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# Bargaining On Monotonic Economic Environments

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## Abstract

Applying the solutions defined in the axiomatic bargaining theory to actual bargaining problems is a challenge when the problem is not described by its Utility Possibility Set (UPS) but as an *economic environment* specifying the set of alternatives and utility profile underlying the UPS. We introduce the axioms of *Independence of Non-Efficient Alternatives* and *Independence of Redundant Alternatives*. A solution satisfying these axioms can be applied to a simplified problem based on any reduced set of alternatives generating the Pareto frontier of the initial problem, and produces the same outcome. We compare our axioms to usual independence axioms, and show that they are satisfied by many usual bargaining solutions. Then, we introduce monotonicity conditions corresponding to the existence of an interest group (i.e., agents ranking the alternatives in the same order). For such monotonic economic environments, we provide a parameterized family of alternatives that generates the Pareto frontier of the bargaining problem, in line with our previous results. Our analysis illustrates that an axiomatic approach can be useful to foster the application of bargaining solutions, in complement to usual computational methods.

**JEL codes:** C7; D7

**keyword :** social choice, axiomatic bargaining theory, alternatives, economic environment, monotonicity

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## Declarations

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# 1 Introduction

The axiomatic bargaining theory and welfare economics are two ways to formalize social choice problems and characterize their solutions. In welfare economics, a social welfare function (SWF) is used to rank alternative options. In the axiomatic bargaining approach, axioms characterize *solutions* that select vectors of utility for the agents within a Utility Possibility Set (UPS). In addition to theoretical developments (Thomson, 2001), the axiomatic bargaining approach has been applied to important issues, such as wage bargaining (McDonald and Solow, 1981; Gerber and Upmann, 2006), monetary equilibrium (Aruoba et al., 2007), and equity issues in health expenditures (Clark, 1995; Cuadras-Morató et al., 2001). Though based on very stylized models, these applications illustrate the potential interest of the axiomatic bargaining approach to shed a new light on social choice problems. Whereas the definition of a SWF relies on functional forms and parameters that induce implicit ethical choices, for example regarding inequality aversion, axiomatic bargaining theory makes these ethical choices explicit through axioms. Policy implications are, however, very sensitive to the choice of the bargaining solution. In fact, Gerber and Upmann (2006) emphasize that the behavior of different bargaining solutions on specific economic problems is not well-understood. It is thus of interest to be able to apply theoretical bargaining solutions to real-life problems, to further study their practical implications.

Axiomatic bargaining theory has, however, not been applied much to complex social choice problems, whereas welfare economics has (e.g., on the climate change issue, Nordhaus, 2007; Stern, 2008). A reason may be that applying a bargaining solution first requires to characterize the UPS of the problem. This abstract framework may be too restrictive to address complex social choice problems, which usually do not aim at allocating utility to agents, but are more practically oriented toward decision-making, for example to select decisions driving the economy. Contrary to the case of the stylized models cited above, the set of alternatives in complex social choice problems may be so large that the agents have no idea of the shape of the UPS when formulating the problem. By focusing on the UPS, the classical bargaining formulation is too far from real situations. A bargaining problem can be enriched by specifying how the set of alternatives and the preferences underlie the UPS and disagreement outcome. In this case, application to real-life issues requires either to define and apply bargaining solutions directly on a set of alternatives or, alternatively, to compute the UPS from the economic description of the problem and apply a classical bargaining solution.

In this paper, we analyze the application of bargaining solutions to problems described as economic environments, in which the information underlying the UPS and disagreement point is explicitly considered. In this framework, a bargaining solution aims at selecting *alternatives* within

a set of feasible ones, instead of allocating utility among agents directly. We provide conditions, in the form of axioms that a bargaining solution may satisfy, under which the set of considered alternatives can be reduced in order to simplify the bargaining problem the agents face, or in order to simplify the computation of the Pareto frontier of the UPS. More specifically, we propose two independence axioms under which any given bargaining problem can be reduced to the bargaining over a subset of alternatives (called a *reduced* economic environment) characterizing the Pareto outcomes of the initial problem. We then provide a parameterized family of such alternatives for economic environments satisfying some monotonicity properties.

Two streams of the literature have departed from the classical formulation of bargaining in the utility space, and formalized bargaining problems directly in the terms of the underlying economic problem. Roemer (1986b, 1990) criticizes the classical axiomatic bargaining theory because it only considers the information contained in the UPS without accounting for the *economic environment* (hereafter, EE when referring to Roemer’s framework), that is, the nature of the goods to be shared and the preferences of the agents. He initiated the study of bargaining problems on EE described by “a collection of agents, an aggregate endowment bundle, and a utility function (or preference relation) for each agent” (Roemer, 1986a). The problem is not to allocate utility among agents, but commodities (distribution problems) from which agents derive utility, with a focus on fairness (Chun and Thomson, 1988). Rubinstein et al. (1992) initiated another approach, in which the bargaining problem is described by a set of alternatives and the preferences of agents on lotteries over the feasible alternatives, the so-called “alternatives-preferences” framework. Even if the two approaches led to quite independent research lines, with few papers referring to both,<sup>1</sup> Sprumont (2013) stresses that both streams of the literature emphasize the limits of the classical bargaining framework and the interest of studying bargaining problems on EE, in particular to interpret axioms and discuss their implications. These streams of the literature mainly focused on the characterization of particular solutions,<sup>2</sup> however, without discussing their applicability.

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<sup>1</sup>An exception is De Clippel (2015) who recovers Nash’s solution using axioms expressed in an EE with lotteries, and discusses his results with respect to the two streams of the literature.

<sup>2</sup>Characterizing a particular solution in the EE framework is more demanding than on the unrestricted domain of outcomes (Roemer, 1988). Roemer (1990) shows that, when expressed properly over EE, Nash’s axioms “hardly restrict the behavior of the bargaining solution at all” (p.296). Nieto (1992) characterizes a resource egalitarian solution corresponding to the lexicographic extension of the maximin criterion defined on EE. Chen and Maskin (1999) enrich the EE by considering the possibility of production, focusing on the egalitarian solution. Ginés and Marhuenda (2000) consider the provision of public goods. Yoshihara (2003) consider the fairness of different resource allocation mechanisms when agents have unequal skills. In the “alternatives-preferences” framework, Rubinstein et al. (1992) characterize Nash’s bargaining solution and its extensions to nonexpected utility preferences. Valenciano and Zarzuelo

Notwithstanding the more realistic description of the bargaining problem, the central element of the classical bargaining theory – the UPS – still needs to be characterized from the set of feasible alternatives in order to actually apply bargaining solutions defined in these frameworks too.

In the context of applied economics, one may usually presume a bargaining solution and obtain the shape of the UPS by computing its Pareto frontier. This approach, which may be challenging from an operational point of view, is relevant to apply solutions that satisfy the *Independence to Irrelevant Alternatives* (IIA) axiom along with Pareto-efficiency (e.g., the Nash solution), but may not be justified for solutions that do not satisfy the strong IIA axiom, unless one assumes the comprehensiveness of the UPS. Our contribution is to provide axiomatic conditions that support the application of bargaining theory, as well as practical methods for computing Pareto frontiers in monotonic economic environments. More specifically, we provide conditions under which it is possible to simplify the bargaining problem described through alternatives to make the application of bargaining solutions easier. Our formalism is more general than the description of EE *à la* Roemer, which mostly considers the division of a fixed aggregate endowment of goods. We consider problems in which the bargaining is over any form of alternatives (and possibly over lotteries), in the spirit of Rubinstein et al. (1992). We assume, however, that the preferences of agents are represented by real-valued functions, as in many studies establishing a link between the classical bargaining approach and the alternatives-preferences approach (see, e.g., Valenciano and Zarzuelo, 1997). We work on the combination of two axioms, *Independence of Non-Efficient Alternatives* (INEA) and *Independence of Redundant Alternatives* (IRA), that have intuitive meaning. We show that, for any given bargaining problem, a solution satisfying these axioms can be applied to any reduced set of alternatives yielding the Pareto-efficient outcomes of the initial problem, without changing the outcome. The underlying idea is to compare the solution of a reduced problem to that of the original problem.

The study of the behavior of bargaining solutions over EE with varying set of alternatives has been used to characterize solutions.<sup>3</sup> It involves either *Independence* axioms or *Monotonicity*

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(1994) show that dropping the symmetry axiom in this framework does not characterize the weighted Nash solution as in the classical bargaining framework, but that the solution is obtained assuming the strong property of *Independence to Isomorphic Transformations*, which requires the bargaining solution to overlook the physical nature of the alternatives and focus exclusively on their positions in terms of the preferences of agents. Grant and Kajii (1995) study other nonexpected utility preferences. Hanany (2007) examines under what conditions on agents' preferences it is possible to axiomatize group preferences (i.e., define a Social Welfare Function), linking the bargaining theory literature with the issues of social choice. Sprumont (2013) studies preference aggregation rules too, to characterize *Relative egalitarianism*.

<sup>3</sup>See, e.g., Nicolò and Perea (2005), who study the role of monotonicity axioms in the classical and alternatives-preferences frameworks, and characterize a solution based on a gradual expansion of the set of alternatives.

axioms, which are linked (Thomson and Myerson, 1980). As such, we discuss the consistency of our axioms with those used in the literature, and how demanding they are. Our axioms are quite weak, and many existing solutions satisfy them. Valenciano and Zarzuelo (1997) establish the links between the classical bargaining framework and the alternatives-preferences framework, and determine the “cost” of traveling from one framework to the other as well as the conditions under which a solution in one framework can be translated in the other. We discuss the cost of our axioms too.

The proposed “reduction” of problems through axioms is a way to foster the application of bargaining theory to real world issues, which may create synergies with the Operations Research approaches based on approximations of the UPS or on the reduction of the payoff dimensions (see, e.g., Lindroth et al., 2010). Another contribution of our paper is to characterize a special case under which our approach is effective. We formulate monotonicity conditions for economic environments under which a subset of efficient alternatives can be computed. When the set of alternatives is equipped with a preorder and the utility of some agents is monotonic with respect to this preorder, they form what we call an *interest group*. Their utility can be represented by a “single dimension” in the set of Pareto outcomes. This low dimensional frontier can be generated by a family of alternatives parameterized by the utility of the outsiders. This provides a practical tool to apply the axiomatic bargaining theory to bargaining problems on monotonic economic environments. This kind of interest group is somehow similar to the alliances of Manzini and Mariotti (2005), whose members can adopt a common position in bargain situations.

The paper is organized as follows. In Sect. 2, we present the framework and definitions. Sect. 3 contains general results and discussion. The case of monotonic economic environments is treated in Sect. 4. Long mathematical proofs are relegated in the Appendix A.

## 2 Framework and definitions

In 2.1, we present the bargaining problem. Then, we recall the notions of efficient outcomes and efficient alternatives in 2.2, and we define two axioms in 2.3.

*Notations:* The cardinality of a finite set  $S$  is denoted by  $|S|$ .  $\mathbb{N}$  is the set of natural numbers,  $\mathbb{N}^*$  the set of positive natural numbers, and  $\mathbb{R}$  the set of real numbers. We denote the power set (i.e., the set of subsets) of a set  $S$  by  $2^S$ . When needed, the set  $\mathbb{R}^q$ , where  $q \in \mathbb{N}^*$ , is equipped with the componentwise order  $y' = (y'_1, \dots, y'_q) \geq y = (y_1, \dots, y_q) \iff y'_i \geq y_i, \forall i = 1, \dots, q$ .

## 2.1 The bargaining problem

DEFINITION 1 (ECONOMIC ENVIRONMENT)

An economic environment  $\xi$  describes a set  $\mathcal{H}$  comprising  $H = |\mathcal{H}| \in \mathbb{N}^*$  agents who bargain over a set  $\mathbb{A}$  of feasible alternatives, given preferences characterized by a utility profile  $U : \mathbb{A} \rightarrow \mathbb{R}^H$  and a disagreement alternative  $a^d \in \mathbb{A}$ . Such an economic environment is denoted by  $\xi = [\mathcal{H}; U; \mathbb{A}; a^d]$ .

By convention, for  $\mathbb{A}' \subset \mathbb{A}$ ,  $U(\mathbb{A}') = \{U(a) \mid a \in \mathbb{A}'\} \subset \mathbb{R}^H$ .

Within this framework, the UPS of a bargaining problem on the economic environment  $\xi$  is  $U(\mathbb{A}) \subset \mathbb{R}^H$  and the disagreement outcome is  $U(a^d)$ .<sup>4</sup>

Characterizing the UPS explicitly is a difficult task in some problems, e.g., when alternatives are complex objects (e.g., items in  $\mathbb{R}^{\mathbb{N}}$  -the set of real sequences-) and  $\mathbb{A}$  is a large (possibly infinite) set. There may be no straightforward way to determine the UPS from the description of the economic environment, except by exploring the whole alternatives, which may not be feasible. To handle such economic environments, we shall consider independence axioms, which represent the readiness of agents to drop alternatives or to identify a subset of alternatives containing the solution without changing the outcome of the bargaining. We will thus consider *reduced economic environments* which differ from a given environment only by their set of considered alternatives.

DEFINITION 2 (REDUCED ECONOMIC ENVIRONMENT)

An economic environment  $\xi'$  is said to be a reduced economic environment of  $\xi = [\mathcal{H}; U; \mathbb{A}; a^d]$  if  $\xi' = [\mathcal{H}; U; \mathbb{A}'; a^d]$ , with  $\mathbb{A}' \subset \mathbb{A}$ .

In classical axiomatic bargaining theory, a bargaining solution allocates utility among agents.<sup>5</sup> In our framework (as well as in the non-welfarist approaches described in the introduction), outcomes are associated with alternatives through the utility profile or preference relationship. We thus define a *bargaining solution* as a process to select alternatives.

DEFINITION 3 (BARGAINING SOLUTION)

A bargaining solution  $\mu$  assigns to an economic environment  $\xi = [\mathcal{H}; U; \mathbb{A}; a^d]$  a subset of acceptable alternatives  $\mu(\xi) \subset \mathbb{A}$  within the set of feasible ones.

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<sup>4</sup> A classical bargaining problem  $(\mathcal{U}, u^d)$ , where  $\mathcal{U} \subset \mathbb{R}^H$  is the UPS and  $u^d$  the disagreement outcome, could be described in this framework by defining alternatives as the possible utility vectors in  $\mathcal{U}$ , and preferences as the identity function  $U(a) = a$ .

<sup>5</sup>Formally, in the classical framework, a bargaining solution is a correspondence  $\mu$  that maps a problem described by a pair  $(\mathcal{U}, u^d)$  to a subset  $\mu(\mathcal{U}, u^d) \subset \mathcal{U}$ . A solution can reduce to a mapping defining a single utility vector  $\mu(\mathcal{U}, u^d) = u^* \in \mathcal{U}$ .



Bargaining solutions are defined on a more or less large set of possible problems, called a *domain*, which influences their characterization (Thomson, 2001).<sup>6</sup> As different solutions have been characterized over different domains in the literature, our interest is to establish results in a “small” problem-specific domain which is consistent with many larger domains. From now on, we will only compare problems derived from a particular economic environment, without changing the set of agents or their preferences, nor the nature of the alternatives they bargain over or the disagreement alternative.

DEFINITION 4 (BARGAINING DOMAIN)

For a given economic environment  $\xi = [\mathcal{H}; U; \mathbb{A}; a^d]$ , the problem-specific bargaining domain  $\Delta_\xi$  is composed of  $\xi$  and all its reduced environments, i.e.,  $\Delta_\xi = \{\xi' = [\mathcal{H}; U; \mathbb{A}'; a^d] \mid \mathbb{A}' \subset \mathbb{A}\}$ .

By defining the bargaining domain this way, we assume that if a bargaining solution  $\mu$  is defined for a given economic environment  $\xi$ , it is also well-defined for its reduced economic environments, i.e.,  $\mu(\xi')$  exists for any  $\xi' \in \Delta_\xi$ . The bargaining solution can, however, be characterized on any larger domain containing the domain  $\Delta_\xi$  of interest.

We will also assume that the set of feasible alternatives  $\mathbb{A}$  is a compact metric space<sup>7</sup> and we will restrict our analysis to economic environments  $\xi$  characterized by a utility profile  $U = (U_1, \dots, U_H)$  composed of upper semicontinuous utility functions  $U_h : \mathbb{A} \rightarrow \mathbb{R}$ ,  $h = 1, \dots, H$ .<sup>8</sup> These restrictions are consistent with many usual domains. This is the case for solutions defined in the classical bargaining framework, whenever the UPS  $\mathcal{U} \subset \mathbb{R}^n$  is compact, as the Identity function is continuous (see footnote 4). This is also the case for problems of commodity sharing in EE *à la* Roemer, in which alternatives are vectors in  $\mathbb{R}^n$  and utility functions are usually assumed to be continuous.

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<sup>6</sup>Solutions in the classical bargaining theory are usually defined on a very large domain corresponding to the unrestricted class of pairs  $(\mathcal{U}, u^d)$  where  $\mathcal{U}$  is some set in  $\mathbb{R}^n$  and  $u^d$  is a point in this set. The choice of such a generic domain makes the characterization of solutions less demanding (see the discussion in Roemer, 1988). Rubinstein et al. (1992, p. 1173) emphasize that “specifying the domain of a solution is a delicate issue” when considering bargaining problems over alternatives, as a given UPS can be generated by many set of alternatives. Several authors impose strict restrictions on the domain in which they characterize solutions. Roemer (1990) examines problems of commodity sharing in which “the utility functions all measure the same kind of utility or outcome, and (...) the social alternatives all involve the distribution of the same kind of goods” (p. 290). Valenciano and Zarzuelo (1994) restrict their analysis to domains in which the set of alternatives and the disagreement alternative are fixed, and the set of alternatives is a compact topological space. Nicolò and Perea (2005) consider a domain in which alternatives are subsets of  $\mathbb{R}^n$ . De Clippel (2015) emphasizes the importance of the domain when considering axioms that relate the solution of different problems, such as monotonicity or independence axioms.

<sup>7</sup>For example,  $\mathbb{A}$  can be a compact subset of  $\mathbb{R}^n$  when  $n \in \mathbb{N}^*$ , or a compact subset of  $\mathbb{R}^{\mathbb{N}}$  (the set of all sequences in  $\mathbb{R}$ ).

<sup>8</sup>For all  $h = 1, \dots, H$ , the function  $U_h : \mathbb{A} \rightarrow \mathbb{R}$  is upper semicontinuous if, for all  $a \in \mathbb{A}$  and for all  $a_k \rightarrow a$ , one has  $\limsup_{k \rightarrow \infty} U_h(a_k) \leq U_h(a)$ .

For problems in the alternatives-preferences setting *à la* Rubinstein et al. (1992), it is also usual to assume that the set of alternatives is compact and the preferences continuous. As such, our results are quite general.

In next sections, as the bargaining problems in the domain  $\Delta_\xi$  differ only with respect to their set of alternatives, whenever no confusion could arise, we simplify the notation of economic environments and write  $\xi = \langle \mathbb{A} \rangle$ .

## 2.2 Efficient outcomes and efficient alternatives

We introduce the properties of (weak and strong) *efficiency* for outcomes (i.e., vectors of utility in  $\mathbb{R}^H$ ). These properties are formulated as correspondences, which can be used either to reduce the set of utility vectors worth considering (i.e., presolutions ; see Thomson, 2001, p. 354) or to characterize solutions with corresponding axioms. Such axioms will be specified later.

An outcome  $u = (u_1, \dots, u_H) \in \mathcal{U} \subset \mathbb{R}^H$  is *weakly Pareto efficient* – weakly efficient for short – on  $\mathcal{U}$  if, for any  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_H)$  such that  $\tilde{u}_h > u_h$  for all  $h = 1, \dots, H$ , one has  $\tilde{u} \notin \mathcal{U}$ . We denote by  $\mathcal{E}^w(\mathcal{U}) \subset \mathcal{U}$  the set of all weakly efficient outcomes of  $\mathcal{U}$ , and call it the *weak Pareto frontier* of  $\mathcal{U}$ .

An outcome  $u = (u_1, \dots, u_H) \in \mathcal{U} \subset \mathbb{R}^H$  is *strongly Pareto efficient* – efficient for short – on  $\mathcal{U}$  if, for any  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_H)$  such that  $\tilde{u}_h \geq u_h$  for all  $h = 1, \dots, H$  and  $\tilde{u}_h > u_h$  for some  $h$ , one has  $\tilde{u} \notin \mathcal{U}$ . We denote by  $\mathcal{E}(\mathcal{U}) \subset \mathcal{U}$  the set of all efficient outcomes of  $\mathcal{U}$ , and call it the *Pareto frontier* of  $\mathcal{U}$ . We have  $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}^w(\mathcal{U})$ . An outcome  $(u_1, \dots, u_H) \in \mathcal{E}^w(\mathcal{U}) \setminus \mathcal{E}(\mathcal{U})$  is dominated in the sense that one can increase the utility of at least one agent without decreasing that of the others.

Efficient outcomes are generated by *efficient alternatives*. We define the set of efficient alternatives for an economic environment  $\xi$  as follows.

### DEFINITION 5 (EFFICIENT ALTERNATIVES)

For an economic environment  $\xi = \langle \mathbb{A} \rangle$ , the set of efficient alternatives is the subset  $\mathbb{A}^* \subset \mathbb{A}$  of alternatives yielding efficient outcomes, that is,  $\mathbb{A}^* = \{a \in \mathbb{A} \mid U(a) \in \mathcal{E}(U(\mathbb{A}))\}$ . The corresponding efficient reduced economic environment, which contains only efficient alternatives  $\mathbb{A}^* \subset \mathbb{A}$ , is denoted by  $\xi^* = \langle \mathbb{A}^* \rangle$ .

The Appendix A gathers lemmas characterizing such efficient alternatives (Lemma 1) and the properties of efficient alternatives for reduced economic environments (Lemma 2). These properties are used to prove our results in the following sections.

### 2.3 Axioms

We use the axiom of *Pareto efficiency* as a well-known benchmark, and we introduce two new axioms: *Independence of Non-Efficient Alternatives* and *Independence of Redundant Alternatives*. These two axioms are specific to our framework in that they are axioms about alternatives (and not about utility outcomes).

AXIOM 1 (PARETO EFFICIENCY)

A bargaining solution  $\mu$  satisfies the axiom of Pareto Efficiency if, for any  $\xi = \langle \mathbb{A} \rangle$ , one has  $U(\mu(\xi)) \subset \mathcal{E}(U(\mathbb{A}))$ . Given Definition 5 of efficient alternatives, this is equivalent to  $\mu(\xi) \subset \mathbb{A}^*$ .

The axiom of Pareto Efficiency has the usual interpretation – a bargaining solution is Pareto-efficient if all the alternatives it selects are associated with Pareto-efficient outcomes.

AXIOM 2 (INDEPENDENCE OF NON-EFFICIENT ALTERNATIVES – INEA)

A bargaining solution  $\mu$  satisfies the axiom of Independence of Non-Efficient Alternatives if, for the economic environment  $\xi = \langle \mathbb{A} \rangle$  and any reduced economic environment  $\xi' = \langle \mathbb{A}' \rangle \in \Delta_\xi$ , we have  $\mathbb{A}^* \subset \mathbb{A}' \subset \mathbb{A} \Rightarrow \mu(\xi') = \mu(\xi)$ .

The INEA axiom requires the solution to be independent of alternatives that do not yield efficient outcomes in the sense that dropping any subset of alternatives that are not efficient does not change the bargaining solution.

AXIOM 3 (INDEPENDENCE OF REDUNDANT ALTERNATIVES – IRA)

A bargaining solution  $\mu$  satisfies the axiom of Independence of Redundant Alternatives if, for the economic environment  $\xi = \langle \mathbb{A} \rangle$  and any reduced economic environment  $\xi' = \langle \mathbb{A}' \rangle \in \Delta_\xi$ , we have  $U(\mathbb{A}') = U(\mathbb{A}) \Rightarrow U(\mu(\xi')) = U(\mu(\xi))$ .

This axiom requires that the solution is independent of redundant alternatives in the sense that dropping any subset of alternatives without reducing the set of feasible utilities does not change the bargained outcomes.

Before discussing these axioms, we provide our general results, so that our discussion is enlightened by the the consequences of the axioms.

## 3 General results and discussion

In 3.1, we prove existence of efficient alternatives and show that a bargaining solution satisfying our two axioms can be applied to some subsets of efficient alternatives. A discussion follows in 3.2.

### 3.1 On the existence of efficient alternatives and their use in bargaining solutions

We first establish conditions for efficient alternatives to exist. These conditions correspond to topological assumptions without which the set of efficient alternatives might be empty. The proof is in the Appendix A.

PROPOSITION 1 (EXISTENCE OF EFFICIENT ALTERNATIVES)

If  $\mathbb{A}$  is a compact metric space, and  $U = (U_1, \dots, U_H) : \mathbb{A} \rightarrow \mathbb{R}^H$  is composed of upper semicontinuous functions, then, for all  $u \in U(\mathbb{A})$ , there exists  $a^* \in \mathbb{A}$  such that  $U(a^*) = u^* \in \mathcal{E}(U(\mathbb{A})) \cap (u + \mathbb{R}_+^H)$ , i.e.,  $U(a^*)$  is an efficient point above  $u$  (that is, larger for the componentwise order). As a consequence, the set  $\mathbb{A}^* = \{a \in \mathbb{A} \mid U(a) \in \mathcal{E}(U(\mathbb{A}))\}$  of efficient alternatives (Definition 5) is not empty.

We now show that, if a solution to a bargaining problem  $\xi$  satisfies the INEA and IRA axioms, this solution can be applied to some reduced economic environments without changing the outcome. The proof (to be found in the Appendix A) relies on the existence of efficient alternatives (Proposition 1), and thus on the associated topological assumptions.

PROPOSITION 2 (BARGAINING OVER A SUBSET OF EFFICIENT ALTERNATIVES)

Consider an economic environment  $\xi = [H; U; \mathbb{A}; a^d]$ , where  $\mathbb{A}$  is a compact metric space, and  $U = (U_1, \dots, U_H) : \mathbb{A} \rightarrow \mathbb{R}^H$  is composed of upper semicontinuous functions, and the associated problem-specific bargaining domain  $\Delta_\xi$  (Definition 4). If a bargaining solution  $\mu$  satisfies the axioms of Independence of Non-Efficient Alternatives and Independence of Redundant Alternatives, then, for any reduced economic environment  $\xi' = \langle \mathbb{A}' \rangle \in \Delta_\xi$ , we have  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}') \Rightarrow U(\mu(\xi')) = U(\mu(\xi))$ .

If the bargaining solution satisfies the INEA and IRA axioms, a bargaining problem can be reduced to a simpler bargaining problem over any subset of alternatives generating the Pareto frontier of the UPS. This result can be used to simplify bargaining problems on economic environments. This is the case for *monotonic* economic environments studied in Section 4.

### 3.2 Discussion

Our INEA and IRA axioms are related to independence axioms used in the literature (and, to a lesser extent, to monotonicity axioms<sup>9</sup>). Independence axioms can be interpreted as ways to reduce

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<sup>9</sup>Independence axioms specify that a solution does not change when the UPS – or the set of alternatives – changes, whereas monotonicity axioms specify how the solution evolves under such changes (Thomson and Myerson,

the set of feasible outcomes in the spirit of pre-solutions (Thomson, 2001), but imposing such axioms restricts the applicable solutions (Roth, 1977). Dropping alternatives from a bargaining problem amounts to drop information, and the use of independence axioms is not innocuous and deserves a discussion. As we want our results to apply to many bargaining solutions, each characterized by different axioms, it is in our interest to examine if our axioms are consistent with usual ones, and to show that they are rather weak, in the sense that there are not too demanding.

The axiom of *Independence to Irrelevant Alternatives* (IIA) is one of the most discussed in the literature.<sup>10</sup> The combination of IIA and Pareto Efficiency has interesting practical consequences, as it makes it possible to discard any dominated alternative from the formulation of the bargaining problem. In terms of application, it means that it is sufficient to characterize the Pareto frontier of the UPS of a given problem to apply a solution satisfying Pareto efficiency and IIA. IIA is a strong axiom, however, as a solution satisfying IIA is insensitive to most changes in the possibilities (in terms of outcomes in the UPS or in terms of alternatives). Even if some independence to irrelevant alternatives makes sense when considering group preferences (Peters and Wakker, 1991), the axiom has been strongly criticized as being too demanding (see the discussion in Imai, 1983) and inducing counter-intuitive or perverse effects (Thomson and Myerson, 1980). Roemer (1990) emphasizes that the IIA axiom is even more demanding in the EE framework. Moreover, many solutions do not satisfy the IIA axiom. For example, the solution characterized by Kalai and Smorodinsky (1975) depends on the ideal point.<sup>11</sup> The IIA axiom is thus very strong. We now investigate how demanding our independence axioms are.

Our INEA axiom is more demanding than Pareto Efficiency in the sense that INEA implies Pareto Efficiency, while imposing independence to some alternatives. As the following lemma shows, this is the consequence of the property that, when a bargaining solution satisfies INEA, bargaining over the set of efficient alternatives  $\mathbb{A}^*$  (Definition 5) leads to the same result as bargaining over the full set of alternatives  $\mathbb{A}$ .

PROPOSITION 3 (INEA IMPLIES PARETO EFFICIENCY)

*If a solution  $\mu$  satisfies the axiom of Independence of Non-Efficient Alternatives, then it satisfies the axiom of Pareto Efficiency.*

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1980). Monotonicity axioms are often imposed or strengthened when independence axioms are relaxed (see, e.g., Kalai and Smorodinsky, 1975; Imai, 1983; De Clippel, 2015).

<sup>10</sup>We here consider the following definition of IIA. A solution  $\mu$  is IIA if, whenever  $\mathbb{A}' \subset \mathbb{A}$  and  $\mu(\xi) \subset \mathbb{A}'$ , it follows that  $\mu(\xi') = \mu(\xi)$ .

<sup>11</sup>The *ideal point* (sometimes called *utopia*) is the virtual vector of payoffs in which the utility of each agent corresponds to the maximal utility he/she could get from the bargaining.

**Proof** The economic environment  $\xi^* = \langle \mathbb{A}^* \rangle$  (Definition 5) is a reduced economic environment of  $\xi = \langle \mathbb{A} \rangle$ , from the Definition 2 with  $\mathbb{A}' = \mathbb{A}^* \subset \mathbb{A}$ . Applying the INEA axiom (Axiom 2) to this reduced environment gives that  $\mu(\xi) = \mu(\xi^*) \subset \mathbb{A}^*$ , implying that  $\mu$  is Pareto efficient (Axiom 1).  $\square$

For Pareto efficient solutions, the INEA axiom is however much less demanding than IIA as any efficient solution satisfying IIA also satisfies INEA.

PROPOSITION 4 (IIA + PARETO EFFICIENCY IMPLY INEA)

*If a Pareto efficient solution  $\mu$  satisfies the axiom of Independence of Irrelevant Alternatives, then it satisfies the axiom of Independence of Non-Efficient Alternatives.*

**Proof** Consider an economic environment  $\xi = \langle \mathbb{A} \rangle$  and any reduced economic environment  $\xi' = \langle \mathbb{A}' \rangle \in \Delta_\xi$  satisfying  $\mathbb{A}^* \subset \mathbb{A}' \subset \mathbb{A}$ . If solution  $\mu$  satisfies the axiom of Pareto Efficiency, then  $\mu(\xi) \subset \mathbb{A}^*$ . Therefore,  $\mu(\xi) \subset \mathbb{A}'$ . If in addition  $\mu$  satisfies the IIA axiom, we conclude that  $\mu(\xi') = \mu(\xi)$  proving that  $\mu$  satisfies INEA.  $\square$

Weaker axioms than IIA have been proposed to characterize solutions which are more sensitive to the information contained in UPS. Roth (1977), for example, proposed the axiom of *Independence to Irrelevant Alternatives other than (disagreement point and) Ideal Point* (IIIA), which allows the bargaining solution to be sensitive to the most favorable outcome of each agent. In the spirit of Proposition 4, it is straightforward to prove that, when combined with Pareto-efficiency, IIIA implies INEA. All these axioms emphasize the role of (some) information contained in the Pareto frontier. Our INEA axiom aims at conserving all the information from the efficiency frontier. As shown in Proposition 4, any Pareto efficient bargaining solution satisfying IIA in the classical bargaining theory framework also satisfies INEA when considering economic environments (e.g., the Nash solution (Nash, 1950)). Some classical solutions that do not satisfy IIA can satisfy INEA, as long as they depend only on efficient outcomes (e.g., the Kalai-Smorodinsky solution<sup>12</sup>). Under INEA, the solution may be sensitive to some nonoptimal alternatives (Karni and Schmeidler, 1976) if these alternatives are associated with efficient outcomes.

Our INEA axiom is thus weaker than usual independence axioms, and is consistent with many bargaining solutions. Even in the case in which the removed alternatives were very favorable to one of the agents, as these alternatives are dominated by some efficient alternatives the new set of alternatives does not necessarily modify the bargaining positions. Not only the ideal point is conserved, but the whole Pareto-efficient outcomes.

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<sup>12</sup>Imai (1983) shows that the axiom of *Restricted Monotonicity* used by Kalai and Smorodinsky (1975) implies IIIA. It thus also implies INEA.

Other streams of the literature refer to an IIA axiom, in particular in the literature on social choice *à la* Arrow. Denicolò (2000) provides an important discussion on the links between IIA and Arrow’s choice axiom, to which the present definition of IIA is close. INEA serves also a purpose close to the property of *Independence of Pareto Irrelevant Alternatives* used by Donaldson and Weymark (1988) and Duggan (1996) and justified by the fact that if “social choices must be Pareto optimal, and sub-optimal alternatives may never be chosen by society, it seems reasonable to reinforce these conditions by excluding sub-optimal alternatives from the determination of social choices.”

The IRA axiom is somehow more demanding, at least if one wants to apply a solution characterized on sets of alternatives, and not a solution characterized in the utility space. Under this axiom, the solution  $\mu$  does not need to select all the alternatives that achieve the solution outcomes.

The authors characterizing solutions on EE *à la* Roemer usually assume full correspondences that select all the alternatives yielding the solution outcomes. In this framework, the IRA axiom may modify the set of alternatives selected by a solution.

Redundant alternatives (more precisely, the absence of redundant alternatives) play an important role in the proof of many results of the alternatives-preferences approach to bargaining. Rubinstein et al. (1992), Valenciano and Zarzuelo (1994) and Grant and Kajii (1995) *assume* that there are no redundant alternatives (i.e., alternatives for which both players are indifferent). In actual problems, though, redundant alternatives may occur, and the way to deal with those has to be explicit. Valenciano and Zarzuelo (1997) provide an important discussion on redundant alternatives. Establishing the links between the classical bargaining framework and the alternatives-preferences framework, they determine the “cost” of traveling from one framework to the other. Among other conditions,<sup>13</sup> they show that it involves a condition close to IRA, and state that it “is a straightforward corollary of the expected utility substitution/independence axiom.” In fact, any solution defined in the classical bargaining theory framework satisfies the IRA axiom when translated in our framework. This is due to the fact that solutions defined on UPS satisfy (universal) *welfarism*, i.e., a form of independence both to the utility profile and to the set of alternatives (Roemer, 1990). Valenciano and Zarzuelo (1997, p. 266) emphasize that the “procedure that consists of defining a solution for bargaining games based only on the geometry of the set of utility vectors associated with a feasible set of alternatives” has a large cost and requires “disregarding any detail concerning the particular nature of the alternatives at stake or the bargaining process and environment.” The cost of discarding redundant alternatives is thus already paid for when one

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<sup>13</sup>This cost mainly involves an *invariance to isomorphic transformations* (ISO).

wants to apply a solution defined in the classical framework to an actual problem.

Our IRA axiom does not imply welfarism (strictly speaking), however, because it is defined on the problem-specific domain  $\Delta_\xi = \{\xi' = [\mathcal{H}; U; \mathbb{A}'; a^d] \mid \mathbb{A}' \subset \mathbb{A}\}$  which depends on the utility profile and on the nature of the considered alternatives. A solution could be characterized within the alternatives-preferences framework on that domain only. The IRA axiom would then only imply some weak form of welfarism (linked to the independence to the set of alternatives, but not to the utility profile) within the domain of all the reduced environments of a given problem.

Other papers refer to an IRA axiom or stronger axioms. Dhillon and Mertens (1999) explicitly introduce an IRA axiom (over lotteries) to characterize *Relative Egalitarianism* from a weakened versions of Arrow’s axioms, claiming that the IRA axiom “is so innocuous that a *much* stronger form, taking the set of feasibility utility-vectors as primitive, is taken for granted in the formulation itself of the bargaining problem (...) without ever having been challenged” (p.479, the emphasis is original ; a footnote and a parenthesis have been deleted). Sprumont (2013) introduces an *Independence of Inessential Expansions* axiom, and shows that it is stronger than IRA in his framework.

Our independence axioms aim at restricting the set of alternatives of interest to simplify the problem’s formulation. The cost of our result is thus the loss of exhaustivity (in the sense that solutions are no more full correspondences). This cost, embedded in the IRA axiom, may seem innocuous when the bargaining solution is characterized in the utility space and simply ‘translated’ in the alternatives-preferences setup. Still, whether or not our axioms are senseful should be discussed with respect to the simplifications agents are ready to make when faced with an actual problem.

## 4 The case of monotonic economic environments

In this section, we study bargaining problems on economic environments satisfying some monotonicity properties (4.1). For these problems, we provide in 4.2 a method to determine the set of efficient outcomes and a set of alternatives that produce these outcomes. We finish in 4.3 by providing examples of monotonic economic environments.

### 4.1 Monotonic economic environments: definition

We now consider that the set  $\mathbb{A}$  of alternatives is a compact metric space equipped with a problem-specific preorder  $\preceq$ . Recall that a preorder is a reflexive and transitive binary relation (whereas an order is an antisymmetric preorder); a preorder is total (resp. partial) if all (resp. not all) alterna-



tives can be compared. When the preorder is an order, one can partially order its alternatives, in the sense that some of them can be ranked. For instance, vectors of quantities of a given bundle of commodities can naturally be ranked by the componentwise order; so do sequences of such vectors. In the same way, intertemporal trajectories of an economy can be ranked according to the infinite sequence of vectors of their state variables. When alternatives are lotteries, a possible order is the second-order stochastic dominance (Diamond and Rothschild, 1978).<sup>14</sup>

Some economic environment can be characterized by the following monotonicity properties on the preferences: the utility of *some* agents may be monotonic in the sense that their utility is higher for “lower” alternatives.<sup>15</sup> We qualify such agents as an *interest group*, the other agents being *outsiders* (of the interest group).

DEFINITION 6 ((*i, o*)-MONEE)

Suppose that the set  $\mathcal{H}$  of agents is partitioned<sup>16</sup> in an interest group  $\mathcal{H}^i$  and a group  $\mathcal{H}^o$  of outsiders, of cardinalities  $|i| = |\mathcal{H}^i|$  and  $|o| = |\mathcal{H}^o|$ . An economic environment  $\xi = \langle \mathbb{A} \rangle$  is said to be a (*i, o*)-monotonic economic environment – (*i, o*)-MonEE – if the groups’ utility profiles  $U = (U^i, U^o)$ ,  $U^i : \mathbb{A} \rightarrow \mathbb{R}^{|i|}$ ,  $U^o : \mathbb{A} \rightarrow \mathbb{R}^{|o|}$  are such that

- I.  $U^i : \mathbb{A} \rightarrow \mathbb{R}^{|i|}$  is nonincreasing<sup>17</sup>, that is,  $a^b \preceq a^\# \Rightarrow U^i(a^b) \geq U^i(a^\#)$ ,
- II. for all  $u^o \in \mathbb{R}^{|o|}$ , the set  $\{a \in \mathbb{A} \mid U^o(a) \geq u^o\}$  is either empty or admits a (not necessarily unique<sup>18</sup>) minimum,<sup>19</sup> that is, an  $\bar{a} \in \mathbb{A}$  such that

$$U^o(\bar{a}) \geq u^o \text{ and } (a \in \mathbb{A}, U^o(a) \geq u^o \Rightarrow \bar{a} \preceq a) . \quad (1)$$

Item I means that if an alternative is lower<sup>20</sup> than others, it is preferred by all the members of the interest group.

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<sup>14</sup>We emphasize that, even if  $\mathbb{A}$  is not equipped with a natural preorder, e.g., a preorder relative to the physical nature of the alternatives, it is at least equipped with preorders compatible with the preferences represented by the utility profile  $U$ . In that case, our analysis below amounts to assume that the utility of all the members of the interest group are compatible with the same preorder, i.e.,  $a^b \preceq a^\# \Leftrightarrow U^i(a^b) \geq U^i(a^\#)$ , a definition close to the definition of *alliances* in Manzini and Mariotti (2005).

<sup>15</sup>The case in which these agents have higher utility for “higher” alternatives is symmetric. We shall mention how the results change in this case in footnotes.

<sup>16</sup> $(\mathcal{H}^i, \mathcal{H}^o)$  is a partition of  $\mathcal{H}$  if  $\mathcal{H}^i$  and  $\mathcal{H}^o$  are subsets of  $\mathcal{H}$ ,  $\mathcal{H}^i \neq \emptyset \neq \mathcal{H}^o$ ,  $\mathcal{H}^i \cup \mathcal{H}^o = \mathcal{H}$ , and  $\mathcal{H}^i \cap \mathcal{H}^o = \emptyset$ .

<sup>17</sup>Respectively, “nondecreasing” if the utility profile of the interest group is nondecreasing in the alternatives ; *mutatis mutandis* in the mathematical conditions.

<sup>18</sup>As we only require  $\mathbb{A}$  to be equipped with a preorder  $\preceq$ , there might be several distinct minima. If  $\preceq$  is an order, the minimum is unique.

<sup>19</sup>Respectively, “a maximum” if the utility profile of the interest group is nondecreasing in the alternatives ; *mutatis mutandis* in the mathematical conditions.

<sup>20</sup>Respectively, “higher” if the utility profile of the interest group is nondecreasing in the alternatives.

Item II means that, for any vector of utility levels  $u^o$  that can be guaranteed to the outsiders (in the sense that their actual utility *can* be at least equal to that level), there is a (subset of) lowest<sup>21</sup> alternative among the subset of alternatives yielding at least these levels of utility.

## 4.2 Bargaining on an efficient reduced economic environment

For monotonic economic environments satisfying the MonEE properties, it is possible to identify a subset of alternatives  $\mathbb{A}^o \subset \mathbb{A}$  yielding efficient outcomes, consistent with Proposition 2. This set of *satisficing alternatives* is defined as follows.

DEFINITION 7 (SATISFICING ALTERNATIVES)

For a  $(i, o)$ -monotone economic environment  $\xi = \langle \mathbb{A} \rangle$ , we define

I. the set of feasible payoffs for outsiders by

$$\mathcal{U}^o = \{u^o \in \mathbb{R}^{|\mathbb{A}^o|} \mid \{a \in \mathbb{A} \mid U^o(a) \geq u^o\} \neq \emptyset\} \subset \mathbb{R}^{|\mathbb{A}^o|}, \quad (2)$$

II. the satisficing alternatives associated with a feasible payoff for outsiders as<sup>22</sup>

$$\forall u^o \in \mathcal{U}^o, \quad a^o(u^o) = \min\{a \in \mathbb{A} \mid U^o(a) \geq u^o\} \subset \mathbb{A}, \quad (3)$$

where  $\min S$  denotes the set of minima of the subset  $S$  of a preordered set, that is, for  $u^o \in \mathcal{U}^o$  the set  $a^o(u^o)$  contains all the alternatives  $\bar{a}$  satisfying (1).<sup>23</sup>

III. the reduced economic environment  $\xi^o = \langle \mathbb{A}^o \rangle$ , where the set of alternatives

$$\mathbb{A}^o = \{a \in \mathbb{A} \mid \exists u^o \in \mathcal{U}^o, \quad a \in a^o(u^o)\} \subset \mathbb{A} \quad (4)$$

is the set of all satisficing alternatives.

A *feasible payoff for outsiders* is a vector of utility levels that can be guaranteed to outsiders in the sense that there is one (or more) alternative(s) yielding at least these levels for them. The set of such payoffs is defined by eq. (2). The *satisficing alternatives* associated with a feasible payoff for outsiders (eq. 3) are the “lowest”<sup>24</sup> alternatives among those yielding at least the given vector of utility levels for outsiders. When  $\mathbb{A}$  is equipped with an order, there is a unique satisficing alternative. Item I of the Definition 6 of monotonic economic environments specifies that the utility of the

<sup>21</sup>Respectively, “highest” if the utility profile of the interest group is nondecreasing in the alternatives.

<sup>22</sup>Replace the min by a max in eq. (3) if the utility profile of the interest group is nondecreasing in the alternatives.

<sup>23</sup>For an ordered set of alternatives, the minimum would be unique.

<sup>24</sup>Respectively, “highest” if the utility profile of the interest group is nondecreasing in the alternatives.

agents in the interest group is nonincreasing<sup>25</sup> in the alternatives. This means that the satisficing alternatives maximize the utility of the members of the interest group given a feasible payoff for the outsiders. These satisficing alternatives are defined for any feasible payoff for outsiders. We can use them to define a subset of alternatives parameterized by the feasible payoffs for outsiders, as well as the associated reduced economic environment (eq. 4).

We show in Proposition 5 that, for a monotonic economic environment  $\xi$ , the outcomes associated with satisficing alternatives include the Pareto-efficient outcomes of  $U(\mathbb{A})$ . The proof is in the Appendix A.

PROPOSITION 5 (EFFICIENT ALTERNATIVES FOR  $(i, o)$ -MONEE)

For a  $(i, o)$ -monotonic economic environment  $\xi = \langle \mathbb{A} \rangle$ , we have

$$\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}^o) \subset \mathcal{E}^w(U(\mathbb{A})),$$

where the set  $\mathbb{A}^o$  of satisficing alternatives is defined in eq. (4).

Note that if there is no weakly-efficient outcome,  $\mathcal{E}(U(\mathbb{A})) = \mathcal{E}^w(U(\mathbb{A}))$ , and thus  $U(\mathbb{A}^o)$  corresponds exactly to the Pareto frontier.

According to Proposition 5, the reduced environment  $\xi^o = \langle \mathbb{A}^o \rangle$  satisfies the assumptions of Proposition 2. We conclude that a bargaining solution satisfying the INEA and IRA axioms can be applied to this reduced economic environment, and produces the same outcome as the initial problem in the sense that  $U(\mu(\xi^o)) = U(\mu(\xi))$ .<sup>26</sup>

The result in Proposition 5 exploits the same idea as the one used by Kalai and Smorodinsky (1975) and Imai (1983) when defining the function that describes the maximum utility an agent could get given the utility of the other, but applied to economic environments and alternatives. Combined with Proposition 2, it provides a practical tool to simplify the formulation of bargaining problems over alternatives.

### Interest group and low dimensional Pareto frontier.

In large-dimensional bargaining problems on monotonic economic environments, if the bargaining solution satisfies the INEA and IRA axioms, the satisficing alternatives of Definition 7 can be used

<sup>25</sup>Respectively, “nondecreasing” if the utility profile of the interest group is nondecreasing in the alternatives.

<sup>26</sup>Note that when  $\mathbb{A}$  is equipped with a preorder, the possible multiplicity of minimal alternatives implies that the set of alternatives  $\mathbb{A}^o$  may contain redundant alternatives, as any two alternatives in  $a^o(u^o)$  have the same utility. (This is a consequence of Lemma 3 in the Appendix.) Selecting a(ny) single satisficing alternative within  $a^o(u^o)$  for each  $u^o \in \mathcal{U}^o$  is then sufficient to generate a set of alternatives yielding  $U(\mathbb{A}^o)$ , and thus  $\mathcal{E}(U(\mathbb{A}))$ .

to construct a computable, parameterized set of alternatives that yields all the efficient outcomes of the bargaining problem, and thus defines a reduced problem to which the bargaining solution can be applied. When  $u^o$  ranges over  $\mathcal{U}^o$ , the outcomes  $U(\mathbb{A}^o) = U(a^o(\mathcal{U}^o))$  include the Pareto frontier  $\mathcal{E}(U(\mathbb{A}))$  of the economic environment  $\xi$  (Proposition 5). As  $\mathcal{U}^o \subset \mathbb{R}^{|o|}$ , we obtain a parameterization of the Pareto frontier with a dimension at most  $|o| = H - |i|$ , the number of outsiders. The larger the interest group, the smaller the dimensions to explore to characterize the reduced environment  $\xi^o$ . The utility of the  $|i|$  agents in the interest group are co-monotonic, making it possible to represent them in the set of Pareto outcomes by a “single dimension.” Moreover, their utility is monotonically non-increasing with the satisficing utility  $u^o$  of outsiders, in the sense that  $u^o \leq \bar{u}^o \Rightarrow U^i(\bar{a}^*) \leq U^i(a^*) \quad \forall \bar{a}^* \in a^o(\bar{u}^o), \forall a^* \in a^o(u^o)$  (see eq. (26c) in Lemma 3 in the Appendix).

Figure 1 is a stylized illustration of how to generate this set, for  $H = 3$  agents,  $i = \{1, 2\}$ , and  $o = \{3\}$ . The Pareto-frontier is parameterized by the utility of agent 3. For any feasible payoff  $u^3$ , an associated satisficing alternative is used to determine the maximal utility of agents 1 and 2, defining one point of the Pareto-frontier of the UPS. This frontier is a curve (1-dimensional object) within the 3-dimensional space of utility, with the utility of agents 1 and 2 non-increasing when that of agent 3 increases. The figure illustrates the fact that the utility of both agent 1 and 2 is maximized by the same alternative, given a utility level for agent 3. Any other feasible alternative in  $\{a \in \mathbb{A} \mid U^o(a) \geq u^o\}$  would reduce the utility of both agents. The UPS is included in  $U(\mathbb{A}^o) - \mathbb{R}_+^H$ . It is not possible to increase the utility of agent 1 by decreasing that of agent 2 (and vice versa) along the Pareto-frontier, and there are no trade-offs between these agents (interest group). There are trade-offs only between the utility of these agents and that of agent 3 (outsider).

This kind of interest group is somehow similar to alliances, as described by Manzini and Mariotti (2005). Members of an alliance have the same preference ordering over the alternatives even if their utility functions differ. They can adopt a common position in bargain situations, based on different internal procedures to establish the alliance’s strategy. Manzini and Mariotti (2005) thus offer a framework to study the outcome of a bargaining problem over monotonic economic environments. In particular, an internal procedure based on unanimity makes the alliance stronger in the bargaining process.<sup>27</sup>

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<sup>27</sup>Note that outsiders could also form an alliance in the sense of Manzini and Mariotti (2005) if they rank the alternatives in the same order, e.g., opposite to the members of the interest group. In this case, the bargaining problem involves two alliances instead of an alliance and individual outsiders.

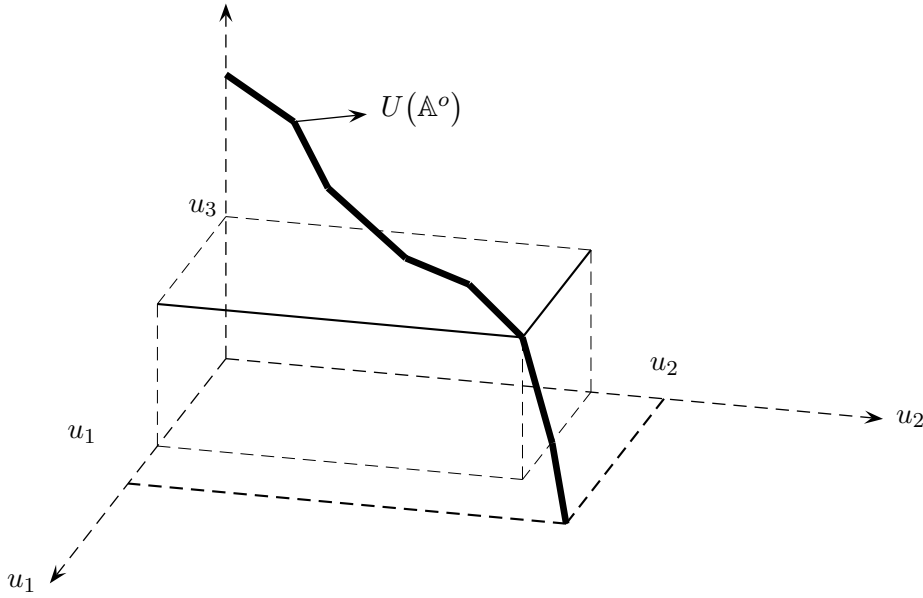


Figure 1: “One-dimensional” Pareto frontier, parameterized by  $u^o$ , with  $(u_1, u_2, u_3) = U(a^*)$  for any  $u^o \in \mathcal{U}^o \subset \mathbb{R}$  and  $a^* \in a(u^o) \subset \mathbb{A}$

### 4.3 Examples of monotonic economic environments

The above description of monotonic economic environments is a bit abstract. To emphasize that such monotonicity is quite current in economic models, we provide some examples, ranging from very simple static sharing problems to complex bargaining problem over the dynamic path of the economy.

In a problem of social choice regarding health, patients all benefit monotonically from health expenditures, in the MonEE sense that their utility does not decrease when health expenditures increase. The medical staff could also benefit from health expenditures, through better working conditions or the satisfaction of doing a better job in treating patients. As such, they can form an interest group in our framework. Other agents (or stakeholders representing other social issues), whose utility does not depend monotonically on health expenditures, would belong to outsiders.

In a companion paper,<sup>28</sup> we study dynamic bargaining problems with maximin intertemporal preferences, and provide conditions under which these problems satisfy the MonEE assumptions. Without entering into details, the following dynamic “climate change” problem is an example of monotonic economic environment. Consider an economy composed of two countries (to ease the description) in which the energy used in the production of goods and services generates local pollution affecting local health as well as global greenhouse gases accumulation. At the initial time, agents

<sup>28</sup>See the second part (from section 4) of the working paper Martinet et al. (2019).

have to jointly decide the world path of greenhouse gases emissions. In each country, a national agent representing economic objectives aims at sustaining national product (i.e., at maximizing the minimal level of production over time), while another agent aims at minimizing the maximum instantaneous level of local pollution (pollution peak), for health-related objectives specific to each country. Last, an international agent aims at minimizing the cumulative amounts of emissions to mitigate climate change. The last three stakeholders have heterogeneous preferences but all exhibit preferences that are nonincreasing in the (national and global) emission levels. As such, they constitute an interest group in our formalism, and could form an alliance aiming at reducing local and global pollutions. The Pareto frontier of the 5-dimensional UPS can be characterized by the utility level of the two outsiders (i.e., the sustained levels of national production).

## 5 Final remarks

We laid out a framework to discuss bargaining problems on economic environments, i.e., problems aiming at selecting alternatives among a set of feasible ones rather than directly allocating utility among agents. We showed that, when a solution satisfies the axioms of *Independence of Non-Efficient Alternatives* and *Independence of Redundant Alternatives*, the bargaining problem can be simplified without changing the outcome, by applying the solution to a reduced set of alternatives yielding the Pareto-efficient outcomes of the initial problem. Additionally, we showed that, for monotonic economic environments, such a reduced set of alternatives is computable. When there is an “interest group,” that is, a subset of agents who rank the alternatives in the same order, the payoffs of the interest group members is comonotonic and the Pareto frontier of the set of feasible payoffs is of a lower dimension than the number of agents. From a practical point of view, this simplifies the computation of the Pareto-efficient solutions.

Our results help characterizing the Pareto frontier of bargaining problems on economic environments. This frontier provides information on the necessary trade-offs among agents. For problems that do not have a countable (or an exhaustively describable) set of alternatives, the critical step of characterizing the Utility Possibility Set is a limit to the application of bargaining solutions defined through axioms. Our results are useful to characterize “Pareto-optimal” (pre)solutions (Thomson, 2001, p.353-354).

The question of the applicability of the solutions characterized in the axiomatic bargaining theory to concrete problems is not only a practical one pertaining to the field of investigation of Operations Research. It can also be an axiomatic one, as we discussed in this paper. We identified two economically-meaningful axioms under which a given bargaining solution can be applied to reduced bargaining problems. We also characterized monotonic problems for which such reduced

bargaining problems are computable. This means that a greater axiomatic characterization of bargaining solutions is useful to set directions of operations research and support the application of the theory to concrete problems. We studied particular monotonicity properties of economic environments that, when combined with specific axioms, can be used with benefits to simplify a bargaining problem. There are certainly other properties of economic environments of the sort that could be identified and matched with axioms, and allow for the simplification of bargaining problems and more frequent application of bargaining solutions.

## A Appendix: proofs of propositions

### A.1 Proof of Proposition 1

Before proving the proposition, we provide technical results on the characterization of efficient alternatives.

LEMMA 1

*Given an economic environment  $\xi = \langle \mathbb{A} \rangle$ , the set  $\mathbb{A}^*$  of efficient alternatives, as in Definition 5, is characterized as follows*

$$\begin{aligned} A \subset \mathbb{A} \text{ and } U(A) \subset \mathcal{E}(U(\mathbb{A})) \text{ and } U(\mathbb{A} \setminus A) \cap \mathcal{E}(U(\mathbb{A})) = \emptyset \\ \iff A = \mathbb{A}^* . \end{aligned} \tag{5}$$

*Furthermore, one has that*

$$A \subset \mathbb{A} \text{ and } \mathcal{E}(U(\mathbb{A})) \subset U(A) \Rightarrow \mathcal{E}(U(\mathbb{A})) \subset U(A \cap \mathbb{A}^*) . \tag{6}$$

*Finally, the following equalities hold true*

$$U(\mathbb{A}^*) = \mathcal{E}(U(\mathbb{A})) \text{ and } U(\mathbb{A} \setminus \mathbb{A}^*) = U(\mathbb{A}) \setminus \mathcal{E}(U(\mathbb{A})) . \tag{7}$$

#### **Proof of Lemma 1.**

Observe that if  $A \subset \mathbb{A}$  is such that  $U(A) \subset \mathcal{E}(U(\mathbb{A}))$ , then  $A \subset \mathbb{A}^*$  from Definition 5 of  $\mathbb{A}^*$ . On the other hand, if  $A \subset \mathbb{A}$  is such that  $U(\mathbb{A} \setminus A) \cap \mathcal{E}(U(\mathbb{A})) = \emptyset$ , one can directly check that  $\mathbb{A}^* \subset A$ . Thus, implication  $\Rightarrow$  in (5) is proven. The other implication  $\Leftarrow$  is straightforward.

For proving (6), let  $A \subset \mathbb{A}$  be such that  $\mathcal{E}(U(\mathbb{A})) \subset U(A)$ . Therefore, any element of  $\mathcal{E}(U(\mathbb{A}))$  can be written as  $U(a)$ , where  $a \in A$ . Now, by the definition of  $\mathbb{A}^*$  (Definition 5), we deduce that  $a \in \mathbb{A}^*$ . We conclude that  $a \in A \cap \mathbb{A}^*$  and that (6) holds true.

Equation (7) is made of two equalities. First, we prove that  $\mathcal{E}(U(\mathbb{A})) = U(\mathbb{A}^*)$  by two inclusions. On the one hand, by the definition of  $\mathbb{A}^*$  (Definition 5), we have that  $U(\mathbb{A}^*) \subset \mathcal{E}(U(\mathbb{A}))$ . On the other hand, the reverse inclusion  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}^*)$  is a consequence of (6) with  $A = \mathbb{A}$ . Now we prove that  $U(\mathbb{A} \setminus \mathbb{A}^*) = U(\mathbb{A}) \setminus \mathcal{E}(U(\mathbb{A}))$  by two inclusions. On the one hand, any element in  $U(\mathbb{A} \setminus \mathbb{A}^*)$  of course belongs to  $U(\mathbb{A})$ . If this element is also in  $\mathcal{E}(U(\mathbb{A}))$  we obtain a contradiction with (5), because in this case  $U(\mathbb{A} \setminus \mathbb{A}^*) \cap \mathcal{E}(U(\mathbb{A})) \neq \emptyset$ . Therefore, we obtain  $U(\mathbb{A} \setminus \mathbb{A}^*) \subset U(\mathbb{A}) \setminus \mathcal{E}(U(\mathbb{A}))$ . On the other hand, consider  $a \in \mathbb{A}$  such that  $U(a) \notin \mathcal{E}(U(\mathbb{A}))$ . By the definition of  $\mathbb{A}^*$  (Definition 5), we deduce that  $a \notin \mathbb{A}^*$ . Therefore,  $a \in \mathbb{A} \setminus \mathbb{A}^*$  and  $U(a) \in U(\mathbb{A} \setminus \mathbb{A}^*)$ . We conclude that  $U(\mathbb{A}) \setminus \mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A} \setminus \mathbb{A}^*)$ . Hence, we have obtained (7).

We now provide the **proof of Proposition 1**.

In the following, for two vectors  $u = (u_1, \dots, u_H)$  and  $u' = (u'_1, \dots, u'_H)$  in  $\mathbb{R}^H$ , we will use the distance  $\delta(u, u') = \sum_{h=1}^H |u_h - u'_h|$ .

The proof is in two steps. In the first part, we prove that, for every  $u \in U(\mathbb{A})$ , there exists  $a^* \in \mathbb{A}$  such that  $u^* = U(a^*) \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$  and

$$\delta(u, u^*) = \alpha_u = \sup_{u' \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)} \delta(u, u') . \quad (8)$$

For this purpose, we consider a sequence  $(u^k)_{k \in \mathbb{N}^*}$  in  $U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$  such that

$$\delta(u, u^k) \rightarrow \alpha_u \text{ as } k \rightarrow \infty . \quad (9)$$

Since  $u^k \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$ , then, for each  $k \in \mathbb{N}^*$ , there exist  $a^k \in \mathbb{A}$  and  $v^k = (v_1^k, \dots, v_H^k) \in \mathbb{R}_+^H$  such that

$$u^k = u + v^k = U(a^k) = (U_1(a^k), \dots, U_H(a^k)) . \quad (10)$$

As the set  $\mathbb{A}$  is metric compact, there exist  $a^* \in \mathbb{A}$  and a subsequence  $(a^{k_j})_{j \in \mathbb{N}^*}$  such that

$$a^{k_j} \rightarrow a^* \text{ as } j \rightarrow \infty . \quad (11)$$

We put

$$u^* = (u_1^*, \dots, u_H^*) \text{ where } u_h^* = U_h(a^*) , \quad \forall h = 1, \dots, H , \quad (12)$$

and we now prove that  $u^* \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$  and solves (8). Observe that, by (12), we have that  $u^* \in U(\mathbb{A})$ . We also have that  $u^* \in (u + \mathbb{R}_+^H)$ . Indeed, from (11) and the upper semicontinuity of the functions  $U_h : \mathbb{A} \rightarrow \mathbb{R}$ , for  $h = 1, \dots, H$ , we obtain that

$$\limsup_{j \rightarrow \infty} U_h(a^{k_j}) \leq U_h(a^*) = u_h^* , \quad \forall h = 1, \dots, H . \quad (13)$$



Therefore, by (10) where  $v^k = (v_1^k, \dots, v_H^k) \in \mathbb{R}_+^H$ , we deduce that

$$0 \leq \limsup_{j \rightarrow \infty} v_h^{k_j} = \limsup_{j \rightarrow \infty} U_h(a^{k_j}) - u_h \leq u_h^* - u_h, \quad (14)$$

for all  $h = 1, \dots, H$ , that is,  $u^* \in (u + \mathbb{R}_+^H)$ .

$$\begin{aligned} \alpha_u &= \lim_{j \rightarrow \infty} \delta(u, u^{k_j}) \text{ by (9)} \\ &= \limsup_{j \rightarrow \infty} \delta(u, u^{k_j}) = \limsup_{j \rightarrow \infty} \sum_{h=1}^H |u_h - u_h^{k_j}| = \limsup_{j \rightarrow \infty} \sum_{h=1}^H |v_h^{k_j}| \text{ by (10)} \\ &= \limsup_{j \rightarrow \infty} \sum_{h=1}^H v_h^{k_j} \text{ because } v^k = (v_1^k, \dots, v_H^k) \in \mathbb{R}_+^H \\ &\leq \sum_{h=1}^H \limsup_{j \rightarrow \infty} v_h^{k_j} \leq \sum_{h=1}^H (u_h^* - u_h) \text{ by (14)} \\ &= \delta(u, u^*). \end{aligned}$$

Thus, we obtain that  $\delta(u, u^*) \geq \alpha_u = \sup_{u' \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)} \delta(u, u')$ . Now, as  $u^* = U(a^*) \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$  by (13), we conclude that (8) holds true.

In the second part, we prove that, for any  $u \in U(\mathbb{A})$  and  $u^* \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$  such that  $\alpha_u = \delta(u, u^*)$  ( $\alpha_u$  defined in (8)), then  $u^* \in \mathcal{E}(U(\mathbb{A}))$ .

The proof is obtained by contradiction. Suppose  $u^* \notin \mathcal{E}(U(\mathbb{A}))$ . Then, there exists  $\tilde{u} \in U(\mathbb{A}) \cap (u^* + \mathbb{R}_+^H \setminus \{0\})$ . Expressing that  $u^* \in (u + \mathbb{R}_+^H)$  and that  $\tilde{u} \in U(\mathbb{A}) \cap (u^* + \mathbb{R}_+^H \setminus \{0\})$ , we get that  $u^* = u + v$  and  $\tilde{u} = u^* + w$  for some  $v = (v_1, \dots, v_H) \in \mathbb{R}_+^H$  and  $w = (w_1, \dots, w_H) \in \mathbb{R}_+^H \setminus \{0\}$  (i.e.,  $w_h > 0$  for some  $h \in \{1, \dots, H\}$ ). We easily deduce that  $\tilde{u} \in U(\mathbb{A}) \cap (u + \mathbb{R}_+^H)$ . Now, we have that  $\alpha_u = \delta(u, u^*) = \sum_{h=1}^H v_h < \sum_{h=1}^H (v_h + w_h) = \delta(u, \tilde{u})$ , which contradicts the definition of  $\alpha_u$  (eq. 8).

We have shown that, for every  $u \in U(\mathbb{A})$ , there exists  $a^* \in \mathbb{A}$  such that  $U(a^*) = u^* \in \mathcal{E}(U(\mathbb{A})) \cap (u + \mathbb{R}_+^H)$ . As a consequence, the set  $\mathbb{A}^* = \{a \in \mathbb{A} \mid U(a) \in \mathcal{E}(U(\mathbb{A}))\}$  is not empty.

## A.2 Proof of Proposition 2

The proof of Proposition 2 will be based on the following results that characterize the efficient alternatives of reduced economic environments.

LEMMA 2

Let  $(\xi', \xi)$  be a couple of economic environments such that

$$\mathbb{A}' \subset \mathbb{A} \tag{16}$$

$$\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}'). \tag{17}$$

Suppose that the set  $\mathbb{A}$  of alternatives is a compact metric space and the utility profile  $U : \mathbb{A} \rightarrow \mathbb{R}^H$  is upper semicontinuous. Then, we have that  $\mathbb{A}'^* \subset \mathbb{A}^*$  and  $U(\mathbb{A}'^*) = U(\mathbb{A}^*)$ .

**Proof of Lemma 2.**

Consider a couple  $(\xi', \xi)$  of economic environments such that  $\mathbb{A}' \subset \mathbb{A}$  and  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}')$ . First, we prove that

$$U(\mathbb{A}' \cap \mathbb{A}^*) = U(\mathbb{A}^*). \tag{18}$$

From (7), we have that  $\mathcal{E}(U(\mathbb{A})) = U(\mathbb{A}^*)$ . As, by assumption (17),  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}')$ , we deduce from (6) that  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}' \cap \mathbb{A}^*)$ . Linking these two results, we obtain  $U(\mathbb{A}^*) = \mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}' \cap \mathbb{A}^*) \subset U(\mathbb{A}^*)$ , and we conclude that all terms are equal, so that  $U(\mathbb{A}' \cap \mathbb{A}^*) = U(\mathbb{A}^*)$ . Second, we prove that

$$U(\mathbb{A}^*) \subset U(\mathbb{A}'^*). \tag{19}$$

For this purpose, we will use the following property, which follows from the definition of Pareto sets:

$$\bar{U} \subset \mathcal{U} \subset \mathbb{R}^H \Rightarrow \mathcal{E}(\mathcal{U}) \cap \bar{U} \subset \mathcal{E}(\bar{U}). \tag{20}$$

We have

$$\begin{aligned} U(\mathbb{A}^*) &= U(\mathbb{A}' \cap \mathbb{A}^*) \text{ by (18)} \\ &\subset U(\mathbb{A}') \cap U(\mathbb{A}^*) = U(\mathbb{A}') \cap \mathcal{E}(U(\mathbb{A})) \text{ by (7)} \\ &\subset \mathcal{E}(U(\mathbb{A}')) \text{ by (20)} \end{aligned}$$

since  $\mathbb{A}' \subset \mathbb{A}$ , by assumption (16), hence  $U(\mathbb{A}') \subset U(\mathbb{A})$ . Therefore, we conclude that  $U(\mathbb{A}^*) \subset \mathcal{E}(U(\mathbb{A}')) = U(\mathbb{A}'^*)$  by (7).

Third, for concluding the proof, we need to show the inclusions  $\mathbb{A}'^* \subset \mathbb{A}^*$  and  $U(\mathbb{A}'^*) \subset U(\mathbb{A}^*)$ . For this purpose, it is sufficient to prove that  $\mathbb{A}'^* = \mathbb{A}' \cap \mathbb{A}^*$ , an equality that we will deduce using the equivalence (5), that requires three conditions. We establish that these three conditions hold true.

1. We establish that  $\mathbb{A}' \cap \mathbb{A}^* \subset \mathbb{A}'$ . This is obvious.

2. We establish that  $U(\mathbb{A}' \cap \mathbb{A}^*) \subset \mathcal{E}(U(\mathbb{A}'))$ . Indeed,

$$\begin{aligned} U(\mathbb{A}' \cap \mathbb{A}^*) &= U(\mathbb{A}^*) \text{ by (18)} \\ &\subset U(\mathbb{A}'^*) \text{ by (19)} \\ &= \mathcal{E}(U(\mathbb{A}')) \text{ by (7)}. \end{aligned}$$

3. We establish that  $U(\mathbb{A}' \setminus (\mathbb{A}' \cap \mathbb{A}^*)) \cap \mathcal{E}(U(\mathbb{A}')) = \emptyset$ . This is where the assumptions that the set  $\mathbb{A}$  of alternatives is compact and that the utility profile  $U : \mathbb{A} \rightarrow \mathbb{R}^H$  is upper semicontinuous play a role, as they make it possible to use Proposition 1. Consider an alternative  $a \in \mathbb{A}' \setminus (\mathbb{A}' \cap \mathbb{A}^*)$ . By Proposition 1, there exists an alternative  $\tilde{a} \in \mathbb{A}$  such that  $U(\tilde{a}) > U(a)$ , with  $U(\tilde{a}) \in \mathcal{E}(U(\mathbb{A}))$ . As  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}')$ , we deduce that there exists an alternative  $\bar{a} \in \mathbb{A}'$  such that  $U(\bar{a}) = U(\tilde{a}) > U(a)$ . As a consequence,  $a \notin \mathcal{E}(U(\mathbb{A}'))$ .

We have thus proven that  $U(\mathbb{A}' \setminus (\mathbb{A}' \cap \mathbb{A}^*)) \cap \mathcal{E}(U(\mathbb{A}')) = \emptyset$ .

We now provide the **proof of Proposition 2**.

Consider a couple  $(\xi, \xi')$  of economic environments such that  $\mathbb{A}' \subset \mathbb{A}$  and  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}')$ . We use the corresponding efficient reduced environments  $\xi^* = \langle \mathbb{A}^* \rangle$  and  $\xi'^* = \langle \mathbb{A}'^* \rangle$  (Definition 5). The assumptions of Lemma 2 are satisfied, as  $\mathbb{A}' \subset \mathbb{A}$  and  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}')$  correspond to equations (16) and (17). From Lemma 2, we obtain that  $\mathbb{A}'^* \subset \mathbb{A}^*$  and  $U(\mathbb{A}'^*) = U(\mathbb{A}^*)$ . As the solution  $\mu$  is IRA, we deduce from Axiom 3 that

$$U(\mu(\xi^*)) = U(\mu(\xi'^*)) . \quad (23)$$

Since the solution  $\mu$  is INEA and  $\mathbb{A}^* \subset \mathbb{A}$ , we deduce from Axiom 2 that

$$\mu(\xi^*) = \mu(\xi) . \quad (24)$$

Repeating the argument with the environment  $\xi'^*$ , we obtain that

$$\mu(\xi'^*) = \mu(\xi') . \quad (25)$$

Therefore, from (23), (24), and (25), we conclude that  $U(\mu(\xi)) = U(\mu(\xi^*)) = U(\mu(\xi'^*)) = U(\mu(\xi'))$ .

### A.3 Proof of Proposition 5

We need a preliminary Lemma.

LEMMA 3

Consider a given  $(i, o)$ -monotonic economic environment  $\xi = \langle \mathbb{A} \rangle$ , as in Definition 6. Then, for all  $(u^o, \bar{u}^o) \in \mathcal{U}^o \times \mathcal{U}^o$  and all  $\bar{a} \in \mathbb{A}$ , we have

$$u^o \leq U^o(\bar{a}) \Rightarrow U^i(\bar{a}) \leq U^i(a^*) \quad \forall a^* \in a^o(u^o), \quad (26a)$$

$$\bar{a}^* \in a^o(\bar{u}^o), u^o \leq U^o(\bar{a}^*) \Rightarrow U^i(\bar{a}^*) \leq U^i(a^*) \quad \forall a^* \in a^o(u^o), \quad (26b)$$

$$u^o \leq \bar{u}^o \Rightarrow U^i(\bar{a}^*) \leq U^i(a^*) \quad \forall \bar{a}^* \in a^o(\bar{u}^o), \forall a^* \in a^o(u^o). \quad (26c)$$

**Proof of Lemma 3** Let  $(u^o, \bar{u}^o) \in \mathcal{U}^o \times \mathcal{U}^o$  and  $\bar{a} \in \mathbb{A}$ .

1. Suppose that  $u^o \leq U^o(\bar{a})$ . By eq. (3) in Definition 7, defining the set  $a^o(u^o)$  composed by minima of the set  $\{a \in \mathbb{A} \mid U^o(a) \geq u^o\}$  (see (1) in Definition 6), we deduce that  $a^* \preceq \bar{a}$  for all  $a^* \in a^o(u^o)$ . Now, the mapping  $U^i : \mathbb{A} \rightarrow \mathbb{R}^{|i|}$  is nonincreasing by item I in the Definition 6 of MonEE. We deduce that  $U^i(\bar{a}) \leq U^i(a^*)$ , for all  $a^* \in a^o(u^o)$ . Hence, (26a) is proven.
2. Use (26a) with  $\bar{a} = \bar{a}^* \in a^o(\bar{u}^o)$  to derive (26b).
3. Let  $u^o \leq \bar{u}^o$ . As  $\bar{u}^o \in \mathcal{U}^o$ , by definition of the set  $\mathcal{U}^o$  of feasible payoffs for outsiders (eq. 2), and by eq. (3), for  $\bar{a}^* \in a^o(\bar{u}^o)$  we have that  $\bar{u}^o \leq U^o(\bar{a}^*)$ . Therefore, we get that  $u^o \leq U^o(\bar{a}^*)$  and we use (26b) to obtain (26c).

**Proof of Proposition 5**

First, we prove that

$$(u^i, u^o) \in U(\mathbb{A}) \Rightarrow u^o \in \mathcal{U}^o, \quad u^i \leq U^i(a^*), \quad u^o \leq U^o(a^*) \quad \forall a^* \in a^o(u^o). \quad (27)$$

By definition of  $U(\mathbb{A})$ , there exists an alternative  $\bar{a} \in \mathbb{A}$ , such that  $(u^i, u^o) = u = U(\bar{a}) = (U^i(\bar{a}), U^o(\bar{a}))$ . On the one hand, as  $u^o = U^o(\bar{a})$ , we deduce from item II in the Definition 6 of MonEE (eq. 2) that  $u^o \in \mathcal{U}^o$ , and that  $u^i = U^i(\bar{a}) \leq U^i(a^*)$  for all  $a^* \in a^o(u^o)$  by (26a). On the other hand, by definition of the set  $\mathcal{U}^o$  of feasible payoffs for outsiders (eq. 2), and by eq. (3), we obtain that  $u^o \in \mathcal{U}^o \Rightarrow u^o \leq U^o(a^*)$  for all  $a^* \in a^o(u^o)$ .

Second, we prove that  $\mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A}^o)$ . Let  $u \in \mathcal{E}(U(\mathbb{A}))$ . By the definition of a Pareto set, we have that  $u \in \mathcal{E}(U(\mathbb{A})) \subset U(\mathbb{A})$ . By (27), we obtain that  $u^i \leq U^i(a^*)$  and that  $u^o \leq U^o(a^*)$  for all  $a^* \in a^o(u^o)$ . Putting  $\bar{u} = (U^i(a^*), U^o(a^*))$  for some (fixed)  $a^* \in a^o(u^o)$ , we deduce that  $u \leq \bar{u}$ . As  $\bar{u} \in U(\mathbb{A})$  and  $u \in \mathcal{E}(U(\mathbb{A}))$ , we obtain that  $\bar{u} = u$ , by definition of a (strong) Pareto set. By definition of  $\mathbb{A}^o$  (eq. 4), we conclude that  $u = \bar{u} = (U^i(a^*), U^o(a^*)) \in U(\mathbb{A}^o)$ .

Third, we prove that  $U(\mathbb{A}^o) \subset \mathcal{E}^w(U(\mathbb{A}))$ . The proof is obtained by contradiction. Let  $u \in U(\mathbb{A}^o)$  and suppose that there exists  $\bar{u} \in U(\mathbb{A})$  such that  $u < \bar{u}$  (i.e, a strict inequality component

by component). On the one hand, by definition of  $\mathbb{A}^o$  (eq. 4), there exists  $\hat{u}^o \in \mathcal{U}^o$  such that  $u = U(\hat{a})$  for some  $\hat{a} \in a^o(\hat{u}^o)$ . On the other hand, we have that  $\bar{u} \leq U(\hat{a})$ , by (27). We deduce that

$$U(\hat{a}) = u < \bar{u} \leq U(\bar{a}^*) \quad \forall \bar{a}^* \in a^o(\bar{u}^o). \quad (28)$$

As already seen, we have that  $\hat{u}^o \leq U^o(\hat{a})$ . Together with inequality (28), this yields  $\hat{u}^o < U^o(\bar{a}^*)$  for all  $\bar{a}^* \in a^o(\bar{u}^o)$ . Now fix  $\bar{a}^* \in a^o(\bar{u}^o)$ . By (26b), we deduce that  $U^i(\bar{a}^*) \leq U^i(\hat{a})$ . Now, using inequality (28), we get that  $\bar{u}^i \leq U^i(\bar{a}^*) \leq U^i(\hat{a}) = u^i$ . However, this contradicts  $u < \bar{u}$ . Therefore, no such  $\bar{u}$  exists and we conclude that  $u \in \mathcal{E}^w(U(\mathbb{A}))$ .

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