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► **To cite this version:**

Ludovic Martaud, Mehdi Badsı, Christophe Berthon, Arnaud Duran, Khaled Saleh. Global entropy stability for a class of unlimited high-order schemes for hyperbolic systems of conservation laws. 2021. hal-03206727

HAL Id: hal-03206727

<https://hal.science/hal-03206727v1>

Preprint submitted on 23 Apr 2021

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1 **GLOBAL ENTROPY STABILITY FOR A CLASS OF UNLIMITED**
2 **HIGH-ORDER SCHEMES FOR HYPERBOLIC SYSTEMS OF**
3 **CONSERVATION LAWS**

4 LUDOVIC MARTAUD*, MEHDI BADSI*, CHRISTOPHE BERTHON*, ARNAUD DURANT†,
5 AND KHALED SALEH†

6 **AMS subject classifications.** 65N06, 35L65, 35L67

7 **Key words.** Systems of conservation laws, High-order finite Volume schemes, Global entropy
8 stability.

9 **Abstract.** We design high-order schemes to approximate the weak solutions of hyperbolic
10 systems of conservation laws. These schemes are based on high order correction of the standard
11 HLL flux. They are proved to satisfy a global entropy stability property under an appropriate CFL
12 condition. These schemes do not involve limitation techniques and thus relevantly preserve the order
13 of accuracy. Numerical experiments illustrate the accuracy and the stability of the proposed schemes.

14 **1. Introduction.** The present work concerns the numerical approximation of
15 the weak solutions of systems made of $d \geq 1$ conservation laws in one space dimension
16 given by

$$\partial_t w + \partial_x f(w) = 0, \quad x \in \mathbb{R}, \quad t > 0. \quad (1.1)$$

17 The unknown state vector $w(x, t)$ is assumed to belong to Ω a non-empty convex open
18 subset of \mathbb{R}^d . Here, $f : \Omega \rightarrow \mathbb{R}^d$ is a given smooth flux function. It is assumed to be
19 such that the $d \times d$ Jacobian matrix $\nabla f(w)$ is diagonalizable in \mathbb{R} so that the system
20 (1.1) is a hyperbolic system of conservation laws. We consider the Cauchy problem
21 for (1.1), that is we prescribe an initial data at time $t = 0$ as follows:

$$w(x, t = 0) = w_0(x), \quad x \in \mathbb{R}, \quad (1.2)$$

22 where $w_0 : \mathbb{R} \rightarrow \Omega$ is a given measurable function. According to [34, 35, 42] (see
23 also [18, 19, 36, 47]), it is well-known that the solutions of (1.1)-(1.2) may develop, in
24 a finite time, discontinuities and that the weak solutions are in general non unique.
25 In order to rule out non-admissible weak solutions, the system (1.1) must be endowed
26 with entropy inequalities. In this regard, we assume the existence of both a strictly
27 convex function $\eta \in C^2(\Omega, \mathbb{R})$, called entropy function, and an entropy flux function
28 $G \in C^2(\Omega, \mathbb{R})$ such that

$$\nabla \eta(w)^T \nabla f(w) = \nabla G(w)^T, \quad \forall w \in \Omega. \quad (1.3)$$

We then note that smooth solutions of (1.1) satisfy the following additional conser-
vation law

$$\partial_t \eta(w) + \partial_x G(w) = 0,$$

29 while weak solutions, containing discontinuities, verify an entropy inequality (for in-
30 stance, see [34, 35, 42]) given by

$$\partial_t \eta(w) + \partial_x G(w) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0, +\infty)). \quad (1.4)$$

*Laboratoire de Mathématiques Jean Leray, CNRS UMR 6629, Université de Nantes, 2 rue de
la Houssinière, BP 92208, 44322 Nantes, France.

†Institut Camille Jordan, CNRS UMR 5208, Université Claude Bernard Lyon 1, 43 boulevard
du 11 novembre 1918, F-69622 Villeurbanne Cedex, France.

A weak solution of (1.1) is called an entropy satisfying solution if and only if the entropy inequality (1.4) holds for any pair entropy-entropy flux (η, G) . Integrating in space the entropy inequality (1.4) results in a global entropy stability inequality,

$$\frac{d}{dt} \int_{\mathbb{R}} \eta(w(x, t)) dx \leq 0.$$

1 As a consequence, provided $\int_{\mathbb{R}} \eta(w_0(x)) dx$ is finite, we have for all $t > 0$

$$\int_{\mathbb{R}} \eta(w(x, t)) dx \leq \int_{\mathbb{R}} \eta(w_0(x)) dx. \quad (1.5)$$

The inequality (1.5) is a global entropy stability inequality. Within the specific context of scalar conservation laws, we may use $\eta(w) = w^2/2$ or for symmetric system of conservation laws $\eta(w) = \frac{1}{2} \sum_{j=1}^d w_j^2$ so that (1.5) reformulates as follows for all $t > 0$

$$\|w(t, \cdot)\|_{L^2} \leq \|w_0\|_{L^2},$$

2 which expresses the decrease of the L^2 -norm satisfied by the solution. For general
3 hyperbolic systems of conservation laws (1.1), the global entropy decreasing property
4 (1.5) is reminiscent of a L^2 weighted type stability since the strict convexity of the
5 entropy function η yields that the hyperbolic system (1.1) is symmetrizable.

6 As for the numerical approximation, we approximate the weak solutions of (1.1),
7 at time t^n , by the following piecewise constant function

$$w_{\Delta}(x, t^n) = w_i^n \quad \text{if } x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad (1.6)$$

where $(x_{i+\frac{1}{2}})_{i \in \mathbb{Z}}$ define the sequence of the mesh nodes. The quantities w_i^n are approximations of the average of the solution over the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ as follows,

$$w_i^n \simeq \frac{1}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, t^n) dx$$

8 where $w(x, t^n)$ naturally belongs to $L^1_{\text{loc}}(\mathbb{R})$. For the sake of simplicity, we consider a
9 uniform mesh made of constant size mesh cells $\Delta x > 0$. As a consequence, we have
10 $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \Delta x$ for all $i \in \mathbb{Z}$. In addition, we introduce the time increment $\Delta t > 0$
11 so that $t^{n+1} = t^n + \Delta t$. Over the past fifty years, numerous strategies have been
12 proposed to evolve in time the approximation (1.6) and to define suitable updated
13 states $(w_i^{n+1})_{i \in \mathbb{Z}}$ (for instance, see [11, 18, 19, 26, 36, 41, 47] and references therein). In
14 the present work, we use conservative finite volume schemes so that the updated state
15 reads

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} \right), \quad (1.7)$$

16 where $\mathcal{F}_{i+\frac{1}{2}} \in \mathbb{R}^d$ is a numerical flux function. According to [19, 26, 47], provided
17 $w_i^n = w$ for all $i \in \mathbb{Z}$, where w denotes here an arbitrary constant state, if we get

$$\mathcal{F}_{i+\frac{1}{2}} = f(w), \quad \forall w \in \Omega, \quad (1.8)$$

1 then the scheme (1.7) is known to be first-order consistent and in conservative form.
 2 As a consequence, we may expect from the famous Theorem by Lax and Wendroff [25],
 3 some convergence results. Namely, if the sequence $(w_i^n)_{i \in \mathbb{Z}, n \in \mathbb{N}}$ converges in a sense to
 4 be defined then the limit function is a weak solution of (1.1). However, the obtained
 5 limit solution is not necessarily entropy satisfying and non-admissible discontinuous
 6 waves may appear (for instance, see [9,31]). To correct such unphysical solutions, one
 7 asks the approximate solution to satisfy, in addition, discrete entropy inequalities in
 8 the form

$$\eta(w_i^{n+1}) \leq \eta(w_i^n) - \frac{\Delta t}{\Delta x} \left(\mathcal{G}_{i+\frac{1}{2}} - \mathcal{G}_{i-\frac{1}{2}} \right), \quad (1.9)$$

9 where $\mathcal{G}_{i+\frac{1}{2}} \in \mathbb{R}^d$ denotes a numerical entropy flux function, which must be consistent;
 10 namely $\mathcal{G}_{i+\frac{1}{2}} = G(w)$ as long as $w_i^n = w$ for all $i \in \mathbb{Z}$ where $G(w)$ is the entropy flux
 11 function given by (1.3). From (1.9), we immediately recover the numerical counterpart
 12 to the global entropy stability condition (1.5) so that

$$\sum_{i \in \mathbb{Z}} \eta(w_i^{n+1}) \Delta x \leq \sum_{i \in \mathbb{Z}} \eta(w_i^n) \Delta x. \quad (1.10)$$

13 The design of numerical schemes able to provide discrete entropy inequalities
 14 (1.9) and thus able to satisfy the global entropy stability (1.10) turns out to be very
 15 challenging. Among the few first-order approaches able to exhibit such estimates, we
 16 refer to the exact Godunov scheme [20,26], the kinetic schemes [4,32], the HLL scheme
 17 [25,26], the HLLC scheme [48], some relaxation schemes such as Suliciu relaxation
 18 approaches [3,5,10] or the numerical strategy introduced by Tadmor [45,46]. Staggered
 19 schemes introduce an appropriate framework with respect to entropy stability, as
 20 illustrated in [27] in the implicit case for instance. In the same formalism, fully
 21 explicit results were proposed in [16,17,28] for the Euler equations, and in [15] for
 22 the shallow water equations. In [8,29,30], the global estimation (1.10) is established
 23 to justify the stability of the derived schemes. Let us however underline that, from a
 24 general viewpoint, time discretization is an important technical obstacle and stability
 25 is often considered in the semi-discrete setting [1,7,45]. Unfortunately, such semi-
 26 discrete entropy inequalities are known not to be sufficient to rule out non-admissible
 27 discontinuities in the converged solutions.

28 As far as high-order numerical approximations are concerned, the situation turns
 29 out to be drastically distinct. We may quote Bouchut [5] page 54, “*It is extremely*
 30 *difficult to obtain second-order schemes that verify an entropy inequality*”. Several
 31 works devoted to high-order schemes attempted to exhibit discrete entropy inequalities
 32 (1.9). For instance, in [46], semi-discrete entropy estimates associated with (1.9), are
 33 established. In [2,6] (see also [39,40]), fully discrete entropy estimates are introduced
 34 as follows:

$$\eta(w_i^{n+1}) \leq \bar{\eta}_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{G}_{i+\frac{1}{2}} - \mathcal{G}_{i-\frac{1}{2}} \right), \quad (1.11)$$

35 for a suitable entropy average $\bar{\eta}_i^n$. Such discrete entropy inequality strategies are not
 36 fully relevant since Lax-Wendroff Theorem [33] cannot be successfully applied. It has
 37 been illustrated in [13] where the authors proved that, in the convergence limit, the
 38 expected entropy inequality (1.4) is satisfied up to a positive measure. In addition,
 39 in [13], numerical experiments exhibited the capture of non-admissible shock solutions
 40 for MUSCL schemes which satisfy (1.11).

1 Hopefully, recent formal developments, proposed in [12, 24, 38], may indicate that
 2 a discrete entropy global stability (1.10) is reachable. The key ingredient in their
 3 strategy consists in a suitable control of the high-order diffusion term in the numerical
 4 fluxes to get the required global numerical entropy stability (1.10). Thus, the aim of
 5 the present work is the design of high-order schemes to approximate the weak solutions
 6 of (1.1) which satisfy the global entropy stability condition (1.10). Although it is a
 7 global stability criterion, a local stability of the approximate solution may be observed
 8 numerically [8, 29, 30].

9 The paper is organized as follows. In the next section, we introduce a class
 10 of high-order schemes. This class is derived from the well-known HLL scheme [26]
 11 complemented with suitable higher-order corrections obtained by ensuring the high
 12 order consistency of the numerical flux function with the physical flux function. For
 13 the sake of conciseness in the paper, we derive second-, third- and fourth-order space
 14 accurate schemes to approximate the weak solutions of (1.1). The reader will be
 15 easily convinced by the possibility of high-order accurate extensions. In Section 3,
 16 we establish (1.10). The proof relies on the design of a relevant CFL-like condition
 17 to restrict the time step, and the use of the large enough dissipation granted by the
 18 first order viscosity of the HLL scheme to control (likely anti-dissipative) high order
 19 corrective terms. In the last section, several numerical experiments are carried out to
 20 illustrate both the stability and the accuracy of the proposed schemes.

21 **2. Unlimited high-order HLL schemes.** We derive high-order space accurate
 22 schemes. The starting point is the original first-order HLL scheme [26] that reads as
 23 follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) - \mathcal{F}_\lambda^{O1}(w_{i-1}^n, w_i^n)), \quad (2.1)$$

24 where the numerical flux function is given by

$$\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) = \frac{1}{2} (f(w_i^n) + f(w_{i+1}^n)) - \frac{\lambda}{2} (w_{i+1}^n - w_i^n). \quad (2.2)$$

Here, $\lambda > 0$ stands for the numerical viscosity coefficient. Under the following CFL conditions:

$$\frac{\Delta t}{\Delta x} \lambda \leq \frac{1}{2} \quad \text{with} \quad \lambda \geq \max_{i \in \mathbb{Z}} (|\mu(w_i^n)|),$$

25 where $\mu(w)$ denotes the spectral radius of $\nabla f(w)$, the scheme (2.1) is known to be
 26 entropy preserving (see [26]). As a consequence, there exists a numerical entropy flux
 27 function $\mathcal{G}_\lambda^{O1}(w_i^n, w_{i+1}^n)$, consistent with the entropy flux function $G(w)$, such that for
 28 all $i \in \mathbb{Z}$ we have

$$\eta(w_i^{n+1}) \leq \eta(w_i^n) - \frac{\Delta t}{\Delta x} (\mathcal{G}_\lambda^{O1}(w_i^n, w_{i+1}^n) - \mathcal{G}_\lambda^{O1}(w_{i-1}^n, w_i^n)), \quad (2.3)$$

29 for all entropy pairs (η, G) .

30 Equipped with this first-order scheme, we are in position to increase the order of
 31 accuracy in space. Before doing so, we first recall the following result that character-
 32 izes the accuracy of finite volume schemes (for instance, see [5] Proposition 2.26 for
 33 the proof).

LEMMA 2.1. *Consider a numerical scheme of the form*

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}(w_{i-\nu}^n, \dots, w_{i+\nu+1}^n) - \mathcal{F}(w_{i-\nu-1}^n, \dots, w_{i+\nu}^n)),$$

where $\nu \geq 0$ is an integer. The scheme is k^{th} -order of space accuracy if, for a fixed $x_{i+\frac{1}{2}}$, we have

$$\mathcal{F}(u_{i-\nu}, \dots, u_{i+\nu+1}) = f(u(x_{i+\frac{1}{2}})) + \mathcal{O}(\Delta x^k),$$

1 where, for a given smooth function $u(x)$, we have set

$$u_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x) dx. \quad (2.4)$$

2
3 Thanks to this result, we easily notice that the numerical flux function (2.2) is
4 first-order. More precisely, with (2.4), a standard Taylor expansion in a neighborhood
5 of $x_{i+\frac{1}{2}}$ gives

$$\begin{aligned} \mathcal{F}_\lambda^{O1}(u_i, u_{i+1}) &= f\left(u(x_{i+\frac{1}{2}})\right) - \frac{\lambda \Delta x}{2} \partial_x u(x_{i+\frac{1}{2}}) \\ &\quad + \frac{\Delta x^2}{8} \left(\partial_{xx} f\left(u(x_{i+\frac{1}{2}})\right) + \frac{1}{3} \nabla f\left(u(x_{i+\frac{1}{2}})\right) \partial_{xx} u(x_{i+\frac{1}{2}}) \right) \\ &\quad - \frac{\lambda \Delta x^3}{24} \partial_{xxx} u(x_{i+\frac{1}{2}}) + \mathcal{O}(\Delta x^4). \end{aligned} \quad (2.5)$$

The main idea is then to define a high-order correction of the numerical flux function \mathcal{F}_λ^{O1} that is based on the Taylor expansion (2.5). We therefore consider numerical flux functions of the form

$$\mathcal{F}_\lambda^{Ok}(w_{i-\nu}^n, \dots, w_{i+\nu+1}^n) = \mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) + \frac{1}{2} (\alpha_i^{Ok} + \alpha_{i+1}^{Ok}).$$

where the superscript Ok refers to the term "kth-order" and will take values in the set $\{O2, O3, O4\}$ according to the space order accuracy of the scheme. From the Taylor expansion (2.5), we observe that the following consistency, in a neighborhood of a fixed $x_{i+\frac{1}{2}}$, must be satisfied by the correction α_i^{Ok} according to the selected order of accuracy:

$$\alpha_i^{O2} = \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^2), \quad (2.6)$$

$$\begin{aligned} \alpha_i^{O3} &= \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) \\ &\quad - \frac{\Delta x^2}{8} \left(\partial_{xx} f\left(w(x_{i+\frac{1}{2}}, t^n)\right) + \frac{1}{3} \nabla f\left(w(x_{i+\frac{1}{2}}, t^n)\right) \partial_{xx} w(x_{i+\frac{1}{2}}, t^n) \right) + \mathcal{O}(\Delta x^3), \end{aligned} \quad (2.7)$$

$$\begin{aligned} \alpha_i^{O4} &= \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) \\ &\quad - \frac{\Delta x^2}{8} \left(\partial_{xx} f\left(w(x_{i+\frac{1}{2}}, t^n)\right) + \frac{1}{3} \nabla f\left(w(x_{i+\frac{1}{2}}, t^n)\right) \partial_{xx} w(x_{i+\frac{1}{2}}, t^n) \right) \\ &\quad + \frac{\lambda \Delta x^3}{24} \partial_{xxx} w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^4), \end{aligned} \quad (2.8)$$

6 where respectively α_i^{O2} is the second-order correction, α_i^{O3} the third-order correction
7 and α_i^{O4} the fourth-order correction. We thereby stress that the high order numerical

1 flux function \mathcal{F}_λ^{Ok} contains both approximation of the term $-\frac{\lambda\Delta x}{2}\partial_x w$ which inherits
 2 from the HLL flux function \mathcal{F}_λ^{O1} and approximation of the same term but with the
 3 opposite sign $+\frac{\lambda\Delta x}{2}\partial_x w$ which inherits from the corrective term α_i^{Ok} . At the con-
 4 tinuous level the sum of these two terms is equal to zero. However at the discrete
 5 level, since these two terms are not discretized within the same stencil, they do not
 6 generally compensate. The difference controls the numerical viscosity of the scheme
 7 and thus its stability.

8 We now give the definition of the corrective terms α_i^{Ok} . For the sake of clarity in
 9 the forthcoming notations, we set

$$\delta_{i+\frac{1}{2}} = w_{i+1}^n - w_i^n. \quad (2.9)$$

10 Concerning the second-order correction, we propose

$$\alpha_i^{O2} = \frac{\lambda}{2} \Delta x \overline{\partial_x w_i}^{O2}, \quad (2.10)$$

11 where

$$\Delta x \overline{\partial_x w_i}^{O2} = \Theta_i^{O2} \delta_{i+\frac{1}{2}} + (I - \Theta_i^{O2}) \delta_{i-\frac{1}{2}}. \quad (2.11)$$

12 Here, I is the $d \times d$ identity matrix while Θ_i^{O2} is a free $d \times d$ diagonal matrix parameter
 13 to be defined. This matrix parameter will play a central role to establish the required
 14 global entropy stability and it will be defined later on. We mention that other dis-
 15 cretizations of the term $\Delta x \overline{\partial_x w_i}^{O2}$ are likely possible and extensions with more general
 16 matrices Θ_i^{O2} could be considered as well. We emphasize that the second-order con-
 17 sistency statement (2.6) is immediately satisfied provided the diagonal matrices Θ_i^{O2}
 18 remains bounded as Δx tends to 0.

19 Next, concerning the third-order correction, we choose

$$\alpha_i^{O3} = \frac{\lambda}{2} \Delta x \overline{\partial_x w_i}^{O3} - \frac{\Delta x^2}{8} \overline{\partial_{xx} f(w)}_i - \frac{\Delta x^2}{24} \overline{\nabla f(w) \partial_{xx} w}_i, \quad (2.12)$$

where

$$\begin{aligned} \Delta x \overline{\partial_x w_i}^{O3} &= \frac{1}{3} \Theta_i^{O3} \left(\delta_{i+\frac{3}{2}} + \delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}} \right) + \frac{1}{3} (I - \Theta_i^{O3}) \left(\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}} + \delta_{i-\frac{3}{2}} \right) \\ &\quad + \frac{1}{2} (I - 2\Theta_i^{O3}) \left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} \right), \end{aligned} \quad (2.13)$$

$$\Delta x^2 \overline{\partial_{xx} f(w)}_i = f(w_{i+1}^n) - 2f(w_i^n) + f(w_{i-1}^n), \quad (2.14)$$

$$\Delta x^2 \overline{\nabla f(w) \partial_{xx} w}_i = \nabla f(w_i^n) \left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} \right). \quad (2.15)$$

20 We end up the definition of the scheme with the fourth-order correction that is given
 21 by

$$\alpha_i^{O4} = \frac{\lambda}{2} \Delta x \overline{\partial_x w_i}^{O4} - \frac{\Delta x^2}{8} \overline{\partial_{xx} f(w)}_i - \frac{\Delta x^2}{24} \overline{\nabla f(w) \partial_{xx} w}_i + \lambda \frac{\Delta x^3}{24} \overline{\partial_{xxx} w}_i^{O4}, \quad (2.16)$$

where we have set

$$\begin{aligned} \Delta x \overline{\partial_x w_i}^{O4} &= \Theta_i^{O4} \Delta x \overline{\partial_x w_{i+\frac{1}{2}}}^{O4} + (I - \Theta_i^{O4}) \Delta x \overline{\partial_x w_{i-\frac{1}{2}}}^{O4} \\ &+ \frac{1}{4} \left(-\Theta_i^{O4} \Delta x^2 \overline{\partial_{xx} w_{i+1}}^{O4} + (I - 2\Theta_i^{O4}) \Delta x^2 \overline{\partial_{xx} w_i}^{O4} + (I - \Theta_i^{O4}) \Delta x^2 \overline{\partial_{xx} w_{i-1}}^{O4} \right) \\ &+ \frac{\Delta x^3}{8} \overline{\partial_{xxx} w_i}^{O4}, \end{aligned} \quad (2.17)$$

$$\Delta x \overline{\partial_x w_{i+\frac{1}{2}}}^{O4} = \frac{1}{24} \left(-\delta_{i+\frac{3}{2}} + 26 \delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} \right), \quad (2.18)$$

$$\Delta x^2 \overline{\partial_{xx} w_i}^{O4} = \delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}, \quad (2.19)$$

$$\Delta x^3 \overline{\partial_{xxx} w_i}^{O4} = (\delta_{i+\frac{3}{2}} - \delta_{i+\frac{1}{2}}) - (\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}). \quad (2.20)$$

1 Note that the terms $\Delta x \overline{\partial_x w_i}^{O2}$, $\Delta x \overline{\partial_x w_i}^{O3}$, $\Delta x \overline{\partial_x w_i}^{O4}$ are consistent with $\Delta x \partial_x w(x_{i+\frac{1}{2}})$
 2 but with different order of consistency according to the consistency relations (2.6)-
 3 (2.8). This is why different formulas are proposed.

4 Equipped with the correction terms α_i^{O2} , α_i^{O3} and α_i^{O4} , we are now able to give
 5 the high-order scheme of interest as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+\frac{1}{2}}^{Ok} - \mathcal{F}_{i-\frac{1}{2}}^{Ok} \right), \quad (2.21)$$

6 where we have set

$$\mathcal{F}_{i+\frac{1}{2}}^{Ok} = \mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) + \frac{1}{2} (\alpha_i^{Ok} + \alpha_{i+1}^{Ok}), \quad (2.22)$$

with

$$\begin{aligned} \alpha_i^{Ok} &= \alpha_i^{O2} && \text{for the second-order scheme,} \\ \alpha_i^{Ok} &= \alpha_i^{O3} && \text{for the third-order scheme,} \\ \alpha_i^{Ok} &= \alpha_i^{O4} && \text{for the fourth-order scheme.} \end{aligned}$$

7 We complete this section by establishing the order of accuracy of the schemes.

PROPOSITION 2.2. *Let be given $u(x)$ a smooth function and define u_i by (2.4). Let the sequence of matrices $(\Theta_i^{Ok})_{i \in \mathbb{Z}}$ be bounded as $\Delta x \rightarrow 0$. For a fixed $x_{i+\frac{1}{2}}$ and $k \in \{2, 3, 4\}$ we have*

$$\mathcal{F}_\lambda^{O1}(u_i, u_{i+1}) + \frac{1}{2} (\alpha_i^{Ok} + \alpha_{i+1}^{Ok}) = f(u(x_{i+\frac{1}{2}})) + \mathcal{O}(\Delta x^k).$$

8 *As a consequence, the high-order scheme (2.21) is space second-, third- or fourth-order*
 9 *according to the selected order of accuracy.*

10 *Proof.* A direct Taylor expansion and the application of Lemma 2.1 achieve the
 11 proof. \square

12 To conclude this section, we highlight that the high-order schemes do not involve
 13 limitations techniques in contrast with other usual approaches (MUSCL technique [49]
 14 or ENO/WENO schemes [37, 43, 44] or DG schemes [14, 29, 44], for instance). We do
 15 not need limitations in the high-order correction terms to establish global entropy
 16 stability.

3. Global entropy stability. In this section we establish the global entropy stability (1.10) satisfied by the high-order scheme (2.21). In order to deal simultaneously with second-, third- and fourth-order of space accuracy, the high-order correction α_i^{Ok} is reformulated as follows:

$$\begin{aligned} \alpha_i^{Ok} &= \frac{\lambda}{2} \overline{\Delta x \partial_x w_i}^{Ok} - \varepsilon^{O3} \frac{\Delta x^2}{8} \overline{\partial_{xx} f(w)}_i - \varepsilon^{O3} \frac{\Delta x^2}{24} \overline{\nabla f(w) \partial_{xx} w_i} \\ &\quad + \lambda \varepsilon^{O4} \frac{\Delta x^3}{24} \overline{\partial_{xxx} w_i}^{O4}, \end{aligned}$$

where

$$\varepsilon^{O3} = \begin{cases} 1 & \text{for third- and fourth-order,} \\ 0 & \text{otherwise,} \end{cases} \quad \varepsilon^{O4} = \begin{cases} 1 & \text{for fourth-order,} \\ 0 & \text{otherwise,} \end{cases}$$

1 where $\overline{\Delta x \partial_x w_i}^{Ok}$ is given by definitions (2.11), (2.13) and (2.17) according to the
2 selected order of accuracy $Ok \in \{O2, O3, O4\}$.

3 For the forthcoming developments, it is convenient to condense the high-order
4 scheme (2.21)-(2.22) in the form

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok}, \quad (3.1)$$

5 where

$$\mathcal{R}_i^{Ok} = -\frac{1}{2} (f(w_{i+1}^n) - f(w_{i-1}^n)) + \frac{\lambda}{2} (\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}) - \frac{1}{2} (\alpha_{i+1}^{Ok} - \alpha_{i-1}^{Ok}). \quad (3.2)$$

6 We now state our main result.

7 **THEOREM 3.1.** Consider $\eta \in C^2(\Omega, \mathbb{R})$ a strictly convex entropy. Let the approx-
8 imation at time t^n , $w_\Delta(\cdot, t^n)$ given by (1.6) being a non zero function in $L^2(\mathbb{R})$ and
9 such that $\int_{\mathbb{R}} \eta(w_\Delta(x, t^n)) dx$ is finite. We assume the following:

- 10 a) There exists a compact set $K \subset \Omega$ such that $w_\Delta(x, t^n) \in K$ for every $x \in \mathbb{R}$.
b) The sequence of bounded (as $\Delta x \rightarrow 0$) diagonal matrices $(\Theta_i^{Ok})_{i \in \mathbb{Z}}$, defined according to the selected order of accuracy, satisfies for all $i \in \mathbb{Z}$ the following condition

$$\begin{aligned} &\sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \\ &\quad - \frac{1}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\overline{\Delta x \partial_x w_i}^{Ok} + \frac{\varepsilon^{O4}}{12} \overline{\Delta x^3 \partial_{xxx} w_i}^{O4} \right) > 0, \end{aligned} \quad (3.3)$$

11 where $\overline{\Delta x \partial_x w_i}^{Ok}$ and $\overline{\Delta x^3 \partial_{xxx} w_i}^{O4}$ linearly depend on Θ_i^{Ok} .
Let μ^{Ok} be a positive bounded (as $\Delta x \rightarrow 0$) constant such that

$$\begin{aligned} &\mu^{Ok} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \\ &\leq \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \\ &\quad - \frac{1}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\overline{\Delta x \partial_x w_i}^{Ok} + \frac{\varepsilon^{O4}}{12} \overline{\Delta x^3 \partial_{xxx} w_i}^{O4} \right). \end{aligned} \quad (3.4)$$

1 Then there exists positive constants, denoted $C_{\eta,n}$, $C_{\eta,f,n}^{Ok}$ independent from λ and
 2 $\frac{\Delta t}{\Delta x}$ and positive constants $r_n(\lambda)$, $C_n^{Ok}(\lambda)$ that depend on $\lambda > 0$ but not on $\frac{\Delta t}{\Delta x}$ such
 3 that if $\lambda > 0$ large enough and $\frac{\Delta t}{\Delta x} > 0$ small enough verify both

$$\lambda \mu^{Ok} \geq 2 \max \left(\max_{i \in \mathbb{Z}} |\mu(w_i^n)|, 4\epsilon^{O4} \frac{C_{\eta,f,n}^{Ok}}{C_{\eta,n}} \right), \quad (3.5)$$

$$\frac{\Delta t}{\Delta x} < \min \left(\frac{1}{\lambda \mu^{Ok}}, \frac{\frac{\lambda \mu^{Ok}}{8} C_{\eta,n} - \epsilon^{O3} C_{\eta,f,n}^{Ok}}{C_n^{Ok}(\lambda)}, \frac{\text{dist}(K, \partial\Omega)}{r_n(\lambda)} \right), \quad (3.6)$$

4 where $\mu(w)$ stands for the spectral radius of $\nabla f(w)$ and $\text{dist}(K, \partial\Omega) > 0$ is the distance
 5 from the compact K to the boundary $\partial\Omega$, then updated approximation given by the
 6 high-order scheme (2.21) verifies $w_\Delta(\cdot, t^{n+1}) \subset \Omega$ and one has the global entropy
 7 stability inequality (1.10), that is

$$\int_{\mathbb{R}} \eta(w_\Delta(x, t^{n+1})) dx \leq \int_{\mathbb{R}} \eta(w_\Delta(x, t^n)) dx. \quad (3.7)$$

8

9 Before going any further in the establishment of the main result, let us comment
 10 on the technical assumptions :

- 11 • The assumption a) has to be understood as an L^∞ bound on the solution. It is
 12 however much stronger since we require that the solution belongs to a compact
 13 subset of Ω where Ω is an open set. It is used for several purposes, namely; to
 14 obtain a lower bound for the smallest eigenvalue of the Hessian of the entropy
 15 $\nabla^2 \eta$, to get a L^∞ bound on the physical flux f and last but not the least
 16 to obtain the robustness of the scheme for $\frac{\Delta t}{\Delta x} > 0$ small enough. Note that
 17 in the case of scalar conservation laws or symmetric system of conservation
 18 laws this assumption could easily be removed since the entropy function to be
 19 considered $\eta(w) = \frac{1}{2}|w|^2$ is a strongly convex function and the admissible set
 20 is $\Omega = \mathbb{R}^d$. In the case of Euler equations with a perfect gas, the admissible
 21 set is $\Omega = \{w = (\rho, \rho u, E) \in \mathbb{R}^3 : \rho > 0, E - \rho u^2/2 > 0\}$. Our assumption
 22 therefore implies that the density ρ and the pressure $p = (\gamma - 1)(E - \rho u^2/2)$,
 23 where $\gamma \in (1, 3)$ stands for the adiabatic constant, are strictly away from
 24 the vacuum and bounded from above. It is a somehow standard assumption
 25 (see [42]).
- 26 • The CFL condition $\frac{\Delta t}{\Delta x} r_n(\lambda) < \text{dist}(K, \partial\Omega)$ with $\text{dist}(K, \partial\Omega) > 0$ is used
 27 to prove the robustness of the scheme, namely; $w_\Delta(\cdot, t^{n+1}) \subset \Omega$. This CFL
 28 condition can be quite restrictive. However in practice we always consider
 29 datum that are far away from the border $\partial\Omega$. We mention that it is difficult
 30 to prove robustness for high order scheme under a less restrictive condition,
 31 except in the case where limitation techniques are used (for instance, see
 32 [2, 40]).
- The assumption b) about the definition of the matrix parameters Θ_i^{Ok} is used
 to control the dissipation of the global entropy. Once again this assumption
 is easily satisfied in the case of scalar conservation laws or symmetric systems
 with the quadratic entropy $\eta(w) = \frac{1}{2}|w|^2$. For this specific entropy, and the
 second order in space scheme O2, the inequality (3.3) reformulates as follows:

$$\sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}} - \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}} \right) \cdot \left(\Theta_i^{O2} \delta_{i+\frac{1}{2}} + (I - \Theta_i^{O2}) \delta_{i-\frac{1}{2}} \right) > 0,$$

which after a translation of indices is equal to

$$\frac{1}{4} \sum_{i \in \mathbb{Z}} \left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} \right)^2 - \frac{1}{2} \sum_{i \in \mathbb{Z}} \Theta_i^{O2} \left(\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}} \right) \cdot \left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} \right) > 0.$$

For instance the following choice:

$$\forall i \in \mathbb{Z}, \quad \Theta_i^{O2} = \text{diag}_{1 \leq j \leq d} \left(-\text{sign} \left((\delta_{i+\frac{1}{2}})_j^2 - (\delta_{i-\frac{1}{2}})_j^2 \right) \right)$$

yields the desired inequality. Moreover, since it is bounded as $\Delta x \rightarrow 0$, it preserves the order of consistency of the scheme. Another possible choice is:

$$\forall i \in \mathbb{Z}, \quad \Theta_i^{O2} = 0 \in \mathcal{M}_d(\mathbb{R}).$$

This choice also gives the desired inequality since for a non trivial solution $w_\Delta(\cdot, t^n) \in L^2(\mathbb{R})$ the sum $\frac{1}{4} \sum_{i \in \mathbb{Z}} \left(\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} \right)^2$ is positive.

- For an arbitrary entropy η , we propose a systematic way to design a sequence of matrices $(\Theta_i^{O2k})_{i \in \mathbb{Z}}$ such that the inequality (3.3) of assumption b) holds (see Proposition (A.1) in the appendix). We did not manage to prove that the proposed sequence of matrices stays bounded as $\Delta x \rightarrow 0$, we however observed it numerically.

3.1. Dissipation estimates. To prove our main result 3.1, we shall need several technical lemmata that arise in the study of the quantity $\sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{O2k}$. This formal quantity with an appropriate CFL condition controls the dissipation rate of the global entropy of the scheme (2.21) and results from the following expansion of the scheme (2.21),

$$\begin{aligned} \eta(w_i^{n+1}) &= \eta(w_i^n) + \frac{\Delta t}{\Delta x} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{O2k} \\ &\quad + \left(\frac{\Delta t}{\Delta x} \right)^2 \int_0^1 (1-s) \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O2k} \right) \mathcal{R}_i^{O2k} \cdot \mathcal{R}_i^{O2k} ds. \end{aligned}$$

The quantity $\sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{O2k}$ thus must necessarily be negative for (3.7) to hold. Actually, we shall be more precise and prove a quantitative estimate that essentially shows that the global entropy dissipation rate of the the high order scheme (2.21) can be controlled by the global dissipation rate granted by the first order HLL scheme. In this respect, we begin with the following lemma. It gives an estimate on how the first order part of the scheme (2.21) dissipates the global entropy.

LEMMA 3.2. *For any sequence $(v_i)_{i \in \mathbb{Z}} \in l^2(\mathbb{Z})$ with values in Ω such that $\sum_{i \in \mathbb{Z}} \eta(v_i) \Delta x$ is finite, for all $\Lambda \geq \max_{i \in \mathbb{Z}} |\mu(v_i)|$ where $\mu(v)$ denotes the sequence of the eigenvalues of $\nabla f(v)$, under the CFL condition*

$$\frac{\Delta t}{\Delta x} \Lambda \leq \frac{1}{2}, \tag{3.8}$$

we have

$$\begin{aligned} & -\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(v_i) \cdot (f(v_{i+1}) - f(v_{i-1})) - \frac{\Lambda}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(v_{i+1}) - \nabla \eta(v_i)) \cdot (v_{i+1} - v_i) \\ & \leq -\frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(v_i + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1,\Lambda} \right) \mathcal{R}_i^{O1,\Lambda} \cdot \mathcal{R}_i^{O1,\Lambda} ds, \end{aligned}$$

1 where

$$\mathcal{R}_i^{O1,\Lambda} = -\frac{1}{2} (f(v_{i+1}) - f(v_{i-1})) + \frac{\Lambda}{2} (v_{i+1} - v_i - (v_i - v_{i-1})). \quad (3.9)$$

2

3 *Proof.* Let $(v_i)_{i \in \mathbb{Z}} \in l^2(\mathbb{Z})$ be an arbitrary sequence with values in Ω . Consider
4 the updated sequence $(\tilde{v}_i)_{i \in \mathbb{Z}}$ determined by the first order HLL scheme (2.1) with a
5 numerical viscosity Λ and a CFL condition given by (3.8). That is

$$\tilde{v}_i = v_i - \frac{\Delta t}{2\Delta x} (f(v_{i+1}) - f(v_{i-1})) + \frac{\Lambda \Delta t}{2\Delta x} (v_{i+1} - v_i - (v_i - v_{i-1})). \quad (3.10)$$

Since $\Lambda \geq \max_{i \in \mathbb{Z}} |\mu(v_i)|$, it is known that the first order HLL scheme verifies $\tilde{v}_i \in \Omega$
(because Ω is convex) and is entropy preserving (see [26]). As a consequence, we get

$$\eta(\tilde{v}_i) \leq \eta(v_i) - \frac{\Delta t}{\Delta x} (\mathcal{G}_\Lambda^{O1}(v_i, v_{i+1}) - \mathcal{G}_\Lambda^{O1}(v_{i-1}, v_i)),$$

6 where \mathcal{G}_Λ^{O1} is the numerical entropy flux function. We then obtain

$$\sum_{i \in \mathbb{Z}} \eta(\tilde{v}_i) - \sum_{i \in \mathbb{Z}} \eta(v_i) \leq 0. \quad (3.11)$$

Besides, since $\tilde{v}_i \in \Omega$, using (3.10) we have

$$\begin{aligned} \eta(\tilde{v}_i) &= \eta(v_i) + \frac{\Delta t}{\Delta x} \nabla \eta(v_i) \cdot \mathcal{R}_i^{O1,\Lambda} \\ &\quad + \left(\frac{\Delta t}{\Delta x} \right)^2 \int_0^1 (1-s) \nabla^2 \eta \left(v_i + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1,\Lambda} \right) \mathcal{R}_i^{O1,\Lambda} \cdot \mathcal{R}_i^{O1,\Lambda} ds, \end{aligned}$$

where $\mathcal{R}_i^{O1,\Lambda}$ is given by (3.9). Considering (3.11), we necessarily have

$$\sum_{i \in \mathbb{Z}} \nabla \eta(v_i) \cdot \mathcal{R}_i^{O1,\Lambda} \leq -\frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(v_i + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1,\Lambda} \right) \mathcal{R}_i^{O1,\Lambda} \cdot \mathcal{R}_i^{O1,\Lambda} ds.$$

Eventually remark that

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \nabla \eta(v_i) \cdot \mathcal{R}_i^{O1,\Lambda} &= -\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(v_i) \cdot (f(v_{i+1}) - f(v_{i-1})) \\ &\quad + \frac{\Lambda}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(v_i) \cdot (v_{i+1} - v_i - (v_i - v_{i-1})), \\ &= -\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(v_i) \cdot (f(v_{i+1}) - f(v_{i-1})) \\ &\quad - \frac{\Lambda}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(v_{i+1}) - \nabla \eta(v_i)) \cdot (v_{i+1} - v_i), \end{aligned}$$

7 which yields the desired inequality and achieves the proof. \square

8 Our second lemma states that the upper bound in (3.4) which controls the numerical
9 viscosity $\mu^{Ok} > 0$ is bounded as $\Delta x \rightarrow 0$ and thus the scheme does not need to be
10 infinitely viscous.

1 LEMMA 3.3. Let the approximation at time t^n , $w_\Delta(\cdot, t^n)$ given by (1.6) being a
 2 non zero function in $L^2(\mathbb{R})$ and such that it verifies the assumption a) of Theorem
 3 3.1. Let $(\Theta_i)_{i \in \mathbb{Z}}$ a sequence of bounded as $(\Delta x \rightarrow 0)$ matrices that verifies (3.3).
 4 Then the upper bound in (3.4), is bounded as $\Delta x \rightarrow 0$.

Proof. First, since $\eta \in C^2(\Omega, \mathbb{R})$ is a strictly convex function and $w_\Delta(\cdot, t^n)$ belongs to a compact set $K \subset \Omega$ there exists a constant $\alpha_{\eta, n} > 0$ such that we have

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} &= \int_0^1 \sum_{i \in \mathbb{Z}} \nabla^2 \eta(w_i^n + s \delta_{i+\frac{1}{2}}) \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}} ds, \\ &\geq \alpha_{\eta, n} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}. \end{aligned}$$

Since $w_\Delta(\cdot, t^n)$ is in $L^2(\mathbb{R})$, the sum $\sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}} > 0$ is convergent. Let us set

$$\begin{aligned} \mathcal{S} &= \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \\ &\quad - \frac{1}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\Delta x \overline{\partial_x w_i^{Ok}} + \frac{\varepsilon^{O4}}{12} \Delta x^3 \overline{\partial_{xxx} w_i^{O4}} \right) \end{aligned}$$

which is positive since the sequence of matrices $(\Theta_i)_{i \in \mathbb{Z}}$ satisfies (3.3). Since η is smooth and $w_\Delta(\cdot, t^n)$ lives in a compact set $K \subset \Omega$ then there also exists a positive constant $\tilde{\beta}_{\eta, n}$ such that

$$\left| \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \right| \leq \tilde{\beta}_{\eta, n} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}.$$

Moreover, since $\Delta x \overline{\partial_x w_i^{Ok}}$ and $\Delta x^3 \overline{\partial_{xxx} w_i^{O4}}$ are linear functions with respect to $(\delta_{i+\nu+\frac{1}{2}})_{-2 \leq \nu \leq +1}$ and with bounded coefficients, there exists positive constant $\bar{\beta}_{\eta, n}$ such that

$$\begin{aligned} &\left| \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\Delta x \overline{\partial_x w_i^{Ok}} + \frac{\varepsilon^{O4}}{12} \Delta x^3 \overline{\partial_{xxx} w_i^{O4}} \right) \right| \\ &\leq \bar{\beta}_{\eta, n} \sum_{i \in \mathbb{Z}} \sum_{\nu=-2}^1 \left(\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}} \right) \cdot \delta_{i+\nu+\frac{1}{2}}, \\ &\leq 8 \bar{\beta}_{\eta, n} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}. \end{aligned}$$

It results in the existence of a positive constant $\beta_{\eta, n} > 0$ such that

$$\mathcal{S} \leq \beta_{\eta, n} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}.$$

As a consequence, we get

$$0 < \frac{\mathcal{S}}{\sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}}} \leq \frac{\beta_{\eta, n}}{\alpha_{\eta, n}},$$

5 and thus, from (3.4), μ^{Ok} remains bounded as $\Delta x \rightarrow 0$. The proof of Lemma 3.3 is
 6 completed. \square

1 Our last lemma is the cornerstone of this work. It is an estimate on how the high
 2 order global entropy dissipation rate can be controlled by the first order dissipation
 3 rate.

4 **LEMMA 3.4.** *Let the approximation at time t^n , $w_\Delta(\cdot, t^n)$ given by (1.6) being a
 5 non zero function in $L^2(\mathbb{R})$. Let the assumption a) and b) of Theorem 3.1 hold. Let
 6 $\mu^{Ok} > 0$ be large enough to satisfy (3.4). Let D^{Ok} be the high order dissipation rate
 7 given by*

$$D^{Ok} = \frac{\lambda}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}) - \frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (\alpha_{i+1}^{Ok} - \alpha_{i-1}^{Ok}). \quad (3.12)$$

Then there exists a positive constant $C_{\eta,f,n}^{Ok}$ which does not depend on λ such that

$$D^{Ok} \leq -\frac{\lambda \mu^{Ok}}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} + \varepsilon^{O3} C_{\eta,f,n}^{Ok} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}},$$

8

Proof. Let us begin with a bound on the second term $|\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (\alpha_{i+1}^{Ok} - \alpha_{i-1}^{Ok})|$.
 The approximation at time t^n , $w_\Delta(\cdot, t^n)$ is in $L^2(\mathbb{R})$ and $w_\Delta(\cdot, t^n) \subset K$ where K is
 a compact set of Ω . Besides, since the functions f and η are smooth there exists a
 positive constant $C_{\eta,f,n}^{Ok}$, which does not depend on λ such that

$$\begin{aligned} & \left| \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \left(\frac{\Delta x^2}{8} \overline{\partial_{xx} f(w)}_{i+1} + \frac{\Delta x^2}{24} \overline{\nabla f(w) \partial_{xx} w}_{i+1} \right. \right. \\ & \quad \left. \left. - \frac{\Delta x^2}{8} \overline{\partial_{xx} f(w)}_{i-1} - \frac{\Delta x^2}{24} \overline{\nabla f(w) \partial_{xx} w}_{i-1} \right) \right| \\ &= \left| \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\frac{\Delta x^2}{8} \overline{\partial_{xx} f(w)}_i + \frac{\Delta x^2}{24} \overline{\nabla f(w) \partial_{xx} w}_i \right) \right|, \\ &\leq C_{\eta,f,n}^{Ok} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}. \end{aligned}$$

Expanding D^{Ok} , we obtain

$$\begin{aligned} D^{Ok} &\leq \frac{\lambda}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}}) - \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (\Delta x \overline{\partial_x w}_{i+1}^{Ok} - \Delta x \overline{\partial_x w}_{i-1}^{Ok}) \\ &\quad - \varepsilon^{O4} \frac{\lambda}{48} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) (\Delta x^3 \overline{\partial_{xxx} w}_{i+1}^{O4} - \Delta x^3 \overline{\partial_{xxx} w}_{i-1}^{O4}) \\ &\quad + \varepsilon^{O3} C_{\eta,f,n}^{Ok} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}, \\ &\leq -\frac{\lambda}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \\ &\quad + \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\Delta x \overline{\partial_x w}_i^{Ok} + \frac{\varepsilon^{O4}}{12} \Delta x^3 \overline{\partial_{xxx} w}_i^{O4} \right) \\ &\quad + \varepsilon^{O3} C_{\eta,f,n}^{Ok} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}. \end{aligned}$$

Multiplying the inequality (3.4) by $-\lambda/2$ one has the bound

$$\begin{aligned} & -\frac{\lambda}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \\ & + \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \left(\Delta x \overline{\partial_x w_i^{Ok}} + \frac{\varepsilon^{O4}}{12} \Delta x^3 \overline{\partial_{xxx} w_i^{O4}} \right) \\ & \leq -\frac{\lambda \mu^{Ok}}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} \end{aligned}$$

1 gathering all the terms together yields the desired inequality. \square

2 **3.2. Proof of the main result.** Let $\lambda > 0$ to be fixed later. We first prove the
3 robustness of the scheme (3.1). Namely, there exists a compact set $K' \subset \Omega$ such that
4 for all $s \in [0, 1]$ and $i \in \mathbb{Z}$, $w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \in K'$ for some small enough $\frac{\Delta t}{\Delta x} > 0$. We
5 argue as follows: since $w_\Delta(\cdot, t^n)$ is assumed to belong to a compact set $K \subset \Omega$, by
6 a standard continuity argument, one can find a positive constant $r_n(\lambda)$ that depends
7 on λ but not on $\frac{\Delta t}{\Delta x}$ such that for all $i \in \mathbb{Z}$ $|\mathcal{R}_i^{Ok}| \leq r_n(\lambda)$. Consequently, one has the
8 following embedding

$$\left\{ w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} : s \in [0, 1], i \in \mathbb{Z} \right\} \subset K + \frac{\Delta t}{\Delta x} \mathcal{B}(r_n(\lambda)) := K'$$

9 where $\mathcal{B}(r_n(\lambda))$ is the ball in \mathbb{R}^d of radius $r_n(\lambda)$. For any $\frac{\Delta t}{\Delta x} > 0$, the set K' is
10 a compact subset of \mathbb{R}^d . Since K is a compact subset of Ω and Ω is an open set,
11 then $\text{dist}(K, \partial\Omega) > 0$. Provided $0 < \frac{\Delta t}{\Delta x} r_n(\lambda) < \text{dist}(K, \partial\Omega)$, one has $K' = K +$
12 $\frac{\Delta t}{\Delta x} \mathcal{B}(r_n(\lambda)) \subset \Omega$ which proves the robustness.

We now prove the global entropy stability. Since $\eta \in C^2(\Omega; \mathbb{R})$ is a smooth enough
function and the updated approximation $w_\Delta(\cdot, t^{n+1})$ given by (3.1) belongs to Ω , one
has using a Taylor expansion,

$$\begin{aligned} \eta(w_i^{n+1}) &= \eta(w_i^n) + \frac{\Delta t}{\Delta x} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} \\ &+ \left(\frac{\Delta t}{\Delta x} \right)^2 \int_0^1 (1-s) \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds. \end{aligned}$$

13 We have to prove the following inequality

$$\sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} + \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds \leq 0. \quad (3.13)$$

We decompose the first term as follows

$$\sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} = -\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (f(w_{i+1}^n) - f(w_{i-1}^n)) + D^{Ok},$$

where \mathcal{R}_i^{Ok} is given by (3.2) and D^{Ok} is given by (3.12). Using Lemma 3.4, the second
term of the right hand side of the above equality can be from bounded above so that

we have,

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} &\leq -\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (f(w_{i+1}^n) - f(w_{i-1}^n)) \\ &\quad - \frac{\lambda \mu^{Ok}}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \delta_{i+\frac{1}{2}} + \varepsilon^{O3} C_{\eta, f, n}^{Ok} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}. \end{aligned}$$

Using Lemma 3.2 with the sequence $(v_i = w_i^n)_{i \in \mathbb{Z}}$ and with the numerical viscosity $\Lambda = \lambda \mu^{Ok}/2$ and the CFL condition (3.8), one can bound the first term of the right hand side of the above inequality as follows,

$$\begin{aligned} &-\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot (f(w_{i+1}^n) - f(w_{i-1}^n)) - \frac{\lambda \mu^{Ok}}{4} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \delta_{i+\frac{1}{2}} \\ &\leq -\frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \right) \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \cdot \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} ds, \end{aligned}$$

where $\mathcal{R}_i^{O1, \lambda \mu^{Ok}/2}$ is given by (3.9). As a consequence, gathering all the terms together we glean

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} &\leq -\frac{\lambda \mu^{Ok}}{4} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \delta_{i+\frac{1}{2}} \\ &\quad - \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \right) \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \cdot \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} ds \\ &\quad + \varepsilon^{O3} C_{\eta, f, n}^{Ok} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}. \end{aligned}$$

Moreover, since η is strictly convex and continuous and $w_\Delta(\cdot, t_n)$ is assumed to live in a compact set $K \subset \Omega$, there exists $C_{\eta, n} > 0$ such that

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} &= \int_0^1 \sum_{i \in \mathbb{Z}} \nabla^2 \eta(w_i^n + s \delta_{i+\frac{1}{2}}) \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}} ds, \\ &\geq C_{\eta, n} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}, \end{aligned}$$

so that we get the intermediate following inequality

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} &\leq \left(-\frac{\lambda \mu^{Ok}}{4} C_{\eta, n} + \varepsilon^{O3} C_{\eta, f, n}^{Ok} \right) \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}} \\ &\quad - \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \right) \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \cdot \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} ds. \end{aligned} \tag{3.14}$$

Adding the term $\frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds$ to the inequality

(3.14) results in

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} + \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds \\
& \leq \left(-\frac{\lambda \mu^{Ok}}{4} C_{\eta, n} + \varepsilon^{O3} C_{\eta, f, n}^{Ok} \right) \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}} \\
& \quad - \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \right) \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \cdot \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} ds \\
& \quad + \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds.
\end{aligned}$$

- 1 To complete the proof, we have to provide an upper bound for the two last terms.
- 2 For the first term, using once again the numerical viscosity $\Lambda = \lambda \mu^{Ok}/2$ and the
- 3 CFL condition (3.8), one has for all $s \in [0, 1]$ and $i \in \mathbb{Z}$, $w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \in \Omega$.
- 4 Therefore by standard continuity argument, there exists a positive constant $C_1(\lambda)$
- 5 that depends on λ such that

$$\int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \right) \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \cdot \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} ds \leq C_1(\lambda) \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}.$$

- 6 We now deal with the second term. Using a continuity argument in the compact
- 7 set $K' \subset \Omega$, there exists a positive constant $C_2(\lambda)$ that depends on λ such that

$$\int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds \leq C_2(\lambda) \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}.$$

We then get with a positive constant $C^{Ok}(\lambda) = C_1(\lambda) + C_2(\lambda)$

$$\begin{aligned}
& - \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \right) \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} \cdot \mathcal{R}_i^{O1, \lambda \mu^{Ok}/2} ds \\
& + \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds \\
& \leq C^{Ok}(\lambda) \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}.
\end{aligned}$$

It eventually yields

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^{Ok} + \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{Ok} \right) \mathcal{R}_i^{Ok} \cdot \mathcal{R}_i^{Ok} ds \\
& \leq \left(-\frac{\lambda \mu^{Ok}}{4} C_{\eta} + \varepsilon^{O3} C_{\eta, f}^{Ok} + \frac{\Delta t}{\Delta x} C^{Ok}(\lambda) \right) \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}} \cdot \delta_{i+\frac{1}{2}}.
\end{aligned}$$

To conclude the proof, it is enough to choose λ large enough to satisfy (3.5) and $\frac{\Delta t}{\Delta x}$ small enough to verify the additional CFL restriction (3.6) so that

$$-\frac{\lambda \mu^{Ok}}{4} C_{\eta} + \varepsilon^{O3} C_{\eta, f}^{Ok} + \frac{\Delta t}{\Delta x} C^{Ok}(\lambda) \leq 0,$$

- 8 and the required inequality (3.13) is satisfied. The proof of Theorem 3.1 is thus
- 9 achieved.

1 **4. Numerical experiments.** In this section, we provide several numerical ex-
 2 amples that illustrate the accuracy and the stability of the proposed schemes. In
 3 order to be complete, some details in the scheme implementation must be given.

4 As far as the time order of accuracy is concerned, the scheme (2.21) is first-
 5 order in time. To increase the time accuracy, we use the well-known SSP Rung-Kutta
 6 methods introduced in [21–23]. Since this high-order time approach is based on convex
 7 combination of first-order time sub-steps, the global entropy stability result (1.10) is
 8 preserved.

9 Let us now explain how the parameters of the scheme are settled. Being given a
 10 strictly convex entropy η , we design the matrix parameter $(\Theta_i^{Ok})_{i \in \mathbb{Z}}$ that satisfies the
 11 criteria (3.3). Then we have to choose an explicit definition of the numerical viscosity
 12 coefficient λ and the time step Δt . For a fixed $\Delta x > 0$, according to Theorem 3.1,
 13 there exists $\lambda > 0$ large enough and $\Delta t > 0$ small enough such that the inequality
 14 (3.13) is satisfied which implies the global entropy stability. From practical point of
 15 view λ and $\Delta t/\Delta x$ are chosen such that $\lambda = \max_i |\mu(w_i^n)|$ and $\frac{\lambda \Delta t}{\Delta x} \leq \frac{1}{2}$. We shall
 16 verify systematically at the numerical level that this choice ensures the decrease of
 17 the total entropy.

18 Equipped with this numerical parameters, we performed numerical simulations
 19 considering mainly the scalar Burgers equation and the Euler equations. For each case,
 20 we propose three different choice of the matrix parameter $\Theta_i^{Ok} \in \{\Theta_{a,i}^{Ok}, \Theta_{b,i}^{Ok}, \Theta_{c,i}^{Ok}\}$.
 21 We systematically measure the error in L^1 , L^2 and L^∞ norms between the numerical
 22 solutions and an exact solution. Plots of the obtained numerical solutions and the total
 23 entropy are also given. A particular attention must be paid on the choice $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$
 24 whose results are surprisingly good, notably because very few oscillations are observed
 25 in the discontinuities.

26 **4.1. Burgers equation.** The Burgers equations consists in taking $w \in \mathbb{R}$ and
 27 the flux function given by $f(w) = w^2/2$. We consider the entropy function $\eta(w) =$
 28 $w^2/2$ so that the global entropy stability (1.10) coincides with a L^2 -decreasing prop-
 29 erty. We shall present several test with the following parameters:

$$\begin{aligned} \Theta_{a,i}^{Ok} &= -\theta \operatorname{sign}((\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n))(\mathcal{A}_i)), \\ \Theta_{b,i}^{Ok} &= \frac{(\delta_{i-1/2}^2 - \delta_{i+1/2}^2)(\delta_{i-1/2}^2 + \delta_{i+1/2}^2)}{(\delta_{i-1/2}^2 + \delta_{i+1/2}^2)^2 + \varepsilon}, \\ \Theta_{c,i}^{Ok} &= \frac{1}{2}, \end{aligned} \quad (4.1)$$

30 where we fix $\theta = -\min(0, S/D)$, with S given by (A.2) and D given by (A.5), and
 31 $\varepsilon = 10^{-12}$. Numerically, we verified that these choices of Θ_i^{Ok} satisfy the criteria
 32 (3.3).

33 **4.1.1. Smooth solution.** We take a smooth initial data $w_0(x) = 0.25 + 0.5 \sin(\pi x)$
 34 over a periodic domain $[-1, 1)$. With a final time small enough, here given by $t = 0.3$,
 35 the exact solution remains smooth so that the order of accuracy can be evaluated.

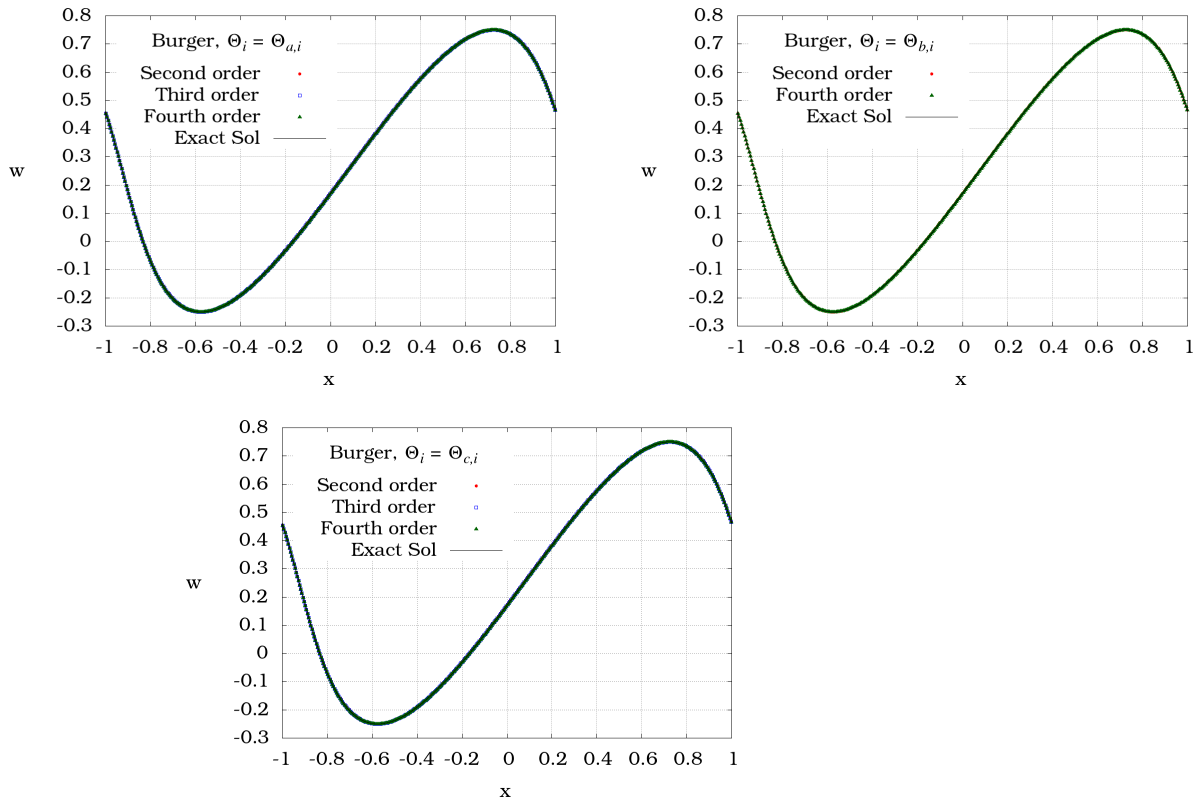


Table 4.1: Second-, third- and fourth-order accurate approximation of the smooth Burgers with a mesh made of 400 cells.

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	5.8E-04	-	6.9E-04	-	8.2E-04	-
200	1.4E-04	2.0	1.7E-04	2.0	2.0E-04	2.0
400	3.6E-05	2.0	4.2E-05	2.0	5.0E-05	2.0
800	8.9E-06	2.0	1.0E-05	2.0	1.2E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	3.1E-06	2.0
Third-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	8.2E-05	-	1.3E-04	-	3.4E-05	-
200	1.0E-05	3.0	1.6E-05	3.0	2.4E-06	3.8
400	1.3E-06	3.0	2.0E-06	3.0	1.9E-07	3.7
800	1.6E-07	3.0	2.5E-07	3.0	1.6E-08	3.6
1600	2.0E-08	3.0	3.1E-08	3.0	1.4E-09	3.6

Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	6.1E-06	-	1.0E-05	-	1.2E-05	-
200	3.8E-07	4.0	6.4E-07	4.0	7.9E-07	4.0
400	2.4E-08	4.0	4.0E-08	4.0	4.9E-08	4.0
800	1.5E-09	4.0	2.5E-09	4.0	3.0E-09	4.0
1600	9.1E-11	4.0	1.5E-10	4.0	1.9E-10	4.0

1

Table 4.2: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the smooth Burgers solution for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

2

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	1.4E-03	-	1.6E-03	-	1.0E-03	-
200	2.5E-04	2.5	2.8E-04	2.5	9.9E-05	3.4
400	3.9E-05	2.7	4.3E-05	2.7	4.8E-05	1.1
800	8.9E-06	2.1	1.0E-05	2.1	1.2E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	3.1E-06	2.0
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	6.0E-06	-	9.8E-06	-	1.2E-05	-
200	3.7E-07	4.0	6.3E-07	4.0	7.6E-07	3.9
400	2.3E-08	4.0	3.9E-08	4.0	4.8E-08	4.0
800	1.5E-09	4.0	2.5E-09	4.0	3.0E-09	4.0
1600	9.1E-11	4.0	1.5E-10	4.0	1.9E-10	4.0

3

4

Table 4.3: Errors and order evaluations for the second- and fourth-order accurate schemes with the smooth Burgers solution for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

5

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	4.5E-04	-	4.8E-04	-	4.5E-04	-
200	1.1E-04	2.0	1.2E-04	2.1	1.1E-04	2.1
400	2.7E-05	2.0	2.8E-05	2.0	2.7E-05	2.0
800	6.6E-06	2.0	6.9E-06	2.0	6.6E-06	2.0
1600	1.6E-06	2.0	1.7E-06	2.0	1.6E-06	2.0
Third-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	8.2E-05	-	1.3E-04	-	1.3E-05	-
200	1.0E-05	3.0	1.6E-05	3.0	1.0E-06	3.7
400	1.3E-06	3.0	2.0E-06	3.0	8.7E-08	3.6
800	1.6E-07	3.0	2.5E-07	3.0	7.6E-09	3.5
1600	2.0E-08	3.0	3.1E-08	3.0	6.8E-10	3.5

6

7

Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-06	-	5.9E-06	-	1.8E-06	-
200	2.0E-07	4.1	3.3E-07	4.1	1.1E-07	4.0
400	1.2E-08	4.1	1.9E-08	4.1	6.6E-09	4.1
800	7.2E-10	4.0	1.2E-09	4.1	4.0E-10	4.0
1600	4.5E-11	4.0	7.1E-11	4.0	2.2E-11	4.2

Table 4.4: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the smooth Burgers solution for $\Theta_i^{O_k} = \Theta_{c,i}^{O_k}$.

The obtained numerical solutions are presented in Fig. 4.1. We notice the very good behavior of the approximations. This remark is completed by Table 4.2, 4.3, 4.4 where the evaluation of the order of accuracy is presented. Since the high-order scheme is unlimited, we get the expected order.

4.1.2. Discontinuous solution. We take a discontinuous initial data over the periodic domain $[-1, 1)$ defined by $w_0(x) = \begin{cases} 1 & \text{if } -0.25 \leq x \leq 0.25, \\ 0 & \text{otherwise.} \end{cases}$

The exact solution is made of both rarefaction and shock waves. With a final time $t = 0.3$, the waves do not interact. In Table 4.5, 4.6, 4.7 we present the evaluated order of accuracy. The obtained approximations are presented in Fig. 4.1. We notice a remarkable behavior of the approximate solutions since very little spurious oscillations appear.

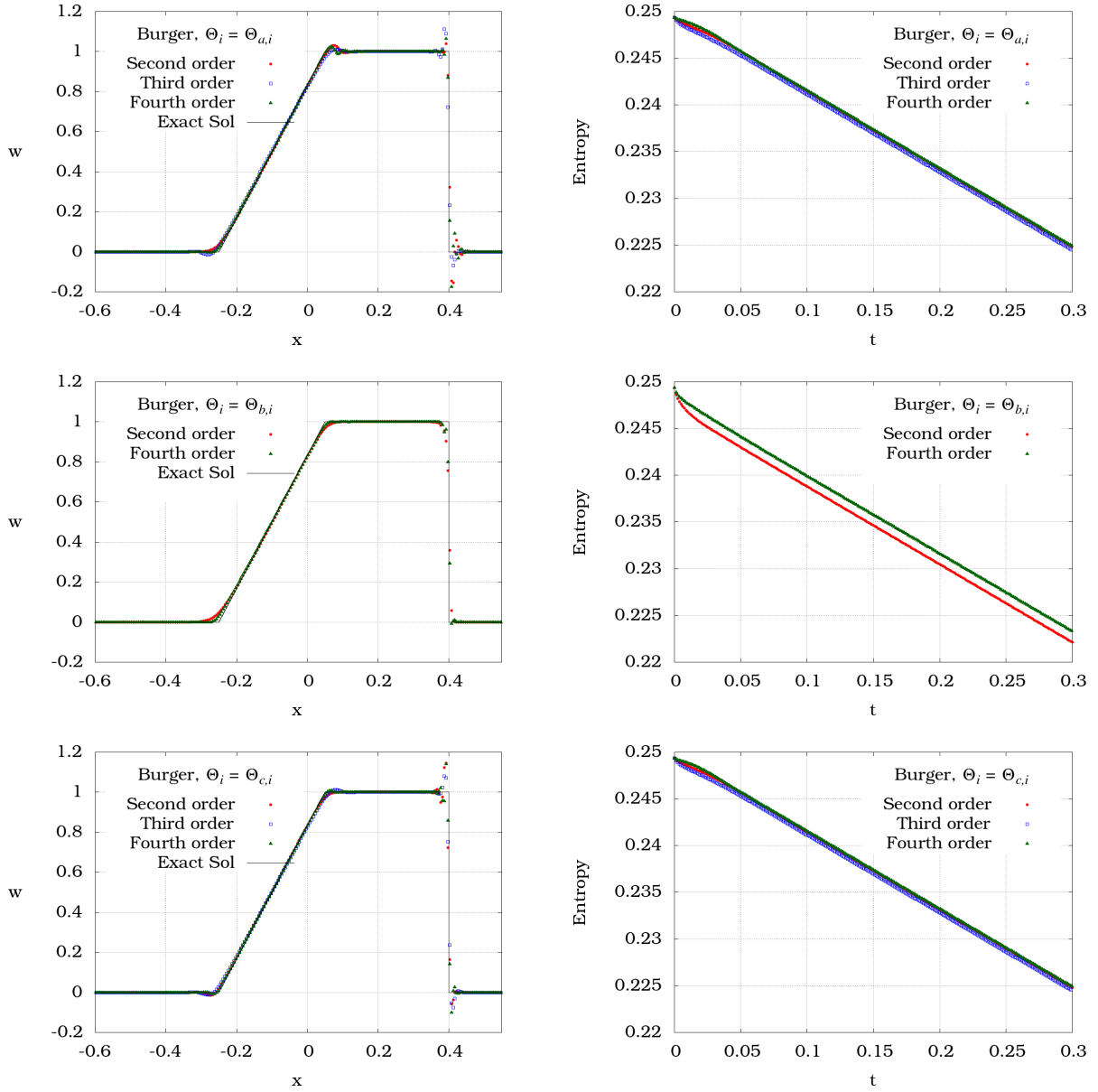


Fig. 4.1: Second-, third- and fourth-order accurate approximation of the Burgers solution made of rarefaction and shock waves with a mesh made of 400 cells.

cells	Second-order scheme errors					
	L^1	order	L^2	order	L^∞	order
100	3.4E-02	-	6.4E-02	-	8.0E-02	-
200	1.7E-02	1.0	4.3E-02	0.6	6.4E-02	0.3
400	8.4E-03	1.0	3.0E-02	0.5	5.0E-02	0.3
800	4.2E-03	1.0	2.1E-02	0.5	4.0E-02	0.3
1600	2.1E-03	1.0	1.5E-02	0.5	3.2E-02	0.3

Third-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.6E-02	-	6.3E-02	-	5.3E-02	-
200	1.7E-02	1.1	4.2E-02	0.6	4.2E-02	0.3
400	8.4E-03	1.0	2.9E-02	0.5	3.3E-02	0.3
800	4.2E-03	1.0	2.0E-02	0.5	2.6E-02	0.3
1600	2.1E-03	1.0	1.4E-02	0.5	2.1E-02	0.3
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	2.6E-02	-	4.5E-02	-	3.7E-02	-
200	1.2E-02	1.1	3.0E-02	0.6	2.9E-02	0.3
400	5.7E-03	1.0	2.1E-02	0.5	2.4E-02	0.3
800	2.8E-03	1.0	1.5E-02	0.5	1.9E-02	0.3
1600	1.4E-03	1.0	1.1E-02	0.5	1.5E-02	0.3

Table 4.5: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for

$$\Theta_i^{Ok} = \Theta_{a,i}^{Ok}.$$

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-02	-	7.0E-02	-	8.6E-02	-
200	1.8E-02	0.9	4.9E-02	0.5	6.8E-02	0.3
400	9.2E-03	1.0	3.3E-02	0.5	5.4E-02	0.3
800	4.6E-03	1.0	2.3E-02	0.5	4.3E-02	0.3
1600	2.3E-03	1.0	1.6E-02	0.5	3.4E-02	0.3
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	2.3E-02	-	5.5E-02	-	7.3E-02	-
200	1.3E-02	0.8	3.9E-02	0.5	5.6E-02	0.4
400	6.5E-03	1.0	2.7E-02	0.5	4.4E-02	0.3
800	3.3E-03	1.0	1.9E-02	0.5	3.5E-02	0.3
1600	1.6E-03	1.0	1.3E-02	0.5	2.8E-02	0.3

Table 4.6: Errors and order evaluations for the second- and fourth-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for

$$\Theta_i^{Ok} = \Theta_{b,i}^{Ok}.$$

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.1E-02	-	5.8E-02	-	6.2E-02	-
200	1.4E-02	1.1	4.0E-02	0.6	5.0E-02	0.3
400	7.1E-03	1.0	2.8E-02	0.5	4.0E-02	0.3
800	3.5E-03	1.0	1.9E-02	0.5	3.1E-02	0.3
1600	1.8E-03	1.0	1.4E-02	0.5	2.5E-02	0.3

Third-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.4E-02	-	5.9E-02	-	3.2E-02	-
200	1.6E-02	1.1	3.9E-02	0.6	2.6E-02	0.3
400	7.8E-03	1.0	2.7E-02	0.5	2.0E-02	0.3
800	3.9E-03	1.0	1.9E-02	0.5	1.6E-02	0.3
1600	1.9E-03	1.0	1.3E-02	0.5	1.3E-02	0.3
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	2.3E-02	-	4.2E-02	-	3.1E-02	-
200	9.7E-03	1.2	2.8E-02	0.6	2.5E-02	0.3
400	4.8E-03	1.0	2.0E-02	0.5	2.0E-02	0.3
800	2.4E-03	1.0	1.4E-02	0.5	1.6E-02	0.3
1600	1.2E-03	1.0	9.7E-03	0.5	1.3E-02	0.3

Table 4.7: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for $\Theta_i^{Ok} = \Theta_{c,i}^{Ok}$.

4.2. Euler system. The second numerical experiment concerns the Euler system for a perfect diatomic gas where the unknown vector and the flux function are given as follows:

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad f(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{pmatrix}, \quad \text{where } p = (\gamma - 1) \left(\rho E - \frac{\rho u^2}{2} \right).$$

We fix $\gamma = 1.4$ and we endow the system with the following entropy:

$$\eta(w) = -\rho \ln \left(\frac{p}{\rho^\gamma} \right). \quad (4.2)$$

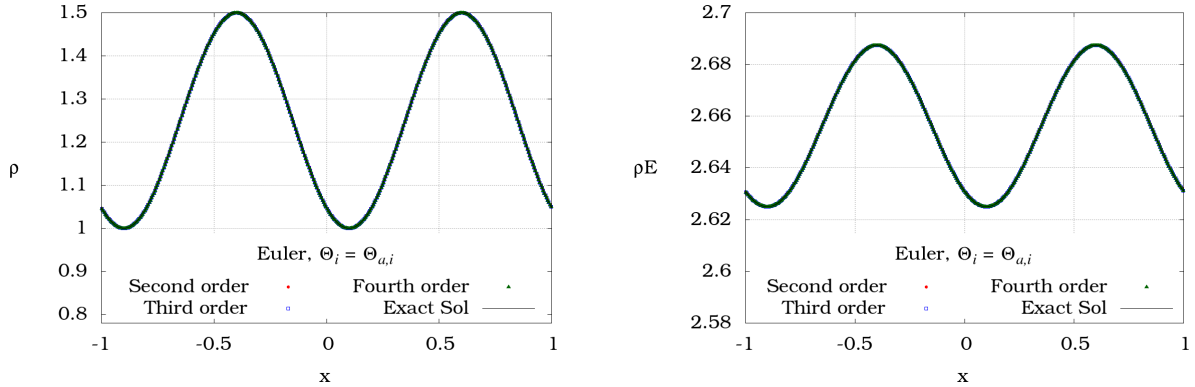
We set the following matrix parameter Θ_i^{Ok} values for the Euler problem

$$\begin{aligned} \Theta_{a,i}^{Ok} &= -\theta \operatorname{diag}_{1 \leq j \leq d} \left(\operatorname{sign} \left((\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n))_j (\mathcal{A}_i)_j \right) \right), \\ \Theta_{b,i}^{Ok} &= \operatorname{diag}_{1 \leq j \leq d} \left(\frac{\left((\delta_{i-1/2})_j^2 - (\delta_{i+1/2})_j^2 \right) \left((\delta_{i-1/2})_j^2 + (\delta_{i+1/2})_j^2 \right)}{\left((\delta_{i-1/2})_j^2 + (\delta_{i+1/2})_j^2 \right)^2 + \varepsilon} \right), \end{aligned} \quad (4.3)$$

where θ and ε are taken equal to $-\min(0, S/D)$ and 10^{-12} respectively. Once again, we perform two numerical simulations respectively concerned with a continuous solution, to relevantly evaluate the order of accuracy, and with a shock tube to illustrate the behavior of the approximate solution within shock waves and the absence of spurious oscillations.

4.2.1. Smooth solution. The initial data is given as follows over the periodic domain $[-1, 1)$: $\rho_0(x) = 1 + 0.5 \sin^2(\pi x)$, $u_0(x) = 0.5$, $p_0(x) = 1$. For such an initial data the Euler equations reduces to a linear transport problem and the solution remains smooth for all time $t > 0$.

1



2

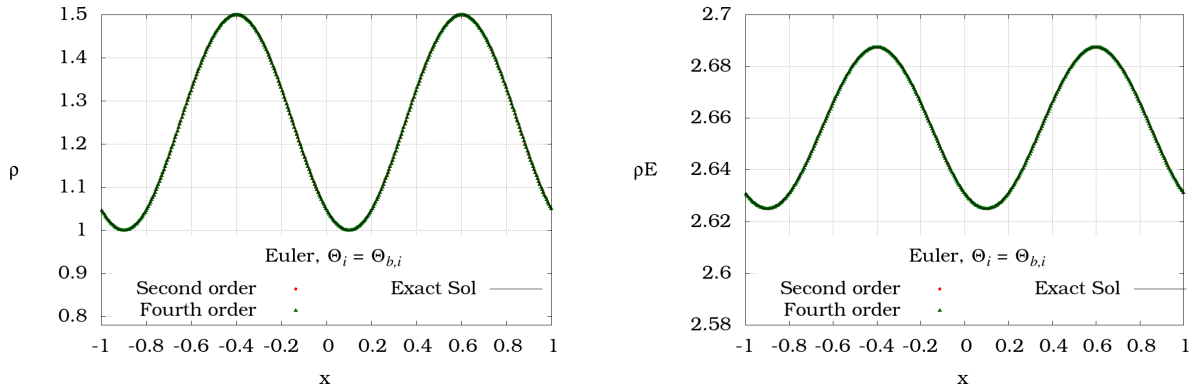


Fig. 4.2: Second-, third- and fourth-order accurate approximation of the smooth Euler solution and entropy with a mesh made of 400 cells.

3

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-03	-	1.9E-03	-	1.7E-03	-
200	8.7E-04	2.0	4.7E-04	2.0	4.2E-04	2.0
400	2.2E-04	2.0	1.2E-04	2.0	1.0E-04	2.0
800	5.4E-05	2.0	2.9E-05	2.0	2.6E-05	2.0
1600	1.4E-05	2.0	7.4E-06	2.0	6.5E-06	2.0
Third-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	6.3E-04	-	3.4E-04	-	3.0E-04	-
200	7.9E-05	3.0	4.3E-05	3.0	3.8E-05	3.0
400	9.9E-06	3.0	5.4E-06	3.0	4.8E-06	3.0
800	1.2E-06	3.0	6.7E-07	3.0	6.0E-07	3.0
1600	1.5E-07	3.0	8.4E-08	3.0	7.5E-08	3.0

4

5

Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	2.0E-05	-	1.1E-05	-	9.6E-06	-
200	1.3E-06	4.0	6.8E-07	4.0	6.0E-07	4.0
400	7.8E-08	4.0	4.3E-08	4.0	3.8E-08	4.0
800	4.9E-09	4.0	2.7E-09	4.0	2.4E-09	4.0
1600	3.1E-10	4.0	1.7E-10	4.0	1.5E-10	4.0

Table 4.8: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the continuous Euler solution and for $\Theta_i^{Ok} = \Theta_{a,i}^{Ok}$.

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	1.2E-02	-	7.6E-03	-	1.1E-02	-
200	2.4E-03	2.3	1.6E-03	2.2	2.9E-03	1.9
400	3.4E-04	2.8	2.0E-04	3.0	3.7E-04	3.0
800	6.0E-05	2.5	3.1E-05	2.7	2.6E-05	3.8
1600	1.4E-05	2.1	7.4E-06	2.1	6.5E-06	2.0
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	5.5E-05	-	3.9E-05	-	7.2E-05	-
200	3.4E-06	4.0	2.7E-06	3.9	6.4E-06	3.5
400	1.2E-07	4.9	6.8E-08	5.3	1.4E-07	5.5
800	5.3E-09	4.4	2.8E-09	4.6	2.4E-09	5.9
1600	3.1E-10	4.1	1.7E-10	4.1	1.5E-10	4.0

Table 4.9: Errors and order evaluations for the second- and fourth-order accurate schemes with the continuous Euler solution and for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

The obtained approximate solutions are displayed in Fig. 4.2. Once again, we notice a very good agreement of the approximate solution when compared to the exact one. This remark is emphasized with Tables 4.8–4.9 where we show that the expected order of accuracy are obtained even surprisingly a greater order for $\Theta_i^{Ok} = \Theta_{b,i}^{Ok}$.

4.2.2. Shock tube solution. We perform a shock tube as described in over the domain $[0, 1]$ where the initial data is given by

$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.125 & \text{otherwise,} \end{cases} \quad u_0(x) = 0, \quad p_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.1 & \text{otherwise,} \end{cases}$$

The final time is 0.2. In order to impose periodic conditions on the boundaries, we work on the domain $[-1, 1]$ and we set a symmetric tube shock problem on $[-1, 0]$.

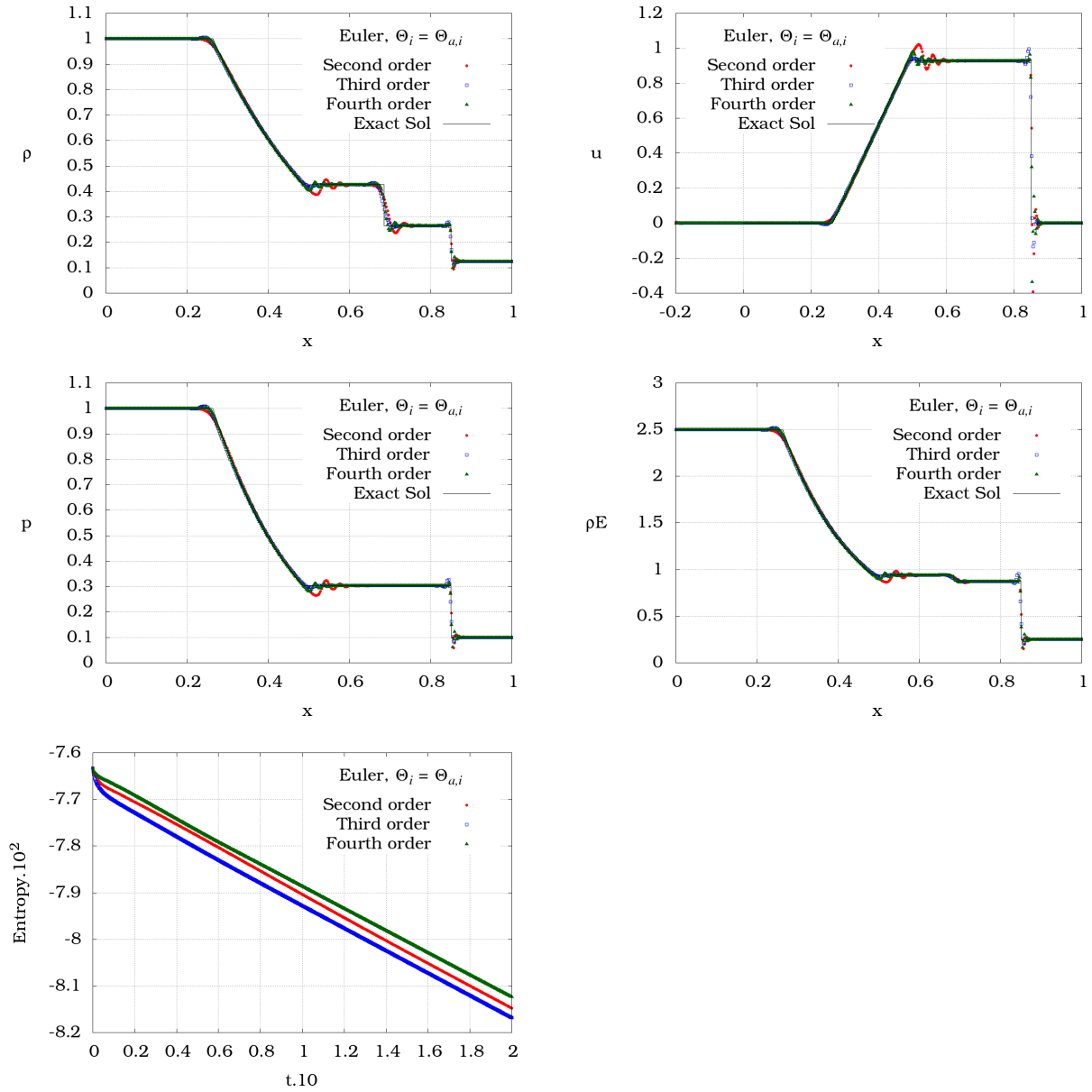


Fig. 4.3: Second-, third- and fourth-order accurate approximation of the shock tube

1 Euler solution and entropy with a mesh made of 400 cells for $\Theta_i^{O^k} = \Theta_{a,i}^{O^k}$.

2

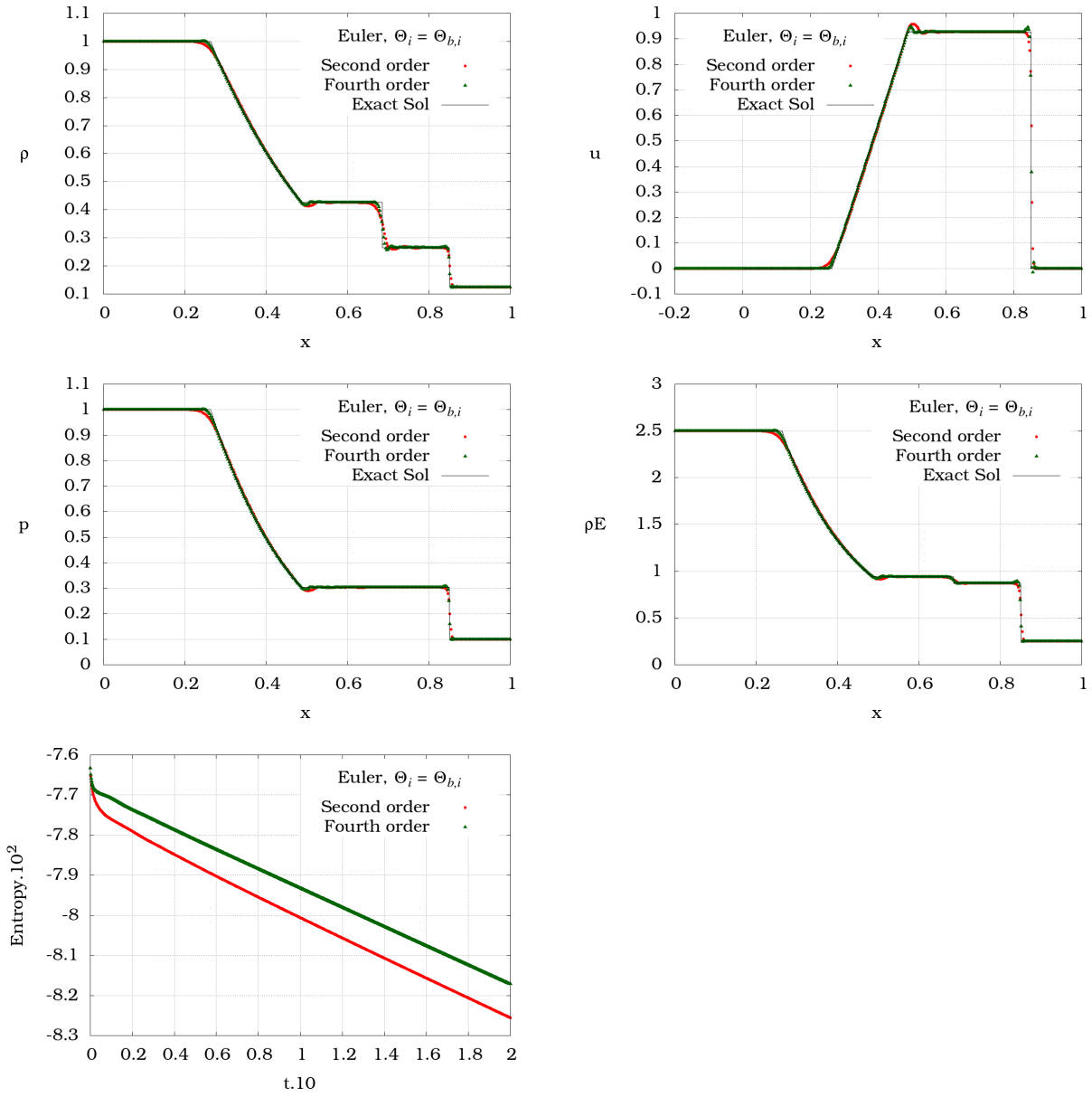


Fig. 4.4: Second-, third- and fourth-order accurate approximation of the shock tube
 1 Euler solution and entropy with a mesh made of 400 cells for $\Theta_i^{Ok} = \Theta_{b,i}^k$.

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	7.2E-02	-	7.1E-02	-	2.7E-01	-
200	4.0E-02	0.8	4.5E-02	0.7	2.4E-01	0.1
400	2.2E-02	0.9	2.9E-02	0.6	2.4E-01	0.0
800	1.2E-02	0.9	1.9E-02	0.6	1.9E-01	0.4
1600	6.4E-03	0.9	1.3E-02	0.6	1.7E-01	0.1
Third-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	4.5E-02	-	5.0E-02	-	2.0E-01	-
200	2.3E-02	0.9	3.1E-02	0.7	1.9E-01	0.1
400	1.2E-02	1.0	2.0E-02	0.7	1.7E-01	0.1
800	5.9E-03	1.0	1.2E-02	0.7	1.3E-01	0.4
1600	3.2E-03	0.9	8.5E-03	0.5	1.0E-01	0.4
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-02	-	3.8E-02	-	1.4E-01	-
200	1.9E-02	0.9	2.4E-02	0.6	1.2E-01	0.3
400	9.9E-03	0.9	1.6E-02	0.6	1.2E-01	0.0
800	4.9E-03	1.0	9.5E-03	0.7	8.1E-02	0.6
1600	2.5E-03	1.0	6.5E-03	0.6	8.5E-02	0.1

Table 4.12: Errors and order evaluations for the second-, third- and fourth-order accurate schemes with the shock tube Euler solution and for $\Theta_i^{O_k} = \Theta_{a,i}^{O_k}$.

Second-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	6.0E-02	-	6.3E-02	-	2.5E-01	-
200	3.2E-02	0.9	4.0E-02	0.7	2.3E-01	0.1
400	1.7E-02	0.9	2.6E-02	0.6	2.4E-01	0.0
800	8.7E-03	1.0	1.6E-02	0.7	1.7E-01	0.5
1600	4.5E-03	0.9	1.1E-02	0.5	2.2E-01	0.4
Fourth-order scheme errors						
cells	L^1	order	L^2	order	L^∞	order
100	3.2E-02	-	3.9E-02	-	1.7E-01	-
200	1.6E-02	1.0	2.5E-02	0.7	1.6E-01	0.1
400	8.0E-03	1.0	1.5E-02	0.7	1.4E-01	0.2
800	3.8E-03	1.1	8.8E-03	0.8	1.1E-01	0.4
1600	2.0E-03	0.9	6.0E-03	0.5	8.9E-02	0.3

Table 4.13: Errors and order evaluations for the second- and fourth-order accurate schemes with the shock tube Euler solution and for $\Theta_i^{O_k} = \Theta_{b,i}^{O_k}$.

The obtained approximate solutions are displayed Fig. 4.3 4.4. Once again, we remark only little spurious oscillations. In Table 4.12 4.13, we detail the evaluated orders of accuracy.

Acknowledgment. The authors thanks J. P. Vila for fruitfully discussions. M. Badsı, C. Berthon and L. Martaud realized this work with the financial support of

1 the ANR project MUFFIN ANR-19-CE46-0004. Arnaud Duran acknowledges financial
 2 support of the French National Research Agency project NABUCO, grant ANR-17-
 3 CE40-0025 and the french INSU-CNRS (Institut National des Sciences de l'Univers
 4 - Centre National de la Recherche Scientifique) program LEFE-MANU (Méthodes
 5 Mathématiques et Numériques), project DWAVE.

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5 **Appendix. Existence of the matrix parameter $(\Theta_i^{Ok})_{i \in \mathbb{Z}}$.**

6 PROPOSITION A.1. Consider $\eta \in C^2(\Omega, \mathbb{R})$ a strictly convex entropy. Let the ap-
7 proximation at time t^n , $w_\Delta(\cdot, t^n)$ given by (1.6) being a non zero function in $L^2(\mathbb{R})$.
8 Assume there exists a compact set $K \subset \Omega$ such that $w_\Delta(x, t^n) \in K$ for every $x \in \mathbb{R}$.
9 Then there exists a sequence of bounded (as $\Delta x \rightarrow 0$) diagonal matrices $(\Theta_i^{Ok})_{i \in \mathbb{Z}}$
10 such that the inequality (3.3) is satisfied.

Proof. According to the selected order of accuracy, we remark that both $\Delta x \overline{\partial_x w_i}^{Ok}$
and $\Delta x^3 \overline{\partial_{xxx} w_i}^{O4}$ are affine functions with respect to Θ_i^{Ok} so that we may write

$$\left(\Delta x \overline{\partial_x w_i}^{Ok} + \frac{\varepsilon^{O4}}{24} \Delta x^3 \overline{\partial_{xxx} w_i}^{O4} \right) = \Theta_i^{Ok} \mathcal{A}_i + \mathcal{B}_i,$$

where \mathcal{A}_i and \mathcal{B}_i are vectors of size d that come with a linear dependency on
 $(\delta_{i+\nu+\frac{1}{2}})_{-2 \leq \nu \leq 1}$ (with respect of $(w_i^n)_{i \in \mathbb{Z}}$). The condition (3.3) then reformulates
as follows

$$\sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} - \frac{1}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot (\Theta_i^{Ok} \mathcal{A}_i + \mathcal{B}_i) > 0$$

11 where the above sums are convergent since $w_\Delta(\cdot, t^n)$ belongs to a compact set $K \subset \Omega$
12 and $w_\Delta(\cdot, t^n)$ belongs to $L^2(\mathbb{R})$. Since the matrices Θ_i^{Ok} are diagonal, it equivalently
13 reformulates

$$S - \frac{1}{2} \sum_{i \in \mathbb{Z}} \Theta_i^{Ok} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \mathcal{A}_i > 0, \quad (\text{A.1})$$

14 where we have set

$$S = \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_i^n)) \cdot \delta_{i+\frac{1}{2}} - \frac{1}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n)) \cdot \mathcal{B}_i. \quad (\text{A.2})$$

15 We now choose $\Theta_i^{Ok} \in \mathcal{M}_d(\mathbb{R})$ under the form

$$\Theta_i^{Ok} = -\theta \operatorname{diag}_{1 \leq j \leq d} \left(\operatorname{sign} \left((\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n))_j (\mathcal{A}_i)_j \right) \right), \quad (\text{A.3})$$

16 with $\theta > 0$ a free constant to be fixed. Using such a formula for Θ_i^{Ok} , the inequality
17 (A.1) now reformulates as follows:

$$S + \theta D > 0, \quad (\text{A.4})$$

where D is positive number (because $w_\Delta(\cdot, t^n)$ is non constant) given by

$$D = \frac{1}{2} \sum_{i \in \mathbb{Z}} \sum_{j=1}^d \left| (\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n))_j (\mathcal{A}_i)_j \right| > 0. \quad (\text{A.5})$$

18 Since $D > 0$, it is therefore sufficient to choose $\theta > 0$ such that $\theta > \frac{S^-}{D}$ where
19 $S^- = -\min(S, 0) \geq 0$ is the negative part of S . \square