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▶ **To cite this version:**

Fernand Pelletier. AN INTEGRABILITY CRITERION FOR A PROJECTIVE LIMIT OF BANACH DISTRIBUTIONS. 2021. hal-03207038

HAL Id: hal-03207038
https://hal.science/hal-03207038v1

Preprint submitted on 23 Apr 2021

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AN INTEGRABILITY CRITERION FOR A PROJECTIVE LIMIT OF BANACH DISTRIBUTIONS

FERNAND PELLETIER

Abstract. We give an integrability criterion for a projective limit of Banach distributions on a Fréchet manifold which is a projective limit of Banach manifolds. This leads to a result of integrability of projective limit of involutive bundles on a projective sequence of Banach manifolds. This can be seen as a version of Frobenius Theorem in Fréchet setting. As consequence, we obtain a version of the third Lie theorem for a Fréchet-Lie group which is a submersive projective limit of Banach Lie groups. We also give an application to a sequence of prolongations of a Banach Lie algebroid.

2010 MSC: 53C30, 53Z05, 46T05. *secondary* 18A30, 58B20, 58B25.
Keywords: Banach manifold, Fréchet manifold, projective limit of Banach manifolds, projective limit of Banach bundles, integrability of projective limit of Banach distributions, projective limit of Banach Lie subalgebras.

1. introduction

In classical differential geometry, a *distribution* on a smooth manifold M, is an assignment $\Delta : x \mapsto \Delta_x \subset T_xM$ on M, where Δ_x is a subspace of T_xM. This distribution is *integrable* if, for any $x \in M$, there exists an immersed submanifold $f : L \to M$ such that $x \in f(L)$ and for any $z \in L$, we have $Tf(T_zL) = \Delta_{f(z)}$. On the other hand, Δ is called *involutive* if, for any vector fields X and Y on M tangent to Δ, their Lie bracket $[X, Y]$ is also tangent to Δ.

On a finite dimensional manifold, when Δ is a subbundle of TM, the classical Frobenius Theorem gives an equivalence between integrability and involutivity. In the other cases, the distribution is *singular* and, even under assumptions of smoothness on Δ, in general, the involutivity is not a sufficient condition for integrability (one needs some more additional local conditions). These problems were clarified and resolved essentially in [20] and [19].

In the context of Banach manifolds, the Frobenius Theorem is again true for distributions which are complemented subbundles in the tangent bundle (cf. [16]). For singular Banach distributions closed and complemented (i.e. Δ_x is a complemented Banach subspace of T_xM) we also have the integrability property under some natural geometrical conditions (see [5] for instance). In a more general way, weak Banach distributions Δ (i.e. Δ_x which is a Banach subspace of T_xM not necessary complemented), the integrability property is again true under some additional geometrical assumptions (see [18] or [4] for more details).

The proof of this last results is essentially based on the existence of the flow of a local vector field. In a more general infinite dimensional context as distributions on convenient manifolds or on locally convex manifolds, in general the local flow for a vector field does not exist. Analog results exists in such previous settings: [12], [13], [8], [21] [2] for instance. But essentially, all these integrability criteria are proved under strong assumptions which, either implies the existence of a family of vector fields which are tangent and generate locally a distribution and each one of these vector fields have a local flow, or implies the

existence of an implicit function theorem in such a setting.

The purpose of this paper is to give an integrability criterion for projective limit of Banach distributions on a Fréchet manifold which is a projective limit of Banach manifolds. The precise assumptions on the distribution are presented in assumptions (*) in Definition 1) and this criterion is formulated in a local way in Theorem 2 and in a global way in Theorem 3. These results are obtained under conditions which permits to used a Theorem of existence and unicity of solution of ODE in a Fréchet space proved in [17], and which can be reformulated in the context of projective limit of Banach spaces (cf. Appendix C). Using such a Theorem, this proof needs, in the one hand, an adaptation of some arguments used in the proof of Theorem 1 in [18] for closed distributions on Banach manifolds, and on the other hand, some properties of the Banach Lie group of "uniformly bounded" automorphisms of a Fréchet space (cf. Appendix B). As application, this criterion permits to obtain a kind of *projective limit of "Banach Frobenius theorem"* for submersive projective limit of involutive bundles on a submersive projective sequence of Banach manifolds (cf. Theorem 4). By the way, as consequence, a submersive projective limit of complemented Banach Lie subalgebras of a submersive projective limit of Banach Lie group algebras is the Lie algebra of a Féchet Lie group (cf. Theorem 5). This result can be considered as *a version of the third Lie Theorem for a Fréchet-Lie group* which is a submersive projective limit of Banach Lie groups. We also give an application to a sequence of prolongations of a Banach Lie algebroid (cf. Theorem 7). We complete these results, by an example of integrable Fréchet distribution which is a projective limit of non integrable distributions but which satisfies the assumptions (*).

This paper is organized as follows. The first paragraph of the next section describes the context and the assumptions of this criterion of integrability used in Theorem 2 and Theorem 3. In order to present these Theorems in a more accessible way for a quick reading, the useful definitions and results take place in Appendix A, and we formulate the assumptions with precise references to this Appendix. The Theorem 3 permits to show that, under some natural conditions, the projective limit E of a submersive projective sequence of involutive subbundles E_i of the tangent bundle TM_i of a submersive projective sequence of Banach manifold M_i, is a Fréchet involutive and integrable subbundle of the tangent bundle TM of the Fréchet manifold $M = \varprojlim M_i$.
A first application of these results, is the existence of a Féchet Lie group whose Lie algebra is a submersive projective limit of complemented Banach Lie subalgebra of the a submersive projective limit of Banach Lie groups Lie algebras (cf. Theorem 5 in § 3.1). In § 3.2, we give an application of Theorem 2 to a sequence of prolongations of a Banach-Lie algebroid (see [3]) and we end this paragraph by the announced contre-exemple of Theorem 2 and Theorem 3.
The proof of the basic Theorem 2 is located in §4.
All properties concerning the set of uniformly bounded endomorphisms of a Fréchet space are developed in Appendix B. Also in a series of other Appendices, we expose all the definitions and results needed in the statements of Theorems and in the proof of Theorem 2.

2. An integrability criterion for a submersive projective limit of Banach distributions

2.1. **The criterion and its corollaries.** *The context needed in this section is detailed in Appendix A.*
Let $\left(M_i, \delta_i^j\right)_{j\geq i}$ be a sequence of projective Banach manifolds with projective limit $M = \varprojlim M_i$ [1]. In order to give a criterion of integrability for projective limits of local Banach

[1] cf. section A.4

bundles on M under some additional assumptions, we need to introduce some notations. Let ν (resp. μ) be a norm on a Banach space $\mathbb{E}$ (resp. $\mathbb{M}$). We denote by $\| \ \|^{\mathrm{op}}$ the associated norm on the linear space of continuous linear mappings $\mathcal{L}(\mathbb{E}, \mathbb{M})$.
Then we have:

Definition 1. *Let $\left(M_i, \delta_i^j\right)_{j \geq i}$ be a projective sequence of Banach manifolds where the maps δ_i^j are submersions and $M = \varprojlim M_i$ its projective limit.*
A closed distribution Δ on M will be called a submersive projective limit of local anchored bundles if the following property is satisfied:

(*) *For any $x = \varprojlim x_i \in M$, there exists an open neighbourhood $U = \varprojlim U_i$ of x, a submersive projective sequence of anchored Banach bundles $(E_i, \pi_i, U_i, \rho_i)$* [2] [3] [4] *fulfilling the following properties for any $z = \varprojlim z_i \in U$:*

1. *$\varprojlim \rho_i((E_i)_{z_i}) = \Delta_z$, for any $z \in U$.*
2. *The kernel of $(\rho_i)_{z_i}$ is complemented in $(E_i)_{z_i}$ and the range of $(\rho_i)_{z_i}$ is closed, for all $i \in \mathbb{N}$.*
3. *There exists a constant $C > 0$ and a Finsler norm $|| \ ||^{E_i}$ (resp. $|| \ ||^{M_i}$) on $(E_i)_{|U_i}$ (resp. $TM_{i|U_i}$) such that:*

$$\forall i \in \mathbb{N}, \ ||(\rho_i)_{z_i}||_i^{\mathrm{op}} \leq C, \ \forall z_i \in U_i.^{5}$$

We have the following criterion of integrability:

Theorem 2. *Let M be a projective limit of a submersive projective sequence $(M_i, \delta_i^j)_{j \geq i}$ of Banach manifolds and Δ be a local projective limit of local Banach bundles on M. Assume that, under the property (*), there exists a Lie bracket $[.,.]_i$ on $(E_i, \pi_i, U_i, \rho_i)$ such that $(E_i, \pi_i, U_i, \rho_i, [.,.]_i)$ is a submersive projective sequence of Banach-Lie algebroids* [6].
Then the distribution Δ is integrable and the maximal integral manifold N through $x = \varprojlim x_i$ is a closed Fréchet submanifold of M which is a submersive projective limit of the set of maximal leaves N_i of $\rho_i(E_i)$ through x_i in M_i.

The proof of Theorem takes place in section 4.

Now, we have the following consequences of Theorem 2:

Theorem 3. *Let $(E_i, \pi_i, M_i, \rho_i, [.,.]_i)$ be a submersive projective sequence of split Lie algebroids* [7]. *Then we have:*

1. *$\left(E := \varprojlim E_i, \pi := \varprojlim \pi_i, M := \varprojlim M_i, \rho = \varprojlim \rho_i\right)$ is Fréchet anchored bundle and $\Delta = \rho(E)$ is a closed distribution on M*
2. *If (ρ_i) satisfies the condition (3) in Definition 1, then Δ is integrable and each leaf L of Δ is a projective limit of leaves L_i of Δ_i.*

Proof The property (1) is a consequence of Proposition 29. From this property, it follows that locally Δ satisfies assumption (1) and (2) of Definition 1 so if the assumption (3) is satisfied, the result is a direct consequence of Theorem 2.

□

Theorem 4. *Let $\left(M_i, \delta_i^j\right)_{j \geq i}$ be a submersive projective sequence of Banach manifold and (E_i, π_i, M_i) an involutive subbundle of TM_i where π_i is the restriction of the natural projection $p_{M_i} : TM_i \to M_i$. Assume that the restriction $T\delta_i^j : E_j \to E_i$ is a surjective*

[2] **see:** Definition 25 and Notations 24 for a submersive sequence of projective Banach bundles
[3] **see:** Definition 27 for an anchored Banach bundle
[4] a sequence of projective anchored bundle $(E_i, \pi_i, M_i, \rho_i)$ is a projective sequence of Banach bundles which satisfies assumption **(PSBLA 2)** in Definition 28
[5] More precisely, $||(\rho_i)_{z_i}||_i^{\mathrm{op}} = \sup\{||(\rho_i)_{z_i}(u)||^{M_i}, \ ||u||^{E_i} \leq 1\}$
[6] cf. Definition 28
[7] cf. Definition 28

bundle morphism for all $i \in \mathbb{N}$ and $j \geq i$. Then (E_i, π_i, M_i) is a submersive projective sequence of Banach bundles, and $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)$ is an integrable Fréchet subbundle of TM whose each leaf L of E in M is a projective limit of leaves L_i of E_i in M_i

Proof Since $\delta_i^j : M_j \to M_i$ is a surjective submersion, so is $T\delta_i^j : TM_j \to TM_i$. If $T\delta_i^j : E_j \to E_i$ is a surjective morphism, this implies that $T\delta_i^j$ is a submersion onto E_i and so (E_i, π_i, M_i) is a submersive projective sequence of Banach bundles. Let $\iota_i : E_i \to TM_i$ the natural inclusion and $[\ ,\ ]_i$ the restriction of Lie bracket of vector fields to (local) sections of E_i. Then $(E_i, M_i, \iota_i, [\ ,\ ]_i)$ is a Banach Lie algebroid and since $T\delta_i^j \circ \iota_j = \iota_i \circ T\delta_i^j$ it follows that $(E_i, M_i, \iota_i, [\ ,\ ]_i)$ is a is a submersive projective sequence of Banach-Lie algebroids. Fix some $x = \varprojlim x_i \in M = \varprojlim M_i$. According to Theorem 3 we have only to show that the condition (3) of Definition 1 is satisfied by ι_i.
Given any norm $||\ ||^{M_i}$ on $T_{x_i}M_i$ we denote by $||\ ||^{E_i}$ the induced norm on the fiber $\{E_i\}_{x_i}$, then, for the associated norm operator we have $||\{\iota_i\}_{x_i}||_i^{\rm op} = 1$. So all the assumption of Theorem 2 are satisfied which ends the proof.

□

3. Some applications and contre-example

3.1. **Application to submersive projective sequence of Banach Lie groups.** Let $\left(G_i, \delta_i^j\right)_{j\geq i}$ be a submersive a projective sequence of Banach-Lie groups where G_i is modelled on $\mathbb{G}_i$ (cf. Definition 19). We denote by $\mathbf{L}(G_i)$ the Lie algebra of G_i. Then $\mathbf{L}(G_i) \equiv T_eG_i$ is isomorphic to $\mathbb{G}_i$. If we set $\bar\delta_i^j := T_e\delta_i^j$, then each $\bar\delta_i^j$ is a surjective linear map from $\mathbf{L}(G_j)$ to $\mathbf{L}(G_i)$ whose kernel is complemented.
Consider a sequence $\mathfrak{h}_i$ of complemented sub-Lie algebra of $\mathbf{L}(G_i)$ such that the restriction of $\bar\delta_i^j$ to $\mathfrak{h}_j$ is a continuous surjective map. Then $(\mathfrak{h}_i, \hat\delta_i^j)_{j\geq i}$ is a submersive projective sequence of Banach Lie algebra and so $\mathfrak{h} = \varprojlim \mathfrak{h}_i$ is a Fréchet Lie algebra (cf. [4] chapter 4). Now from classic result on Banach Lie groups (cf. [16]), by left translation each $\mathfrak{h}_i$ gives rise to a complemented involutive subbundle of $\mathcal{H}_i$ of TG_i and the leaf H_i through the neutral e_i in G_i has a structure of connected Banach Lie group so that the inclusion $\iota_i : H_i \to G_i$ is a Banach Lie morphism. Note that $\mathfrak{i}_i := T_{e_i}\iota_i$ is nothing but else that the inclusion of $\mathfrak{h}_i$ in $\mathbf{L}(G_i)$ and which induces the natural inclusion $\hat\imath_i$ of $\mathcal{H}_i$ in TG_i.
Moreover, since $(\mathfrak{h}_i, \hat\delta_i^j)_{j\geq i}$ is a submersive projective sequence of Banach Lie algebra and $\left(G_i, \delta_i^j\right)_{j\geq i}$ be a submersive a projective sequence of Banach-Lie groups, it follows that $(\mathcal{H}_i, G_i, \widehat{\mathfrak{i}}_i, [\ ,\ ]_i)$ [8] is a submersive projective sequence of split Banach Lie algebroids.

On the other hand, from Theorem 21 the Lie algebra $\mathbf{L}(G)$ of $G = \varprojlim G_i$ is $\varprojlim \mathbf{L}G_i$ and so $\mathfrak{h} = \varprojlim \mathfrak{h}_i$ is a closed complemented Lie subalgebra of $\mathbf{L}(G)$. As in the context of Banach setting, by left translation, $\mathfrak{h}$ gives rise to an involutive Fréchet subbundle $\mathcal{H}$ of TG which is clearly the projective limit of the submersive sequence $(\mathcal{H}_i, G_i, \widehat{\mathfrak{i}}_i, [\ ,\ ]_i)$. So from Theorem 4 by same arguments as in Banach Lie groups, we obtain:

Theorem 5. *Let $G = \varprojlim G_i$ be a the projective limit of a submersive projective sequence of Banach-Lie groups $\left(G_i, \delta_i^j\right)_{j\geq i}$ and for each $i \in \mathbb{N}$, consider a closed complemented Banach Lie subalgebra $\mathfrak{h}_i$ of $\mathbf{L}(G_i)$. Assume that the restriction of $\bar\delta_i^j$ to $\mathfrak{h}_j$ is a continuous surjective map. Then there exists a Fréchet Lie group H in G such that $\mathbf{L}(H)$ is isomorphic to $\mathfrak{h}$ and H is the projective limit of $(H_i, \delta^j_{i\,|H_j})_{j\geq i}$.*

Remark 6. *The reader will also find an application of Theorem 2 in the proof of Theorem 8.23 on submersive projective limit of a projective sequence of Banach groupoids in*

[8] here $[\ ,\ ]_i$ denote again the restriction to sections of $\mathcal{H}_i$ of the Lie bracket of vector fields on G_i

[4] *which is a kind of generalization of Theorem 5 to Lie groupoids setting .*

3.2. **Application to sequences of prolongations of a Banach-Lie algebroid over a Banach manifold.** Consider an anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ with typical fiber $\mathbb{A}$. Let $V\mathcal{A} \subset T\mathcal{A}$ be the vertical subbundle of $p_{\mathcal{A}} : T\mathcal{A} \to \mathcal{A}$. If $\mathcal{A}_x := \pi^{-1}(x)$ is the fiber over $x \in M$, according to [3], the prolongation $\mathbf{T}\mathcal{A}$ of the anchored Banach bundle $(\mathcal{A}, \pi, M, \rho)$ over $\mathcal{A}$ is the set $\{(x, a, b, c), (x, b) \in \mathcal{A}_x, \ \ (x, a, c) \in V_{(x,a)}\mathcal{A}\}$. It is a Banach vector bundle $\hat{\mathbf{p}} : \mathbf{T}\mathcal{A} \to \mathcal{A}$ with typical fiber $\mathbb{A} \times \mathbb{A}$ and we have an anchor $\hat{\rho} : \mathbf{T}\mathcal{A} \to T\mathcal{A}$ given by

$$\hat{\rho}(x, a, b, c) = (x, a, \rho_x(b), c) \in T_{(x,a)}\mathcal{A}.$$

From now on, we fix a Banach Lie algebroid $(\mathcal{A}, \pi, M, \rho, [., .]_{\mathcal{A}})$ such that the typical fiber $\mathbb{A}$ of $\mathcal{A}$ is finite dimensional. By the way, we have a Banach Lie algebroid structure $(\mathbf{T}\mathcal{A}, \hat{\mathbf{p}}, \mathcal{A}, \hat{\rho}, [., .]_{\mathbf{T}\mathcal{A}})$ (cf. [3] Corollary 44)

We denote $(\mathcal{A}_1, \pi_1, \mathcal{A}_0, \rho_1, [., .]_1)$ the Banach-Lie algebroid $(\mathcal{A}, \pi, M, \rho, [., .]_{\mathcal{A}})$ over a Banach manifold $\mathcal{A}^0 = M$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\rho_1} & T\mathcal{A}_0 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle p_{\mathcal{A}_0}} & \\ & \mathcal{A}_0 & \end{array} \tag{1}$$

According to the notations of Theorem 43 and Corollary 44 in [3], we set $\mathcal{A}_2 = \mathbf{T}\mathcal{A}_1$, $\rho_2 = \hat{\rho}$, $[., .]_2 = [., .]_{\mathbf{T}\mathcal{A}_1}$ and $\pi_2 = \hat{\mathbf{p}}$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A}_2 & \xrightarrow{\rho_2} & T\mathcal{A}_1 & & \\ {\scriptstyle \pi_2} \searrow & & {\scriptstyle p_{\mathcal{A}_1}} \swarrow \quad \searrow {\scriptstyle T\pi_1} & & \\ & \mathcal{A}_1 & \xrightarrow{\rho_1} & T\mathcal{A}_0 & \\ & {\scriptstyle \pi_1} \searrow & & \swarrow {\scriptstyle p_{\mathcal{A}_0}} & \\ & & \mathcal{A}_0 & & \end{array} \tag{2}$$

Fix some $x \in \mathcal{A}_0$ and a norm $||\ ||_0$ (resp.$||\ ||_1$) on the fibre $T_x\mathcal{A}_0 \equiv \mathbb{A}_0$ (resp. $\mathcal{A}_x = \pi_1^{-1}(x) \equiv \mathbb{A}_1$). Since the fiber $\mathbf{T}_{(x,a)}\mathcal{A}_1$ (resp. $T_{(x,a)}\mathcal{A}_1$) is isomorphic to $\mathbb{A}_1 \times \mathbb{A}_1$ (resp. $\mathbb{A}_0 \times \mathbb{A}_1$), it follows that $\sup\{||\ ||_1 , ||\ ||_1\}$ (resp. $\sup\{||\ ||_0 , ||\ ||_1\}$ gives rise to a norm on $\mathbf{T}_{(x,a)}\mathcal{A}_1$ (resp. $T_{(x,a)}\mathcal{A}_1$). Then for the associated operator norm $||\ ||^{\mathrm{op}}$ we have

$$||(\rho_2)_{(x,a)}||^{\mathrm{op}} \leq \sup(||(\rho_1)_x||^{\mathrm{op}}, 1) \tag{3}$$

By induction, for $i \geq 1$, again according to notations of Theorem 43 and Corollary 44 in [3] , we we set

$\mathcal{A}^{i+1} = \mathbf{T}^{\mathcal{A}_i}\mathcal{A}_i$, $\rho_{i+1} = \hat{\rho}_i$, $[., .]_{i+1} = [., .]_{\mathbf{T}^{\mathcal{A}_i}\mathcal{A}_i}$ and $\pi_{i+1} = \hat{\mathbf{p}}$ and, as before, we have the following commutative diagrams:

$$\begin{array}{ccccc} \mathcal{A}_{i+1} & \xrightarrow{\rho_{i+1}} & T\mathcal{A}_i & & \\ {\scriptstyle \pi_{i+1}} \searrow & & {\scriptstyle p_{\mathcal{A}}} \swarrow \quad \searrow {\scriptstyle T\pi_i} & & \\ & \mathcal{A}_i & \xrightarrow{\rho_i} & T\mathcal{A}_{i-1} & \\ & {\scriptstyle \pi_i} \searrow & & \swarrow {\scriptstyle p_{\mathcal{A}_{i-1}}} & \\ & & \mathcal{A}_{i-1} & & \end{array} \tag{4}$$

Also, by same arguments as for (3) we obtain:

$$||(\rho_{i+1})_{(x,a_1,\ldots,a_{i+1})}||^{\mathrm{op}} \leq \sup(||(\rho_i)_{(x,a_1,\ldots,a_i)}||^{\mathrm{op}}, 1) \tag{5}$$

It follows that we have a submersive projective sequence of Banach-Lie algebroids $(\mathcal{A}_i, \mathcal{A}_{i-1}, \rho_i, [.,.]_i)$ over a submersive projective sequence of Banach manifolds $(\mathcal{A}_i)_{i\in\mathbb{N}}$ which satisfies the assumptions of Theorem 2. Thus, we obtain:

Theorem 7. *Under the previous context,*

$$(\mathcal{A} = \varprojlim_{i\geq 1}\mathcal{A}_i, \mathcal{M} = \varprojlim_{j\geq 0}\mathcal{A}_j, \rho = \varprojlim_{i\geq 1}\rho_i, [.,.] = \varprojlim_{i\geq 1}[.,.]_i)$$

is a Fréchet Lie algebroid on the Fréchet manifold $\mathcal{M}$ and the distribution $\rho(\mathcal{A})$ is integrable. Each leaf L is a projective limit of a projective sequence of leaves of type (L_i) defined by induction in the following way:

L_0 is a leaf of $\rho_1(\mathcal{A}_1)$ and if L_i is a leaf of $\rho_i(\mathbb{A}_i)$ then $L_{i+1} = (\mathcal{A}_i)_{|L_i}$.

3.3. **A contre-example.** In this subsection we give a Example of an integrable distribution on a Fréchet bundle over a finite dimensional manifold which satisfies the assumptions (*) in Definition 1 but is a projective limit of a submersive sequence of Banach **not integrable** distributions.

Let $E = M \times \mathbb{R}^m$ the trivial bundle over a manifold M of dimension n. The set $J^k(E)$ of the k-jets of section of E over M is a finite dimensional manifold which is a vector bundle $\pi^k : J^k(E) \to M$ and whose typical fiber is is the space $\prod_{j=0}^{k} \mathcal{L}^j_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^m)$ where $\mathcal{L}^j_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^m)$ is the space of continuous j-linear symmetric mappings $\mathbb{R}^n \to \mathbb{R}^m$. Then each projection $\pi^l_k : J^l(E) \to J^k(E)$ defined, for $l \geq k$, by

$$\pi^l_k \left[j^l(s)(x) \right] = j^k(s)(x)$$

is a smooth surjection.

Proposition 8. *([4]) $\left(J^k(E), \pi^l_k\right)$ is a submersive projective sequence of Banach vector bundles and the projective limit*

$$J^\infty(E) = \varprojlim J^k(E)$$

can be endowed with a structure of Fréchet vector bundle whose fibre is isomorphic to the Fréchet space $\prod_{j=0}^{\infty} \mathcal{L}^j_{\mathrm{sym}}(\mathbb{R}^n, \mathbb{R}^m)$.

Let s be a section of π on a neighbourhood U of $x \in M$. For $\xi = j^k(s)(x) \in J^k(E)$, the n-dimensional subspace $R(s,x)$ of $T_\xi J^k(E)$ equals to the tangent space at ξ to the submanifold $j^k(s)(U) \subset J^k(\pi)$ is called an *R-plane*.

The Cartan subspace $\mathcal{C}^k(E)$ of $T_\xi J^k(E)$ is the linear subspace spanned by all R-planes $R(s',x)$ such that $j^k(s')(x) = \xi$. So it is the hull of the union of $\left(j^k(s)\right)_*(x)(T_xM)$ where s is any local section of π around de x.

The Cartan subspaces form a smooth distribution on $J^k(\pi)$ called *Cartan distribution*and denoted $\mathcal{C}^k$. Then $\mathcal{C}^k$ is a regular distribution which a contact distribution and so is not integrable (cf. [14]). We have a submersive projective limit of bundle $(\mathcal{C}^k, T\pi^l_k, j^k(E))$ whose projective limit $\mathcal{C} = \varprojlim \mathcal{C}^k$ is called the Cartan distribution on $J^\infty(E)$. In fact $\mathcal{C}$ is integrable (cf. [14]). Note that since $\mathcal{C}^k$ is a subbundle of $TJ^k(E)$, from the proof of Theorem 4, the condition (3) of Definition 1 is satisfied.

4. Proof of Theorem 2

Fix some $x \in M$. According to the property (*) and the Definition 1, we can choose a submersive projective sequence of charts $\left(U_i, \delta^j_{i\,|U_j}\right)_{j\geq i}$ and a submersive projective sequence of Banach bundles $(E_i, \lambda^j_i)_{j\geq i}$, such that:

- $U = \varprojlim U_i$ is an open neighbourhood of x in M;
- If $x = \varprojlim(x_i)$, each U_i is the contractile domain of a chart (U_i, ϕ_i) around x_i in M_i and $(U = \varprojlim(U_i), \phi = \varprojlim(\phi_i))$ is a projective limit chart in M around x.
- the projective sequence of Banach bundles $(E_i, \lambda^j_i)_{j\geq i}$ satisfies the assumption (*) on U.

Step 1: The kernel of ρ_x is supplemented.
There exists a trivialization $\tau_i : E_i \to U_i \times \mathbb{E}_i$ which satisfies the compatibility condition:

$$(\delta^j_i \times \overline{\lambda^j_i}) \circ \tau_j = \tau_i \circ \lambda^j_i \tag{6}$$

where $(\mathbb{E}_i, \overline{\lambda^j_i})_{j\geq i}$ is the projective sequence of Banach spaces on which $(E_i, \lambda^j_i)_{j\geq i}$ is modeled.
Under these conditions, without loss of generality we may assume that, for each $i \in \mathbb{N}$, we have

- $x_i \equiv 0 \in \mathbb{M}_i$;
- U_i is an open subset of $\mathbb{M}_i$ and so $TU_i = U_i \times \mathbb{M}_i$;
- $E_i = U_i \times \mathbb{E}_i$.

The projection δ^j_i at point $x_j \equiv 0$ (resp. λ^j_i in restriction to the fibre of E_j over $x_j \equiv 0$) is denoted d^j_i (resp. ℓ^j_i). The morphism ρ_i in restriction to the fibre E_i over $x_i \equiv 0$ is denoted r_i and ρ_x is denoted r, so that $r = \varprojlim r_i$. Now, according to the context of Assumption 1 in Definition 1, we have the following result:

Lemma 9. *There exists a decomposition $\mathbb{E} = \ker r \oplus \mathbb{F}'$ with the following property: if (ν'_n) (resp. (μ_n) is the graduation on $\mathbb{F}'$ (resp. (μ_n) on $\mathbb{M}$) induced by the norm $||\ ||^{\mathbb{E}_i}$ (resp. $||\ ||^{\mathbb{M}_i}$) on $(E_i)_{x_i}$ (resp. $T_{x_i}M_i$), then the restriction of r to $\mathbb{F}'$ is a closed uniformly bounded operator according to these graduations* [9].

Proof of Lemma 9

At first, in such a context we have $T_{x_j}\delta^j_i \equiv d^j_i$ and the following compatibility condition:

$$d^j_i \circ r_j = r_i \circ \ell^j_i. \tag{7}$$

We set $\mathbb{F}_i = r_i(\mathbb{E}_i)$ for all $i \in \mathbb{N}$. From Definition 1 assumption 1, $\mathbb{F}_i$ is a Banach subspace of $\mathbb{E}_i$ and there exists a decomposition $\mathbb{E}_i = \ker r_i \oplus \mathbb{F}'_i$. Thus the restriction r'_i of r_i to $\mathbb{F}'_i$ is an isomorphism onto $\mathbb{F}_i$. Now, from (7), we have

$$\forall (i,j) \in \mathbb{N}^2 : j > i,\ d^j_i \circ r'_j = r'_i \circ \ell^j_i.$$

But since each r'_i is an isomorphism, the restriction $(\ell^j_i)'$ of ℓ^j_i to $\mathbb{F}'_j$ takes values in $\mathbb{F}'_i$ for all $(i,j) \in \mathbb{N}^2$ such that $j \geq i$. Moreover, as δ^j_i is surjective, according to (7) again, this implies that $d^j_i(\mathbb{F}_j) = \mathbb{F}_i$ and so $\ell^j_i(\mathbb{F}'_j) = \mathbb{F}'_i$. Since $(\mathbb{E}_i, \ell^j_i)_{j\geq i}$ is a projective sequence, this implies that $(\mathbb{F}'_i, (\ell^j_i)')_{j\geq i}$ is a surjective projective system. The vector space $\mathbb{F}' = \varprojlim \mathbb{F}'_i$ is then a Fréchet subspace of $\mathbb{E}$.
On the other hand, let $(\ell^j_i)''$ be the restriction of ℓ^j_i to $\ker r_j$. Always from (7), we have

$$\forall (i,j) \in \mathbb{N}^2 : j > i,\ (\ell^j_i)''(\ker r_j) \subset \ker r_i.$$

[9] cf. Appendix B

By same argument, it implies that $(\ker r_i, (\ell_i^j)'')_{j\geq i}$ is a projective sequence and $\ker r = \varprojlim \ker r_i$. Moreover, since $\mathbb{E}_i = \ker r_i \oplus \mathbb{F}'_i$, it follows that $\mathbb{E} = \ker r \oplus \mathbb{F}'$ and also the restriction r' of r to $\mathbb{F}'$ is obtained as $r' = \varprojlim r'_i$ and r' is an injective continuous operator $r' : \mathbb{F}' \to \mathbb{M}$ whose range is the closed subspace $\mathbb{F} = \varprojlim \mathbb{F}_i$. It remains to show that r' is uniformly bounded.
According to the context of assumption 2 of Definition 1, there exists a constant $C > 0$ and, for each $i \in \mathbb{N}$, we have a norm $\| \ \|^{\mathbb{E}_i}$ on $\mathbb{E}_i$ and a norm $\| \ \|^{\mathbb{M}_i}$ on $\mathbb{M}_i$ such that $\|r_i\|_i^{\mathrm{op}} \leq C$. As $\mathbb{F}'_i$ is a closed Banach subspace of $\mathbb{E}_i$, it follows that for the induced norm on $\mathbb{F}'_i$ we have

$$\forall i \in \mathbb{N},\ ||r'_i||_i^{\mathrm{op}} \leq C. \tag{8}$$

Set $\ell_i = \varprojlim \ell_i^j$ and $d_i = \varprojlim d_i^j$. By construction we have $\ell_i(\mathbb{F}') = \mathbb{F}'_i$ and $d_i(\mathbb{M}) = \mathbb{M}_i$. The norm $\| \ \|^{\mathbb{E}_i}$ on $\mathbb{E}_i$ induces a norm $\| \ \|^{\mathbb{F}'_i}$ on the Banach subspace $\mathbb{F}'_i$ and we get a natural graduation (ν'_i) on $\mathbb{F}'$ given by $\nu'_i(u) = ||\ell_i(u)||^{\mathbb{F}'_i}$ (cf. Appendix B (29)). In the same way, the norm $\| \ \|^{\mathbb{M}_i}$ induces a graduation (μ_i) on $\mathbb{M}$ given by $\mu_i(v) = ||d_i(v)||^{\mathbb{M}_i}$. Now (8) implies that r' is uniformly bounded and $r'(\mathbb{F}') = r(\mathbb{E}) = \Delta_x$ is closed by assumption. Therefore, the proof of Lemma 9 is complete.

□

Step 2: There exists a neighbourhood $V \subset U$ of x such that the map $\rho' = \rho_{|U\times\mathbb{F}'}$ takes values in $\mathcal{IH}_b(\mathbb{F}', \mathbb{M})$[10] and is K-Lipschitz on V for some $K > 0$.

Since $E_i = U_i \times \mathbb{E}_i$, it follows that $E = U \times \mathbb{E}$ and so $\rho : E \to TU$ can be seen as a smooth map from U into $\mathcal{H}(\mathbb{E}, \mathbb{M})$. Let ρ' be the restriction of ρ to $U \times \mathbb{F}'$ and so consider ρ' as a smooth map from U to $\mathcal{H}(\mathbb{F}', \mathbb{M})$. From the definition of a Finsler norm, Assumption 2 in Definition 1 and Lemma 9, the map $x \mapsto \rho'_x$ takes value in $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$.

Lemma 10. *There exists a neighbourhood $V_1 \subset V$ of 0 such that the map $\rho' : V_1 \to \mathcal{H}_b(\mathbb{F}', \mathbb{M})$ is Lipschitz, that is:*
there exists $K > 0$ such that

$$\hat{\mu}_i^{op}(\rho'_z - \rho'_x) \leq K\hat{\mu}_i(z - z'),\ \forall (z, z') \in V_1^2$$

[11]

Proof of Lemma 10 Let $(\hat{\nu'}_i)_{i\in\mathbb{N}}$ (resp. $(\hat{\mu}_i)_{i\in\mathbb{N}}$) be the canonical increasing graduation associated to the graduation $(\nu'_i)_{i\in\mathbb{N}}$ on $\mathbb{M}$ (resp. $(\hat{\mu}_i)_{i\in\mathbb{N}}$) (cf. Appendix (B 29)). Since the map $x \mapsto \rho'_x$ is a smooth map from U to $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ it follows that for each x the differential map $d_x\rho'$ is a continuous linear map from $\mathbb{M}$ to the Banach space $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$ and so there exists $i_0 \in \mathbb{N}$ and a constant $A_x > 0$ such that

$$\|d_x\rho'(u)\|_\infty \leq A_x\ \hat{\nu'}_{i_0}(u) \tag{9}$$

for all $u \in \mathbb{F}'$ and so

$$\|d_x\rho'(u)\|_\infty \leq A_x\ \hat{\nu'}_i(u) \tag{10}$$

for all $u \in \mathbb{F}'$ and and $i \geq i_0$ and according to Remark 32 we set

$$||d_x\rho'||_i^{op} := \sup\{\hat{\mu}_i(d_x\rho'(u)) :\ \hat{\nu'}_i(u) \leq 1\} \leq A_x,\ \forall i \geq i_0 \tag{11}$$

On the other hand, for $1 \leq i < i_0$ we set $C_x^i = ||d_x\rho'||_i^{op}$. Then $d_x\rho'$ belongs to $\mathcal{H}_b\left(\mathbb{M}, \mathcal{H}_b(\mathbb{F}', \mathbb{M})\right)$ for all $x \in U$ since we have

$$||d_x\rho'||_\infty := \sup_{i\in\mathbb{N}} ||d_x\rho'||_i^{op} \leq \sup\{A_x, C_x^1, \cdots, C_x^{i_0-1}\}.$$

[10] cf. Appendix B
[11] cf. Remark 32

We set $C = ||d_0\rho'||_\infty$. By continuity, there exists an open neighbourhood V_1 of 0 such that

$$||d_x\rho'||_\infty \leq 2C \tag{12}$$

By choosing $K = 2C$, from the definition of $||d_x\rho'||_\infty$ it implies the announced result.

□

Step 3: Local flow of the vector field $X_u = \rho'(u)$.

Consider a neighbourhood V as announced in step 2. As δ_i is surjective, $V_i = \delta_i(V)$ is an open set of U_i and so we have $V = \varprojlim V_i$. For each $u \in \mathbb{F}'$, let $X_u = \rho'(u)$ be the vector field on V. If $u = \varprojlim u_i$ with $u_i \in \mathbb{F}'_i$, then $X_{u_i} = \rho'_i(u_i)$ is a vector field on V_i and $X_u = \varprojlim X_{u_i}$. Now since ρ' takes values in $\mathcal{IH}_b(\mathbb{F}', \mathbb{M})$, from our assumption, there exists a constant $C > 0$ such that $||(\rho'_i)_{z_i}||_i^{\mathrm{op}} \leq C$ for all $z_i \in V_i$. Therefore, if u belongs to $\mathbb{F}'$ and $u_i = \lambda(u)$, then $\|X_{u_i}\|^{\mathbb{M}_i} \leq C\|u\|_i^{\mathbb{E}}$ and, from Lemma 10 and the definition of $\hat{\mu}_i^{\mathrm{op}}$ we have

$$\forall \left(x_i,\ x'_i\right) \in (\delta_i(V))^2\,,\ \|X_{u_i}(x_i) - X_{u_i}(x'_i)\|^{\mathbb{M}_i} \leq K\|u_i\|^{\mathbb{E}_i}||x_i - x'_i||^{\mathbb{M}_i}. \tag{13}$$

Now, recall that we have provided $\mathbb{F}'$ and $\mathbb{M}$ with seminorms $(\hat{\nu'}_n)$ and $(\hat{\mu}_n)$ respectively defined by

$$\nu_n(u) = \|\lambda_i(u)\|^{\mathbb{E}_i} \text{ and } \mu_n(x) = \|\delta_i(x)\|^{\mathbb{M}_i}.$$

Since $\delta_i(X_u)(x) = X_{u_i}(\delta_i(x))$ and in this way, from (13), for all $n \in \mathbb{N}$, we have

$$\forall (x, x') \in V^2,\ \mu_n(X_u(x) - X_u(x')) \leq K\nu_n(u)\mu_n(x - x'). \tag{14}$$

Therefore, X_u satisfies the assumption of Corollary 37. Let $\epsilon > 0$ such that the pseudo-ball

$$B_{\mathbb{M}}(0, 2\epsilon) = \{x \in \mathbb{M},\ :\ \hat{\mu}_{n_i}(x) < 2\epsilon, 1 \leq i \leq k\}$$

is contained in V and set

$$C_1 := \max_{1\leq i\leq k}\{K\nu_{n_i}(u)\} = K \max_{1\leq i\leq k} \nu_{n_i}(u);$$

$$C_2 := \sup_{x\in B_{\mathbb{M}}(0,\epsilon)} \left\{\max_{1\leq i\leq k} \mu_{n_i}(X(x))\right\} \leq C \max_{1\leq i\leq k} \nu_{n_i}(u).$$

By application of Corollary 37, for any u such that $\max_{1\leq i\leq k} \nu_{n_i}(u) \leq 1$, there exists $\alpha > 0$ such that

$$\alpha e^{2\alpha K} \leq \frac{\epsilon}{2C},$$

such that the local flow Fl_t^u is defined on the pseudo-ball $B_{\mathbb{M}}(0, \epsilon)$ for all $t \in [-\alpha, \alpha]$ for all u which satisfied the previous inequality

Note that for any $s \in \mathbb{R}$ we have $X_{su} = sX_u$. Therefore, from the classical properties of a flow of a vector field,

there exists $\eta > 0$ such that the local flow Fl_t^u is defined on $[-1, 1]$, for all u in the open pseudo-ball

$$B_{\mathbb{F}'}(0, \eta) := \left\{u \in \mathbb{F},\ :\quad \nu'_{n_i}(u) \leq \eta,\ \ 1 \leq i \leq k\right\}.$$

(cf. for instance proof of Corollary 4.2 in [5]).

We set $B_{\mathbb{M}_i}(0, \epsilon) = \delta_i(B_{\mathbb{M}}(0, \epsilon))$. Then X_{u_i} is a vector field on $V_i = \delta_i(V)$ and $\mathrm{Fl}_t^{u_i} := \delta_i \circ (\mathrm{Fl}_t^u) \circ \delta_i$ is the local flow of X_{u_i} which is defined on $B_{\mathbb{M}_i}(0, \epsilon)$ for all $t \in [-1, 1]$ and $\mathrm{Fl}_t^{u_i}(x_i)$ belongs to V_i for all $x_i \in B_{\mathbb{M}_i}(0, \epsilon)$ and $t \in [-1, 1]$ and, from (15), we have

$$\mathrm{Fl}_t^u = \varprojlim \mathrm{Fl}_t^{u_i}\,. \tag{15}$$

Step 4: Existence of an integral manifold.

Since $(E_i, \pi_i, U_i, \rho_i, [.,.]_i)$ is a Banach-Lie algebroid, the Lie bracket $[X_{u_i}, X_{u'_i}]$ is tangent to Δ_i and so by Definition 3.2 and Lemma 3.6 in [18] we have

$$\forall t \in [-1, 1],\ (T\,\mathrm{Fl}_t^{u_i})((\Delta_i)_{x_i}) = (\Delta_i)_{\mathrm{Fl}_t^{u_i}(x_i)}.$$

Therefore, according to the notations at the end of step 3, for each $i \in \mathbb{N}$, we set:

$$\forall u_i \in B_{\mathbb{F}'_i}(0,\eta) = \lambda_i(B_{\mathbb{F}'}(0,\eta)),\ \Phi_i(u_i) = (\mathrm{Fl}_1^{u_i})(0);$$

$$\forall u \in B_{\mathbb{F}'}(0,\eta),\ \Phi(u) = (\mathrm{Fl}_1^{u})(0);.$$

Lemma 11. *$\Phi = \varprojlim \Phi_i$ is smooth and there exists $0 < \eta' \leq \eta$ such that the restriction of Φ to $B_{\mathbb{F}'}(0,\eta')$ is injective and $T_u\Phi$ belongs to $\mathcal{IH}_b(\mathbb{F}', \mathbb{M})$ for all $u \in B_{\mathbb{F}'}(0,\eta')$.*

Recall that, for each $i \in \mathbb{N}$, $B_{\mathbb{F}'_i}(0,\eta'))$ is an open ball in $\mathbb{F}'_i$ and $\delta_i(V)$ is open neighbourhood of $0 \in \mathbb{M}_i$. From Lemma 11, since Φ is injective and each differential $T_u\Phi$ is injective, it follows that the same is true for each $\Phi_i : B_{\mathbb{F}_i}(0,\eta')) \to \delta_i(V)$. Thus we can apply the proof of Theorem 1 in [18] for Φ_i. By the way, $\mathcal{U}_i = \Phi_i(B_{\mathbb{F}'_i}(0,\eta'))$ is an integral manifold of $\rho_i(\mathbb{F}'_i)$ and $(\mathcal{U}_i, \Phi_i^{-1})$ is a (global) chart for this integral manifold modeled on $\mathbb{F}'_i$. As $\Phi = \varprojlim \Phi_i$, $B_{\mathbb{F}'}(0,\eta') = \varprojlim B_{\mathbb{F}'_i}(0,\eta'))$ $\Phi_i \circ \lambda_i^j = \delta_i^j \circ \Phi_j$ for all $j \geq i$, it follows that $(\mathcal{U}_i, \delta_i^j)_{j\geq i}$ is a surjective projective sequence and so $\mathcal{U} = \varprojlim \mathcal{U}_i$ is a Fréchet manifold modeled on $\mathbb{F}'$. This last result clearly **ends the proof of Theorem 2**.

Proof of Lemma 11
According to step 3, Φ_i is well defined on $B_{\mathbb{F}'_i}(0,\eta)$ and $\Phi = \varprojlim \Phi_i$.
Now, for every $x \in V$, ρ'_x belongs to $\mathcal{IH}_b(\mathbb{F}', \mathbb{M})$ and so is injective; after shrinking V, if necessary, there exists some $M > 0$ such that

$$\forall x \in V,\ \|\rho'_x\|_\infty \leq M. \tag{16}$$

Now, by construction, we have $T_0\Phi(u) = \rho'_0(u)$ and so $T_0\Phi$ is injective.

Claim 12. *The map $u \mapsto T_u\Phi$ is a smooth map from $B'(0,\eta)$ to $\mathcal{H}_b(\mathbb{F}', \mathbb{M})$.*

Proof of Claim 12
We will use some argument of the proof of Lemma 2.12 of [18]. We fix the index i and, for any $y \in B_{\mathbb{F}'_i}(0,\epsilon)$, $v \in \mathbb{F}'_i$ we set

- : $X_v(y) = \rho'_i(y, v)$
- : $\varphi(t, v) = \mathrm{Fl}_t^{X_v}(0)$
- : $A(t) = \partial_1\rho'_i(\varphi(t, v), v)$ (partial derivative relative to the first variable)
- : $B(t) = \rho'_i(\varphi(t, v), .)$

Note that A and B are smooth fields on $[0, 1]$ of operators in $\mathcal{L}(\mathbb{M}_i, \mathbb{M}_i)$ and $\mathcal{L}(\mathbb{F}'_i, \mathbb{M}_i)$ respectively. Therefore the differential equation

$$\dot{S} = A \circ S + B$$

has a unique solution S_t with initial condition $S_0 = 0$ given by

$$S_t = \int_0^t G_{t-s} \circ B(s)ds \tag{17}$$

where G_t is the unique solution of

$$\dot{G} = A \circ G$$

with initial condition $G_0 = \mathrm{Id}_{\mathbb{M}}$. Given by

$$G_t = \mathrm{Id}_{\mathbb{M}} + \int_0^t A \circ G_s ds. \tag{18}$$

Under these notations, from [6], Chapter X § 7, we have $\partial_2\varphi(t, v)(.) = S_t$.

On the one hand, from the choice of ϵ, $\varphi(t,v)$ belongs to V_i and so from (12), we have $\|A(t)\|_i^{\mathrm{op}} \leq K$ for any $t \in [-1,1]$ and from (16) we have $\|B(t)\|_i^{\mathrm{op}} \leq M$. Thus from (18), using Gronwal equality, we obtain

$$\|G_t\|_i^{\mathrm{op}} \leq e^K,$$

And so from (17) we obtain

$$\|S_t\|_i^{\mathrm{op}} \leq Me^K$$

This implies that

$$\|\partial_2\varphi(t,v)\|_i^{\mathrm{op}} \leq Me^K.$$

We set $M_1 = Me^K$. Since $\Phi_i(u_i) = \varphi(1,u_i)$ it follows that:

$$\|T_{u_i}\Phi(v_i)\|^{\mathbb{M}_i} \leq M_1\|v_i\|^{\mathbb{E}_i} \tag{19}$$

from (19) we obtain:

$$\mu_i(T_u\Phi(v)) \leq M_1\nu_i(v). \tag{20}$$

But, since $T_u\Phi(\mathbb{F}') = \Delta_{\Phi(u)} \subset \{\Phi(u)\} \times \mathbb{M}$, it follows that the map $u \mapsto T_u\Phi$ can be considered as a continuous linear map from $\mathbb{F}'$ to $\mathbb{M}$ which takes values in $\mathcal{H}_b(\mathbb{F}',\mathbb{M})$. Now as each Φ_i is a smooth map form $B_{\mathbb{F}_i}(0,\eta'))$ to $\delta_i(V)$ and $\Phi = \varprojlim \Phi_i$, this imply that Φ is a smooth map on $B_{\mathbb{F}'}(0,\eta')) = \varprojlim B_{\mathbb{F}'_i}(0,\eta'))$ to $V = \varprojlim \delta_i(V)$ which ends the proof of the Claim.

□

End of the proof of Lemma 10.
At first, from Claim 12, the map $u \mapsto T_u\Phi$ takes values in the Banach space $\mathcal{H}_b(\mathbb{F}',\mathbb{M})$, as in step 2 for ρ, we can show that this map is Lipschitz on $B_{\mathbb{F}'}(0,\eta)$ for η small enough. As $T_0\Phi = \rho_0'$, from Proposition 34, it follows that, again for η small enough, $T\Phi$ is injective on $B_{\mathbb{F}}(0,\eta')$, and we have (cf. (19))

$$\forall u \in B_{\mathbb{F}}(0,\eta'),\ \forall v \in \mathbb{F}'\ \mu_i(T_u\Phi(v)) \leq M_1\nu_i(v) \tag{21}$$

using the fact that the range of $T_u\Phi$ is always closed for $u \in B_{\mathbb{F}'}(0,\eta')$. Moreover, for $u \in B_{\mathbb{F}'}(0,\eta')$, as for the relation (33) in the proof of Proposition 34, we obtain:

$$\frac{1}{\ell_u}\nu_i(v) \leq \mu_i(T_u\Phi(v))) \leq \ell_u.\nu_n(v) \tag{22}$$

for all $i \in \mathbb{N}$, where $\ell_u = \|T_u\Phi\|_\infty \leq M_1$.
Finally we obtain:

$$\forall i \in \mathbb{N},\ \frac{1}{M_1}\nu_i(v) \leq \mu_i(T_u\Phi(v)) \leq M_1.\nu_i(u) \tag{23}$$

Suppose that, for any $0 \leq \eta' \leq \eta$, the restriction of each Φ_i to $B_{\mathbb{F}'_i}(0,\eta)$ is not injective. Consider any pair $(u,v) \in [B_{\mathbb{F}'}(0,\eta)]^2$ such that $u \neq v$ but $\Phi(u) = \Phi(v)$, we set $h = v-u$. For any $\alpha \in \mathbb{M}^*$, we consider the smooth curve $c_\alpha : [0,1] \to \mathbb{R}$ defined by:

$$c_\alpha(t) =< \alpha, (\Phi(u+th) - \Phi(u)) > .$$

Of course, we have $\dot{c}_\alpha(t) =< \alpha, T_{u+th}\Phi(h) >$.
Denote by $]u,v[$ the set of points $\{w = u+th, t \in]0,1[\}$. As we have $c_\alpha(0) = c_\alpha(1) = 0$, from Rolle's Theorem, there exists $u_\alpha \in]u,v[$ such that

$$< \alpha, \delta_l(T_{u_\alpha}\Phi(h)) >= 0 \tag{24}$$

Note that, for any $t \in \mathbb{R}$, this relation is also true for any th. From our assumption, it follows that, for each $k \in \mathbb{N} \setminus \{0\}$, there exists u_k and v_k in $B_{\mathbb{F}}(0,\frac{\eta}{k})$ so that $u_k \neq v_k$ but with $\Phi(u_k) = \Phi(v_k)$. So from the previous argument, for any $\alpha \in \mathbb{M}^*$, we have

$$< \alpha, (T_{u_{\alpha,k}}\Phi(h_k)) >= 0 \tag{25}$$

for some $u_{\alpha,k} \in]u_k, v_k[$ and $h_k = v_k - u_k$. From (25), for any $i \in \mathbb{N}$, any $t \in \mathbb{R}$ and any $\alpha_i \in \mathbb{M}_i^*$, if $\alpha = \delta_i^*(\alpha_i)$, we have:

$$| < \alpha, T_0\Phi(th_k) > | = | < \alpha_i, \delta_i([T_0\Phi - T_{u_\alpha}\Phi](th_k)) > | = | < \alpha_i, [T_0\Phi_i - T_{\delta_i(u_\alpha)}\Phi_i](\lambda_i(th_k))|$$

Denote by $\|.\|^{\mathbb{M}_i^*}$ the canonical norm on $\mathbb{M}_i^*$ associated to $\|.\|^{\mathbb{M}_i}$. Then from (25) and since $u \mapsto T_u\Phi$ is K_1-Lipschitz on $B'(0,\eta)$ (for some constant K_1) we obtain:

$$\begin{aligned} | < \alpha, T_0\Phi(th_k) > | &\leq (||\alpha_i||^{\mathbb{M}_i^*}.K_1.||\delta_i(u_\alpha)||^{\mathbb{M}_i}||.||\lambda_i(th_k)||^{\mathbb{E}_i} \\ &\leq (||\alpha_i||^{\mathbb{M}_i^*}.K_1.||\lambda_i(th_k)||^{\mathbb{E}_i}\frac{\eta}{k}. \end{aligned} \tag{26}$$

Since $u_k \neq v_k$, there must exist at least one integer $i \in \mathbb{N}$ such that $\lambda_i(h_k) \neq 0$. Thus, by taking $t = \dfrac{u_k - v_k}{\nu_i((u_k - v_k))}$ in (26), we may assume $t = 1$ and $\|\lambda_i(h_k)\|^{\mathbb{E}_i} = 1$ In this way, for this choice of h_k, we have a 1-form $\bar{\beta}_{k,i}$ on the linear space generated by $\lambda_i(h_k)$ in $\mathbb{F}_i'$ such that $< \bar{\beta}_{k,i}, \lambda_i(h_k) >= 1$ and with $(\|\bar{\beta}_{k,i}\|^{\mathbb{E}_i})^* = 1$. From the Hahn-Banach Theorem, we can extend this linear form to a form $\beta_{k,i} \in \mathbb{M}_i^*$ such that $< \beta_{k,i}, \lambda_i(h_k) >= 1$ and $(\|\beta_{k,i}\|^{\mathbb{E}_i})^* = 1$. But since each $T_0\Phi$ is injective, this implies that $T_0\Phi_i$ is injective and so the adjoint $T_0^*\Phi_i$ is surjective. This implies that there exists $\alpha_{k,i} \in \mathbb{M}_i^*$ such that $T_0^*\Phi_i(\alpha_{k,i}) = \beta_{k,i}$. Thus from (26) we obtain

$$\begin{aligned} 1 = | < \beta_{k,i}, \lambda_i(h_k) > | &= | < \alpha_{k,i}, T_0\Phi_i(\lambda_i(h_k)) > | \\ &= | < \delta_i^*\alpha_{k,i}, T_0\Phi(h_k) > | \leq \|\alpha_{k,i}\|^{\mathbb{M}_i^*} K_1 \frac{\eta}{k} \end{aligned} \tag{27}$$

But, on the other hand since the operation "adjoint" is an isometry, from (23) we have

$$\frac{1}{M_1}\|\alpha_{k,i}\|^{\mathbb{M}_i^*} \leq \|T_0^*\Phi\alpha_{k,i}\|^{\mathbb{M}_i^*} = 1 \tag{28}$$

which gives a contradiction with (27) for k large enough.

□

Appendix A. Projective limits

A.1. **Projective limits of topological spaces.**

Definition 13. *A projective sequence of topological spaces is a sequence* $\left(\left(X_i, \delta_i^j\right)\right)_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ *where*

(PSTS 1): *For all $i \in \mathbb{N}$, X_i is a topological space;*
(PSTS 2): *For all $(i,j) \in \mathbb{N}^2$ such that $j \geq i$, $\delta_i^j : X_j \to X_i$ is a continuous map;*
(PSTS 3): *For all $i \in \mathbb{N}$, $\delta_i^i = Id_{X_i}$;*
(PSTS 4): *For all $(i,j,k) \in \mathbb{N}^3$ such that $k \geq j \geq i$, $\delta_i^j \circ \delta_j^k = \delta_i^k$.*

Notation 14. *For the sake of simplicity, the projective sequence* $\left(\left(X_i, \delta_i^j\right)\right)_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ *will be denoted* $\left(X_i, \delta_i^j\right)_{j\geq i}$.

An element $(x_i)_{i\in\mathbb{N}}$ of the product $\prod_{i\in\mathbb{N}} X_i$ is called a *thread* if, for all $j \geq i$, $\delta_i^j(x_j) = x_i$.

Definition 15. *The set $X = \varprojlim X_i$ of all threads, endowed with the finest topology for which all the projections $\delta_i : X \to X_i$ are continuous, is called the projective limit of the sequence* $\left(X_i, \delta_i^j\right)_{j\geq i}$.

A basis of the topology of X is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where U_i is an open subset of X_i (and so δ_i is open whenever δ_i is surjective).

Definition 16. *Let* $\left(X_i, \delta_i^j\right)_{j\geq i}$ *and* $\left(Y_i, \gamma_i^j\right)_{j\geq i}$ *be two projective sequences whose respective projective limits are* X *and* Y*.*

A sequence $(f_i)_{i\in\mathbb{N}}$ *of continuous mappings* $f_i : X_i \to Y_i$*, satisfying, for all* $(i,j)\in\mathbb{N}^2$*,* $j\geq i$*, the coherence condition*

$$\gamma_i^j \circ f_j = f_i \circ \delta_i^j$$

is called a projective sequence of mappings.

The projective limit of this sequence is the mapping

$$\begin{array}{rccc} f: & X & \to & Y \\ & (x_i)_{i\in\mathbb{N}} & \mapsto & \left(f_i\left(x_i\right)\right)_{i\in\mathbb{N}} \end{array}$$

The mapping f is continuous if all the f_i are continuous (cf. [1]).

A.2. **Projective limits of Banach spaces.** Consider a projective sequence $\left(\mathbb{E}_i, \delta_i^j\right)_{j\geq i}$ of Banach spaces.

Remark 17. *Since we have a countable sequence of Banach spaces, according to the properties of bonding maps, the sequence* $\left(\delta_i^j\right)_{(i,j)\in\mathbb{N}^2,\ j\geq i}$ *is well defined by the sequence of bonding maps* $\left(\delta_i^{i+1}\right)_{i\in\mathbb{N}}$*.*

A.3. **Projective limits of differential maps.** The following proposition (cf. [9], Lemma 1.2 and [4], Chapter 4) is essential

Proposition 18. *Let* $\left(\mathbb{E}_i, \delta_i^j\right)_{j\geq i}$ *be a projective sequence of Banach spaces whose projective limit is the Fréchet space* $\mathbb{F} = \varprojlim \mathbb{E}_i$ *and* $(f_i : \mathbb{E}_i \to \mathbb{E}_i)_{i\in\mathbb{N}}$ *a projective sequence of differential maps whose projective limit is* $f = \varprojlim f_i$*. Then the following conditions hold:*

1. *f is smooth in the convenient sense (cf.* [15]*)*
2. *For all* $x = (x_i)_{i\in\mathbb{N}}$*,* $df_x = \varprojlim (df_i)_{x_i}$*.*
3. $df = \varprojlim df_i$*.*

A.4. **Projective limits of Banach manifolds and Banach Lie groups.**

Definition 19. [9] *The projective sequence* $\left(M_i, \delta_i^j\right)_{j\geq i}$ *is called projective sequence of Banach manifolds if*

(PSBM 1): *M_i is a manifold modelled on the Banach space* $\mathbb{M}_i$*;*

(PSBM 2): $\left(\mathbb{M}_i, \overline{\delta_i^j}\right)_{j\geq i}$ *is a projective sequence of Banach spaces;*

(PSBM 3): *For all* $x = (x_i) \in M = \varprojlim M_i$*, there exists a projective sequence of local charts* $(U_i, \xi_i)_{i\in\mathbb{N}}$ *such that* $x_i \in U_i$ *where one has the relation*

$$\xi_i \circ \delta_i^j = \overline{\delta_i^j} \circ \varphi_j;$$

(PSBM 4): $U = \varprojlim U_i$ *is a non empty open set in* M*.*

Under the assumptions **(PSBM 1)** and **(PSBM 2)** in Definition 19, the assumptions **(PSBM 3)**] and **(PSBM 4)** around $x \in M$ is called *the projective limit chart property* around $x \in M$ and $(U = \varprojlim U_i, \phi = \varprojlim \phi_i)$ is called a *projective limit chart.*

The projective limit $M = \varprojlim M_i$ has a structure of Fréchet manifold modelled on the Fréchet space $\mathbb{M} = \varprojlim \mathbb{M}_i$ and is called a PLB-*manifold*. The differentiable structure is defined *via* the charts (U, φ) where $\varphi = \varprojlim \xi_i : U \to \left(\xi_i\left(U_i\right)\right)_{i\in\mathbb{N}}$.
φ is a homeomorphism (projective limit of homeomorphisms) and the charts changings $\left(\psi\circ\varphi^{-1}\right)_{|\varphi(U)} = \varprojlim \left(\left(\psi_i \circ (\xi_i)^{-1}\right)_{|\xi_i(U_i)}\right)$ between open sets of Fréchet spaces are smooth in the sense of convenient spaces.

Definition 20. [9] $\left(G_i, \delta_i^j\right)_{j\geq i}$ *is a projective sequence of Banach-Lie groups where* G_i *is modelled on* $\mathbb{G}_i$ *if , for all* $i \in \mathbb{N}$*, there exists a chart* (U_i, φ_i) *centered at the unity* $e_i \in G_i$ *such that:*

(PLBLG 1): $\forall (i,j) \in \mathbb{N}^2 : j \geq i,\ \delta_i^j(U_j) \subset U_i$*;*

(PLBLG 2): $\forall (i,j) \in \mathbb{N}^2 : j \geq i,\ \delta_i^j \circ \varphi_j = \varphi_j \circ \delta_i^j$*;*

(PLBLG 3): $\varprojlim \varphi_i(U_i)$ *is a non empty open set of* $\mathbb{G}$ *and* $\varprojlim U_i$ *is open in* G *according to the projective limit topology.*

A projective sequence of Banach-Lie groups $\left(G_i, \delta_i^j\right)_{j\geq i}$ is *submersive* if each δ_i^j is a surjective submersion.

Theorem 21. [9] *Let* $G = \varprojlim G_i$ *be a the projective limit of a projective sequence of Banach-Lie groups* $\left(G_i, \delta_i^j\right)_{j\geq i}$*. Then we have the following properties:*

1. *G is a Fréchet-Lie group.*
2. *If* $\mathbf{L}(G_i)$ *is the Lie algebra of* G_i *then* $\mathbf{L}(G) = \varprojlim \mathbf{L}G_i$*.*
3. *If* $\exp_{G_i}$ *is the exponential map for* G_i*, then* $\exp_G = \varprojlim \exp_{G_i}$ *is the exponential map of the Fréchet-Lie group* G*.*

A.5. **Projective limits of Banach vector bundles.** Let $\left(M_i, \delta_i^j\right)_{j\geq i}$ be a projective sequence of Banach manifolds where each manifold M_i is modeled on the Banach space $\mathbb{M}_i$.
For any integer i, let (E_i, π_i, M_i) be the Banach vector bundle whose type fibre is the Banach vector space $\mathbb{E}_i$ where $\left(\mathbb{E}_i, \lambda_i^j\right)_{j\geq i}$ is a projective sequence of Banach spaces.

Definition 22. $\left((E_i, \pi_i, M_i), \left(\xi_i^j, \delta_i^j\right)\right)_{j\geq i}$*, where* $\xi_i^j : E_j \to E_i$ *is a morphism of vector bundles, is called a projective sequence of Banach vector bundles on the projective sequence of manifolds* $\left(M_i, \delta_i^j\right)_{j\geq i}$ *if, for all* (x_i)*, there exists a projective sequence of trivializations* (U_i, τ_i) *of* (E_i, π_i, M_i)*, where* $\tau_i : (\pi_i)^{-1}(U_i) \to U_i \times \mathbb{E}_i$ *are local diffeomorphisms, such that* $x_i \in U_i$ *(open in* M_i*) and where* $U = \varprojlim U_i$ *is a non empty open set in* M *where, for all* $(i,j) \in \mathbb{N}^2$ *such that* $j \geq i$*, we have the compatibility condition*

(PLBVB): $\left(\delta_i^j \times \lambda_i^j\right) \circ \tau_j = \tau_i \circ \xi_i^j$*.*

With the previous notations, $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ is called a *projective bundle chart limit*. The triple of projective limit $(E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i))$ is called a *projective limit of Banach bundles* or PLB-bundle for short.

The following proposition generalizes the result of [10] about the projective limit of tangent bundles to Banach manifolds (cf. [7] and [4]).

Proposition 23. *Let* $\left((E_i, \pi_i, M_i), \left(\xi_i^j, \delta_i^j\right)\right)_{j\geq i}$ *be a projective sequence of Banach vector bundles.*
Then $\left(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i\right)$ *is a Fréchet vector bundle.*

Notation 24. *For the sake of simplicity, the projective sequence* $\left((E_i, \pi_i, M_i), \left(\xi_i^j, \delta_i^j\right)\right)_{j\geq i}$ *will be denoted* (E_i, π_i, M_i)*.*

Definition 25. *A sequence* (E_i, π_i, M_i) *is called a submersive projective sequence of Banach vector bundles if* $\left(M_i, \delta_i^j\right)_{j\geq i}$*) is a submersive projective sequence of Banach manifolds and if around each* $x \in M = \varprojlim M_i$*, there exists a projective limit chart bundle* $(U = \varprojlim U_i, \tau = \varprojlim \tau_i)$ *such that for all* $i \in \mathbb{N}$*, we have a decomposition* $\mathbb{E}_{i+1} = \ker \bar{\lambda}_i^{i+1} \oplus \mathbb{E}'_i$ *such that the condition* (**PLBVB**) *is true.*

The projective limit (E,π,M) of a projective sequence of Banach vector bundles (E_i,π_i,M_i) is called a *submersive projective limit of Banach bundles* or *submersive* PLB*-bundle* for short.

Now, we have the following result:

Proposition 26. *Let (E_i,π_i,M_i) be a submersive projective sequence of Banach bundles. Then, for each $i\in\mathbb{N}$, the maps $\delta_i : M\to M_i$ and $\lambda_i : E\to E_i$ are submersions.*

A.6. **Projective limit of Banach Lie algebroids.**

Definition 27. *Let $\pi : E\to M$ be a Banach bundle.*

(1) *an anchor is a vector bundle morphism $\rho : E\to TM$ and (E,ρ) is called an anchored bundle*

(2) *An almost Lie bracket $[.,.]_E$ on an anchored bundle E is a sheaf of antisymmetric bilinear maps*

$$[.,.]_{E_U} : \Gamma(E_U)\times\Gamma(E_U)\to\Gamma(E_U)$$

for any open set $U\subseteq M$ and which satisfies the following properties

(AL 1) *the Leibniz identity:*

$$\forall(\mathfrak{a}_1,\mathfrak{a}_2)\in\Gamma(E_U)^2,\forall f\in C^\infty(M),\ [\mathfrak{a}_1,f\mathfrak{a}_2]_{E_U}=f.[\mathfrak{a}_1,\mathfrak{a}_2]_{E_U}+df(\rho(\mathfrak{a}_1)).\mathfrak{a}_2.$$

(AL 2) *For any open set $U\subseteq M$ and any $(\mathfrak{a}_1,\mathfrak{a}_2)\in\Gamma(E_U)^2$, the map*

$$(\mathfrak{a}_1,\mathfrak{a}_2)\mapsto[\mathfrak{a}_1,\mathfrak{a}_2]_{E_U}$$

only depends on the 1-jets of $\mathfrak{a}_1$ and $\mathfrak{a}_2$.

(3) *An anchored bundle (E,ρ) provided with an almost Lie bracket $[.,.]_E$ which satisfies the Jacobi identity*

$$[[\mathfrak{a}_1,\mathfrak{a}_2]_E,\mathfrak{a}_3]_E+[[\mathfrak{a}_2,\mathfrak{a}_3]_E,\mathfrak{a}_1]_E+[[\mathfrak{a}_3,\mathfrak{a}_1]_E,\mathfrak{a}_2]_E=0$$

$$\forall(\mathfrak{a}_1,\mathfrak{a}_2\mathfrak{a}_3)\in\Gamma(E_U)^3$$

is called a Lie algebroid.

Definition 28. *$(E_i,\pi_i,M_i,\rho_i,[.,.]_i)$ is called a submersive projective sequence of split Lie algebroids if*

(PSBLA 1): *$\left(E_i,\xi_i^j\right)_{j\geq i}$ is a submersive projective sequence of Banach vector bundles $(\pi_i : E_i\to M_i)_{i\in\mathbb{N}}$ over the projective sequence of manifolds $\left(M_i,\delta_i^j\right)_{j\geq i}$;*

(PSBLA 2): *For all $(i,j)\in\mathbb{N}^2$ such that $j\geq i$, one has*

$$\rho_i\circ\xi_i^j=T\delta_i^j\circ\rho_j$$

(PSBLA 3): *For all $(i,j)\in\mathbb{N}^2$ such that $j\geq i$, one has*

$$\xi_i^j([.,.]_j)=[\xi_i^j(.),\xi_i^j(.)]_i$$

(PSBLA 4): *For all $i\in\mathbb{N}$ and $x_i\in M$ the kernel $\ker(\rho_i)_{x_i}$ is complemented in the fiber E_{x_i}.*

Proposition 29. *([4]) Let $(E_i,\pi_i,M_i,\rho_i,[.,.]_i)$ be a submersive projective sequence of split Lie algebroids. Then $\left(E:=\varprojlim E_i,\pi:=\varprojlim\pi_i,M:=\varprojlim M_i,\rho=\varprojlim\rho_i\right)$ is Fréchet anchored bundle and $\Delta=\rho(E)$ is a closed distribution on E*

Remark 30. *Under the assomptions of Proposition 29 , unfortunately $[.,.]=\varprojlim[.,.]_i$ does not define a Lie bracket on the set of all local sections of (E,π,M) but only on section which are projective limit of section of (E_i,π_i,M_i). Therefore $(E,\pi,M,\rho,[.,.])$ does not have a Fréchet Lie algebroid structure.*

Appendix B. The Banach space $\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$

Any Fréchet space $\mathbb{F}$ can be realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$. Following [7], 2.3, we can identify $\mathbb{F}$ with the projective limit of the projective sequence

$$\left(\widehat{\mathbb{B}}_n = \{x = (x_i) \in \prod_{0 \leq i \leq n} \mathbb{B}_i : \ \forall j \geq i, \ x_i = \lambda_i^j(x_j) \ \}, \ \widehat{\lambda}_n^m = (\lambda_n^n, \ldots, \lambda_n^m) \right)_{m \geq n}.$$

We denote by $\lambda_n : \mathbb{F} \to \mathbb{B}_n$ and $\widehat{\lambda}_n : \mathbb{F} \to \widehat{\mathbb{B}}_n$ the canonical surjective projections. Let $(|| \ ||_n)_{n \in \mathbb{N}}$ be a sequence where $|| \ ||_n$ is a norm on $\mathbb{B}_n$. In this way,

$$\widehat{|| \ ||}_n = \sup_{0 \leq i \leq n} || \ ||_i$$

defines a norm on $\widehat{\mathbb{B}}_n$. Then

$$\hat{\nu}_n = \widehat{|| \ ||}_n \circ \widehat{\lambda}_n \ (\text{resp. } \nu_n = || \ ||_n \circ \lambda_n) \tag{29}$$

is the semi-norm on $\mathbb{F}$ associated to the sequence $(\widehat{|| \ ||}_n)$ (resp.$(|| \ ||_n)$). Moreover, we have $\hat{\nu}_n = \max_{0 \leq i \leq n} \nu_i$ and the topology of $\mathbb{F}$ is defined by $(\hat{\nu}_n)$ or (ν_n).

Let $(\mathbb{F}_1, \nu_n^1)$ (resp. $(\mathbb{F}_2, \nu_n^2)$) be a graded Fréchet space.

Recall that a linear map $L : \mathbb{F}_1 \to \mathbb{F}_2$ is *continuous* if

$$\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}, \exists C_n > 0 : \forall x \in \mathbb{F}_1, \nu_2^n(L.x) \leq C_n \nu_1^{k_n}(x).$$

The space $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ of continuous linear maps between both these Fréchet spaces generally drops out of the Fréchet category. Indeed, $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ is a Hausdorff locally convex topological vector space whose topology is defined by the family of semi-norms $\{p_{n,B}\}$:

$$p_{n,B}(L) = \sup_{x \in B} \left\{ \nu_n^2(L.x) \right\}$$

where $n \in \mathbb{N}$ and B is any bounded subset of $\mathbb{F}_1$. This topology is not metrizable since the family $\{p_{n,B}\}$ is not countable.
So $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$ will be replaced, under certain assumptions, by a projective limit of appropriate functional spaces as introduced in [10].

We denote by $\mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n)$ the space of linear continuous maps (or equivalently bounded linear maps because $\mathbb{B}_1^n$ and $\mathbb{B}_2^n$ are normed spaces). We then have the following result ([7], Theorem 2.3.10).

Theorem 31. *The space of all continuous linear maps between $\mathbb{F}_1$ and $\mathbb{F}_2$ which can be represented as projective limits*

$$\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2) = \left\{ (L_n) \in \prod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{B}_1^n, \mathbb{B}_2^n) : \varprojlim L_n \ \textit{exists} \right\}$$

is a Fréchet space.

For this sequence $(L_n)_{n \in \mathbb{N}}$ of linear maps, for any integer $0 \leq n \leq m$, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{B}_1^n & \xleftarrow{(\delta_1)_n^m} & \mathbb{B}_1^m \\ {\scriptstyle L_n}\downarrow & & \downarrow{\scriptstyle L_m} \\ \mathbb{B}_2^n & \xleftarrow{(\delta_2)_n^m} & \mathbb{B}_2^m \end{array}$$

On $\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)$, the topology can be defined by the sequence of seminorms p_n given by

$$p_n(L) = \max_{0 \leq k \leq n} \sup \left\{ \nu_k^2(L.x), x \in \mathbb{F}_1, \ \nu_k^1(x) = 1 \right\}$$

so that $(\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2),p_n)$ is a graded Fréchet space.

Remark 32. *For $l\in\{1,2\}$, given a graduation $\left(\nu_n^l\right)$ on a Fréchet space $\mathbb{F}_l$, let $\mathbb{B}_l^n$ be the associated local Banach space and $\delta_l^n:\mathbb{F}_l\to\mathbb{B}_l^n$ the canonical projection.*
The quotient norm $\tilde{\nu}_n^l$ associated to ν_n^l is defined by

$$\tilde{\nu}_n^l(\delta_n(z))=\sup\{\nu_n^l(y):\ \delta_n(y)=\delta_n(z)\}. \tag{30}$$

We denote by $(\tilde{\nu}_n^2)^{\mathrm{op}}$ the corresponding operator norm on $\mathcal{L}(\mathbb{B}_1^n,\mathbb{B}_2^n)$.
If $L=\varprojlim L_n$ where $L_n:\mathbb{B}_1^n\to\mathbb{B}_2^n$, then we have

$$(\tilde{\nu}_n^2)^{\mathrm{op}}(L)=\sup\{\tilde{\nu}_n^2(L_n.x),\ \ x\in\mathbb{B}_1^n\ \ \tilde{\nu}_n^1(x)\leq 1\}=\sup\{\nu_n^2(L.x),x\in\mathbb{F}_1,\nu^1(x)\leq 1\}.$$

This implies that

$$p_n(L)=\max_{0\leq i\leq n}(\tilde{\nu}_i^2)^{\mathrm{op}}(L_n).$$

Definition 33. *Let $(\mathbb{F}_1,\nu_n^1)$ and $(\mathbb{F}_2,\nu_n^2)$ be graded Fréchet spaces. A linear map $L:\mathbb{F}_1\to\mathbb{F}_2$ is called a uniformly bounded operator, if*

$$\exists C>O:\forall n\in\mathbb{N},\ \nu_n(L(x))\leq C\mu_n(x).$$

We denote by $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$ the set of uniformly bounded operators. Of course $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$ is contained in $\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2)$ and $L\in\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2)$ belongs to $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$ if and only if $||L||_\infty:=\sup_{n\in\mathbb{N}}p_n(L)<\infty$ and so

$$\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)=[\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2)]_b:=\{L\in\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2):\ ||L||_\infty<\infty\}$$

When $\mathbb{F}=\mathbb{F}_1=\mathbb{F}_2$ and $\nu_n^1=\nu_n^2$ for all $n\in\mathbb{N}$, the set $\mathcal{H}(\mathbb{F},\mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F},\mathbb{F})$) is simply denoted $\mathcal{H}(\mathbb{F})$ (resp. $\mathcal{H}_b(\mathbb{F})$).

We denote by $\mathcal{IH}_b(\mathbb{F}_1,\mathbb{F}_2)$ (resp. $\mathcal{SH}_b(\mathbb{F}_1,\mathbb{F}_2)$) the set of injective (resp. surjective) operators of $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$ with closed range.

Proposition 34. *([4])*

1. *Each operator $L\in\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2)$ has a closed range if and only if, for each $n\in\mathbb{N}$, the induced operator $L_n:\mathbb{B}_1^n\to\mathbb{B}_2^n$ has a closed range.*
2. *$\mathcal{IH}_b(\mathbb{F}_1,\mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$.*
3. *$\mathcal{SH}_b(\mathbb{F}_1,\mathbb{F}_2)$ is an open subset of $\mathcal{H}_b(\mathbb{F}_1,\mathbb{F}_2)$.*

We will give the sketch of the proof of Point (2) since some arguments used in this proof are also useful for the proof of Theorem 2:

Proof
(2) Consider an injective operator $L\in\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2)$. According to the representation $\mathbb{F}_i=\varprojlim\mathbb{B}_i^n$ as a projective limit of a projective Banach sequence $(\mathbb{B}_i^n,(\delta_i)_n^m)_{m\geq n}$, we have a sequence of linear operators $L_n:\mathbb{B}_1^n\to\mathbb{B}_2^n$ such that $L=\varprojlim L_n$ (cf. Theorem 31). Considering each

$$\mathbb{F}_i=\{(x_n)\in\prod_{n\in\mathbb{N}}\mathbb{B}_i^n:\ \forall m\geq n,\ x_n=(\delta_2)_n^m(x_m)\}$$

, then if $x=(x_n)\in\mathbb{F}_1$ then $L(x)=(L_n(x_n))\in\mathbb{F}_2$. Thus it is clear that L is injective if and only if L_n is injective for all $n\in\mathbb{N}$.

Now if $L\in\mathcal{IH}_b(\mathbb{F}_1,\mathbb{F}_2)$, then L_n is an isomorphism from $\mathbb{B}_1^n$ onto its range and so we have

$$\frac{1}{\ell_n}.\tilde{\nu}_n^1(x)\leq\tilde{\nu}_n^2(L_n(x)\leq\ell_n.\tilde{\nu}_n^1(x) \tag{31}$$

for all $x \in \mathbb{B}_1^n$, all $n \in \mathbb{N}$, where $\tilde{\nu}_n^i$ is the quotient norm of ν_n^i on $\mathbb{B}_i^n$ for $i \in 1, 2$, and

$$\ell_n = (\tilde{\nu}_n^2)^{\mathrm{op}}(L) = \sup\{\frac{\tilde{\nu}_n^2(L_n(x))}{\tilde{\nu}_n^1(x)} \ : \ x \neq 0\}.$$

Since δ_n^2 is the canonical projection of $\mathbb{F}_2$ on $\mathbb{B}_2^n$ and $\nu_n^2 \circ \delta_n = \tilde{\nu}_n^2$, we obtain

$$\frac{1}{\ell_n}\nu_n^1(x) \leq \nu_n^2(L(x)) \leq \ell_n.\nu_n^1(x) \tag{32}$$

for all $x \in \mathbb{F}_1$ and $n \in \mathbb{N}$. But we have $\ell_n \leq ||L||_\infty$ and we finally obtain

$$\frac{1}{\ell}\nu_n^1(x) \leq \nu_n^2(L(x)) \leq \ell.\nu_n^1(x) \tag{33}$$

for all $x \in \mathbb{F}_1$, all $n \in \mathbb{N}$ and where $\ell = ||L||_\infty$.

Fix some $L \in \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)$ and set $\ell = ||L||_\infty$, we consider the open set

$$W = \{T \in \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2), \ : \ ||T - L||_\infty < \frac{\ell}{2}\}$$

Fix some $n \in \mathbb{N}$. For any $x \in \mathbb{F}_1$ and $T \in W$, we have

$$\nu_n^1(x) - \nu_n^1(T(x)) \leq \nu_n^1(T - L)(x) \leq p_n(T - L).\nu_n^1(x) \leq ||T - L||_\infty.\nu_n^1(x) \leq \frac{\ell}{2}\nu_n^1(x).$$

This implies that

$$\nu_n^2(T(x)) \geq \frac{\ell}{2}\nu_n^1(x). \tag{34}$$

Since (ν_n^i) is a separating sequence of semi-norms, it follows that L is injective.
Now taking in account inequality (34) and relation $\tilde{\nu}_n^i = \nu_n^i \circ (\delta_i)_n$, for $T \in W$ and each $n \in \mathbb{N}$, we have

$$\tilde{\nu}_n^2(T_n(x)) \leq \frac{3\ell}{2}\nu_n^1(x) \leq 3\tilde{\nu}_n^2(T_n(x))$$

for all $x \in \mathbb{B}_1^n$. It follows that T_n is closed and so T is closed (cf. 1.). Finally, W is an open neighbourhood of L contained in $\mathcal{IH}_b(\mathbb{F}_1, \mathbb{F}_2)$, which ends the proof of (2).

□

From this Proposition we have

Theorem 35. *([4])*

1. *The Banach space $\mathcal{H}_b(\mathbb{F})$ has a Banach-Lie algebra structure and the set $\mathcal{GH}_b(\mathbb{F})$ of uniformly bounded isomorphisms of $\mathbb{F}$ is open in $\mathcal{H}_b(\mathbb{F})$.*
2. *$\mathcal{GH}_b(\mathbb{F})$ has a structure of Banach-Lie group whose Lie algebra is $\mathcal{H}_b(\mathbb{F})$.*
3. *If $\mathbb{F}$ is identified with the projective $\varprojlim \mathbb{B}^n$ we denote by $\exp_n : \mathcal{L}(\mathbb{B}_n) \to \mathcal{GL}(\mathbb{B}_n)$, then we a have a well defined smooth map $\exp := \varprojlim \exp_n : \mathcal{H}_b(\mathcal{F}) \to \mathcal{GH}_b(\mathbb{F})$ which is a diffeomorphism from an open set of $0 \in \mathcal{H}_b(\mathcal{F})$ onto a a neighbourhood of $\mathrm{Id}_\mathbb{F}$.*

Appendix C. A theorem of existence of ODE

The following result is in fact a reformulation in our context of Theorem 1 in [17].

Theorem 36. *Let $\mathbb{F}$ a Fréchet space realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$ whose topology is defined by the sequence of seminorms $(\nu_n)_{n \in \mathbb{N}}$. Let I be an open interval in $\mathbb{R}$ and U be an open set of $I \times \mathbb{F}$. Then U is a surjective projective limit of open sets $U_n \subset I \times \mathbb{B}_n$. Consider a smooth map $f = \varprojlim f_n : U \to \mathbb{F}$, projective limit of maps $f_n : U_n \to \mathbb{B}_n$. [12] Assume that for every point $(t, x) \in U$, and every $n \in \mathbb{N}$, there exists an integrable function $K_n > 0$ such that*

$$\forall \left((t, x), (t, x')\right) \in U^2, \ \nu_n(f(t, x) - f(t, x')) \leq K_n(t)\nu_n(x - x'). \tag{35}$$

[12] This means that we have: $\forall m \geq n, \ \lambda_n^m \circ f_m = f_n \circ (Id_\mathbb{R} \times \lambda_n^m)$

and consider the differential equation:

$$\dot{x} = \phi\,(t, x)\,. \tag{36}$$

(1) *For any $(t_0, x_0) \in U$, there exists $\alpha > 0$ with $I_\alpha = [t_0 - \alpha, t_0 + \alpha] \subset I$, an open pseudo-ball $V = B(x_0, r) \subset U$ and a map $\Phi : I_\alpha \times I_\alpha \times V \to \mathbb{F}$ such that*

$$t \mapsto \Phi(t, \tau, x)$$

is the unique solution of (36) with initial condition $\Phi(\tau, \tau, x) = x$ for all $x \in V$.

(2) *V is the projective limit of the open balls V_n of $\mathbb{B}_n$. For each $n \in \mathbb{N}$, the curve $t \mapsto \lambda_n \circ \Phi(t, \tau, \lambda_n(x))$ is the unique solution $\gamma : I_\alpha \to \mathbb{B}_n$ of the differential equation $\dot{x}_n = \phi_n\,(t, x_n)$ with initial condition $\gamma(\tau) = \lambda_n(x)$.*

From this theorem we obtain easily:

Corollary 37. *Let $U = \varprojlim U_n$ be an open subset of $\mathbb{F}$ and $X = \varprojlim X_n : U \to \mathbb{F}$ a projective limit of smooth maps $X_n : U_n \to \mathbb{B}_n$. Assume that for every $n \in \mathbb{N}$ we have*

$$\forall\,\big((t, x), (t, x')\big) \in U^2,\ \nu_n(X(x) - X(x')) \leq K_n \nu_n(x - x'). \tag{37}$$

For $x_0 \in U$, let $B(x_0, 2r) = \{x \in \mathbb{F},\ :\ \nu_{n_i}(x - x_0) < 2r,\ 1 \leq i \leq k\}$ be a pseudo-ball contained in U. Let us set

- $C_1 = \max_{1 \leq i \leq k} K_{n_i}$
- $C_2 = \sup\limits_{z \in B(x_0, r)} \left\{ \max\limits_{1 \leq i \leq k} \nu_{n_i}(f(z)) \right\}$.

Then for any $\alpha > 0$ such that $\alpha e^{2\alpha C_1} \leq \frac{r}{2C_2}$, there exists a neighbourhood $V = B(x_0, r)$ and a smooth map $\phi_\alpha : I_\alpha \times V$ such that $t \mapsto \phi^\alpha(t, x)$ is the unique solution of $\dot{x} = X(x)$ defined on I_α with initial condition $\phi^\alpha(0, x) = x$. Moreover if $V_n = \lambda_n(V)$, consider $\phi_n^\alpha : I_\alpha \times V_n \to \mathbb{B}_n$ defined by $\phi_n^\alpha = \lambda_n \circ \phi^\alpha$; For each $z \in V_n$, the map $t \mapsto \phi^\alpha(t, z)$ is the unique solution of the differential equation $\dot{x}_n = X_n(x_n)$ defined on I_α with initial condition $\phi^\alpha(0, z) = z$

Remark 38. *If $X = \varprojlim X_n$ is a smooth vector field defined on an open set $U = \varprojlim U_n$ of $\mathbb{F}$, which satisfies assumption (37), as classically, according to Corollary 37, the map $\mathrm{Fl}_t^X := \mathrm{Fl}^X(t,\ )$ is* the local flow of *X that is Fl_t^X fullfils the properties of a 1-parameter group:*

- $\mathrm{Fl}_0^X = Id_V$
- $\mathrm{Fl}_t^X \circ \mathrm{Fl}_s^X = \mathrm{Fl}_{s+t}^X$ *if s,t and $s + t$ belong to I_α.*

In particular Fl_t^X is a diffeomorphism from V onto it range and its inverse is Fl_{-t}^X. Moreover $\mathrm{Fl}_t^{X_n} = \lambda_n \circ \mathrm{Fl}_t^X \circ \lambda_n$ is local flow of $X_n = \lambda_n \circ X \circ \lambda_n$ and we have $\mathrm{Fl}_t^X = \varprojlim \mathrm{Fl}_t^{X_n}$.

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