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Bijjective proofs for Eulerian numbers of types B and D*

Dedicated to Maurice Pouzet on the occasion of his 75th birthday

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Let $\langle \binom{n}{k} \rangle$, $\langle \binom{B_n}{k} \rangle$, and $\langle \binom{D_n}{k} \rangle$ be the Eulerian numbers of the types A, B, and D, respectively—that is, the number of permutations of n elements with k descents, the number of signed permutations (of n elements) with k type B descents, the number of even-signed permutations (of n elements) with k type D descents. Let $S_n(t) = \sum_{k=0}^{n-1} \langle \binom{n}{k} \rangle t^k$, $B_n(t) = \sum_{k=0}^n \langle \binom{B_n}{k} \rangle t^k$, and $D_n(t) = \sum_{k=0}^n \langle \binom{D_n}{k} \rangle t^k$. We give bijective proofs of the identity

$$B_n(t^2) = (1+t)^{n+1} S_n(t) - 2^n t S_n(t^2)$$

and of Stembridge's identity

$$D_n(t) = B_n(t) - n 2^{n-1} t S_{n-1}(t).$$

These bijective proofs rely on a representation of signed permutations as paths. Using this representation we also establish a bijective correspondence between even-signed permutations and pairs (w, E) with $([n], E)$ a threshold graph and w a degree ordering of $([n], E)$, which we use to obtain bijective proofs of enumerative results for threshold graphs.

Keywords: signed permutation, weak order, threshold graph

1 Introduction

The Eulerian numbers $\langle \binom{n}{k} \rangle$ count the number of permutations in the symmetric group S_n that have k descent positions. Let us recall that, for a permutation $w = w_1 w_2 \cdots w_n \in S_n$ (thus, with $w_i \in \{1, \dots, n\}$ and $w_i \neq w_j$ for $i \neq j$), a *descent* of w is an index (or position) $i \in \{1, \dots, n-1\}$ such that $w_i > w_{i+1}$.

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This is only one of the many interpretations that we can give to these numbers, see e.g. (Petersen, 2015), yet it is intimately order-theoretic. The set S_n can be endowed with a lattice structure, known as the weak (Bruhat) ordering on permutations or Permutohedron, see e.g. (Guilbaud and Rosenstiehl, 1963; Caspard et al., 2016). Descent positions of $w \in S_n$ are then bijection with its lower covers, so the Eulerian numbers $\langle n \rangle_k$ can also be taken as counting the number of permutations in S_n with k lower covers. In particular, $\langle n \rangle_1 = 2^n - n - 1$ is the number of join-irreducible elements in S_n . A subtler order-theoretic interpretation is given in (Barnard, 2016): since the S_n are (join-)semidistributive as lattices, every element can be written canonically as the join of join-irreducible elements (Freese et al., 1995); the numbers $\langle n \rangle_k$ count then the permutations $w \in S_n$ that can be written canonically as the join of k join-irreducible elements.

The symmetric group S_n is a particular instance of a Coxeter group, see (Björner and Brenti, 2005), since it yields a concrete realization of the Coxeter group A_{n-1} in the family A. Notions of length, descent, inversion, and also a weak order, can be defined for elements of an arbitrary finite Coxeter group (Björner, 1984). We shift the focus to the families B and D of Coxeter groups. More precisely, this paper concerns the Eulerian numbers in the types B and D. The Eulerian number $\langle B_n \rangle_k$ (resp., $\langle D_n \rangle_k$) counts the number of elements in the group B_n (resp., D_n) with k descent positions. Order-theoretic interpretations of these numbers, analogous to the ones mentioned for the standard Eulerian numbers, are still valid. As the abstract group A_{n-1} has a concrete realization as the symmetric group S_n , the group B_n (resp., D_n) has a realization as the hyperoctahedral group of signed permutations (resp., the group of even-signed permutations). Starting from these concrete representations of Coxeter groups of type B and D, we pinpoint some new representations of signed permutations. Relying on these representations we provide bijective proofs of known formulas for Eulerian numbers of the types B and D. These formulas allow us to compute the Eulerian numbers of the types B and D from the better-known Eulerian numbers of type A.

Let $S_n(t)$ and $B_n(t)$ be the Eulerian polynomials of the types A and B:

$$S_n(t) := \sum_{k=0}^{n-1} \langle n \rangle_k t^k, \quad B_n(t) := \sum_{k=0}^n \langle B_n \rangle_k t^k. \quad (1)$$

In (Petersen, 2015, §13, p. 215) the following polynomial identity is stated:

$$2B_n(t^2) = (1+t)^{n+1}S_n(t) + (1-t)^{n+1}S_n(-t). \quad (2)$$

Considering that, for $f(t) = \sum_{k \geq 0} a_k t^k$,

$$f(t) + f(-t) = 2 \sum_{k \geq 0} a_{2k} t^{2k},$$

the polynomial identity (2) amounts to the following identity among coefficients:

$$\langle B_n \rangle_k = \sum_{i=0}^{2k} \langle n \rangle_i \binom{n+1}{2k-i}. \quad (3)$$

We present a bijective proof of (3) and also establish the identity

$$2^n \left\langle n \atop k \right\rangle = \sum_{i=0}^{2k+1} \left\langle n \atop i \right\rangle \binom{n+1}{2k+1-i}. \quad (4)$$

Considering that, for $f(t) = \sum_{k \geq 0} a_k t^k$,

$$f(t) - f(-t) = 2 \sum_{k \geq 0} a_{2k+1} t^{2k+1},$$

the identity (4) yields the polynomial identity:

$$2^{n+1} t S_n(t^2) = (1+t)^{n+1} S_n(t) - (1-t)^{n+1} S_n(-t).$$

More importantly, (3) and (4) jointly yield the polynomial identity

$$(1+t)^{n+1} S_n(t) = B_n(t^2) + 2^n t S_n(t^2). \quad (5)$$

Let now $D_n(t)$ be the Eulerian polynomial of type D:

$$D_n(t) := \sum_{k=0}^n \left\langle D_n \atop k \right\rangle t^k.$$

Investigating further the terms $2^n S_n(t)$, we could find a simple bijective proof, that we present here, of Stembridge's identity (Stembridge, 1994, Lemma 9.1)

$$D_n(t) = B_n(t) - n 2^{n-1} t S_{n-1}(t), \quad (6)$$

which, in terms of the Eulerian numbers of type D, amounts to

$$\left\langle D_n \atop k \right\rangle = \left\langle B_n \atop k \right\rangle - n 2^{n-1} \left\langle n-1 \atop k-1 \right\rangle.$$

The proofs presented here differ from known proofs of the identities (2) and (6). As suggested in (Petersen, 2015), the first identity may be derived by computing the f -vector of the type B Coxeter complex and then by applying the transform yielding the h -vector from the f -vector. A similar method is used in (Stembridge, 1994) to prove the identity (6). Our proofs directly rely on the combinatorial properties of signed/even-signed permutations and on two representations of signed permutations that we call one the *path representation* and, the other, the *simply barred permutation* representation. The idea is that a signed permutation of $[n]$ can be organised into a discrete path from $(0, n)$ to $(n, 0)$ that only uses East and South steps and that, by projecting onto the x -axis, we obtain a permutation divided into blocks, as suggested in Figure 1. As a byproduct of these representations, we also obtain a bijection between even-signed permutations of $[n]$ and pairs (w, E) where $([n], E)$ is a threshold graph and w is a permutation or, better, a linear ordering of $[n]$ that is a degree ordering for $([n], E)$. Under the bijection, the ordering of D_n is coordinatewise, that is, we have $(w_1, E_1) \leq (w_2, E_2)$ if and only if $w_1 \leq w_2$ in S_n and $E_1 \subseteq E_2$. It is in the scope of future research to shed some light on the lattice structure of the weak order on the Coxeter

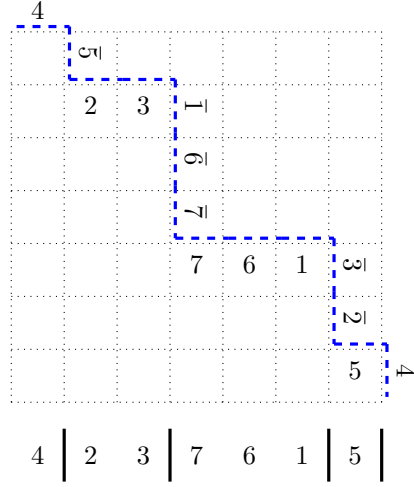


Fig. 1: Signed permutations as paths and as barred permutations

groups of type D using this representation of the ordering. Our confidence that this is indeed possible relies on our progress studying these lattices, which yielded the discovery of the path representation of signed permutations. In the meantime, the two representations of signed permutations, together with the bijection between even-signed permutations and pairs (w, E) as mentioned above, also yield a representation of threshold graphs as specific simply barred permutations. We exemplify once more the convenience of these representations by deducing enumerative results for threshold graphs (Simpson Beissinger and Peled, 1987; Spiro, 2020).

2 Background

The notation used is chosen to be consistent with (Petersen, 2015). We use $[n]$ for the set $\{1, \dots, n\}$ and S_n for the set of permutations of $[n]$. We use $[n]_0$ for the set $\{0, 1, \dots, n\}$, $[-n]$ for $\{-n, \dots, -1\}$, and $[\pm n]$ for $\{-n, \dots, -1, 1, \dots, n\}$. We write a permutation $w \in S_n$ as a word $w = w_1 w_2 \cdots w_n$, with $w_i \in [n]$. For $w \in S_n$, its set of descents and its set of inversions⁽ⁱ⁾ are defined as follows:

$$\begin{aligned} \text{Des}(w) &:= \{i \in \{1, \dots, n-1\} \mid w_i > w_{i+1}\}, \\ \text{Inv}(w) &:= \{(i, j) \mid 1 \leq i < j \leq n, w^{-1}(i) > w^{-1}(j)\}. \end{aligned} \quad (7)$$

Then, we let

$$\text{des}(w) := |\text{Des}(w)|. \quad (8)$$

Notice that a descent i of a permutation $w_1 w_2 \cdots w_n$ is uniquely identified by the two contiguous letters $w_i w_{i+1}$ such that $w_i > w_{i+1}$. Therefore, we shall often identify such a descent by these two contiguous

⁽ⁱ⁾ It is also possible to define $\text{Inv}(w)$ as the set $\{(i, j) \mid 1 \leq i < j \leq n, w_i > w_j\}$. The definition given above is better suited for the order-theoretic approach.

letters. The Eulerian number $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$, counting the number of permutations of n elements with k descents, can be formally defined as follows:

$$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle := |\{ w \in S_n \mid \text{des}(w) = k \}|.$$

Let us define a *signed permutation* of $[n]$ as a permutation u of $[\pm n]$ such that, for each $i \in [\pm n]$, $u_{-i} = -u_i$. We use B_n for the set of signed permutations of $[n]$. When writing a signed permutation u as a word $u_{-n} \cdots u_{-1} u_1 \cdots u_n$, we prefer writing $u_i = \bar{x}$ in place of $-x$ if $u_i < 0$ and $|u_i| = x$. Also, we often write $u \in B_n$ in *window notation*, that is, we only write the suffix $u_1 u_2 \cdots u_n$; indeed, the prefix $u_{-n} u_{-n-1} \cdots u_{-1}$ is determined as the mirror of the suffix $u_1 u_2 \cdots u_n$ up to exchanging the signs. The set B_n is a subgroup of the group of permutations of the set $[\pm n]$ and, as mentioned before, it is the standard model for the Coxeter group in the family B with n generators. General notions from the theory of Coxeter groups (descent, inversion) apply to signed permutations.

The Cayley graph of a Coxeter group (which, by definition, comes with a set of generators) can always be oriented by increasing length, where the length of an element is defined as the number of its inversions. The oriented graph is then the Hasse diagram of a poset where descents of an element mark its lower covers. While for permutations the notions of descent, inversion, and length are customary from elementary combinatorics, for signed permutations these notions are subtler yet well studied, we refer to standard monographs such as (Björner and Brenti, 2005; Petersen, 2015). We present below, as definitions, the well-known explicit formulas for the descent and inversion sets of $u \in B_n$. We let

$$\begin{aligned} \text{Des}_B(u) &:= \{ i \in \{0, \dots, n-1\} \mid u_i > u_{i+1} \}, \\ \text{Inv}_B(u) &:= \{ (i, j) \mid 1 \leq |i| \leq j \leq n, u^{-1}(i) > u^{-1}(j) \}, \end{aligned} \quad (9)$$

where we set $u_0 := 0$, so 0 is a descent of u if and only if $u_1 < 0$,

$$\begin{aligned} \text{des}_B(u) &= |\text{Des}_B(u)|, \\ \left\langle \begin{smallmatrix} B_n \\ k \end{smallmatrix} \right\rangle &:= |\{ u \in B_n \mid \text{des}_B(u) = k \}|. \end{aligned} \quad (10)$$

The definition of the Eulerian polynomials in the types A and B appears in (1). Let us mention that the type A Eulerian polynomial is often (for example in (Bóna, 2012)) defined as follows:

$$A_n(t) := \sum_{k=1}^n \left\langle \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\rangle t^k = t S_n(t).$$

We exclusively manipulate the polynomials $S_n(t)$ and never the $A_n(t)$. Notice that $S_n(t)$ has degree $n-1$ and $B_n(t)$ has degree n .

We shall introduce later even-signed permutations and their groups, as well as related notions arising from the fact that these groups are standard models for Coxeter groups in the family D.

For $u \in B_n$ we let $\text{Des}_B^+(u) := \text{Des}_B(u) \setminus \{0\}$, that is, $\text{Des}_B^+(u)$ is the set of strictly positive descents of u . Let us observe the following:

Lemma 2.1. $|\{u \in \mathbb{B}_n \mid |\text{Des}_{\mathbb{B}}^+(u)| = k\}| = 2^n \binom{n}{k}$.

Proof: Recall that the window notation of a signed permutation u is the word $u_1 \cdots u_n$. We identify the window notation of u with the mapping $\tilde{u} : [n] \rightarrow [\pm n]$ such that $\tilde{u}(i) = u_i$, for each $i \in [n]$.

We claim that maps arising as window notation of a signed permutation are in bijection with pairs (w, ι) where $w \in \mathbb{S}_n$ and $\iota : [n] \rightarrow [\pm n]$ is an order preserving injection such that $x \in \iota([n])$ iff $-x \notin \iota([n])$. The bijection goes as follows. Given a signed permutation u , the image $\tilde{u}([n]) \subseteq [\pm n]$ of its window notation is a subset of integers of cardinality n with the linear ordering inherited from integers. Thus, there exists a unique order preserving bijection $\psi : \tilde{u}([n]) \rightarrow [n]$. We associate to \tilde{u} the pair (w, ι) where $w = \psi \circ \tilde{u}$ and where ι is the composition of $\psi^{-1} : [n] \rightarrow \tilde{u}([n])$ and the inclusion $\tilde{u}([n]) \subseteq [\pm n]$. Notice that $\tilde{u} = \iota \circ w$, from which it follows that $\tilde{u}([n]) = \iota([n])$ and that $x \in \iota([n])$ iff $-x \notin \iota([n])$. Let us argue that this decomposition is unique. Notice that $\tilde{u}([n]) = \iota([n])$ uniquely determining ι . Since moreover ι is injective, if $\iota \circ w = \tilde{u} = \iota \circ w'$, then $w = w'$ as well. Also, given such a pair (w, ι) , $\iota \circ w$ is a preimage of (w, ι) via the correspondence, which is therefore surjective and a bijection.

Next, the order preserving injections $\iota : [n] \rightarrow [\pm n]$ such that $x \in \iota([n])$ iff $-x \notin \iota([n])$ are uniquely determined by their positive image $\iota([n]) \cap [n]$, so there are 2^n such mappings. Finally, let us argue that, for $i = 1, \dots, n-1$, $u_i > u_{i+1}$ if and only if $w_i > w_{i+1}$: $u_i > u_{i+1}$ iff $\tilde{u}(i) > \tilde{u}(i+1)$ iff $\iota(w(i)) > \iota(w(i+1))$ iff $w_i > w_{i+1}$, since ι preserves and reflects the ordering. It follows that $\text{Des}_{\mathbb{B}}^+(u) = \text{Des}(w)$, thus yielding the statement of the lemma. \square

Example 2.2. Consider the signed permutation $u := 3\bar{4}1\bar{2}\bar{5}$. Then $\tilde{u} = \iota \circ w$ with $w = 52431$ and ι the order preserving map $\bar{5}\bar{4}\bar{2}13$ with $\iota([n]) \cap [n] = \{1, 3\}$. \diamond

Let us end this section with a notational remark. We can index cells or points of a grid such as the one in Figure 1 in two different ways. Either we consider them as being part of a matrix and index them by row (we count here rows from the bottom to the top) and column, thus using the letters i, j . Or we can index them using the abscissa and ordinate of the two-dimensional plane. We shall prefer the latter method when the axes are ordered in the standard way, and the first method when the axes are ordered according to a permutation. For example, in Figure 1, the dashed path makes an East-South turn at point (x, y) with $x = 3$ and $y = 6$. However, if we relabel the axes by means of the permutation $w = 4237615$, then it makes sense to say that the path makes an East-South turn at row $i = w(6) = 1$ and column $j = w(3) = 3$. The use of both kind of indexing shall be unavoidable. The reader should be aware that rows correspond to the ordinate and columns to the abscissa, so a point (x, y) yields the cell $M_{w(y), w(x)}$ of a matrix M and that a cell $M_{i,j}$ is located at point $(w^{-1}(j), w^{-1}(i))$.

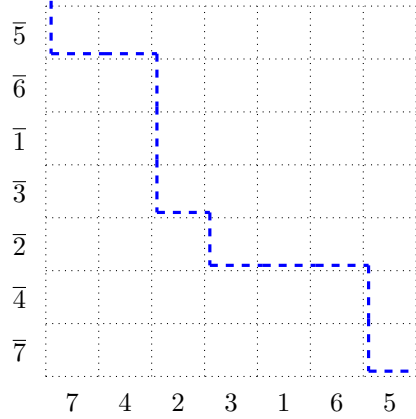
3 Path representation of signed permutations

We present here our main combinatorial tool to deal with signed permutations, the path representation.

Definition 3.1. The *path representation* of $u \in \mathbb{B}_n$ is a triple $(\pi^u, \lambda_x^u, \lambda_y^u)$ where π^u is a discrete path, drawn on a grid $[n]_0 \times [n]_0$ and joining the point $(0, n)$ to the point $(n, 0)$, $\lambda_x^u : [n] \rightarrow [n]$, and $\lambda_y^u : [n] \rightarrow [-n]$. The triple $(\pi^u, \lambda_x^u, \lambda_y^u)$ is constructed from u according to the following algorithm: (i) u is written in full notation as a word and scanned from left to right: each positive letter yields an East step (a length 1 step along the x -axis towards the right), and each negative letter yields a South step (a length 1 step along the y -axis towards the bottom); (ii) the labelling $\lambda_x^u : [n] \rightarrow [n]$ is obtained by

projecting each positive letter on the x -axis, (iii) the labelling $\lambda_y^u : [n] \rightarrow [-n]$ is obtained by projecting each negative letter on the y -axis.

Example 3.2. Consider the signed permutation $u := \overline{2}316\overline{4}75$, in window notation, that is, $\overline{5}74\overline{6}1\overline{3}2\overline{2}316\overline{4}75$, in full notation. Applying the algorithm above, we draw the path π^u and the labellings λ_x^u, λ_y^u as follows:



Therefore, π^u is the dashed path, λ_x^u is the permutation 7423165, and λ_y^u is $\overline{7}\overline{4}\overline{2}\overline{3}\overline{1}\overline{6}\overline{5}$. \diamond

It is easily seen that, for an arbitrary $u \in \mathcal{B}_n$, $(\pi^u, \lambda_x^u, \lambda_y^u)$ has the following properties:

- (i) π^u is symmetric along the diagonal,
- (ii) $\lambda_x^u \in \mathcal{S}_n$ and, moreover, it is the subword of u of positive letters,
- (iii) for each $x \in [n]$, $\lambda_y^u(x) = \overline{\lambda_x^u(x)}$ and, moreover, λ_y^u is the mirror of the subword of u of negative letters.

In particular, we see that the data $(\pi^u, \lambda_x^u, \lambda_y^u)$ is redundant since λ_y^u is completely determined from λ_x^u .

Proposition 3.3. *The mapping $u \mapsto (\pi^u, \lambda_x^u)$ is a bijection from the set of signed permutations \mathcal{B}_n to the set of pairs (π, w) , where $w \in \mathcal{S}_n$ and π is a discrete path from $(0, n)$ to $(n, 0)$ with East and South steps which, moreover, is symmetric along the diagonal.*

We leave it to the reader to verify the above statement. Next, we argue for the interest of this representation by looking at the inversion set of a signed permutation. According to the definition in (9), the type B inversions of a signed permutation can be split into its positive inversions, the pairs (i, j) with $1 \leq i < j \leq n$, and the negative ones, those of the form (i, j) with $i \leq 0$ and $1 \leq |i| \leq j \leq n$. We claim that the positive inversions of u are the type A inversions of λ_x^u and that its negative inversions are of the form $(\lambda_y^u(y), \lambda_x^u(x))$ such that $1 \leq y \leq x \leq n$ and the cell (x, y) lies below π^u . This idea is exemplified in Figure 2 with the signed permutation $\overline{2}316\overline{4}75$ from Example 3.2. Notice that, when $1 \leq x < y \leq n$ and (x, y) lies below π^u , then (y, x) lies below π^u as well (since π^u is symmetric along the diagonal) and therefore, according to our claim, $(\lambda_y^u(x), \lambda_x^u(y))$ is a negative inversion of u . If we identify, when $x < y$, the pair $(\lambda_y^u(y), \lambda_x^u(x))$ with $(\lambda_y^u(x), \lambda_x^u(y))$, then we can simply say that the negative inversions of u are of the form $(\lambda_y^u(y), \lambda_x^u(x))$ for (x, y) below π^u .

We collect these observations in a formal statement.

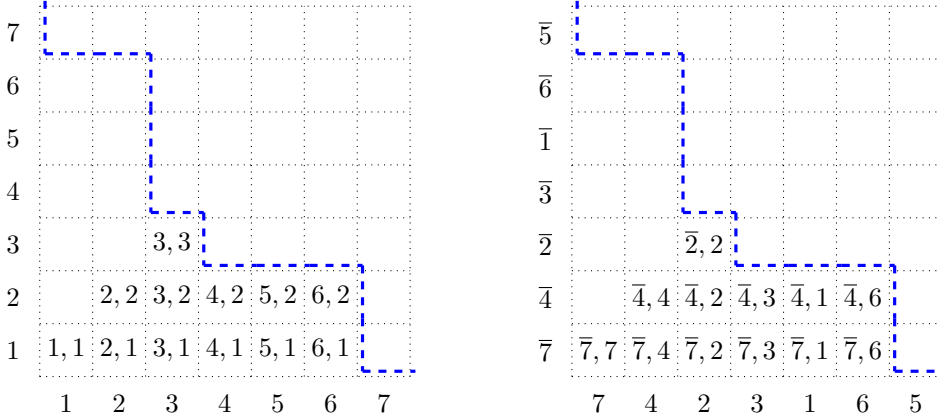


Fig. 2: Negative inversions of $\bar{2}31\bar{6}\bar{4}\bar{7}5$, indexed on the left by the abscissa and ordinate and on the right by λ_y^u, λ_x^u

Proposition 3.4. *Let $u \in \mathcal{B}_n$. For each i, j with $1 \leq |i| \leq j \leq n$, $(i, j) \in \text{Inv}_B(u)$ if and only if either $1 \leq i < j \leq n$ and $(i, j) \in \text{Inv}(\lambda_x^u)$ or $i < 0$ and $((\lambda_x^u)^{-1}(-i), (\lambda_x^u)^{-1}(j))$ lies below the path π^u .*

Proof: Consider a pair (i, j) such that $1 \leq |i| \leq j \leq n$ and such that, if $0 < i$, then $i < j$.

If $0 < i < j$, then both i and j appear in λ_x^u , which is the subword of u (written in full notation) of positive integers. Then $u^{-1}(i) > u^{-1}(j)$ if and only if $(\lambda_x^u)^{-1}(i) > (\lambda_x^u)^{-1}(j)$, that is, $(i, j) \in \text{Inv}_B(u)$ if and only if $(i, j) \in \text{Inv}(\lambda_x^u)$.

We suppose next that $i < 0$. Observe that, as suggested in Figure 3, for $j > 0$ and $i < 0$, the cell identified by λ_y^u, λ_x^u as (i, j) is below π^u if and only if the letter j appears before the letter i in u . Also, for such a pair, j appears before i in u if and only if $(i, j) \in \text{Inv}(u)$, where u is considered as a permutation of the set $[\pm n]$ and the set of inversions is computed w.r.t the standard linear order on this set.

Therefore, if (i, j) with $i < 0$ and $1 \leq |i| \leq j \leq n$, then $(i, j) \in \text{Inv}_B(u)$ if and only if $(i, j) \in \text{Inv}(u)$ if and only if the cell identified by λ_y^u, λ_x^u as (i, j) is below π^u . If, instead of using λ_y^u and λ_x^u to identify cells, we use the abscissa and ordinate, this happens when $((\lambda_y^u)^{-1}(i), (\lambda_x^u)^{-1}(j)) = ((\lambda_x^u)^{-1}(-i), (\lambda_y^u)^{-1}(j))$ is below π_u . \square

Remark 3.5. The fact that (x, y) lies below π^u if and only if (y, x) lies below π^u suggests to look at negative inversions of u as unordered pairs of the form $\{\lambda_x^u(x), \lambda_x^u(y)\}$ (doubletons or singletons) such that (x, y) lies below π^u . We shall explore this graph-theoretic approach in Section 7. We illustrate this with the signed permutation of Example 3.2: we can identify the set of type B inversions of $\bar{2}31\bar{6}\bar{4}\bar{7}5$ with the disjoint union of the set of type A inversions of 7423165 and the set of unordered pairs

$$\{ \{7, 7\}, \{7, 4\}, \{7, 2\}, \{7, 3\}, \{7, 1\}, \{7, 6\}, \{4, 4\}, \{4, 2\}, \{4, 3\}, \{4, 1\}, \{4, 6\}, \{2, 2\} \}.$$

4 Simply barred permutations

We consider now a second way of representing signed permutations. We mostly consider simply barred permutations as shorthands for path representations of signed permutations. While less informative than

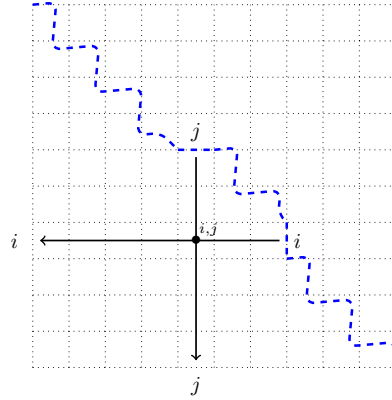


Fig. 3: Characterizing inversions of the form (i, j) with i negative

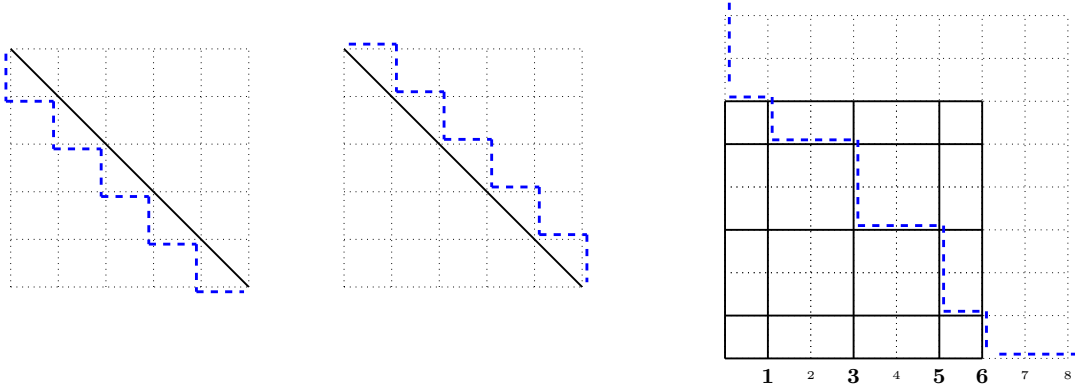
path representations, we shall observe that the enumerative results of the following chapters mostly rely on this representation.

Definition 4.1. A *simply barred permutation* of $[n]$ is a pair (w, B) where $w \in S_n$ and $B \subseteq \{1, \dots, n\}$. We let SBP_n be the set of *simply barred permutations* of $[n]$.

We think of B as a set of positions of w , the barred positions or walls. We have added the adjective “simply” to “barred permutation” since we do not require that B is a superset of $\text{Des}(w)$, as for example in (Gessel and Stanley, 1978).

Example 4.2. We write a simply barred permutation (w, B) as a permutation divided into *blocks* by the bars, placing a vertical bar after w_i for each $i \in B$. For example, $(w, B) = (7423165, \{2, 4, 6\})$ is written $74|23|16|5$. Notice that we allow a bar to appear in the last position, for example $34|1|265|7|$ stands for the simply barred permutation $(3412657, \{2, 3, 6, 7\})$. Thus, a bar appears in the last position if and only if the last block is empty. The last block is indeed the only block that can be empty, which amounts to saying that consecutive bars are not allowed in simply barred permutations. This contrasts with other notions of barred permutation, for example the one appearing in the proof of the alternating sum formula for the Eulerian numbers (Bóna, 2012, Theorem 1.11). \diamond

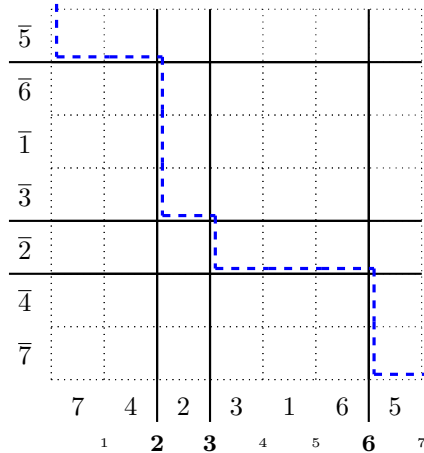
Next, we describe a bijection—that we call ψ —from the set SBP_n to B_n . Let us notice that, in order to establish equipotence of these two sets, other more straightforward bijections are available. In the definition below, for $w = w_1w_2 \cdots w_n$, we let $\bar{w} := \bar{w}_1\bar{w}_2 \cdots \bar{w}_n$. Moreover, we need to explain what we mean for the *upper antidiagonal* of a subgrid. Notice that, in a grid $[n]_0 \times [n]_0$, we have two discrete paths that are closest to the the line $y = n - x$. We call them the lower and upper (discrete) antidiagonal, respectively. This is suggested below with the lower and the upper antidiagonal on the left and on the center, respectively.



Next, a subset $B \subseteq [n]$ determines a subgrid $(B \cup \{0\}) \times (B \cup \{0\})$ which comes with its own upper antidiagonal. We can extend this path with South steps before and East steps after, so to obtain a path from $(0, n)$ to $(n, 0)$. We call this path the *upper antidiagonal of the subgrid*. An example appears above on the right, where the subgrid is determined by $B = \{1, 3, 5, 6\}$.

Definition 4.3. For $(w, B) \in \text{SBP}_n$, we define the signed permutation $\psi(w, B) \in \text{B}_n$ according to the following algorithm: (i) draw the grid $[n]_0 \times [n]_0$; (ii) since $B \subseteq [n]$, $(B \cup \{0\}) \times (B \cup \{0\})$ defines a subgrid of $[n]_0 \times [n]_0$, construct the upper antidiagonal π of this subgrid; (iii) $\psi(w, B)$ is the signed permutation u whose path representation $(\pi^u, \lambda_x^u, \lambda_y^u)$ equals to (π, w, \bar{w}) .

Example 4.4. The construction just described can be understood as raising the bars and transforming them into a grid. For example, for the simply barred permutation $7\bar{4}|2|3\bar{1}\bar{6}|5$ (that is, (w, B) with $w = 7423165$ and $B = \{2, 3, 6\}$) the construction is as follows:



The upper antidiagonal of the subgrid yields the dashed path above. The resulting signed permutation $\psi(w, B)$ is $\bar{2}3\bar{1}\bar{6}\bar{4}\bar{7}5$ as from Example 3.2. \diamond

The inverse image of ψ can be constructed according to the following algorithm: for $u \in \text{B}_n$ (i) construct the path representation $(\pi^u, \lambda_x^u, \lambda_y^u)$ of u , (ii) insert a bar in w at each vertical step of π^u (and

remove consecutive bars), (iii) remove a bar at position 0 if it exists. Said otherwise, $(w, B) = \psi^{-1}(u)$ is obtained from u by transforming each negative letter into a bar, by removing consecutive bars, and then by removing a bar at position 0 if needed.

Even if we consider simply barred permutations as shorthands for path representations of signed permutations, some remarks are due now:

Lemma 4.5. *If $u = \psi(w, B)$, then there is a bijection between the set B of bars and the set of East-South turns of π^u .*

Lemma 4.6. *We have $0 \in \text{Des}_B(\psi(w, B))$ if and only if $|B|$ is odd.*

The lemma can immediately be verified by considering that $0 \in \text{Des}_B(u)$ if and only if the first letter in the window notation of u is negative, if and only if, in the path representation of $\psi(w, B)$, the first step of π^u after meeting the diagonal is along the y -axis, in which case (and only in this case) π^u makes an East-South turn on the diagonal. This happens exactly when π^u has an odd number of East-South turns.

5 Descents from simply barred permutations

We start investigating how the type B descent set can be recovered from a simply barred permutation.

Proposition 5.1. *For a simply barred permutation (w, B) , we have*

$$\text{des}_B(\psi(w, B)) = |\text{Des}(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil. \quad (11)$$

Proof: Write $u = \psi(w, B)$ in window notation and divide it in maximal blocks of consecutive letters having the same sign. If the first block has negative sign, add an empty positive block in position 0. Each change of sign $+ -$ among consecutive blocks yields a descent. These changes of sign bijectively correspond to East-South turns of π^u that lie on or below the diagonal. By Lemma 4.5, each bar determines an East-South turn and, by symmetry of π^u along the diagonal, the number of East-South turns that are on or below the diagonal is $\left\lceil \frac{|B|}{2} \right\rceil$. Therefore this quantity counts the number of descents determined by a change of sign.

The other descents of $\psi(w, B)$ are either of the form $w_i w_{i+1}$ with $w_i > w_{i+1}$ and w_i, w_{i+1} belonging to the same positive block, or of the form $\overline{w_{i+1}} \overline{w_i}$ with $w_i > w_{i+1}$ and $\overline{w_i}, \overline{w_{i+1}}$ belonging to the same negative block. These descents are in bijection with the descent positions of w that do not belong to the set B . \square

For each $k \in \{0, 1, \dots, n\}$, we let in the following $\text{SBP}_{n,k}$ be the set of simply barred permutations $(w, B) \in \text{SBP}_n$ such that $|\text{Des}(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k$.

Corollary 5.2. *The set $\text{SBP}_{n,k}$ is in bijection with the set of signed permutations of $[n]$ with k descents.*

We introduce next loosely barred permutations only as a tool to index simply barred permutations independently of the even/odd cardinalities of their set of bars.

Definition 5.3. *A loosely barred permutation of $[n]$ is a pair (w, B) where w is a permutation of $[n]$ and $B \subseteq \{0, \dots, n\}$ is a set of positions (the bars). We let LBP_n be the set of loosely barred permutations of $[n]$.*

For $D \subseteq [n]$, let $\xi_D : P([n]_0) \longrightarrow P([n])$ be the map defined by

$$\xi_D(B) := (D\Delta B) \setminus \{0\} = D\Delta(B \setminus \{0\}),$$

where Δ stands for the symmetric difference in $P([n]_0)$. Then, we define $\Theta_n : \text{LBP}_n \longrightarrow \text{SBP}_n$ by

$$\Theta_n(w, B) := (w, \xi_{\text{Des}(w)}(B)).$$

We shall investigate properties of the map Θ_n , for which we first need to collect properties of the map ξ_D . These are listed in the following lemmas.

Lemma 5.4. *The map ξ_D is a surjective two-to-one map. That is, each $C \subseteq [n]$ has exactly two preimages, $B_1 := D\Delta C$ and $B_2 = (D\Delta C) \cup \{0\}$.*

In view of $D \setminus \xi_D(B) = D \cap B$, the following relation holds if $0 \notin B$:

$$|D| + |B| = 2|D \setminus \xi_D(B)| + |\xi_D(B)|.$$

Given this relation, the reader shall have no difficulties verifying the properties stated in the next lemma.

Lemma 5.5. *The parity of $|\xi_D(B)|$ can be computed from $|D|$ and $|B|$ according to the following rules:*

1. *if $|D| + |B| = 2k$ and $0 \notin B$, then $|D \setminus \xi_D(B)| + \left\lceil \frac{|\xi_D(B)|}{2} \right\rceil = k$ and, in this case, $|\xi_D(B)|$ is even;*
2. *if $|D| + |B| = 2k$ and $0 \in B$, then $|D \setminus \xi_D(B)| + \left\lceil \frac{|\xi_D(B)|}{2} \right\rceil = k$ and, in this case, $|\xi_D(B)|$ is odd;*
3. *if $|D| + |B| = 2k + 1$ and $0 \notin B$, then $|D \setminus \xi_D(B)| + \left\lceil \frac{|\xi_D(B)|}{2} \right\rceil = k + 1$ and, in this case, $|\xi_D(B)|$ is odd;*
4. *if $|D| + |B| = 2k + 1$ and $0 \in B$, then $|D \setminus \xi_D(B)| + \left\lceil \frac{|\xi_D(B)|}{2} \right\rceil = k$ and, in this case, $|\xi_D(B)|$ is even.*

The next lemma restates these properties on the side of preimages.

Lemma 5.6. *For $C \subseteq [n]$, let B_1, B_2 be the two preimages of C via ξ_D . Then, the following statements hold:*

1. *if $|D \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k$ and $|C|$ is even, then the two preimages B_1, B_2 of C satisfy $|D| + |B_1| = 2k$ and $|D| + |B_2| = 2k + 1$;*
2. *if $|D \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k$ and $|C|$ is odd, then the two preimages B_1, B_2 of C satisfy $|D| + |B_1| = 2k - 1$ and $|D| + |B_2| = 2k$.*

Definition 5.7. For each $n \geq 0$ and $k \in [2n]_0$, we let $\text{LBP}_{n,k}$ be the set of loosely barred permutations (w, B) such that $|\text{Des}(w)| + |B| = k$.

Proposition 5.8. *For each $n \geq 0$ and $k \in [n]_0$, the restriction of Θ_n to $\text{LBP}_{n,2k}$ yields a bijection $\Theta_{n,k}$ from $\text{LBP}_{n,2k}$ to $\text{SBP}_{n,k}$.*

Proof: By the first two items of Lemma 5.5, the restriction of Θ_n to $\text{LBP}_{n,2k}$ takes values in $\text{SBP}_{n,k}$. The restriction map is injective. Indeed, if $\Theta_n(w, B) = \Theta_n(w', B')$, then $w = w'$ and, for $D = \text{Des}(w)$, $C = \xi_D(B) = \xi_D(B')$. Lemma 5.6 states that each $C \subseteq [n]$ has at most one ξ_D -preimage B satisfying $|D| + |B| = 2k$, whence $B = B'$. This map is also surjective: using Lemma 5.6, if $(w, C) \in \text{SBP}_{n,k}$, $D = \text{Des}(w)$, and $B_1 \neq B_2$ are such that $0 \notin B_1$ and $\xi_D(B_1) = \xi_D(B_2) = C$, then (w, B_1) is a preimage of (w, C) if $|C|$ is even, and (w, B_2) is a preimage of (w, C) if $|C|$ is odd. \square

Let us recall that, for $u \in \mathbb{B}_n$, $\text{Des}_{\mathbb{B}}^+(u)$ denotes the set of strictly positive descents of u , see Lemma 2.1.

Definition 5.9. For each $k \in [n-1]_0$, we let SBP_n^k be the set of simply barred permutations $(w, B) \in \text{SBP}_n$ such that $|\text{Des}_{\mathbb{B}}^+(\psi(w, B))| = k$.

Let us pinpoint the following characterization of the set SBP_n^k :

Lemma 5.10. For each simply barred permutation $(w, C) \in \text{SBP}_n$,

$$(w, C) \in \text{SBP}_n^k \quad \text{iff} \quad \begin{cases} |C| \text{ is even and } |\text{Des}(w) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k, \text{ or} \\ |C| \text{ is odd and } |\text{Des}(w) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k + 1. \end{cases}$$

Proof: We have

$$|\text{Des}_{\mathbb{B}}^+(\psi(w, C))| = k \quad \text{iff} \quad \begin{cases} 0 \notin \text{Des}_{\mathbb{B}}(\psi(w, C)) \text{ and } \text{des}_{\mathbb{B}}(\psi(w, C)) = k, \text{ or} \\ 0 \in \text{Des}_{\mathbb{B}}(\psi(w, C)) \text{ and } \text{des}_{\mathbb{B}}(\psi(w, C)) = k + 1. \end{cases}$$

The statement of the lemma follows using Lemma 4.6 and Proposition 5.1. \square

Proposition 5.11. For each $k \in [n-1]_0$, the restriction of Θ_n to $\text{LBP}_{n,2k+1}$ yields a bijection Θ_n^k from $\text{LBP}_{n,2k+1}$ to SBP_n^k .

Proof: By items 3. and 4. in Lemma 5.5, and also using Lemma 5.10, the restriction of Θ_n to $\text{LBP}_{n,2k+1}$ takes values in SBP_n^k . The restriction map is injective. Indeed, if $\Theta_n(w, B) = \Theta_n(w', B')$, then $w = w'$ and, for $D = \text{Des}(w)$, $C = \xi_D(B) = \xi_D(B')$. Lemma 5.6 states that each $C \subseteq [n]$ has at most one preimage B satisfying $|D| + |B| = 2k + 1$, whence $B = B'$. This map is also surjective. Let $(w, C) \in \text{SBP}_n^k$, $D = \text{Des}(w)$, and $B_1 \neq B_2$ be such that $0 \notin B_1$ and $\xi_D(B_1) = \xi_D(B_2) = C$. By Lemma 5.6, if $|C|$ is even, then (w, C) has the preimage (w, B_2) , and if $|C|$ is odd, then (w, C) has the preimage (w, B_1) . \square

To end this section, we collect the consequences of the bijections established so far.

Theorem 5.12. The following relations hold:

$$\left\langle \mathbb{B}_n \right\rangle_k = \sum_{i=0}^{2k} \left\langle n \right\rangle_i \binom{n+1}{2k-i}, \quad (3)$$

$$2^n \left\langle n \right\rangle_k = \sum_{i=0}^{2k+1} \left\langle n \right\rangle_i \binom{n+1}{2k+1-i}. \quad (4)$$

Proof: We have seen that signed permutations $u \in \mathbb{B}_n$ such that $\text{des}_{\mathbb{B}}(u) = k$ are in bijection (via the mapping ψ of Definition 4.3) with simply barred permutations in $\text{SBP}_{n,k}$. Next, this set is in bijection (see Proposition (5.8)) with the set $\text{LBP}_{n,2k}$ of loosely barred permutations $(w, B) \in \text{LBP}_n$ such that $\text{des}(w) + |B| = 2k$. The cardinality of $\text{LBP}_{n,2k}$ is the right-hand side of equality (3).

The left-hand side of equality (4) is the cardinality of the set of signed permutations u such that $|\text{Des}_{\mathbb{B}}^+(u)| = k$, see Lemma 2.1. This set is in bijection with the set SBP_n^k (via ψ defined in 4.3 and by the definition of SBP_n^k) which, in turn, is in bijection (see Proposition (5.11)) with the set $\text{LBP}_{n,2k+1}$ of loosely barred permutations $(w, B) \in \text{LBP}_n$ such that $\text{des}(w) + |B| = 2k + 1$. The cardinality of this set is the right-hand side of equality (4). \square

Theorem 5.13. *The following relation holds:*

$$B_n(t^2) = (1+t)^{n+1}S_n(t) - 2^n t S_n(t^2). \quad (12)$$

Proof: By (3), $\left\langle \begin{smallmatrix} \mathbb{B}_n \\ k \end{smallmatrix} \right\rangle$, which is the coefficient of t^{2k} in the polynomial $B_n(t^2)$, is also the coefficient of t^{2k} in $(1+t)^{n+1}S_n(t)$. By (4), $2^n \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$ is the coefficient of t^{2k+1} in the polynomials $2^n t S_n(t^2)$ and $(1+t)^{n+1}S_n(t)$. Therefore

$$B_n(t^2) + 2^n t S_n(t^2) = (1+t)^{n+1}S_n(t), \quad (5)$$

whence equation (12). \square

6 Stembridge's identity for Eulerian numbers of type D

We recall that a signed permutation $u \in \mathbb{B}_n$ is *even signed* if the number of negative letters in its window notation is even. The even-signed permutations of \mathbb{B}_n form a subgroup \mathbb{D}_n of \mathbb{B}_n and in fact the groups \mathbb{D}_n are standard models for the abstract Coxeter groups of type D.

Definitions analogous to those given in Section 2 for the types A and B can be given for type D. Namely, for $u \in \mathbb{D}_n$, we set

$$\text{Des}_{\mathbb{D}}(u) := \{i \in \{0, 1, \dots, n-1\} \mid u_i > u_{i+1}\}, \quad (13)$$

where we have set $u_0 = -u_2$,

$$\text{des}_{\mathbb{D}}(u) := |\text{Des}_{\mathbb{D}}(u)|, \quad \left\langle \begin{smallmatrix} \mathbb{D}_n \\ k \end{smallmatrix} \right\rangle := |\{u \in \mathbb{D}_n \mid \text{des}_{\mathbb{D}}(u) = k\}|, \quad D_n(t) := \sum_{k=0}^n \left\langle \begin{smallmatrix} \mathbb{D}_n \\ k \end{smallmatrix} \right\rangle t^k.$$

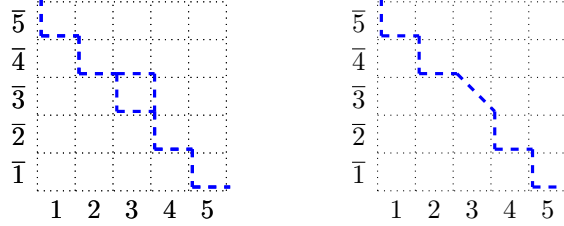
The formula in (13) is the standard one, see e.g. (Björner and Brenti, 2005, §8.2) or (Bagno et al., 2018). The reader will have no difficulties verifying that, up to renaming 0 by -1 , the type D descent set of u can also be defined as follows, see (Petersen, 2015, §13):

$$\text{Des}_{\mathbb{D}}(u) := \{i \in \{-1, 1, \dots, n-1\} \mid u_i > u_{|i|+1}\}, \quad (14)$$

where now $u_{-1} = -u_1$, as normal if u is written in full notation.

It is convenient to consider a more flexible representation of elements of D_n . If $u \in B_n$, then its mate is the signed permutation $\underline{u} \in B_n$ that differs from u only for the sign of the first letter. Notice that $\underline{\underline{u}} = u$. We define a *forked signed permutation* (see (Petersen, 2015, §13)) as an unordered pair of the form $\{u, \underline{u}\}$ for some $u \in B_n$. Clearly, just one of the mates is even signed and therefore forked signed permutations are combinatorial models of D_n .

The path representation of a forked signed permutation is insensitive of how the diagonal is crossed, either from the West, or from the North. The following are possible ways to draw a forked signed permutation on a grid:



Even if the formulas in (13) and (14) have been defined for even-signed permutations, they still can be computed for all signed permutations. The formula in (14) is not invariant under taking mates, however the following lemma shows that this formula suffices to compute the number of type D descents of a forked signed permutation and therefore the Eulerian numbers $\left\langle \begin{smallmatrix} D_n \\ k \end{smallmatrix} \right\rangle$.

Lemma 6.1. *For each $u \in B_n$, $1 \in \text{Des}_D(u)$ if and only if $-1 \in \text{Des}_D(\underline{u})$. Therefore $\text{des}_D(u) = \text{des}_D(\underline{u})$.*

Proof: Suppose $1 \in \text{Des}_D(u)$, that is $u_1 > u_2$. Then $\underline{u}_{-1} = -(-u_1) = u_1 > u_2$, and so $-1 \in \text{Des}_D(\underline{u})$. The opposite entailment is proved similarly.

For the last statement, observe that $\text{Des}_D(u) = \Delta_u \cup \{i \in \{2, \dots, n-1\} \mid u_i > u_{i+1}\}$ with $\Delta_u := \{i \in \{1, -1\} \mid u_i > u_{|i|+1}\}$ and, by what we have just remarked, we have $|\Delta_u| = |\Delta_{\underline{u}}|$. It follows that $|\text{Des}_D(u)| = |\text{Des}_D(\underline{u})|$. \square

Our next aim is to derive Stembridge's identity

$$D_n(t) = B_n(t) - n2^{n-1}tS_{n-1}(t), \quad (15)$$

see (Stembridge, 1994, Lemma 9.1), which, in term of the coefficients of these polynomials, amounts to

$$\left\langle \begin{smallmatrix} D_n \\ k \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} B_n \\ k \end{smallmatrix} \right\rangle - n2^{n-1} \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle. \quad (16)$$

Definition 6.2. A signed permutation u is *smooth* if u_1, u_2 have equal sign and, otherwise, it is *non-smooth*.

The reason for naming a signed permutation smooth arises again from the path representation of a signed permutation: the smooth signed permutation is, between the two mates, the one minimizing the turns nearby the diagonal, as suggested in Figure 4 with two pairs of mates as examples.

Lemma 6.3. *If $u \in B_n$ is smooth, then $-1 \in \text{Des}_D(u)$ if and only if $0 \in \text{Des}_B(u)$ and therefore $\text{des}_D(u) = \text{des}_B(u)$.*

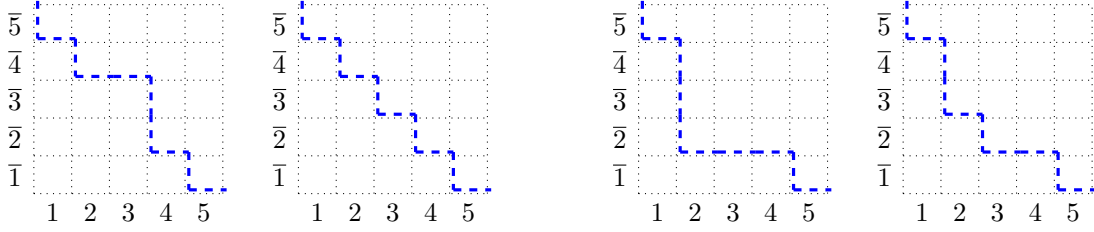


Fig. 4: Two pairs of mates, the smooth mates are on the left

Proof: Suppose $0 \in \text{Des}_{\mathbb{B}}(u)$, so $u_1 < 0$ and $u_2 < 0$ as well, since u is smooth. Then $u_{-1} = -u_1 > 0 > u_2$, so $-1 \in \text{Des}_{\mathbb{D}}(u)$. Conversely, suppose $-1 \in \text{Des}_{\mathbb{D}}(u)$, that is, $u_{-1} > u_2$. If $u_1 > 0$, then $0 > -u_1 = u_{-1} > u_2$, so u_1, u_2 have different sign, a contradiction. Therefore $u_1 < 0$ and $0 \in \text{Des}_{\mathbb{B}}(u)$. \square

Corollary 6.4. *There is a bijection between the set of smooth signed permutations in \mathbb{B}_n with k type B descents and the set of even-signed permutations in \mathbb{D}_n with k type D descents.*

Indeed, if $u \in \mathbb{B}_n$ is smooth and even signed, then we let $v = u$, so $\text{Des}_{\mathbb{B}}(u) = \text{Des}_{\mathbb{D}}(v)$, by Lemma 6.3. If u is smooth but not even signed, then its mate \underline{u} is even signed. We let then $v = \underline{u}$ and then $\text{Des}_{\mathbb{B}}(u) = \text{Des}_{\mathbb{D}}(u) = \text{Des}_{\mathbb{D}}(\underline{u})$, using Lemmas 6.1 and 6.3.

Next, we consider the correspondence—let us call it χ —sending a non-smooth signed permutation $u \in \mathbb{B}_n$ to the pair $(|u_1|, u')$, where u' is obtained from $u_2 \cdots u_n$ by normalising this sequence, so that it takes absolute values in the set $[n-1]$. For example $\chi(6\overline{123475}) = (6, \overline{123465})$ and $\chi(2\overline{316475}) = (2, \overline{215364})$, as suggested below:

$$6\overline{123475} \rightsquigarrow (6, \overline{123475}) \rightsquigarrow (6, \overline{123465}), \quad 2\overline{316475} \rightsquigarrow (2, \overline{316475}) \rightsquigarrow (2, \overline{215364}).$$

Notice that this transformation is reversible. Consider for example the pair $(3, \overline{15423})$. We can first rename $\overline{15423}$ so 3 is not the absolute value of any letter, thus obtaining $\overline{16524}$. We can then add ± 3 in front of this word, having two choices, $3\overline{16524}$ and $\overline{316524}$. There is exactly one choice yielding a non-smooth signed permutation, namely $3\overline{16524}$.

The process of normalizing the sequence $u_2 \cdots u_n$ can be understood as applying to each letter of this sequence the unique order preserving bijection $N_{n,x} : [\pm n] \setminus \{x, \bar{x}\} \rightarrow [\pm n - 1]$ where, in general, $x \in [n]$ and, in this case, $x = |u_1|$.

Lemma 6.5. *Let $n \geq 2$. For each pair (x, v) with $x \in [n]$ and $v \in \mathbb{B}_{n-1}$, there exists a unique non-smooth $u \in \mathbb{B}_n$ such that $\chi(u) = (x, v)$.*

Proof: We construct u firstly by renaming v to v' so that none of x, \bar{x} appears in v' (that is, we apply to each letter of v the inverse of $N_{n,x}$) and then by adding in front of v' either x or \bar{x} , according to the sign of the first letter of v' . \square

Lemma 6.6. *The correspondence χ restricts to a bijection from the set of non-smooth signed permutations $u \in \mathbb{B}_n$ such that $\text{des}_{\mathbb{B}}(u) = k$ to the set of pairs (x, v) where $x \in [n]$ and $v \in \mathbb{B}_{n-1}$ is such that $|\text{Des}_{\mathbb{B}}^+(v)| = k - 1$.*

Proof: We have already argued that χ is a bijection from the set of non-smooth signed permutations u of $[n]$ to the set of pairs (x, v) with $x \in [n]$ and $v \in \mathbb{B}_{n-1}$. Therefore, we are left to argue that, for a non-smooth u and v such that $\chi(u) = (x, v)$, $\text{des}_{\mathbb{B}}(u) = k$ if and only if $|\text{Des}_{\mathbb{B}}^+(v)| = k - 1$. Said otherwise, we need to argue that, for such u and v , $|\text{Des}_{\mathbb{B}}^+(v)| = \text{des}_{\mathbb{B}}(u) - 1$. To this end, observe that (i) $|\text{Des}_{\mathbb{B}}(u) \cap \{0, 1\}| = 1$, since u_1, u_2 have different sign, (ii) $\text{Des}_{\mathbb{B}}^+(v) = \{i - 1 \mid i \in \text{Des}_{\mathbb{B}}(u) \cap \{2, \dots, n - 1\}\}$, from which the relation $|\text{Des}_{\mathbb{B}}^+(v)| = \text{des}_{\mathbb{B}}(u) - 1$ follows. \square

Theorem 6.7. *The following relations hold:*

$$\left\langle \begin{matrix} \mathbb{B}_n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathbb{D}_n \\ k \end{matrix} \right\rangle + n2^{n-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle, \quad B_n(t) = D_n(t) + n2^{n-1}tS_{n-1}(t).$$

Proof: Every signed permutation is either smooth or non-smooth. By Corollary 6.4, the smooth signed permutations with k type B descents are in bijection with the even-signed permutations with k type D descents. By Lemma 6.6, the non-smooth signed permutations $u \in \mathbb{B}_n$ with k type B descents are in bijection with the pairs $(x, v) \in [n] \times \mathbb{B}_{n-1}$ such that $|\text{Des}_{\mathbb{B}}^+(v)| = k - 1$. Using Lemma 2.1, the number of these pairs is $n2^{n-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle$. \square

Example 6.8. We end this section exemplifying the use of formulas (3) and (16) by which computation of the Eulerian numbers of type B and D is reduced to computing Eulerian numbers of type A. Let us mention that our interest in Eulerian numbers originates from our lattice-theoretic work on the lattice variety of Permutohedra (Santocanale and Wehrung, 2018) and its possible extensions to generalized forms of Permutohedra (Pouzet et al., 1995; Santocanale and Wehrung, 2016; Gouveia and Santocanale, 2021). Among these generalizations, we count lattices of finite Coxeter groups in the types B and D (Björner, 1984). While it is known that the lattices \mathbb{B}_n span the same lattice variety of the permutohedra, see (Caspard et al., 2016, Exercise 1.23), characterizing the lattice variety spanned by the lattices \mathbb{D}_n is an open problem. A first step towards solving this kind of problem is to characterize (and count) the join-irreducible elements of a class of lattices. In our case, this amounts to characterizing the elements u in \mathbb{B}_n (resp., in \mathbb{D}_n) such that $\text{des}_{\mathbb{B}}(u) = 1$ (resp., such that $\text{des}_{\mathbb{D}}(u) = 1$). The numbers $\left\langle \begin{matrix} \mathbb{B}_n \\ 1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} \mathbb{D}_n \\ 1 \end{matrix} \right\rangle$ are known to be equal to $3^n - n - 1$ and $3^n - n - 1 - n2^{n-1}$ respectively, see (Petersen, 2015, Propositions 13.3 and 13.4). Let us see how to derive these identities using the formulas (3) and (16). To this end, we also need the alternating sum formula for Eulerian numbers, see e.g. (Bóna, 2012, Theorem 1.11) or (Petersen, 2015, page 12):

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n. \quad (17)$$

For type B, we have

$$\begin{aligned}
\langle \mathbf{B}_n \rangle &= \langle n \rangle \binom{n+1}{2} + \langle n \rangle \binom{n+1}{1} + \langle n \rangle \binom{n+1}{0} \\
&= \binom{n+1}{2} + (2^n - n - 1)(n+1) + \langle n \rangle \\
&= \binom{n+1}{2} + (2^n - n - 1)(n+1) + 3^n - 2^n(n+1) + \binom{n+1}{2}, \quad \text{by (17)} \\
&= 3^n - (n+1)^2 + 2 \binom{n+1}{2} = 3^n - (n+1)(n+1-n) = 3^n - n - 1.
\end{aligned}$$

The computation of type D is then immediate from Stembridge's identity (16):

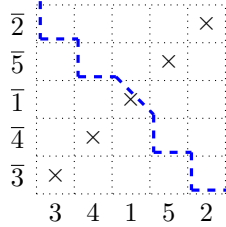
$$\langle \mathbf{D}_n \rangle = \langle \mathbf{B}_n \rangle - n2^{n-1} \langle n-1 \rangle = 3^n - n - 1 - n2^{n-1}. \quad \diamond$$

7 Threshold graphs and their degree orderings

Besides presenting the bijective proofs, a goal of this paper is to illustrate the path representation of signed permutations and exemplify its potential. The attentive reader might object that the path representation is not in use within Section 6. Indeed, after discovering the bijective proof of Stembridge's identity via the path representation, we realized that the proof could be simplified and reach a larger audience by avoiding mentioning the representation. It might be asked then whether the path representation yields more information, in particular with respect to the lattices of the Coxeter groups D_n . We answer this question in this section. The type D set of inversions of an even-signed permutation can be defined as follows:

$$\text{Inv}_D(u) := \text{Inv}_B(u) \setminus \{(-i, i) \mid i \in [n]\},$$

which, graphically, amounts to ignoring cells on the diagonal:



As mentioned in Remark 3.5, we can identify the set of inversions of a signed permutation u with the disjoint union of $\text{Inv}(\lambda_x^u)$ and a set of unordered pairs. For even-signed permutations, this identification yields:

$$\begin{aligned}
\text{Inv}_D(u) &= \text{Inv}(\lambda_x^u) \cup E^u \\
\text{with } E^u &:= \{ \{i, j\} \mid i, j \in [n], i \neq j, ((\lambda_x^u)^{-1}(i), (\lambda_x^u)^{-1}(j)) \text{ lies below } \pi^u \}. \quad (18)
\end{aligned}$$

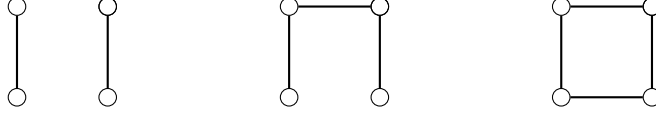


Fig. 5: The (unlabelled) graphs $2K_2$, P_3 , and C_4

Therefore, we consider $([n], E^u)$ as a simple graph on the set of vertices $[n]$. Let us observe that the definition of the set of edges E^u in (18) makes sense for all signed permutations, not just for an even-signed permutations. Moreover, for mates u and \underline{u} , we have $E^u = E^{\underline{u}}$. Before exploring further the graph $([n], E^u)$, we recall some standard graph-theoretic concepts. For an arbitrary simple graph (V, E) and a vertex $v \in V$, we let:

$$N_E(v) := \{u \in V \mid \{v, u\} \in E\}, \quad \deg_E(v) := |N_E(v)|, \quad N_E[v] := N_E(v) \cup \{v\}.$$

$N_E(v)$ is the *neighbourhood* of the vertex v , $\deg_E(v)$ is its *degree*, and $N_E[v]$ is often called the *star* of the vertex v . A linear ordering v_1, \dots, v_n of V is a *degree ordering* of (V, E) if $\deg_E(v_1) \geq \deg_E(v_2) \geq \dots \geq \deg_E(v_n)$. A *preorder* on a set V is a reflexive and transitive binary relation on V . The *vicinal preorder* of a graph (V, E) , denoted \triangleleft_E , is defined by $v \triangleleft_E u$ iff $N_E(v) \subseteq N_E[u]$. Notice that the relation $N_E(v) \subseteq N_E[u]$ is equivalent to $N_E(v) \setminus \{u\} \subseteq N_E(u) \setminus \{v\}$. The vicinal preorder is indeed a preorder, see e.g. (Mahadev and Peled, 1995). For completeness, we add a statement and a proof of this fact.

Lemma 7.1. *The relation \triangleleft_E on a simple graph (V, E) is reflexive and transitive.*

Proof: Reflexivity is obvious. For transitivity, let us consider $u, v, w \in V$ such that $u \triangleleft_E v \triangleleft_E w$. If u, v, w are not pairwise distinct, then $u \triangleleft_E w$ immediately follows. Therefore, let us assume that u, v, w are pairwise distinct with $N_E(u) \subseteq N_E[v]$ and $N_E(v) \subseteq N_E[w]$. Let $x \in N_E(u)$. If $x \neq v$, then $x \in N_E(v) \subseteq N_E[w]$. If $x = v$, then $v \in N_E(u)$, thus $u \in N_E(v) \subseteq N_E[w]$ and since $u \neq w$, $u \in N_E(w)$; thus $w \in N_E(u) \subseteq N_E[v]$, so $w \in N_E(v)$ and $x = v \in N_E(w)$. Therefore $N_E(u) \subseteq N_E[w]$. \square

Next, we take Theorem 1 in (Chvátal and Hammer, 1977) as the definition of the class of threshold graphs and consider, among the possible characterizations of this class, the one that uses the vicinal preorder.

Definition 7.2. A graph (V, E) is *threshold* if it does not contain an induced subgraph isomorphic to one among $2K_2$, P_3 and C_4 (these graphs are illustrated in Figure 5).

A binary relation R on V is *total* if and only if, for each $v, u \in V$, vRu or uRv .

Proposition 7.3 (see e.g. (Mahadev and Peled, 1995, Theorem 1.2.4)). *A graph (V, E) is threshold if and only if the vicinal preorder is total.*

We develop next a few considerations on threshold graphs.

Lemma 7.4. *For a simple graph (V, E) and a total ordering $<$ on V , the following conditions are equivalent:*

- (i) (V, E) is a threshold graph and $<$ is a degree ordering,
- (ii) $u < v$ implies $v \triangleleft_E u$, for each $v, u \in V$.

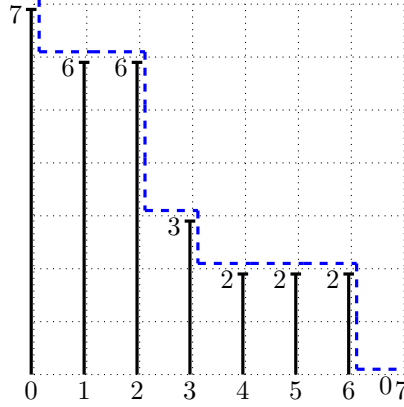


Fig. 6: Paths as height functions

If any of the above conditions hold, then, for each $u \in V$, $N_E(u)$ is a downset in the following sense: if $v \in N_E(u)$ and $w \neq u$ is such that $w < v$, then $w \in N_E(u)$.

Proof: We observe firstly that $v \triangleleft_E u$ implies $\deg_E(v) \leq \deg_E(u)$. Indeed, this follows from the fact that $v \triangleleft_E u$ amounts to $N_E(v) \setminus \{u\} \subseteq N_E(u) \setminus \{v\}$ and that $u \in N_E(v)$ if and only if $v \in N_E(u)$. Notice also that the same argument can be used to argue that if $v \triangleleft_E u$ and $u \not\triangleleft_E v$, then $\deg_E(v) < \deg_E(u)$.

Let therefore (V, E) and $<$ be as stated. By the remark above, if $<$ satisfies (ii) then it is a degree ordering and the relation \triangleleft_E is total, since if $u \not\triangleleft_E v$, then $u \not< v$, so $v \leq u$ and $v \triangleleft_E u$; thus (V, E) is a threshold graph. Suppose next (V, E) is a threshold graph and that $<$ is a degree ordering, so $u < v$ implies $\deg_E(v) \leq \deg_E(u)$. Let $u < v$ and suppose that $v \not\triangleleft_E u$. Since the vicinal preorder is total, we have then $u \triangleleft_E v$ and so $\deg_E(u) < \deg_E(v)$, contradicting $\deg_E(v) \leq \deg_E(u)$.

For the last statement, for such u, v, w , the relation $w < v$ implies $N_E(v) \setminus \{w\} \subseteq N_E(w) \setminus \{v\}$. Since $v \in N_E(u)$, then $u \in N_E(v) \setminus \{w\}$ and $u \in N_E(w) \setminus \{v\}$, so $w \in N_E(u)$. \square

We establish now the connection between threshold graphs on the set of vertices $[n]$, paths, and even-signed permutation. We achieve this through the order-theoretic notion of Galois connection, see e.g. (Davey and Priestley, 2002, §7) or (Nelson, 1976). A Galois connection on $[n]_0$ is a pair of functions $f, g : [n]_0 \rightarrow [n]_0$ such that, for each $x, y \in [n]_0$, $y \leq f(x)$ if and only if $x \leq g(y)$. We say that a map $f : [n]_0 \rightarrow [n]_0$ is a *height function* (we shall discover few lines below the reason for the naming) if it is antitone—that is, $x \leq y$ implies $f(y) \leq f(x)$, for each $x, y \in [n]_0$ —and moreover $f(0) = n$. Observe that, for a Galois connection (f, g) , f is a height function. It is part of elementary order theory that a map $f : [n]_0 \rightarrow [n]_0$ is part of a Galois connection exactly when it is a height function. Moreover, for $f : [n]_0 \rightarrow [n]_0$, there is at most one function $g : [n]_0 \rightarrow [n]_0$ such that (f, g) is a Galois connection.

Paths from $(0, n)$ to $(n, 0)$ that are composed only by East and South steps bijectively correspond to height functions. The bijection is realized by the correspondence sending a path π to $\text{height}_\pi : [n]_0 \rightarrow [n]_0$ such that $\text{height}_\pi(x)$ is the height of π after x East steps; Figure 6 illustrates this correspondence. We refer the reader to (Santocanale, 2019) for the correspondence between paths and this kind of functions in the discrete setting. For a height function $f : [n]_0 \rightarrow [n]_0$, we say that x is *negative* if $x \leq f(x)$ and that

x is positive if $f(x) \leq x$. Notice that x is both positive and negative if and only if it is fixed point of f . Let N_f (resp., P_f) be the set of negative (resp., positive) elements of f . Observe that $N_f \neq \emptyset$, so we can define γ_f , the center of f , as the maximum of this set, $\gamma_f := \max N_f$.

Lemma 7.5. *For a height function $f : [n]_0 \rightarrow [n]_0$, N_f is a downset, P_f is an upset, and $|N_f \cap P_f| \leq 1$. In particular, f has at most one fixed point, necessarily γ_f .*

Proof: If $x \leq y \leq f(y)$, then $f(y) \leq f(x)$, so N_f is a downset. Similarly, P_f is an upset. Also, if x is positive, then $f(x)$ is negative, and if x is negative, then $f(x)$ is positive. Next, the intersection $N_f \cap P_f$ can have at most one element. Indeed, if x, y are distinct fixed points and $x < y$, then $y = f(y) \leq f(x) = x$, a contradiction. \square

Lemma 7.6. *For $f = \text{height}_\pi$, (γ_f, γ_f) is the (necessarily unique) intersection point of π with the diagonal.*

Proof: A straightforward geometric argument shows that the intersection point of π with the diagonal exists and is unique. Let $x = \gamma_f$, so $x \leq \text{height}_\pi(x)$ and $\text{height}_\pi(x+1) \leq x$. Thus, at time x , π moves from $(x, \text{height}_\pi(x))$ to $(x, \text{height}_\pi(x+1))$ through a sequence of South steps. Since $\text{height}_\pi(x+1) \leq x \leq \text{height}_\pi(x)$, the path π meets the point (x, x) . \square

Let us say now that a height function f is *self-adjoint* if (f, f) is a Galois connection. That is, f is self-adjoint if $y \leq f(x)$ is equivalent to $x \leq f(y)$, for each $x, y \in [n]_0$. We say f is *fixed-point-free* if $f(x) \neq x$, for each $x \in [n]_0$.

Lemma 7.7. *For a height function $f : [n]_0 \rightarrow [n]_0$, let π be the unique path such that $\text{height}_\pi = f$. Then f is self-adjoint if and only if π is symmetric along the diagonal, in which case f is fixed-point-free if and only if its first step after meeting the diagonal is an East step.*

Proof: For a path π , let π' be the path obtained from π by reflecting it along the diagonal. It is straightforward that $y \leq \text{height}_\pi(x)$ if and only if the point (x, y) lies below and on the left of π , if and only if the point (y, x) lies below and on the left of π' , if and only if $x \leq \text{height}_{\pi'}(y)$. Therefore, by identifying paths with height functions, the adjoint of π is the path obtained by reflecting along the diagonal. In particular, π is self-adjoint if and only if π is symmetric along the diagonal.

Let us argue that, for $f = \text{height}_\pi$ self-adjoint, f is fixed-point-free if and only if π 's first step after meeting the diagonal is towards East. Let (γ_f, γ_f) be the intersection point of π with the diagonal of $[n]_0$. We shall verify whether γ_f is a fixed point, since, by Lemma 7.5, it is the only candidate with this property. If the following step is a South step, then the last step before meeting the diagonal is an East step, which implies that $\gamma_f = \text{height}_\pi(\gamma_f)$, thus f has a fixed point. If the following step is an East step, then the last step before meeting the diagonal is a South step, which implies that $\gamma_f < \text{height}_\pi(\gamma_f) = f(\gamma_f)$, so f is fixed-point-free. \square

Proposition 7.8. *For $f : [n]_0 \rightarrow [n]_0$ a self-adjoint height function, define*

$$E_f := \{ \{x, y\} \mid x, y \in [n], x \neq y, y \leq f(x) \}.$$

Then $([n], E_f)$ is a threshold graph and $<$ is a degree ordering of $([n], E_f)$. The mapping $f \mapsto E_f$ restricts to a bijection from the set of self-adjoint height functions on $[n]_0$ to the set of threshold graphs of the form $([n], E)$ such that the standard linear ordering of $[n]$ is a degree ordering.

Proof: If $y < x$ and $z \leq f(x)$, then $z \leq f(x) \leq f(y)$, since f is antitone. As a consequence, if $y < x$, then $N_{E_f}(x) \subseteq N_{E_f}(y) \cup \{y\} = N_{E_f}[y]$, so $([n], E_f)$ is a threshold graph and $<$ is a degree ordering, by Lemma 7.4.

Conversely, let $([n], E)$ be a threshold graph for which the standard ordering is a degree ordering. As we have seen, $N_E(x)$ is a downset: if $y \in N_E(x)$ and $z \neq x$ is such that $z < y$, then $z \in N_E(x)$. Define then $f_E(x) := \max N_E(x)$, with the conventions that $\max \emptyset = 0$ and $N_E(0) = [n]_0$, so $f_E : [n]_0 \rightarrow [n]_0$. Observe that the following equivalences holds, by the definition of f_E and the fact that $N_E(x)$ is a downset: $\{x, y\} \in E$ if and only if $y \in N_E(x)$ if and only if $x \neq y$ and $y \leq f_E(x)$. It immediately follows that $y \leq f_E(x)$ if and only if $x \leq f_E(y)$, so f_E is self-adjoint; f_E is fixed-point-free since $x \notin N_E(x)$.

It is easily seen that $E_{f_E} = E$ and, whenever f is fixed-point-free, $f_{E_f} = f$ so under the latter hypothesis the two transformations are inverse to each other. \square

Let in the following TG_n^{dgo} be the set of pairs (w, E) such that $([n], E)$ is a threshold graph and $w \in S_n$ is a degree ordering of $([n], E)$. We can state now the main result of this section:

Theorem 7.9. *The mapping sending u to (λ_x^u, E^u) restricts to a bijection from D_n to TG_n^{dgo} .*

Proof: Firstly, we claim that the pair (λ_x^u, E^u) is constructed through intermediate steps, as suggested in the following diagram (the notation being used is explained immediately after):

$$\begin{array}{ccccc}
 u \in D_n & \longmapsto & (\lambda_x^u, \pi^u) & \longmapsto & (\lambda_x^u, \pi^u) \in S_n \times \text{Pi}_{\text{E,S}}^+([n]_0, [n]_0) \\
 \downarrow & & & & \downarrow \\
 & & & & (\lambda_x^u, \text{height}_{\pi^u}) \in S_n \times \text{HF}_{\text{fpf sa}}([n]_0) \\
 & & & & \downarrow \\
 (\lambda_x^u, E^u) & \xlongequal{\hspace{10em}} & & & (\lambda_x^u, \lambda_x^u \circ E_{\text{height}_{\pi^u}}) \in \text{TG}_n^{\text{dgo}}
 \end{array}$$

We also claim that each step in the upper leg of the diagram yields a bijection. In a second time, we shall verify that the pairs that may appear in the bottom right corner are exactly the elements of TG_n^{dgo} .

We explain the notation used in the diagram. For $u \in D_n$, we let $\underline{u} \in \{u, \underline{u}\}$ be such that $\underline{u}_1 > 0$. The first step of $\pi^{\underline{u}}$ after crossing the diagonal is an East step and therefore the height function corresponding to $\pi^{\underline{u}}$ is fixed-point-free. We let $\text{Pi}_{\text{E,S}}^+([n]_0, [n]_0)$ denote the set of East and South step paths from $(0, n)$ to $(n, 0)$ that make an East step after meeting the diagonal, and that are symmetric along the diagonal. We let $\text{HF}_{\text{fpf sa}}([n]_0)$ denote the set of fixed-point-free self-adjoint height functions of $[n]_0$. Then the height function is a bijection from $\text{Pi}_{\text{E,S}}^+([n]_0, [n]_0)$ to $\text{HF}_{\text{fpf sa}}([n]_0)$. For $f \in \text{HF}_{\text{fpf sa}}([n]_0)$, E_f is the set of edges defined in the statement of Lemma 7.8. Finally, if E is a set of edges on the vertices $[n]$ and $\sigma \in S_n$, then we let

$$\sigma \circ E := \{ \{\sigma(x), \sigma(y)\} \mid \{x, y\} \in E \} = \{ \{i, j\} \mid \{\sigma^{-1}(i), \sigma^{-1}(j)\} \in E \}.$$

We justify now the equality on the bottom line of the diagram. Notice that

$$E^u = \lambda_x^u \circ E_{\pi^u} \quad \text{with} \quad E_{\pi^u} := \{ \{x, y\} \mid x, y \in [n], x \neq y, (x, y) \text{ lies below } \pi^u \}$$

and that $E_{\pi^u} = E_{\pi^v} = E_f$ where $f = \text{height}_{\pi^u}$. Indeed, the condition that (x, y) lies below π^u amounts to saying that y is less than the height of π^u after x East steps. This shows that $E^u = \lambda_x^u \circ E_f$ with $f = \text{height}_{\pi^u}$.

Finally, the following equivalences are clear: f is a fixed-point-free self-adjoint height function on $[n]_0$ if and only if $<$ (the ordering given by the identity permutation) is a degree ordering of the threshold graph $([n], E_f)$ (by Proposition 7.8), if and only if the ordering given by the permutation σ is a degree ordering for the threshold graph $\sigma \circ E$. Thus, in the right bottom corner of the above diagram we have all the pairs (w, E) such that $([n], E)$ is a threshold graph and the linear ordering given by the permutation $w \in S_n$ is among its degree orderings. \square

Remark 7.10. Let f be a choice of mates, that is, a function $f : B_n \rightarrow B_n$ such that $f(u) = f(\underline{u}) \in \{u, \underline{u}\}$. Let $D_n^f = \{f(u) \mid u \in B_n\}$. In view of $E^u = E^{\underline{u}}$ and $\lambda_x^u = \lambda_x^{\underline{u}}$, the same argument appearing in the proof of Theorem 7.9 shows that the mapping $u \mapsto (\lambda_x^u, E^u)$ restricts to a bijection from D_n^f to TG_n^{dgo} . In particular, we shall consider the function $sm : B_n \rightarrow B_n$ picking the unique smooth mate $sm(u) \in \{u, \underline{u}\}$. Then D_n^{sm} is in bijection with TG_n^{dgo} .

Theorem 7.9 yields a natural representation of the weak ordering on D_n as follows. Order TG_n^{dgo} by saying that $(w_1, E_1) \leq (w_2, E_2)$ if and only if $w_1 \leq w_2$ in the weak ordering of S_n and, moreover, $E_1 \subseteq E_2$, so TG_n^{dgo} is clearly a poset.

Theorem 7.11. *The poset TG_n^{dgo} is a lattice isomorphic to the weak ordering of the Coxeter group D_n .*

Notice that TG_n^{dgo} is only loosely related to the lattice of threshold graphs of (Merris and Roby, 2005) where unlabeled (that is, up to isomorphism) threshold graphs are considered. While many are the remarks that we already could develop using this characterisation of the weak order on D_n , it is in the scope of future research to complete them and to give a satisfying description of this ordering.

8 Counting threshold graphs

Proposition 7.8 and Theorem 7.9, originally conceived for achieving a better understating of the structure of the lattices of Coxeter groups of type D, can also be used to enumerate threshold graphs. As the number of paths from $(0, n)$ to $(n, 0)$ that are symmetric along the diagonal and begin with an East step is easily seen to be 2^{n-1} , Proposition 7.8 yields a simple proof that the number of unlabeled threshold graphs is 2^{n-1} . Enumeration results for labeled threshold graphs appear in (Simpson Beissinger and Peled, 1987) and, more recently, in (Spiro, 2020; Galvin et al., 2022) (see also (Stanley, 2007, Exercise 5.25) and (Stanley, 2012, Exercise 3.115)). Theorem 7.9 can be used to give bijective proofs of these results. The ideas that we expose next have been suggested by the bijection described in (Spiro, 2020, §1), that we adapt here to fit the correspondences between threshold graphs coming with a degree ordering, signed and even-signed permutations, and simply barred permutations. Our starting point is the following observation:

Lemma 8.1. *Let $([n], E)$ be a threshold graph. Then, there exists a unique degree ordering w of $([n], E)$ such that, if $\deg_E(i) = \deg_E(j)$ and $i < j$, then $w^{-1}(i) < w^{-1}(j)$.*

Recall that $w^{-1}(i) < w^{-1}(j)$ means that i occurs before j in the permutation w , written as a word. The proof of the Lemma, straightforward, amounts to the remark that, given a degree ordering, we can permute vertices of equal degree and, in doing so, obtain a degree ordering. We call the ordering of Lemma 8.1 the *canonical degree ordering* of $([n], E)$.

In order to count threshold graphs we can count simply barred permutations (w, B) that, along the ideas developed in the previous section, bijectively correspond to some (w, E) such that w is the canonical degree ordering of $([n], E)$. To this goal, recall that, for (w, B) a simply barred permutation, the bars B split $[n]$ into blocks, the last of which might be empty. Observe that $i, j \in [n]$ appear in the same block of (w, B) if and only if they have equal height, by which we mean that, with $u = \psi(w, B)$ (cf. Definition 4.3), $\text{height}_{\pi^u}(w^{-1}(i)) = \text{height}_{\pi^u}(w^{-1}(j))$.

Definition 8.2. We say that a simply barred permutation (w, B) is *normal* if, whenever i, j belong to the same block and $i < j$, then $w^{-1}(i) < w^{-1}(j)$.

Observe from now that an ordered partition of $[n]$ yields two normal simply barred permutations. The partition is written as a word, where the blocks, already ordered, are written in increasing order and separated by a bar. This is the standard construction allowing to compute the number of partitions of $[n]$ into k blocks from the Eulerian numbers, see e.g. (Gessel and Stanley, 1978) or (Bóna, 2012, Theorem 1.17). If we add to the same word a bar in the last position n , then we obtain a second simply barred permutation. The following definition, more involved, shall receive a more intuitive meaning with the lemma that immediately follows.

Definition 8.3. The *central block* of a simply barred permutation (w, B) is the k -th block, with $k = \left\lceil \frac{|B|+1}{2} \right\rceil$.

Lemma 8.4. Let (w, B) be a simply barred permutation, let $u = \psi(w, B)$, and consider the path π^u . If π^u makes an East-South turn when meeting the diagonal, then the central block is the block of equal height immediately before the diagonal. If π^u makes a South-East turn when meeting the diagonal, then the central block is the block of equal height immediately after the diagonal.

Proof: Let, as before, $f = \text{height}_{\pi^u}$ and γ_f be such that π^u meets the diagonal in (γ_f, γ_f) .

Suppose that π^u makes an East-South turn when meeting the diagonal. Since bars bijectively correspond to East-South turns (cf. Lemma 4.6) and π^u is symmetric along the diagonal, then $|B|$ is odd, say $|B| = 2k - 1$. The path π^u therefore makes $k - 1$ East-South turns strictly on the left of the diagonal. Thus, there are $k - 1$ vertical bars strictly on the left of γ_f and therefore, for $k = \left\lceil \frac{|B|+1}{2} \right\rceil$, the k -th block is the group of equal height immediately before meeting the diagonal—that is, the block containing $w(\gamma_f)$.

If π^u makes a South-East turn when meeting the diagonal, then $|B|$ is even. Say $|B| = 2k - 2$, so $k = \left\lceil \frac{|B|+1}{2} \right\rceil$. The path π^u makes $k - 1$ East-South turns before meeting the diagonal, and therefore there are $k - 1$ bars on the left of γ_f . Since π^u makes a South-East turn when meeting the diagonal, the last of these bars is in position γ_f . The k -th block immediately occurs after γ_f : it is the block of equal height containing $w(\gamma_f + 1)$. \square

Definition 8.5. We say that a simply barred permutation (w, B) is *compatible* if, for $u = \psi(w, B)$ and $i, j \in [n]$, $\text{deg}_{E^u}(i) = \text{deg}_{E^u}(j)$ if and only if i, j appear in the same block of B .

Lemma 8.6. For a simply barred permutation (w, B) , the following are equivalent:

1. (w, B) is compatible,
2. $\psi(w, B)$ is smooth,

3. the central block has at least two elements.

Proof: Let us argue that 1. is equivalent to 2. For $u = \psi(w, B)$, consider the path representation $(\pi^u, \lambda_x^u, \lambda_y^u)$ and the threshold graph $([n], E^u)$, so $w = \lambda_x^u$. For readability, we also let $f = \text{height}_{\pi^u}$. Recall that i, j appear in the same block of B if and only if $f(w^{-1}(i)) = f(w^{-1}(j))$. On the other hand, $\deg_{E^u}(i) = f(w^{-1}(i)) - 1$, if $w^{-1}(i) \leq f(w^{-1}(i))$ and, otherwise, $\deg_{E^u}(i) = f(i)$. That is, up to a renaming of vertices, the degree is computed as the height modulo a normalisation by one before meeting the diagonal. Therefore (w, B) is not compatible, if and only if, for some x, y such that $x \leq f(x)$ and $f(y) < y$, we have $f(x) - 1 = f(y)$. Considering that $x < y$ and that f is antitone, this happens exactly when $f(\gamma_f) - 1 = f(\gamma_f + 1)$. The latter condition holds exactly when π^u , immediately after meeting the diagonal, either takes a South step followed by an East step, or takes an East step followed by a South step. In turn, this condition amounts to saying that u is not smooth.

Let us argue that 2. is equivalent to 3. If π^u makes an East-South turn when meeting the diagonal, then u is smooth if and only if also its second step after meeting the diagonal is a South step, if and only if, before meeting the diagonal, π^u takes two East steps, if and only if the central block has at least two elements. If π^u makes a South-East turn when meeting the diagonal, then u is smooth if and only if also its second step after meeting the diagonal is an East step, if and only if the central block has at least two elements. \square

Theorem 8.7. *There is a bijection between the set of threshold graphs on the vertex set $[n]$ and the normal simply barred permutations whose central block has at least two elements. The bijection sends vertices of equal degree to vertices in the same block.*

Proof: Given the threshold graph $([n], E)$, let w be its canonical degree ordering.

For $(w, E) \in \text{TG}_n^{\text{dgo}}$, we let $u \in \mathbb{B}_n$ be the unique smooth signed permutation corresponding to (w, E) under the bijection described in Theorem 7.9 and Remark 7.10, and let $(w', B) = \psi^{-1}(u)$. Notice that $w = w'$. Then, by Lemma 8.6, the central block of (w, B) has cardinality at least two, since it corresponds to a smooth signed permutation. Since two vertices have equal degree if and only if they belong to the same block, (w, B) is normal if and only if w is the canonical degree ordering of $([n], E)$. \square

The following corollary allows us to simplify the counting arguments that follow.

Corollary 8.8. *There is a bijection between the set of threshold graphs on the set $[n]$ and the set of normal simply barred permutations such that the first block has at least two elements.*

Proof: Given a normal simply barred permutation whose central block has at least two elements, we move this central block in first position by permuting the blocks. This transformation is reversible, since we can determine the position where to move back the first block from the cardinality of B . \square

We can count then the labeled threshold graphs on $[n]$ according to the number i of blocks of equal degree. By the previous considerations, this amounts to counting normal simply barred permutations whose first block has at least two elements. Considering that normal simply barred permutations are sort of ordered partitions, recall that $\left\{ \begin{smallmatrix} n \\ i \end{smallmatrix} \right\}$, the Stirling number of the second kind, counts the number of unordered partitions of an n -element set into i blocks. We immediately recover the formula from

(Simpson Beissinger and Peled, 1987, §3) for the number $T_{n,i}$ of threshold graphs on n vertices with i different degrees:

$$T_{n,i} = 2 \cdot (i! \cdot \left\{ \begin{matrix} n \\ i \end{matrix} \right\} - n \cdot (i-1)! \left\{ \begin{matrix} n-1 \\ i-1 \end{matrix} \right\}) = 2 \cdot (i-1)! \cdot (i \cdot \left\{ \begin{matrix} n \\ i \end{matrix} \right\} - n \cdot \left\{ \begin{matrix} n-1 \\ i-1 \end{matrix} \right\}).$$

The formula can be understood as follows: out of all the ordered partitions of $[n]$ into i blocks, eliminate those whose first block is a singleton; transform then such an ordered partition into a simply barred permutation by adding or not a bar in the last position. Notice that, under the bijection, there is a bar in the last position if and only if no vertex is isolated (i.e. has degree 0).

It is well known that an ordered partition, when transformed into a simply barred permutation as before, is determined by its set of descent positions (that are always barred), and by the set of the other barred positions (ascent positions, those that are not descent positions). Whence, we can count simply barred permutations according to the number of descents. This yields the formula from (Spiro, 2020) for counting the number T_n of threshold graphs as the sum of the numbers $\tau_{n,k}$:

$$T_n = \sum_{k=0, \dots, n-2} \tau_{n,k}, \quad \text{with} \quad \tau_{n,k} = 2 \cdot P(n, k) \cdot 2^{n-2-k} = P(n, k) \cdot 2^{n-1-k}.$$

Above $P(n, k) = (k+1) \cdot \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle$ is the number of permutations of the set $[n]$ having exactly k descents, and whose first block has at least two elements, see (Spiro, 2020, Lemma 6). The number $\tau_{n,k}$ can be interpreted as the number of threshold graphs whose ordered partition determined by equal degree has exactly k descents. The explicit formula for $\tau_{n,k}$ stems from the fact that, in order to construct such a partition, we can choose a permutation whose first block has at least two elements, and then add other bars at ascent positions except for the first ascent position.

That threshold graphs are related to the families B and D in the theory of Coxeter groups has already been observed, see e.g. (Edelman and Reiner, 1994), (Stanley, 2007, Exercise 5.25), and (Stanley, 2012, Exercise 3.115). It needs to be emphasized, however, that the way we came up with threshold graphs is orthogonal to the way threshold graphs are being used in these works. As part of future research, we wish to investigate the bijections presented in Theorems 7.9 and 8.7 (which can be adapted to fit the type B) with the goal of understanding whether they play any role with respect to the problem, dealt with in (Edelman and Reiner, 1994), of characterizing free sub-arrangements of the Coxeter arrangements of type B. We also aim to understand whether the connection with threshold graphs established here can shed some light on the problem of giving combinatorial characterisations of the notion of free arrangement (Stanley, 2007, §4).

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References

E. Bagno, R. Biagioli, and D. Garber. Stirling and eulerian numbers of types B and D . In L. Ferrari and M. Vamvakari, editors, *Proceedings of the 11th International Conference on Random and*

- Exhaustive Generation of Combinatorial Structures, GASCom 2018, Athens, Greece, June 18-20, 2018*, volume 2113 of *CEUR Workshop Proceedings*, pages 53–59. CEUR-WS.org, 2018. URL <http://ceur-ws.org/Vol-2113/paper4.pdf>.
- E. Barnard. The canonical join complex. Preprint, retrievable from <https://arxiv.org/abs/1610.05137>, 2016.
- A. Björner. Orderings of Coxeter groups. In *Combinatorics and Algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 175–195. Amer. Math. Soc., Providence, RI, 1984.
- A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005. ISBN 978-3540-442387; 3-540-44238-3.
- M. Bóna. *Combinatorics of permutations*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2012. ISBN 978-1-4398-5051-0. doi: 10.1201/b12210. URL <https://doi.org/10.1201/b12210>. With a foreword by Richard Stanley.
- N. Caspard, L. Santocanale, and F. Wehrung. Permutohedra and associahedra. In *Lattice theory: special topics and applications. Vol. 2*, pages 215–286. Birkhäuser/Springer, Cham, 2016.
- V. Chvátal and P. L. Hammer. Aggregation of inequalities in integer programming. In *Studies in integer programming (Proc. Workshop, Bonn, 1975)*, pages 145–162. Ann. of Discrete Math., Vol. 1, 1977.
- B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002. ISBN 0-521-78451-4.
- P. H. Edelman and V. Reiner. Free hyperplane arrangements between A_{n-1} and B_n . *Math. Z.*, 215(3):347–365, 1994. ISSN 0025-5874. doi: 10.1007/BF02571719. URL <https://doi.org/10.1007/BF02571719>.
- R. Freese, J. Ježek, and J. Nation. *Free Lattices*, volume 42 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1995. ISBN 0-8218-0389-1.
- D. Galvin, G. Wesley, and B. Zacovic. Enumerating threshold graphs and some related graph classes. *J. Integer Seq.*, 25(2):article 22.2.7, 36, 2022. ISSN 1530-7638. URL cs.uwaterloo.ca/journals/JIS/VOL25/Galvin/galvin2.html.
- I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.
- M. J. Gouveia and L. Santocanale. The continuous weak order. *Journal of Pure and Applied Algebra*, 225(2):106472, 2021. ISSN 0022-4049. doi: <https://doi.org/10.1016/j.jpaa.2020.106472>. URL <https://www.sciencedirect.com/science/article/pii/S0022404920301730>.
- G. Guilbaud and P. Rosenstiehl. Analyse algébrique d’un scrutin. *Math. Sci. Hum.*, 4:9–33, 1963.
- N. V. R. Mahadev and U. N. Peled. *Threshold graphs and related topics*, volume 56 of *Annals of Discrete Mathematics*. North-Holland Publishing Co., Amsterdam, 1995. ISBN 0-444-89287-7.
- R. Merris and T. Roby. The lattice of threshold graphs. *JIPAM. J. Inequal. Pure Appl. Math.*, 6(1):Article 2, 21, 2005. ISSN 1443-5756.

- E. Nelson. Galois connections as left adjoint maps. *Comment. Math. Univ. Carolinae*, 17(3):523–541, 1976. ISSN 0010-2628.
- T. K. Petersen. *Eulerian numbers*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2015. ISBN 978-1-4939-3090-6; 978-1-4939-3091-3. doi: 10.1007/978-1-4939-3091-3. URL <https://doi.org/10.1007/978-1-4939-3091-3>. With a foreword by Richard Stanley.
- M. Pouzet, K. Reuter, I. Rival, and N. Zaguia. A generalized permutahedron. *Algebra Universalis*, 34(4):496–509, 1995. ISSN 0002-5240. doi: 10.1007/BF01181874. URL <https://doi.org/10.1007/BF01181874>.
- L. Santocanale. On discrete idempotent paths. In R. Mercaş and D. Reidenbach, editors, *WORDS 2019*, volume 11682 of *Lecture Notes in Comput. Sci.*, pages 312–325. Springer, Cham, 2019.
- L. Santocanale. Bijective proofs for Eulerian numbers in types B and D. In M. Couceiro, P. Monnin, and A. Napoli, editors, *Proceedings of the 1st International Conference on Algebras, Graphs and Ordered Sets (ALGOS 2020)*, pages 81–94, Aug. 2020. Retrievable from <https://hal.archives-ouvertes.fr/hal-02918958/>.
- L. Santocanale and F. Wehrung. Generalizations of the permutohedron. In *Lattice Theory: Special Topics and Applications. Vol. 2*, pages 287–397. Birkhäuser, 2016.
- L. Santocanale and F. Wehrung. The equational theory of the weak Bruhat order on finite symmetric groups. *Journal of the European Mathematical Society*, 20(8):1959–2003, 2018.
- J. Simpson Beissinger and U. N. Peled. Enumeration of labelled threshold graphs and a theorem of Frobenius involving Eulerian polynomials. *Graphs Comb.*, 3:213–219, 1987. ISSN 0911-0119. doi: 10.1007/BF01788543.
- S. Spiro. Counting labeled threshold graphs with Eulerian numbers. *Australas. J. Comb.*, 77:249–255, 2020. ISSN 1034-4942. URL ajc.maths.uq.edu.au/pdf/77/ajc_v77_p249.pdf.
- R. P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.
- R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012. ISBN 978-1-107-60262-5.
- J. R. Stembridge. Some permutation representations of Weyl groups associated with the cohomology of toric varieties. *Adv. Math.*, 106(2):244–301, 1994. ISSN 0001-8708. doi: 10.1006/aima.1994.1058. URL <https://doi.org/10.1006/aima.1994.1058>.