proposition 29, unfortunately $[\cdot, \cdot] = \varprojlim [\cdot, \cdot]_i$ does not define a Lie bracket on the set of all local sections of (E, π, M) but only on section which are projective limit of section of (E_i, π_i, M_i) . Therefore $(E, \pi, M, \rho, [\cdot, \cdot])$ does not have a Fréchet Lie algebroid structure.*

APPENDIX B. THE BANACH SPACE $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$

Any Fréchet space \mathbb{F} can be realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$. Following [7], 2.3, we can identify \mathbb{F} with the projective limit of the projective sequence

$$\left(\widehat{\mathbb{B}}_n = \left\{ x = (x_i) \in \prod_{0 \leq i \leq n} \mathbb{B}_i : \forall j \geq i, x_i = \lambda_i^j(x_j) \right\}, \widehat{\lambda}_n^m = (\lambda_n^m, \dots, \lambda_n^m) \right)_{m \geq n}.$$

We denote by $\lambda_n : \mathbb{F} \rightarrow \mathbb{B}_n$ and $\widehat{\lambda}_n : \mathbb{F} \rightarrow \widehat{\mathbb{B}}_n$ the canonical surjective projections. Let $(\|\cdot\|_n)_{n \in \mathbb{N}}$ be a sequence where $\|\cdot\|_n$ is a norm on \mathbb{B}_n . In this way,

$$\widehat{\|\cdot\|}_n = \sup_{0 \leq i \leq n} \|\cdot\|_i$$

defines a norm on $\widehat{\mathbb{B}}_n$. Then

$$\widehat{\nu}_n = \widehat{\|\cdot\|}_n \circ \widehat{\lambda}_n \quad (\text{resp. } \nu_n = \|\cdot\|_n \circ \lambda_n) \quad (29)$$

is the semi-norm on \mathbb{F} associated to the sequence $(\widehat{\|\cdot\|}_n)$ (resp. $(\|\cdot\|_n)$). Moreover, we have $\widehat{\nu}_n = \max_{0 \leq i \leq n} \nu_i$ and the topology of \mathbb{F} is defined by $(\widehat{\nu}_n)$ or (ν_n) .

Let (\mathbb{F}_1, ν_n^1) (resp. (\mathbb{F}_2, ν_n^2)) be a graded Fréchet space.

Recall that a linear map $L : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ is *continuous* if

$$\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}, \exists C_n > 0 : \forall x \in \mathbb{F}_1, \nu_n^2(Lx) \leq C_n \nu_1^{k_n}(x).$$

The space $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ of continuous linear maps between both these Fréchet spaces generally drops out of the Fréchet category. Indeed, $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ is a Hausdorff locally convex topological vector space whose topology is defined by the family of semi-norms $\{p_{n,B}\}$:

$$p_{n,B}(L) = \sup_{x \in B} \{\nu_n^2(Lx)\}$$

where $n \in \mathbb{N}$ and B is any bounded subset of \mathbb{F}_1 . This topology is not metrizable since the family $\{p_{n,B}\}$ is not countable.

So $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ will be replaced, under certain assumptions, by a projective limit of appropriate functional spaces as introduced in [10].

We denote by $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$ the space of linear continuous maps (or equivalently bounded linear maps because \mathbb{B}_1^n and \mathbb{B}_2^n are normed spaces). We then have the following result ([7], Theorem 2.3.10).

Theorem 31. *The space of all continuous linear maps between \mathbb{F}_1 and \mathbb{F}_2 which can be represented as projective limits*

$$\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) = \left\{ (L_n) \in \prod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n) : \varprojlim L_n \text{ exists} \right\}$$

is a Fréchet space.

For this sequence $(L_n)_{n \in \mathbb{N}}$ of linear maps, for any integer $0 \leq n \leq m$, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{B}_1^n & \xleftarrow{(\delta_1)_n^m} & \mathbb{B}_1^m \\ L_n \downarrow & & \downarrow L_m \\ \mathbb{B}_2^n & \xleftarrow{(\delta_2)_n^m} & \mathbb{B}_2^m \end{array}$$

On $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$, the topology can be defined by the sequence of seminorms p_n given by

$$p_n(L) = \max_{0 \leq k \leq n} \sup \{ \nu_k^2(Lx), x \in \mathbb{F}_1, \nu_k^1(x) = 1 \}$$

so that $(\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2), p_n)$ is a graded Fréchet space.

Remark 32. For $l \in \{1, 2\}$, given a graduation (ν_n^l) on a Fréchet space \mathbb{F}_l , let \mathbb{B}_l^n be the associated local Banach space and $\delta_l^n : \mathbb{F}_l \rightarrow \mathbb{B}_l^n$ the canonical projection. The quotient norm $\tilde{\nu}_n^l$ associated to ν_n^l is defined by

$$\tilde{\nu}_n^l(\delta_n(z)) = \sup\{\nu_n^l(y) : \delta_n(y) = \delta_n(z)\}. \quad (30)$$

We denote by $(\tilde{\nu}_n^2)^{\text{op}}$ the corresponding operator norm on $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$. If $L = \varprojlim L_n$ where $L_n : \mathbb{B}_1^n \rightarrow \mathbb{B}_2^n$, then we have

$$(\tilde{\nu}_n^2)^{\text{op}}(L) = \sup\{\tilde{\nu}_n^2(L_n.x), x \in \mathbb{B}_1^n, \tilde{\nu}_n^1(x) \leq 1\} = \sup\{\nu_n^2(L.x), x \in \mathbb{F}_1, \nu^1(x) \leq 1\}.$$

This implies that

$$p_n(L) = \max_{0 \leq i \leq n} (\tilde{\nu}_i^2)^{\text{op}}(L_n).$$

Definition 33. Let (\mathbb{F}_1, ν_n^1) and (\mathbb{F}_2, ν_n^2) be graded Fréchet spaces. A linear map $L : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ is called a uniformly bounded operator, if

$$\exists C > 0 : \forall n \in \mathbb{N}, \nu_n(L(x)) \leq C \mu_n(x).$$

We denote by $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ the set of uniformly bounded operators. Of course $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ is contained in $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ and $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ belongs to $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ if and only if $\|L\|_\infty := \sup_{n \in \mathbb{N}} p_n(L) < \infty$ and so

$$\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2) = [\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)]_b := \{L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) : \|L\|_\infty < \infty\}$$

When $\mathbb{F} = \mathbb{F}_1 = \mathbb{F}_2$ and $\nu_n^1 = \nu_n^2$ for all $n \in \mathbb{N}$, the set $\mathcal{H}(\mathbb{F}, \mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F}, \mathbb{F})$) is simply denoted $\mathcal{H}(\mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F})$).

We denote by $\mathcal{IH}_b(\mathbb{F}_1, \mathbb{F}_2)$ (resp. $\mathcal{SH}_b(\mathbb{F}_1, \mathbb{F}_2)$) the set of injective (resp. surjective) operators of $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ with closed range.

Proposition 34. ([4])

- (1) Each operator $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$ has a closed range if and only if, for each $n \in \mathbb{N}$, the induced operator $L_n : \mathbb{B}_1^n \rightarrow \mathbb{B}_2^n$ has a closed range.
- (2) $\mathcal{IH}_b(\mathbb{F}_1, \mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$.
- (3) $\mathcal{SH}_b(\mathbb{F}_1, \mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$.

We will give the sketch of the proof of Point (2) since some arguments used in this proof are also useful for the proof of Theorem 2:

Proof

(2) Consider an injective operator $L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$. According to the representation $\mathbb{F}_i = \varprojlim \mathbb{B}_i^n$ as a projective limit of a projective Banach sequence $(\mathbb{B}_i^n, (\delta_i)_n^m)_{m \geq n}$, we have a sequence of linear operators $L_n : \mathbb{B}_1^n \rightarrow \mathbb{B}_2^n$ such that $L = \varprojlim L_n$ (cf. Theorem 31). Considering each

$$\mathbb{F}_i = \{(x_n) \in \prod_{n \in \mathbb{N}} \mathbb{B}_i^n : \forall m \geq n, x_n = (\delta_2)_n^m(x_m)\}$$

, then if $x = (x_n) \in \mathbb{F}_1$ then $L(x) = (L_n(x_n)) \in \mathbb{F}_2$. Thus it is clear that L is injective if and only if L_n is injective for all $n \in \mathbb{N}$.

Now if $L \in \mathcal{IH}_b(\mathbb{F}_1, \mathbb{F}_2)$, then L_n is an isomorphism from \mathbb{B}_1^n onto its range and so we have

$$\frac{1}{\ell_n} \cdot \tilde{\nu}_n^1(x) \leq \tilde{\nu}_n^2(L_n(x)) \leq \ell_n \cdot \tilde{\nu}_n^1(x) \quad (31)$$

for all $x \in \mathbb{B}_1^n$, all $n \in \mathbb{N}$, where $\tilde{\nu}_n^i$ is the quotient norm of ν_n^i on \mathbb{B}_i^n for $i \in 1, 2$, and

$$\ell_n = (\tilde{\nu}_n^2)^{\text{op}}(L) = \sup\left\{\frac{\tilde{\nu}_n^2(L_n(x))}{\tilde{\nu}_n^1(x)} : x \neq 0\right\}.$$

Since δ_n^2 is the canonical projection of \mathbb{F}_2 on \mathbb{B}_2^n and $\nu_n^2 \circ \delta_n = \tilde{\nu}_n^2$, we obtain

$$\frac{1}{\ell_n} \nu_n^1(x) \leq \nu_n^2(L(x)) \leq \ell_n \cdot \nu_n^1(x) \quad (32)$$

for all $x \in \mathbb{F}_1$ and $n \in \mathbb{N}$. But we have $\ell_n \leq \|L\|_\infty$ and we finally obtain

$$\frac{1}{\ell} \nu_n^1(x) \leq \nu_n^2(L(x)) \leq \ell \cdot \nu_n^1(x) \quad (33)$$

for all $x \in \mathbb{F}_1$, all $n \in \mathbb{N}$ and where $\ell = \|L\|_\infty$.

Fix some $L \in \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ and set $\ell = \|L\|_\infty$, we consider the open set

$$W = \{T \in \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2), : \|T - L\|_\infty < \frac{\ell}{2}\}$$

Fix some $n \in \mathbb{N}$. For any $x \in \mathbb{F}_1$ and $T \in W$, we have

$$\nu_n^1(x) - \nu_n^1(T(x)) \leq \nu_n^1(T - L)(x) \leq p_n(T - L) \cdot \nu_n^1(x) \leq \|T - L\|_\infty \cdot \nu_n^1(x) \leq \frac{\ell}{2} \nu_n^1(x).$$

This implies that

$$\nu_n^2(T(x)) \geq \frac{\ell}{2} \nu_n^1(x). \quad (34)$$

Since (ν_n^i) is a separating sequence of semi-norms, it follows that L is injective.

Now taking in account inequality (34) and relation $\tilde{\nu}_n^i = \nu_n^i \circ (\delta_i)_n$, for $T \in W$ and each $n \in \mathbb{N}$, we have

$$\tilde{\nu}_n^2(T_n(x)) \leq \frac{3\ell}{2} \nu_n^1(x) \leq 3\tilde{\nu}_n^2(T_n(x))$$

for all $x \in \mathbb{B}_1^n$. It follows that T_n is closed and so T is closed (cf. 1.). Finally, W is an open neighbourhood of L contained in $\mathcal{I}\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$, which ends the proof of (2). □

From this Proposition we have

Theorem 35. ([4])

- (1) The Banach space $\mathcal{H}_b(\mathbb{F})$ has a Banach-Lie algebra structure and the set $\mathcal{GH}_b(\mathbb{F})$ of uniformly bounded isomorphisms of \mathbb{F} is open in $\mathcal{H}_b(\mathbb{F})$.
- (2) $\mathcal{GH}_b(\mathbb{F})$ has a structure of Banach-Lie group whose Lie algebra is $\mathcal{H}_b(\mathbb{F})$.
- (3) If \mathbb{F} is identified with the projective $\varprojlim \mathbb{B}^n$ we denote by $\exp_n : \mathcal{L}(\mathbb{B}_n) \rightarrow \mathcal{GL}(\mathbb{B}_n)$, then we have a well defined smooth map $\exp := \varprojlim \exp_n : \mathcal{H}_b(\mathcal{F}) \rightarrow \mathcal{GH}_b(\mathbb{F})$ which is a diffeomorphism from an open set of $0 \in \mathcal{H}_b(\mathcal{F})$ onto a neighbourhood of $\text{Id}_{\mathbb{F}}$.

APPENDIX C. A THEOREM OF EXISTENCE OF ODE

The following result is in fact a reformulation in our context of Theorem 1 in [17].

Theorem 36. Let \mathbb{F} a Fréchet space realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$ whose topology is defined by the sequence of seminorms $(\nu_n)_{n \in \mathbb{N}}$. Let I be an open interval in \mathbb{R} and U be an open set of $I \times \mathbb{F}$. Then U is a surjective projective limit of open sets $U_n \subset I \times \mathbb{B}_n$. Consider a smooth map $f = \varprojlim f_n : U \rightarrow \mathbb{F}$, projective limit of maps $f_n : U_n \rightarrow \mathbb{B}_n$.¹² Assume that for every point $(t, x) \in U$, and every $n \in \mathbb{N}$, there exists an integrable function $K_n > 0$ such that

$$\forall ((t, x), (t, x')) \in U^2, \nu_n(f(t, x) - f(t, x')) \leq K_n(t) \nu_n(x - x'). \quad (35)$$

¹²This means that we have: $\forall m \geq n, \lambda_n^m \circ f_m = f_n \circ (\text{Id}_{\mathbb{R}} \times \lambda_n^m)$

and consider the differential equation:

$$\dot{x} = \phi(t, x). \quad (36)$$

- (1) For any $(t_0, x_0) \in U$, there exists $\alpha > 0$ with $I_\alpha = [t_0 - \alpha, t_0 + \alpha] \subset I$, an open pseudo-ball $V = B(x_0, r) \subset U$ and a map $\Phi : I_\alpha \times I_\alpha \times V \rightarrow \mathbb{F}$ such that

$$t \mapsto \Phi(t, \tau, x)$$

is the unique solution of (36) with initial condition $\Phi(\tau, \tau, x) = x$ for all $x \in V$.

- (2) V is the projective limit of the open balls V_n of \mathbb{B}_n . For each $n \in \mathbb{N}$, the curve $t \mapsto \lambda_n \circ \Phi(t, \tau, \lambda_n(x))$ is the unique solution $\gamma : I_\alpha \rightarrow \mathbb{B}_n$ of the differential equation $\dot{x}_n = \phi_n(t, x_n)$ with initial condition $\gamma(\tau) = \lambda_n(x)$.

From this theorem we obtain easily:

Corollary 37. Let $U = \varprojlim U_n$ be an open subset of \mathbb{F} and $X = \varprojlim X_n : U \rightarrow \mathbb{F}$ a projective limit of smooth maps $X_n : U_n \rightarrow \mathbb{B}_n$. Assume that for every $n \in \mathbb{N}$ we have

$$\forall ((t, x), (t, x')) \in U^2, \nu_n(X(x) - X(x')) \leq K_n \nu_n(x - x'). \quad (37)$$

For $x_0 \in U$, let $B(x_0, 2r) = \{x \in \mathbb{F}, : \nu_{n_i}(x - x_0) < 2r, 1 \leq i \leq k\}$ be a pseudo-ball contained in U . Let us set

$$\begin{aligned} &: C_1 = \max_{1 \leq i \leq k} K_{n_i} \\ &: C_2 = \sup_{z \in B(x_0, r)} \left\{ \max_{1 \leq i \leq k} \nu_{n_i}(f(z)) \right\}. \end{aligned}$$

Then for any $\alpha > 0$ such that $\alpha e^{2\alpha C_1} \leq \frac{r}{2C_2}$, there exists a neighbourhood $V = B(x_0, r)$ and a smooth map $\phi_\alpha : I_\alpha \times V$ such that $t \mapsto \phi^\alpha(t, x)$ is the unique solution of $\dot{x} = X(x)$ defined on I_α with initial condition $\phi^\alpha(0, x) = x$. Moreover if $V_n = \lambda_n(V)$, consider $\phi_n^\alpha : I_\alpha \times V_n \rightarrow \mathbb{B}_n$ defined by $\phi_n^\alpha = \lambda_n \circ \phi^\alpha$; For each $z \in V_n$, the map $t \mapsto \phi^\alpha(t, z)$ is the unique solution of the differential equation $\dot{x}_n = X_n(x_n)$ defined on I_α with initial condition $\phi^\alpha(0, z) = z$.

Remark 38. If $X = \varprojlim X_n$ is a smooth vector field defined on an open set $U = \varprojlim U_n$ of \mathbb{F} , which satisfies assumption (37), as classically, according to Corollary 37, the map $\text{Fl}_t^X := \text{Fl}^X(t, \cdot)$ is the local flow of X that is Fl_t^X fulfils the properties of a 1-parameter group:

$$\begin{aligned} &: \text{Fl}_0^X = Id_V \\ &: \text{Fl}_t^X \circ \text{Fl}_s^X = \text{Fl}_{s+t}^X \text{ if } s, t \text{ and } s+t \text{ belong to } I_\alpha. \end{aligned}$$

In particular Fl_t^X is a diffeomorphism from V onto its range and its inverse is Fl_{-t}^X . Moreover $\text{Fl}_t^{X_n} = \lambda_n \circ \text{Fl}_t^X \circ \lambda_n$ is local flow of $X_n = \lambda_n \circ X \circ \lambda_n$ and we have $\text{Fl}_t^X = \varprojlim \text{Fl}_t^{X_n}$.

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