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BIJECTIVE PROOFS FOR EULERIAN NUMBERS IN TYPES B AND D

DEDICATED TO MAURICE POUZET ON THE OCCASION OF HIS 75TH BIRTHDAY

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ABSTRACT

Let $\langle n \rangle_k$, $\langle B_n \rangle_k$, and $\langle D_n \rangle_k$ be the Eulerian numbers in the types A, B, and D, respectively—that is, the number of permutations of n elements with k descents, the number of signed permutations (of n elements) with k type B descents, the number of even signed permutations (of n elements) with k type D descents. Let $S_n(t) = \sum_{k=0}^{n-1} \langle n \rangle_k t^k$, $B_n(t) = \sum_{k=0}^n \langle B_n \rangle_k t^k$, and $D_n(t) = \sum_{k=0}^n \langle D_n \rangle_k t^k$. We give bijective proofs of the identity

$$B_n(t^2) = (1+t)^{n+1} S_n(t) - 2^n t S_n(t^2)$$

and of Stembridge's identity

$$D_n(t) = B_n(t) - n 2^{n-1} t S_{n-1}(t).$$

These bijective proofs rely on a representation of signed permutations as paths. Using this representation we also establish a bijective correspondence between even signed permutations and pairs (w, E) with $([n], E)$ a threshold graph and w a degree ordering of $([n], E)$.

1 Introduction

The Eulerian numbers $\langle n \rangle_k$ count the number of permutations in the symmetric group S_n that have k descent positions. Let us recall that, for a permutation $w = w_1 w_2 \dots w_n \in S_n$ (thus, with $w_i \in \{1, \dots, n\}$ and $w_i \neq w_j$ for $i \neq j$), a *descent* of w is an index (or position) $i \in \{1, \dots, n-1\}$ such that $w_i > w_{i+1}$.

This is only one of the many interpretations that we can give to these numbers, see e.g. [16], yet it is intimately order-theoretic. The set S_n can be endowed with a lattice structure, known as the weak (Bruhat) ordering on permutations or Permutohedron, see e.g. [12, 6]. Descent positions of $w \in S_n$ are then bijection with its lower covers, so the Eulerian numbers $\langle n \rangle_k$ can also be taken as counting the number of permutations in S_n with k lower covers. In particular, $\langle n \rangle_1 = 2^n - n - 1$ is the number of join-irreducible elements in S_n . A subtler order-theoretic interpretation is given in [2]: since the S_n are (join-)semidistributive as lattices, every element can be written canonically as the join of join-irreducible elements [9]; the numbers $\langle n \rangle_k$ count then the permutations $w \in S_n$ that can be written canonically as the join of k join-irreducible elements.

The symmetric group S_n is a particular instance of a Coxeter group, see [4], since it yields a concrete realization of the Coxeter group A_{n-1} in the family A. Notions of length, descent, inversion, and also a weak order, can be defined for elements of an arbitrary finite Coxeter group [3]. We shift the focus to the families B and D of Coxeter groups. More precisely, this paper concerns the Eulerian numbers in the types B and D. The Eulerian number $\langle B_n \rangle_k$ (resp., $\langle D_n \rangle_k$) counts the number of elements in the group B_n (resp., D_n) with k descent positions. Order-theoretic interpretations

* This is a revised version of the manuscript [19] appeared in the proceedings of the conference ALGOS 2020

of these numbers, analogous to the ones mentioned for the standard Eulerian numbers, are still valid. As the abstract group A_{n-1} has a concrete realization as the symmetric group S_n , the group B_n (resp., D_n) has a realization as the hyperoctahedral group of signed permutations (resp., the group of even signed permutations). Starting from these concrete representations of Coxeter groups in the types B and D, we pinpoint some new representations of signed permutations. Relying on these representations we provide bijective proofs of known formulas for Eulerian numbers in the types B and D. These formulas allow to compute the Eulerian numbers in the types B and D from the better known Eulerian numbers in the type A.

Let $S_n(t)$ and $B_n(t)$ be the Eulerian polynomials in the types A and B:

$$S_n(t) := \sum_{k=0}^{n-1} \left\langle n \atop k \right\rangle t^k, \quad B_n(t) := \sum_{k=0}^n \left\langle B_n \atop k \right\rangle t^k. \quad (1)$$

In [16, §13, p. 215] the following polynomial identity is stated:

$$2B_n(t^2) = (1+t)^{n+1}S_n(t) + (1-t)^{n+1}S_n(-t). \quad (2)$$

Considering that, for $f(t) = \sum_{k \geq 0} a_k t^k$,

$$f(t) + f(-t) = 2 \sum_{k \geq 0} a_{2k} t^{2k},$$

the polynomial identity (2) amounts to the following identity among coefficients:

$$\left\langle B_n \atop k \right\rangle = \sum_{i=0}^{2k} \left\langle n \atop i \right\rangle \binom{n+1}{2k-i}. \quad (3)$$

We present a bijective proof of (3) and also establish the identity

$$2^n \left\langle n \atop k \right\rangle = \sum_{i=0}^{2k+1} \left\langle n \atop i \right\rangle \binom{n+1}{2k+1-i}. \quad (4)$$

Considering that, for $f(t) = \sum_{k \geq 0} a_k t^k$,

$$f(t) - f(-t) = 2 \sum_{k \geq 0} a_{2k+1} t^{2k+1},$$

the identity (4) yields the polynomial identity:

$$2^{n+1}tS_n(t^2) = (1+t)^{n+1}S_n(t) - (1-t)^{n+1}S_n(-t).$$

More importantly, (3) and (4) jointly yield the polynomial identity

$$(1+t)^{n+1}S_n(t) = B_n(t^2) + 2^n t S_n(t^2). \quad (5)$$

Let now $D_n(t)$ be the Eulerian polynomial in type D:

$$D_n(t) := \sum_{k=0}^n \left\langle D_n \atop k \right\rangle t^k.$$

Investigating further the terms $2^n S_n(t)$, we could find a simple bijective proof, that we present here, of Stembridge's identity [24, Lemma 9.1]

$$D_n(t) = B_n(t) - n2^{n-1}tS_{n-1}(t), \quad (6)$$

which, in terms of the Eulerian numbers in type D, amounts to

$$\left\langle D_n \atop k \right\rangle = \left\langle B_n \atop k \right\rangle - n2^{n-1} \left\langle n-1 \atop k-1 \right\rangle.$$

The proofs presented here differ from known proofs of the identities (2) and (6). As suggested in [16], the first identity may be derived by computing the f -vector of the type B Coxeter complex and then by applying the transform yielding h -vector from the f -vector. A similar method is used in [24] to prove the identity (6). Our proofs directly rely on the combinatorial properties of signed/even signed permutations and on two representations of these objects that we call the path representation of a signed permutation and simply barred permutations. The idea is that a signed permutation of $[n]$ can be organised into a discrete path from $(n, 0)$ to $(0, n)$ that uses East and South steps and that, by projecting onto the x axis, we obtain a permutation divided into blocks, as suggested in Figure 1. As a byproduct of these representations, we also obtain a bijection between even signed permutations of $[n]$ and pairs (w, E) where $([n], E)$ is a threshold graph and w is a permutation or, better, a linear ordering of $[n]$ that is a degree ordering for $([n], E)$. Under the bijection, the ordering of D_n is coordinatewise, that is, we have $(w_1, E_1) \leq (w_2, E_2)$ if and only if $w_1 \leq w_2$ in S_n and $E_1 \subseteq E_2$. We are confident that such representation of the ordering will shed some light on the lattice structure of the weak order on the Coxeter groups of type D, our main goal since the inception of this research.

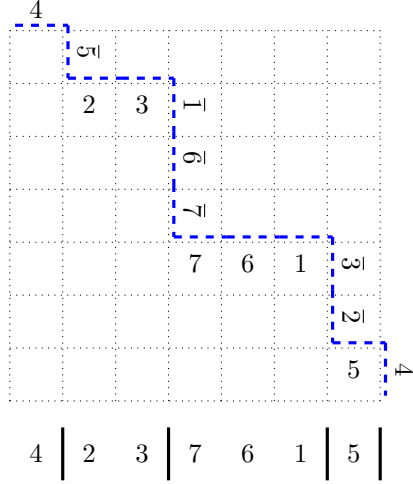


Figure 1: Signed permutations as paths and as barred permutations

2 Notation, elementary definitions, and facts

The notation used is chosen to be consistent with [16]. We use $[n]$ for the set $\{1, \dots, n\}$ and S_n for the set of permutations of $[n]$. We use $[n]_0$ for the set $\{0, 1, \dots, n\}$, $[-n]$ for $\{-n, \dots, -1\}$, and $[\pm n]$ for $\{-n, \dots, -1, 1, \dots, n\}$. We write a permutation $w \in S_n$ as a word $w = w_1 w_2 \dots w_n$, with $w_i \in [n]$. For $w \in S_n$, its set of descents and its set of inversions¹ are defined follows:

$$\text{Des}(w) := \{i \in \{1, \dots, n-1\} \mid w_i > w_{i+1}\}, \quad \text{Inv}(w) := \{(i, j) \mid 1 \leq i < j \leq n, w^{-1}(i) > w^{-1}(j)\}.$$

Then, we let

$$\text{des}(w) := |\text{Des}(w)|.$$

The Eulerian number $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$, counting the number of permutations of n elements with k descents, can be formally defined as follows:

$$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle := |\{w \in S_n \mid \text{des}(w) = k\}|.$$

Let us define a *signed permutation of $[n]$* as a permutation u of $[\pm n]$ such that, for each $i \in [\pm n]$, $u_{-i} = -u_i$. We use B_n for the set of signed permutations of $[n]$. When writing a signed permutation u as a word $u_{-n} \dots u_{-1} u_1 \dots u_n$, we prefer writing $u_i = \bar{x}$ in place of $-x$ if $u_i < 0$ and $|u_i| = x$. Also, we often write $u \in B_n$ in *window notation*, that is, we only write the suffix $u_1 u_2 \dots u_n$; indeed, the prefix $u_{-n} u_{-n-1} \dots u_{-1}$ is determined as the mirror of the suffix $u_1 u_2 \dots u_n$ up to exchanging the signs. The set B_n is a subgroup of the group of permutations of the set $[\pm n]$ and, as mentioned before, it is the standard model for the Coxeter group in the family B with n generators. Therefore, general notions from the theory of Coxeter groups (descent, inversion) apply to signed permutations. We present below, as definitions, the well-known explicit formulas for the descent and inversion sets of $u \in B_n$. We let

$$\text{Des}_B(u) := \{i \in \{0, \dots, n-1\} \mid u_i > u_{i+1}\}, \quad \text{Inv}_B(u) := \{(i, j) \mid 1 \leq |i| \leq j \leq n, u^{-1}(i) > u^{-1}(j)\},$$

where we set $u_0 := 0$, so 0 is a descent of u if and only if $u_1 < 0$,

$$\text{des}_B(u) = |\text{Des}_B(u)|, \quad \left\langle \begin{smallmatrix} B_n \\ k \end{smallmatrix} \right\rangle := |\{u \in B_n \mid \text{des}_B(u) = k\}|.$$

The definition of the Eulerian polynomials in the types A and B appears in (1). Let us mention that the type A Eulerian polynomial is often (for example in [5]) defined as follows:

$$A_n(t) := \sum_{k=1}^n \left\langle \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\rangle t^k = t S_n(t).$$

¹It is also possible to define $\text{Inv}(w)$ as the set $\{(i, j) \mid 1 \leq i < j \leq n, w_i > w_j\}$. The definition given above is better suited for the order-theoretic approach.

We exclusively manipulate the polynomials $S_n(t)$ and never the $A_n(t)$. Notice that $S_n(t)$ has degree $n - 1$ and $B_n(t)$ has degree n .

We shall introduce later even signed permutations and their groups, as well as related notions arising from the fact that these groups are standard models for Coxeter groups in the family D.

For $u \in B_n$ we let $\text{Des}_B^+(u) := \text{Des}_B(u) \setminus \{0\}$, that is, $\text{Des}_B^+(u)$ is the set of strictly positive descents of u . Let us observe the following:

Lemma 2.1. $|\{u \in B_n \mid \text{Des}_B^+(u) = k\}| = 2^n \binom{n}{k}$.

Proof. By considering its window notation, a signed permutation u yields a mapping $\tilde{u} : [n] \rightarrow [\pm n]$. This mapping has a unique decomposition of the form $\tilde{u} = \iota \circ w$ with $w \in S_n$ and $\iota : [n] \rightarrow [\pm n]$ an order preserving injection such that $x \in \iota([n])$ iff $-x \notin \iota([n])$. The monotone injections with this property are uniquely determined by their positive image $\iota([n]) \cap [n]$, so there are 2^n such injections. Moreover, for $i = 1, \dots, n - 1$, $w_i > w_{i+1}$ if and only if $u_i > u_{i+1}$, so $|\text{Des}_B^+(\iota \circ w)| = |\text{Des}(w)|$. \square

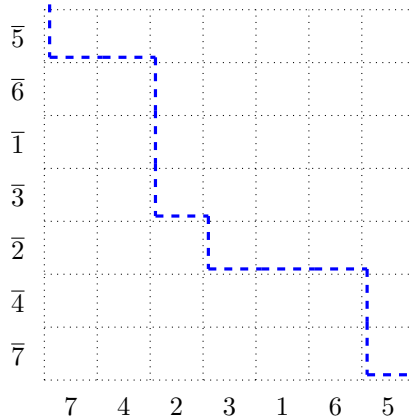
Example 2.2. Consider the signed permutation $u := 3\bar{4}1\bar{2}\bar{5}$. Then $\tilde{u} = \iota \circ w$ with $w = 52431$ and ι the order preserving map $\bar{5}\bar{4}\bar{2}\bar{1}3$ with $\iota([n]) \cap [n] = \{1, 3\}$. \diamond

3 Path representation of signed permutations, simply barred permutations

We present here our combinatorial tools to deal with signed permutations, the path representation and the simply barred permutations.

Definition 3.1. The *path representation* of $u \in B_n$ is a triple $(\pi^u, \lambda_x^u, \lambda_y^u)$ where π^u is a discrete path, drawn on a grid $[n]_0 \times [n]_0$ and joining the point $(0, n)$ to the point $(n, 0)$, $\lambda_x^u : [n] \rightarrow [n]$, and $\lambda_y^u : [n] \rightarrow [-n]$. The triple $(\pi^u, \lambda_x^u, \lambda_y^u)$ is constructed from u according to the following algorithm: (i) u is written in full notation as a word and scanned from left to right: each positive letter yields an East step (a length 1 step along the x -axis towards the right), and each negative letter yields a South step (a length 1 step along the y -axis towards the bottom); (ii) the labelling $\lambda_x^u : [n] \rightarrow [n]$ is obtained by projecting each positive letter on the x -axis, (iii) the labelling $\lambda_y^u : [n] \rightarrow [-n]$ is obtained by projecting each negative letter on the y -axis.

Example 3.2. Consider the signed permutation $u := \bar{2}316\bar{4}\bar{7}\bar{5}$, in window notation, that is, $\bar{5}7\bar{4}\bar{6}\bar{1}3\bar{2}\bar{2}316\bar{4}\bar{7}\bar{5}$, in full notation. Applying the algorithm above, we draw the path π^u and the labellings λ_x^u, λ_y^u as follows:



Therefore, π^u is the dashed blue path, λ_x^u is the permutation 7423165, and λ_y^u is $\bar{7}\bar{4}\bar{2}\bar{3}\bar{1}\bar{6}\bar{5}$. \diamond

It is easily seen that, for an arbitrary $u \in B_n$, $(\pi^u, \lambda_x^u, \lambda_y^u)$ has the following properties:

- (i) π^u is symmetric along the diagonal,
- (ii) $\lambda_x^u \in S_n$ and, moreover, it is the subword of u of positive letters,
- (iii) for each $i \in [n]$, $\lambda_y^u(i) = \overline{\lambda_x^u(i)}$ and, moreover, λ_y^u is the mirror of the subword of u of negative letters.

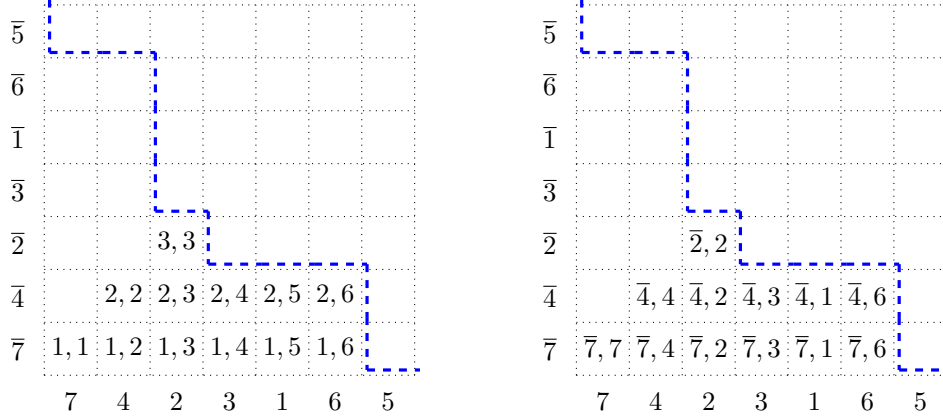


Figure 2: Negative inversions of u , indexed on the left by the identity and, on the right, by λ_y^u, λ_x^u

In particular, we see that the data $(\pi^u, \lambda_x^u, \lambda_y^u)$ is redundant since λ_y^u is completely determined from λ_x^u .

Proposition 3.3. *The mapping $u \mapsto (\pi^u, \lambda_x^u)$ is a bijection from the set of signed permutations B_n to the set of pairs (π, w) , where $w \in S_n$ and π is a discrete path from $(0, n)$ to $(n, 0)$ with East and South steps which, moreover, is symmetric along the diagonal.*

We leave the reader verify the above statement. Next we argue for the interest of this representation by looking at the inversion set of a signed permutation. We claim that the type B inversions of u can be identified as the type A inversions of λ_x^u (the positive inversions of u) and the ordered pairs $(\lambda_y^u(i), \lambda_x^u(j))$ such that the cell (i, j) —where $1 \leq i \leq j \leq n$, i is the row number and j is the column number, lines being indexed from bottom to top—lies below π^u (we call these the negative inversions of u). This is exemplified in Figure 2 with the signed permutation $\overline{2}31\overline{6}4\overline{7}5$ from Example 3.2. It might be the case that $\lambda_x^u(j) < |\lambda_y^u(j)|$, in which case we identify the pair $(\lambda_y^u(i), \lambda_x^u(j))$ with $(\lambda_y^u(j), \lambda_x^u(i))$.

More in general, notice that, for $1 \leq i \leq j \leq n$, the cell (i, j) lies below π^u if and only if (j, i) does, by symmetry of π^u along the diagonal. We can therefore identify negative inversions of u with the unordered pairs $\{\lambda_y^u(i), \lambda_x^u(j)\}$ (thus doubletons or singletons) such that (i, j) lies below π^u . This observation opens the way to the graph theoretic approach of Section 6. Thus, for the signed permutation $\overline{2}31\overline{6}4\overline{7}5$ of Example 3.2, we identify the set of type B inversions of $\overline{2}31\overline{6}4\overline{7}5$ with the disjoint union of the set of type A inversions of 7423165 and the set of unordered pairs

$$\{\{7, 7\}, \{7, 4\}, \{7, 2\}, \{7, 3\}, \{7, 1\}, \{7, 6\}, \{4, 4\}, \{4, 2\}, \{4, 3\}, \{4, 1\}, \{4, 6\}, \{2, 2\}\}.$$

Let us argue for this formally.

Proposition 3.4. *Let $u \in B_n$. For each i, j with $1 \leq |i| \leq j \leq n$, $(i, j) \in \text{Inv}_B(u)$ if and only if either $1 \leq i < j \leq n$ and $(i, j) \in \text{Inv}(\lambda_x^u)$ or $i < 0$ and $((\lambda_x^u)^{-1}(-i), (\lambda_x^u)^{-1}(j))$ lies below the path π^u .*

Proof. Let us consider a pair (i, j) such that $1 \leq |i| \leq j \leq n$ and such that, if $0 < i$, then $i < j$.

If $0 < i < j$, then both i and j appear in λ_x^u , which is the subword of u (written in full notation) of positive integers. Then $u^{-1}(i) > u^{-1}(j)$ if and only if $(\lambda_x^u)^{-1}(i) > (\lambda_x^u)^{-1}(j)$, that is, $(i, j) \in \text{Inv}_B(u)$ if and only if $(i, j) \in \text{Inv}(\lambda_x^u)$.

We suppose next that $i < 0$. Observe that, as suggested in Figure 3, if $x > 0$ and $\overline{y} < 0$, then the cell identified by λ_y^u, λ_x^u as (\overline{y}, x) is below π^u if and only if the letter x appears before the letter \overline{y} in u . Also, for such a pair, x appears before \overline{y} in u if and only if $(\overline{y}, x) \in \text{Inv}(u)$, where u is considered as a permutation of the set $[\pm n]$ and the set of inversions is computed w.r.t the standard linear order on this set.

Therefore, if (i, j) with $i < 0$ and $1 \leq |i| \leq j \leq n$, then $(i, j) \in \text{Inv}_B(u)$ if and only if $(i, j) \in \text{Inv}(u)$ if and only if the cell identified by λ_y^u, λ_x^u as (i, j) is below π^u . If, instead of using λ_y^u and λ_x^u to identify cells, we use the identity permutation, this happens when $((\lambda_y^u)^{-1}(i), (\lambda_x^u)^{-1}(j)) = ((\lambda_x^u)^{-1}(-i), (\lambda_y^u)^{-1}(j))$ is below π_u . \square

We consider next a second representation of signed permutations.

Definition 3.5. *A simply barred permutation of $[n]$ is a pair (w, B) where $w \in S_n$ and $B \subseteq \{1, \dots, n\}$. We let SBP_n be the set of simply barred permutations of $[n]$.*

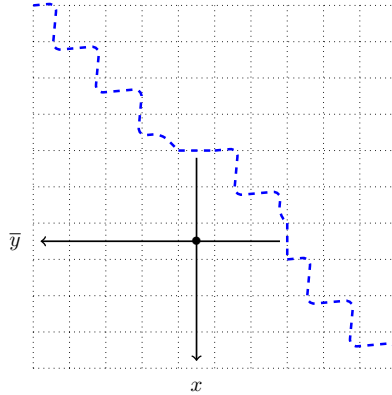


Figure 3: Characterizing inversions of the form (i, j) with i negative

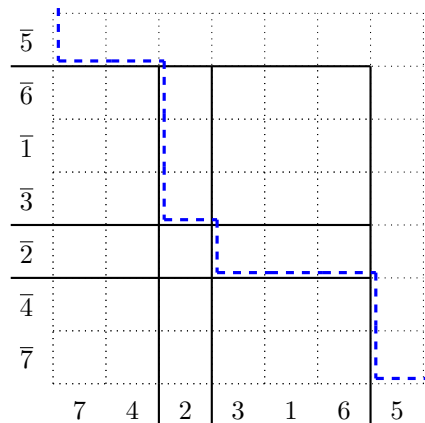
We think of B as a set of positions of w , the barred positions or walls. We have added the adjective “simply” to “barred permutation” since we do not require that B is a superset of $\text{Des}(w)$, as for example in [10].

Example 3.6. We write a simply barred permutation (w, B) as a permutation divided into blocks by the bars, placing a vertical bar after w_i for each $i \in B$. For example, $(w, B) = (7423165, \{2, 4, 6\})$ is written $74|23|16|5$. Notice that we allow a bar to appear in the last position, for example $34|1|265|7|$ stands for the simply barred permutation $(3412657, \{2, 3, 6, 7\})$. Thus, a bar appears in the last position if and only if the last block is empty. The last block is indeed the only block that can be empty, which amounts to saying that consecutive bars are not allowed in simply barred permutations. This contrasts with other notions of barred permutations, see for example those appearing in the proof of the alternating sum formula for the Eulerian numbers [5, Theorem 1.11]. \diamond

Next, we describe a bijection—that we call ψ —from the set SBP_n to \mathbb{B}_n . Let us notice that, in order to establish equipotence of these two sets, other more straightforward bijections are available.

Definition 3.7. For $(w, B) \in \text{SBP}_n$, we define the signed permutation $\psi(w, B) \in \mathbb{B}_n$ according to the following algorithm: (i) draw the grid $[n]_0 \times [n]_0$; (ii) since $B \subseteq [n]$, $B \times B$ defines a subgrid of $[n]_0 \times [n]_0$, construct the upper anti-diagonal π of this subgrid; (iii) $\psi(w, B)$ is the signed permutation u whose path representation $(\pi^u, \lambda_x^u, \lambda_y^u)$ equals to (π, w, \bar{w}) .

Example 3.8. The construction just described can be understood as raising the bars and transforming them into a grid. For example, for the simply barred permutation $74|2|316|5$ (that is, (w, B) with $w = 7423165$ and $B = \{2, 3, 6\}$) the construction is as follows:



The dashed path is the upper anti-diagonal of the subgrid. The resulting signed permutation $\psi(w, B)$ is $\bar{2}316\bar{4}\bar{7}5$ as from Example 3.2. \diamond

The inverse image of ψ can be constructed according to the following algorithm: for $u \in \mathbb{B}_n$ (i) construct the path representation $(\pi^u, \lambda_x^u, \lambda_y^u)$ of u , (ii) insert a bar in w at each vertical step of π^u (and remove consecutive bars),

(iii) remove a bar at position 0 if it exists. Said otherwise, $(w, B) = \psi^{-1}(u)$ is obtained from u by transforming each negative letter into a bar, by removing consecutive bars, and then by removing a bar at position 0 if needed.

In the following chapters we shall deal mostly with simply barred permutations to carry out computations. We consider simply barred permutations as shorthands for path representations of even signed permutations. Yet, some remarks are due now:

Lemma 3.9. *If $u = \psi(w, B)$, then there is a bijection between the set B of bars and the set of East-South turns of π^u .*

Lemma 3.10. *We have $0 \in \text{Des}_B(\psi(w, B))$ if and only if $|B|$ is odd.*

The lemma can immediately be verified by considering that $0 \in \text{Des}_B(u)$ if and only if, in the path representation of $\psi(w, B)$, the first step of π^u is along the y -axis. In this case (and only in this case), π^u makes an East-South turn on the diagonal. This happens exactly when π^u has an odd number of East-South turns.

4 Descents from simply barred permutations

We start investigating how the type B descent set can be recovered from a simply barred permutation.

Proposition 4.1. *For a simply barred permutation (w, B) , we have*

$$\text{des}_B(\psi(w, B)) = |\text{Des}(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil. \quad (7)$$

Proof. Write $u = \psi(w, B)$ in window notation and divide it in maximal blocks of consecutive letters having the same sign. If the first block has negative sign, add an empty positive block in position 0. Each change of sign $+ -$ among consecutive blocks yields a descent. These changes of sign bijectively correspond to East-South turns of π^u that lie on or below the diagonal. By Lemma 3.9, each bar determines an East-South turn and, by symmetry of π^u along the diagonal, the number of East-South turns that are on or below the diagonal is $\left\lceil \frac{|B|}{2} \right\rceil$. Therefore this quantity counts the number of descents determined by a change of sign.

The other descents of $\psi(w, B)$ are either of the form $w_i w_{i+1}$ with $w_i > w_{i+1}$ and w_i, w_{i+1} belonging to the same positive block, or of the form $\overline{w_{i+1} w_i}$ with $w_i > w_{i+1}$ and $\overline{w_i}, \overline{w_{i+1}}$ belonging to the same negative block. These descents are in bijection with the descent positions of w that do not belong to the set B . \square

For each $k \in \{0, 1, \dots, n\}$, in the following we let $\text{SBP}_{n,k}$ be the set simply barred permutations $(w, B) \in \text{SBP}_n$ such that $|D(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k$.

Corollary 4.2. *The set $\text{SBP}_{n,k}$ is in bijection with the set of signed permutations of n with k descents.*

Definition 4.3. A *loosely barred permutation* of $[n]$ is a pair (w, B) where w is a permutation of $[n]$ and $B \subseteq \{0, \dots, n\}$ is a set of positions (the bars). We let LBP_n be the set of *loosely barred permutations* of $[n]$.

Loosely barred permutations are being introduced only as a tool to index simply barred permutations independently of the even/odd cardinalities of their set of bars. Namely, for a loosely barred permutation (w, B) , let us define its simplification $\zeta(w, B)$ by

$$\zeta(w, B) := (w, B \setminus \{0\}).$$

Then $\zeta(w, B)$ is a simply barred permutation whose set of bars has even (resp., odd) cardinality if either $0 \in B$ and $|B|$ is odd (resp., even), or $0 \notin B$ and $|B|$ is even (resp., odd). For a loosely barred permutations (w, B) , we shall often need to evaluate the expression $\left\lceil \frac{|B \setminus \{0\}|}{2} \right\rceil$. We record this value once for all in the lemma below.

Lemma 4.4. *For a loosely barred permutation (w, B) we have*

$$\left\lceil \frac{|B \setminus \{0\}|}{2} \right\rceil = \begin{cases} \left\lceil \frac{|B|}{2} \right\rceil, & \text{if } |B| \text{ is even,} \\ \left\lceil \frac{|B|-1}{2} \right\rceil, & \text{if } |B| \text{ is odd and } 0 \in B, \\ \left\lceil \frac{|B|+1}{2} \right\rceil, & \text{if } |B| \text{ is odd and } 0 \notin B. \end{cases} \quad (8)$$

Next, we define an involution—that we name θ_n —from the set of loosely barred permutations of $[n]$ to itself. For a loosely barred permutation (w, B) , $\theta_n(w, B)$ is defined by:

$$\theta_n(w, B) := (w, \text{Des}(w) \Delta B), \quad (9)$$

where Δ stands for symmetric difference. Let us insist that this involution is defined for all loosely barred permutations, not just for the simply barred permutations.

Lemma 4.5. *If $(u, C) = \theta_n(w, B)$, then*

$$|\text{Des}(w)| + |B| = 2|\text{Des}(u) \setminus C| + |C|. \quad (10)$$

Proof. Recall that $u = w$ and $C = \text{Des}(w)\Delta B$, so $\text{Des}(u) \setminus C = \text{Des}(w) \cap B$. Equation (10) follows since $|\text{Des}(w)| + |B| = |\text{Des}(w)\Delta B| + 2|\text{Des}(w) \cap B|$. \square

We define now a variant Θ_n of the correspondences θ_n defined in (9), as follows:

$$\Theta_n(w, B) := \varsigma(\theta_n(w, B)) = (w, \text{Des}(w)\Delta B \setminus \{0\}).$$

Notice that, in the definition of Θ_n , $(\text{Des}(w)\Delta B) \setminus \{0\} = \text{Des}(w)\Delta(B \setminus \{0\})$, since $0 \notin \text{Des}(w)$. This time Θ_n yields simply barred permutations, that is, we have $\Theta_n : \text{LBP}_n \rightarrow \text{SBP}_n$.

Definition 4.6. For each $n \geq 0$ and $k \in [2n]_0$, we let $\text{LBP}_{n,k}$ be the set of loosely barred permutations (w, B) such that $|\text{Des}(w)| + |B| = k$.

Proposition 4.7. *For each $n \geq 0$ and $k \in [n]_0$, the restriction of Θ_n to $\text{LBP}_{n,2k}$ yields a bijection $\Theta_{n,k}$ from $\text{LBP}_{n,2k}$ to $\text{SBP}_{n,k}$.*

Proof. Let $(w, B) \in \text{LBP}_{n,2k}$, so $|\text{Des}(w)| + |B| = 2k$. Let also $(w, C) = \theta_n(w, B)$, so $\Theta_n(w, B) = (w, C \setminus \{0\})$. Then, by (10),

$$2k = 2|\text{Des}(w) \setminus C| + |C|$$

and, in particular, $|C|$ is even. Therefore, using this relation and equation (8), we obtain

$$|\text{Des}(w) \setminus (C \setminus \{0\})| + \left\lceil \frac{|C \setminus \{0\}|}{2} \right\rceil = |\text{Des}(w) \setminus C| + \frac{|C|}{2} = k.$$

The transformation $\Theta_{n,k}$ is injective. If $\Theta_n(w, B) = \Theta_n(w', B')$, then $w = w'$ and therefore $\text{Des}(w) = \text{Des}(w')$. Moreover, from $\text{Des}(w)\Delta B \setminus \{0\} = \text{Des}(w)\Delta B' \setminus \{0\}$ we deduce $B \setminus \{0\} = B' \setminus \{0\}$. If moreover $(w, B), (w', B') \in \text{LBP}_{n,2k}$, then $|B| = 2k - |\text{Des}(w)| = |B'|$. Then, from $|B| = |B'|$ and $B \setminus \{0\} = B' \setminus \{0\}$ it follows $B = B'$.

In order to show that the transformation $\Theta_{n,k}$ is surjective, let us fix $(v, C) \in \text{SBP}_{n,k}$, so $|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k$. If $|C|$ is even, then $(v, B) = \theta_n(v, C)$ is such that $\Theta_n(v, B) = \theta_n(v, B) = (v, C)$ and, using equations (8) and (10), $(v, B) \in \text{LBP}_{n,2k}$:

$$|\text{Des}(v)| + |B| = 2|\text{Des}(v) \setminus C| + |C| = 2(|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil) = 2k.$$

If $|C|$ is odd, then $(v, B) = \theta_n(v, C \cup \{0\})$ is such that $\Theta_n(v, B) = (v, C)$ and $(v, B) \in \text{LBP}_{n,2k}$:

$$\begin{aligned} |\text{Des}(v)| + |B| &= 2|\text{Des}(v) \setminus (C \cup \{0\})| + |C \cup \{0\}| = 2|\text{Des}(v) \setminus C| + 2\frac{|C|+1}{2} \\ &= 2(|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil) = 2k. \end{aligned} \quad \square$$

Let us recall that, for $u \in \mathbb{B}_n$, $\text{Des}_B^+(u)$ denotes the set of strictly positive descents of u .

Definition 4.8. For each $k \in [n-1]_0$, we let SBP_n^k be the set of simply barred permutations $(w, B) \in \text{SBP}_n$ such that $|\text{Des}_B^+(\psi(w, B))| = k$.

Let us remark the following characterization of the set SBP_n^k :

Lemma 4.9. *For each simply barred permutation $(w, B) \in \text{SBP}_n$,*

$$(w, B) \in \text{SBP}_n^k \quad \text{iff} \quad \begin{cases} |B| \text{ is even and } |D(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k, \text{ or} \\ |B| \text{ is odd and } |D(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k + 1. \end{cases}$$

Proof. We have

$$|\text{Des}_B^+(\psi(w, B))| = k \quad \text{iff} \quad \begin{cases} 0 \notin \text{Des}_B(\psi(w, B)) \text{ and } \text{des}_B(\psi(w, B)) = k, \text{ or} \\ 0 \in \text{Des}_B(\psi(w, B)) \text{ and } \text{des}_B(\psi(w, B)) = k + 1. \end{cases}$$

The statement of the lemma follows using Lemma 3.10 and Proposition 4.1. \square

Proposition 4.10. *For each $k \in [n - 1]_0$, the restriction of Θ_n to $\text{LBP}_{n,2k+1}$ yields a bijection Θ_n^k from $\text{LBP}_{n,2k+1}$ to SBP_n^k .*

Proof. Let $(w, B) \in \text{LBP}_{n,2k+1}$ and $(w, C) = \theta_n(w, B)$, so $\Theta_n(w, B) = (w, C \setminus \{0\})$. Thus $|\text{Des}(w)| + |B| = 2k + 1$ and, by (10),

$$2k = 2|\text{Des}(w) \setminus C| + |C| - 1,$$

so in particular $|C|$ is odd. Using this relation and equation (8), we obtain

$$|\text{Des}(w) \setminus C| + \left\lceil \frac{|C \setminus \{0\}}{2} \right\rceil = \begin{cases} |\text{Des}(w) \setminus C| + \frac{|C|-1}{2} = k, & 0 \in C, \\ |\text{Des}(w) \setminus C| + \frac{|C|+1}{2} = k + 1, & 0 \notin C. \end{cases}$$

Considering that $|C|$ is odd, we have the following equality

$$|\text{Des}(w) \setminus (C \setminus \{0\})| + \left\lceil \frac{|C \setminus \{0\}}{2} \right\rceil = \begin{cases} k, & |C \setminus \{0\}| \text{ is even,} \\ k + 1, & |C \setminus \{0\}| \text{ is odd,} \end{cases}$$

which, by Lemma 4.9 amounts to $\Theta_n(w, B) = (w, C \setminus \{0\}) \in \text{SBP}_n^k$.

The transformation Θ_n^k is injective (the argument being similar to the one for $\Theta_{n,k}$). If $\Theta_n(w, B) = \Theta_n(w', B')$ then $w = w'$ and $\text{Des}(w) = \text{Des}(w')$. Moreover, from $\text{Des}(w) \Delta B \setminus \{0\} = \text{Des}(w) \Delta B' \setminus \{0\}$ we deduce $B \setminus \{0\} = B' \setminus \{0\}$. Considering now that $|B| = 2k + 1 - |\text{Des}(w)| = |B'|$ and $B \setminus \{0\} = B' \setminus \{0\}$, we infer $B = B'$.

In order to show that the transformation Θ_n^k is surjective, let us fix $(v, C) \in \text{SBP}_n^k$, so either (i) $|C|$ is even and $|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k$ or (ii) $|C|$ is odd and $|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k + 1$.

Let us suppose (i). Then $(v, B) = \theta_n(v, C \cup \{0\})$ is such that $\Theta_n(v, B) = (v, C)$ and, using equations (8) and (10), $(v, B) \in \text{LBP}_{n,2k+1}$:

$$\begin{aligned} |\text{Des}(v)| + |B| &= 2|\text{Des}(v) \setminus (C \cup \{0\})| + |C \cup \{0\}| = 2|\text{Des}(v) \setminus C| + 2\frac{|C|+1}{2} \\ &= 2(|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil + \frac{1}{2}) = 2k + 1. \end{aligned}$$

Let us suppose (ii). Then $(v, B) := \theta_n(v, C)$ is such that $\Theta_n(v, B) = \theta_n(v, B) = (v, C)$ and, using equations (8) and (10), $(v, B) \in \text{LBP}_{n,2k+1}$:

$$\begin{aligned} |\text{Des}(v)| + |B| &= 2|\text{Des}(v) \setminus C| + |C| = 2|\text{Des}(v) \setminus C| + 2\frac{|C|}{2} \\ &= 2(|\text{Des}(v) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil - \frac{1}{2}) = 2(k + 1) - 1 = 2k + 1. \end{aligned} \quad \square$$

To end this section, we collect the consequences of the bijections established so far.

Theorem 4.11. *The following relations hold:*

$$\left\langle \text{B}_n \right\rangle_k = \sum_{i=0}^{2k} \left\langle n \right\rangle_i \binom{n+1}{2k-i}, \quad (3)$$

$$2^n \left\langle n \right\rangle_k = \sum_{i=0}^{2k+1} \left\langle n \right\rangle_i \binom{n+1}{2k+1-i}. \quad (4)$$

Proof. We have seen that signed permutations $u \in \mathcal{B}_n$ such that $\text{des}_{\mathcal{B}}(u) = k$ are in bijection (via the mapping ψ of Definition 3.7) with simply barred permutations in $\text{SBP}_{n,k}$. Next, this set is in bijection (see Proposition (4.7)) with the set $\text{LBP}_{n,2k}$ of loosely barred permutations $(w, B) \in \text{LBP}_n$ such that $\text{des}(w) + |B| = 2k$. The cardinality of $\text{LBP}_{n,2k}$ is the right-hand side of equality (3).

The left-hand side of equality (4) is the cardinality of the set of signed permutations u such that $|\text{Des}_{\mathcal{B}}^+(u)| = k$, see Lemma 2.1. This set is in bijection with the set SBP_n^k (via ψ defined in 3.7 and by the definition of SBP_n^k) which, in turn, is in bijection (see Proposition (4.10)) with the set $\text{LBP}_{n,2k+1}$ of loosely barred permutations $(w, B) \in \text{LBP}_n$ such that $\text{des}(w) + |B| = 2k + 1$. The cardinality of this set is the right-hand side of equality (4). \square

Theorem 4.12. *The following relation holds:*

$$B_n(t^2) = (1+t)^{n+1}S_n(t) - 2^n t S_n(t^2). \quad (11)$$

Proof. By (3), $\langle \mathcal{B}_n \rangle_k$, which is the coefficient of t^{2k} in the polynomial $B_n(t^2)$, is also the coefficient of t^{2k} in $(1+t)^{n+1}S_n(t)$. By (4), $2^n \langle \mathcal{B}_n \rangle_k$ is the coefficient of t^{2k+1} in the polynomials $2^n t S_n(t^2)$ and $(1+t)^{n+1}S_n(t)$. Therefore

$$B_n(t^2) + 2^n t S_n(t^2) = (1+t)^{n+1}S_n(t), \quad (5)$$

whence equation (11). \square

5 Stembridge's identity for Eulerian numbers in type D

We recall that a signed permutation $u \in \mathcal{B}_n$ is *even signed* if the number of negative letters in its window notation is even. The even signed permutations of \mathcal{B}_n form a subgroup \mathcal{D}_n of \mathcal{B}_n and in fact the groups \mathcal{D}_n are standard models for the abstract Coxeter groups of type D.

Definitions analogous to those given in Section 2 for the types A and B can be given for type D. Namely, for $u \in \mathcal{D}_n$, we set

$$\text{Des}_{\mathcal{D}}(u) := \{i \in \{0, 1, \dots, n-1\} \mid u_i > u_{i+1}\}, \quad (12)$$

where we have set $u_0 = -u_2$,

$$\text{des}_{\mathcal{D}}(u) := |\text{Des}_{\mathcal{D}}(u)|, \quad \langle \mathcal{D}_n \rangle_k := |\{u \in \mathcal{D}_n \mid \text{des}_{\mathcal{D}}(u) = k\}|, \quad D_n(t) := \sum_{k=0}^n \langle \mathcal{D}_n \rangle_k t^k.$$

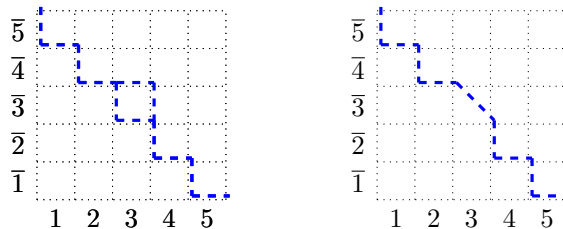
The formula in (12) is the standard one, see e.g. [4, §8.2] or [1]. The reader will have no difficulties verifying that, up to renaming 0 by -1 , the type D descent set of u can also be defined as follows, see [16, §13]:

$$\text{Des}_{\mathcal{D}}(u) := \{i \in \{-1, 1, \dots, n-1\} \mid u_i > u_{|i|+1}\}, \quad (13)$$

where now $u_{-1} = -u_1$, as normal if u is written in full notation.

It is convenient to consider a more flexible representation of elements of \mathcal{D}_n . If $u \in \mathcal{B}_n$, then its mate is the signed permutation $\underline{u} \in \mathcal{B}_n$ that differs from u only for the sign of the first letter. Notice that $\underline{\underline{u}} = u$. We define a *forked signed permutation* (see [16, §13]) as an unordered pair of the form $\{u, \underline{u}\}$ for some $u \in \mathcal{B}_n$. Clearly, just one of the mates is even signed and therefore forked signed permutations are combinatorial models of \mathcal{D}_n .

The path representation of a forked signed permutation is insensitive of how the diagonal is crossed, either from the West, or from the North. The following are possible ways to draw a forked signed permutation on a grid:



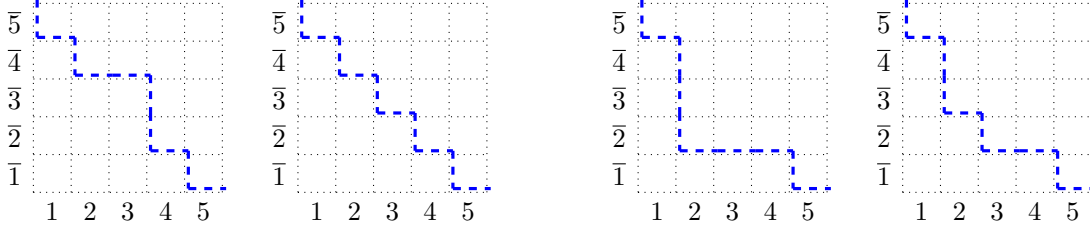


Figure 4: Two pairs of mates, the smooth mates are on the left

Even if the formulas in (12) and (13) have been defined for even signed permutations, they still can be computed for all signed permutations. The formula in (13) is not invariant under taking mates, however the following lemma shows that this formula suffices to compute the number of type D descents of a forked signed permutation and therefore the Eulerian numbers $\langle D_n^k \rangle$.

Lemma 5.1. *For each $u \in B_n$, $1 \in \text{Des}_D(u)$ if and only if $-1 \in \text{Des}_D(\underline{u})$. Therefore $\text{des}_D(u) = \text{des}_D(\underline{u})$.*

Proof. Suppose $1 \in \text{Des}_D(u)$, that is $u_1 > u_2$. Then $\underline{u}_{-1} = -(-u_1) = u_1 > u_2$, and so $-1 \in \text{Des}_D(\underline{u})$. The opposite entailment is proved similarly.

For the last statement, observe that $\text{Des}_D(u) = \Delta_u \cup \{i \in \{2, \dots, n-1\} \mid u_i > u_{i+1}\}$ with $\Delta_u := \{i \in \{1, -1\} \mid u_i > u_{|i|+1}\}$ and, by what we have just remarked, we have $|\Delta_u| = |\Delta_{\underline{u}}|$. It follows that $|\text{Des}_D(u)| = |\text{Des}_D(\underline{u})|$. \square

Our next aim is to derive Stembridge's identity

$$D_n(t) = B_n(t) - n2^{n-1}tS_{n-1}(t), \quad (14)$$

see [24, Lemma 9.1], which, in term of the coefficients of these polynomials, amounts to

$$\langle D_n^k \rangle = \langle B_n^k \rangle - n2^{n-1} \langle n-1 \rangle_{k-1}. \quad (15)$$

Definition 5.2. A signed permutation u is *smooth* if u_1, u_2 have equal sign and, otherwise, it is *non-smooth*.

The reason for naming a signed permutation smooth arises again from the path representation of a signed permutation: the smooth signed permutation is, between the two mates, the one minimizing the turns nearby the diagonal, as suggested in Figure 4 with two pairs of mates as examples.

Lemma 5.3. *If $u \in B_n$ is smooth, then $-1 \in \text{Des}_D(u)$ if and only if $0 \in \text{Des}_B(u)$ and therefore $\text{des}_D(u) = \text{des}_B(u)$.*

Proof. Suppose $0 \in \text{Des}_B(u)$, so $u_1 < 0$ and $u_2 < 0$ as well, since u is smooth. Then $u_{-1} = -u_1 > 0 > u_2$, so $-1 \in \text{Des}_D(u)$. Conversely, suppose $-1 \in \text{Des}_D(u)$, that is, $u_{-1} > u_2$. If $u_1 > 0$, then $0 > -u_1 = u_{-1} > u_2$, so u_1, u_2 have different sign, a contradiction. Therefore $u_1 < 0$ and $0 \in \text{Des}_B(u)$. \square

Next, we consider the correspondence—let us call it χ —sending a non-smooth signed permutation $u \in B_n$ to the pair $(|u_1|, u')$, where u' is obtained from $u_2 \dots u_n$ by normalising this sequence, so that it takes absolute values in the set $[n-1]$. For example $\chi(6\overline{1}234\overline{7}5) = (6, \overline{1}234\overline{6}5)$ and $\chi(\overline{2}3164\overline{7}5) = (2, 215\overline{3}64)$, as suggested below:

$$6\overline{1}234\overline{7}5 \rightsquigarrow (6, \overline{1}234\overline{7}5) \rightsquigarrow (6, \overline{1}234\overline{6}5), \quad \overline{2}3164\overline{7}5 \rightsquigarrow (2, 3164\overline{7}5) \rightsquigarrow (2, 215\overline{3}64).$$

The process of normalizing the sequence $u_2 \dots u_n$ can be understood as applying to each letter of this sequence the unique order preserving bijection $N_{n,x} : [\pm n] \setminus \{x, \bar{x}\} \rightarrow [\pm n - 1]$ where, in general, $x \in [n]$ and, in this case, $x = |u_1|$.

Lemma 5.4. *Let $n \geq 2$. For each pair (x, v) with $x \in [n]$ and $v \in B_{n-1}$, there exists a unique non-smooth $u \in B_n$ such that $\chi(u) = (x, v)$.*

Proof. We construct u firstly by renaming v to v' so that none of x, \bar{x} appears in v' (that is, we apply to each letter of v the inverse of $N_{n,x}$) and then by adding in front of v' either x or \bar{x} , according to the sign of the first letter of v' . \square

Lemma 5.5. *The correspondence χ restricts to a bijection from the set of non-smooth signed permutations $u \in \mathbb{B}_n$ such that $\text{des}_{\mathbb{B}}(u) = k$ to the set of pairs (x, v) where $x \in [n]$ and $v \in \mathbb{B}_{n-1}$ is such that $|\text{Des}_{\mathbb{B}}^+(v)| = k - 1$.*

Proof. We have already argued that χ is a bijection from the set of non-smooth signed permutations u of $[n]$ to the set of pairs (x, v) with $x \in [n]$ and $v \in \mathbb{B}_{n-1}$. Therefore, we are left to argue that, for a non-smooth u and v such that $\chi(u) = (x, v)$, $\text{des}_{\mathbb{B}}(u) = k$ if and only if $|\text{Des}_{\mathbb{B}}^+(v)| = k - 1$. Said otherwise, we need to argue that, for such u and v , $|\text{Des}_{\mathbb{B}}^+(v)| = \text{des}_{\mathbb{B}}(u) - 1$. To this end, observe that (i) $|\text{Des}_{\mathbb{B}}(u) \cap \{0, 1\}| = 1$, since u_1, u_2 have different sign, (ii) $\text{Des}_{\mathbb{B}}^+(v) = \{i - 1 \mid i \in \text{Des}_{\mathbb{B}}(u) \cap \{2, \dots, n - 1\}\}$, from which the relation $|\text{Des}_{\mathbb{B}}^+(v)| = \text{des}_{\mathbb{B}}(u) - 1$ follows. \square

Theorem 5.6. *The following relations hold:*

$$\left\langle \begin{matrix} \mathbb{B}_n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathbb{D}_n \\ k \end{matrix} \right\rangle + n2^{n-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle, \quad B_n(t) = D_n(t) + n2^{n-1}tS_{n-1}(t).$$

Proof. Every signed permutation is either smooth or non-smooth. By Lemma 5.3, the smooth signed permutations with k type B descents are in bijection with the even signed permutations with k type D descents. By Lemma 5.5, the non-smooth signed permutations $u \in \mathbb{B}_n$ with k type B descents are in bijection with the pairs $(x, v) \in [n] \times \mathbb{B}_{n-1}$ such that $|\text{Des}_{\mathbb{B}}^+(v)| = k - 1$. Using Lemma 2.1, the number of these pairs is $n2^{n-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle$. \square

Example 5.7. We end this section exemplifying the use of formulas (3) and (15) by which computation of the Eulerian numbers in type B and D is reduced to computing Eulerian numbers in type A. Let us mention that our interest in Eulerian numbers originates from our lattice theoretic work on the lattice variety of Permutohedra [21] and its possible extensions to generalized forms of Permutohedra [17, 20, 11]. Among these generalizations, we count lattices of finite Coxeter groups in the types B and D [3]. While it is known that the lattices \mathbb{B}_n span the same lattice variety of the permutohedra, see [6, Exercice 1.23], characterizing the lattice variety spanned by the lattices \mathbb{D}_n is an open problem. A first step towards solving this kind of problem is to characterize (and count) the join-irreducible elements of a class of lattices. In our case, this amounts to characterizing the elements u in \mathbb{B}_n (resp., in \mathbb{D}_n) such that $\text{des}_{\mathbb{B}}(u) = 1$ (resp., such that $\text{des}_{\mathbb{D}}(u) = 1$). The numbers $\left\langle \begin{matrix} \mathbb{B}_n \\ 1 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} \mathbb{D}_n \\ 1 \end{matrix} \right\rangle$ are known to be equal to $3^n - n - 1$ and $3^n - n - 1 - n2^{n-1}$ respectively, see [16, Propositions 13.3 and 13.4]. Let us see how to derive these identities using the formulas (3) and (15). To this end, we also need the alternating sum formula for Eulerian numbers, see e.g. [5, Theorem 1.11] or [16, page 12]:

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n. \quad (16)$$

For type B, we have

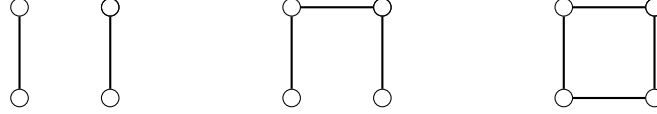
$$\begin{aligned} \left\langle \begin{matrix} \mathbb{B}_n \\ 1 \end{matrix} \right\rangle &= \left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle \binom{n+1}{2} + \left\langle \begin{matrix} n \\ 1 \end{matrix} \right\rangle \binom{n+1}{1} + \left\langle \begin{matrix} n \\ 2 \end{matrix} \right\rangle \binom{n+1}{0} \\ &= \binom{n+1}{2} + (2^n - n - 1)(n+1) + \left\langle \begin{matrix} n \\ 2 \end{matrix} \right\rangle \\ &= \binom{n+1}{2} + (2^n - n - 1)(n+1) + 3^n - 2^n(n+1) + \binom{n+1}{2}, \quad \text{by (16)} \\ &= 3^n - (n+1)^2 + 2 \binom{n+1}{2} = 3^n - (n+1)(n+1-n) = 3^n - n - 1. \end{aligned}$$

The computation in type D is then immediate from Stembridge's identity (15):

$$\left\langle \begin{matrix} \mathbb{D}_n \\ 1 \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathbb{B}_n \\ 1 \end{matrix} \right\rangle - n2^{n-1} \left\langle \begin{matrix} n-1 \\ 0 \end{matrix} \right\rangle = 3^n - n - 1 - n2^{n-1}. \quad \diamond$$

6 Threshold graphs and their degree orderings

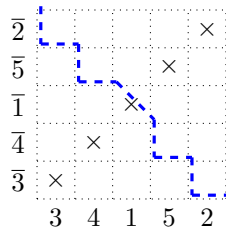
Besides presenting the bijective proofs, a goal of this paper is to illustrate the path representation of signed permutations and exemplify its potential. The attentive reader might object that the path representation is not really in use


 Figure 5: The (unlabelled) graphs $K_{2,2}$, P_3 , and C_4

within Section 5. Indeed, after discovering the bijective proof of Stembridge's identity via the path representation, we realized that the proof could be simplified and reach a larger audience by avoiding mentioning the representation. It might be asked then whether the path representation yields more information, in particular with respect to the lattices of the Coxeter groups D_n . We answer this question in this section. The type D set of inversions of an even signed permutation can be defined as follows:

$$\text{Inv}_D(u) := \text{Inv}_B(u) \setminus \{(-i, i) \mid i \in [n]\},$$

which, graphically, amounts to ignoring cells on the diagonal:



As mentioned in Proposition 3.4, we can identify the set of inversions of a signed permutation u with the disjoint union of $\text{Inv}(\lambda_x^u)$ and a set of unordered pairs. For even signed permutations, this identification yields:

$$\text{Inv}_D(u) = \text{Inv}(\lambda_x^u) \cup E^u \quad \text{with} \quad E^u := \{ \{i, j\} \mid i, j \in [n], i \neq j, ((\lambda_x^u)^{-1}(i), (\lambda_x^u)^{-1}(j)) \text{ lies below } \pi^u \}.$$

Therefore, we consider $([n], E^u)$ as a simple graph on the set of vertices $[n]$. Before taking this route, let us recall some standard definitions that apply to an arbitrary simple graph (V, E) and to a vertex $v \in V$:

$$N_E(v) := \{u \in V \mid \{v, u\} \in E\}, \quad \deg_E(v) := |N_E(v)|, \quad N_E[v] := N(v) \cup \{v\}.$$

A linear ordering v_1, \dots, v_n of V is a *degree ordering* of (V, E) if $\deg_E(v_1) \geq \deg_E(v_2) \geq \dots \geq \deg_E(v_n)$. The *vicinal preorder* of a graph (V, E) , noted \triangleleft_E , is defined by saying that $v \triangleleft_E u$ iff $N_E(v) \subseteq N_E[u]$. Notice that the relation $N_E(v) \subseteq N_E[u]$ is equivalent to $N_E(v) \setminus \{u\} \subseteq N_E(u) \setminus \{v\}$. That the vicinal preorder is indeed a preorder is well-known, see e.g. [13]. For completeness, let us add a statement and a proof of this fact.

Lemma 6.1. *The relation \triangleleft_E on a simple graph (V, E) is reflexive and transitive.*

Proof. Reflexivity is obvious. For transitivity, let $u, v, w \in V$ be distinct and such that $N_E(u) \subseteq N_E[v]$ and $N_E(v) \subseteq N_E[w]$. Let $x \in N_E(u)$. If $x \neq u$, then $x \in N_E(v) \subseteq N_E[w]$. If $x = v$, then $v \in N_E(u)$, thus $u \in N_E(v) \subseteq N_E[w]$ and since $u \neq w$, $x = u \in N_E(w)$. Therefore $N_E(u) \subseteq N_E[w]$. \square

Next, we take Theorem 1 in [7] as the definition of the class of threshold graphs and consider, among the possible characterizations of this class, the one that uses the vicinal preorder.

Definition 6.2. A graph (V, E) is *threshold* if it does not contain an induced subgraph isomorphic to one among $K_{2,2}$, P_3 and C_4 (these graphs are illustrated in Figure 5).

Proposition 6.3 (see e.g. [13, Theorem 1.2.4]). *A graph (V, E) is threshold if and only if the vicinal preorder is total.*

We develop next a few considerations on threshold graphs.

Lemma 6.4. *For a simple graph (V, E) and $<$ a total ordering of V , the following conditions are equivalent:*

- (i) (V, E) is a threshold graph and $<$ is a degree ordering,
- (ii) $u < v$ implies $v \triangleleft_E u$, for each $v, u \in V$.

If any of the above conditions hold, then, for each $v \in V$, $N_E(v)$ is a downset in the following sense: if $u \in N_E(v)$ and $w \neq v$ is such that $w < v$, then $w \in N_E(v)$.

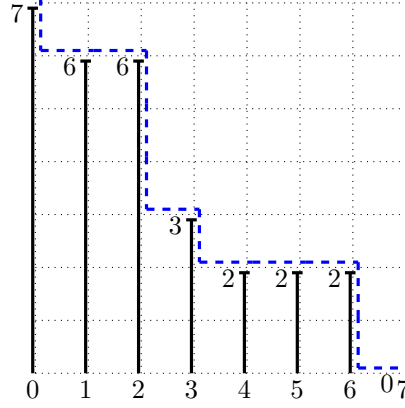


Figure 6: Paths and Galois connections

Proof. We observe firstly that $v \triangleleft_E u$ implies $\deg_E(v) \leq \deg_E(u)$. Indeed, this follows from the fact that $v \triangleleft_E u$ amounts to $N_E(v) \setminus \{u\} \subseteq N_E(u) \setminus \{v\}$ and that $u \in N_E(v)$ if and only if $v \in N_E(u)$. Notice also that the same argument can be used to argue that if $v \triangleleft_E u$ and $u \not\triangleleft_E v$, then $\deg_E(v) < \deg_E(u)$.

Let therefore (V, E) and $<$ be as stated. By the remark above, if $<$ satisfies (ii) then it is a degree ordering and also (V, E) is a threshold graph, since if $u \not\triangleleft_E v$, then $u \not\leq v$, so $v \leq u$ and $v \triangleleft_E u$. Suppose next (V, E) is a threshold graph and that $<$ is a degree ordering, so $u < v$ implies $\deg_E(v) \leq \deg_E(u)$. Let $u < v$ and suppose that $v \not\triangleleft_E u$. Since the vicinal preorder is total, then we have $u \triangleleft_E v$ and so $\deg_E(u) < \deg_E(v)$, contradicting $\deg_E(v) \leq \deg_E(u)$.

For the last statement, for such v, u, w , we have $v \in N_E(u) \setminus \{w\} \subseteq N_E(w) \setminus \{u\}$, so $v \in N_E(w)$ and $w \in N_E(v)$. \square

We establish now a connection between threshold graphs whose set of vertices is $[n]$ and paths via the notion of Galois connection. Let us recall that paths from $(0, n)$ to $(n, 0)$ that are composed only by East and South steps bijectively correspond to *Galois connections* on $[n]_0$, that is, to functions $f : [n]_0 \rightarrow [n]_0$, such that, for some other (necessarily unique) function $g : [n]_0 \rightarrow [n]_0$, $y \leq f(x)$ if and only if $x \leq g(y)$, for each $x, y \in [n]_0$. The correspondence sends a path π to the function height_π such that $\text{height}_\pi(x)$ is the height of π after x East steps, we illustrate this in Figure 6. We refer the reader to [15] for Galois connections and [18] for the correspondence between paths and sup-preserving functions in the discrete setting. Such a Galois connection f is *self-adjoint* if $y \leq f(x)$ is equivalent to $x \leq f(y)$, for each $x, y \in [n]_0$ and it is *fixed-point free* if $f(x) \neq x$, for each $x \in [n]_0$. Notice that a Galois connection f is antitone, that is, we have $x \leq y$ implies $f(y) \leq f(x)$, for each $x, y \in [n]_0$. Since $[n]_0$ is a finite chain, a map $f : [n]_0 \rightarrow [n]_0$ is a Galois connection if and only if it is antitone and $f(0) = n$.

Lemma 6.5. *For $f : [n]_0 \rightarrow [n]_0$ a fixed-point free self-adjoint Galois connection, define*

$$E_f := \{ \{x, y\} \mid x, y \in [n], x \neq y, y \leq f(x) \}.$$

Then $([n], E_f)$ is a threshold graph and $<$ is a degree ordering of $([n], E_f)$. The mapping $f \mapsto E_f$ is a bijection from the set of fixed-point free self-adjoint Galois connection of $[n]_0$ to the set of threshold graphs of the form $([n], E)$ such that the standard linear ordering of $[n]$ is a degree ordering.

Proof. If $y < x$ and $z \leq f(x)$, then $z \leq f(x) \leq f(y)$, since f is antitone. As a consequence, if $y < x$, then $N_{E_f}(x) \subseteq N_{E_f}(y) \cup \{y\} = N_{E_f}[y]$, so $([n], E_f)$ is a threshold graph and $<$ is a degree ordering, by Lemma 6.4.

Conversely, let $([n], E)$ be a threshold graph for which the standard ordering is a degree ordering. As we have seen, $N_E(x)$ is a downset: if $y \in N_E(x)$ and $z \neq x$ is such that $z < y$, then $z \in N_E(x)$. Define then $f_E(x) := \max N_E(x)$, with the conventions that $\max \emptyset = 0$ and $N_E(0) = [n]_0$, so $f_E : [n]_0 \rightarrow [n]_0$. Observe that the following equivalences holds, by the definition of f_E and the fact that $N_E(x)$ is a downset: $\{x, y\} \in E$ if and only if $y \in N_E(x)$ if and only if $x \neq y$ and $y \leq f_E(x)$. It immediately follows that $y \leq f_E(x)$ if and only if $x \leq f_E(y)$, so f_E is self-adjoint; f_E is fixed-point free since $x \notin N_E(x)$.

It is easily seen that $f_{E_f} = f$ and that $E_{f_E} = E$, so the two transformations are inverse to each other. \square

With these tools available, we can state the main result of this section:

Theorem 6.6. *The mapping sending u to (λ_x^u, E^u) is a bijection from the set D_n to the set of pairs (w, E) such that $([n], E)$ is a threshold graph and the linear ordering given by the permutation $w \in S_n$ is a degree ordering for it.*

Proof. Firstly, we claim that the pair (λ_x^u, E^u) is constructed through intermediate steps, as suggested in the following diagram (the notation being used is explained immediately after):

$$\begin{array}{ccc}
 u \in D_n & \longmapsto & (\lambda_x^u, \pi^u) \longmapsto (\lambda_x^u, \pi^u) \in S_n \times \text{Pi}_{E,S}^+([n]_0, [n]_0) \\
 \downarrow & & \downarrow \\
 & & (\lambda_x^u, \text{height}_{\pi^u}) \in S_n \times \text{GC}_{\text{fpfsa}}([n]_0) \\
 & & \downarrow \\
 (\lambda_x^u, E^u) & \longlongequal{\hspace{10em}} & (\lambda_x^u, \lambda_x^u \circ E_{\text{height}_{\pi^u}}) \in ?
 \end{array}$$

We also claim that each step in the upper leg of the diagram yields a bijection. Therefore, in a second time, we shall be left to characterise the pairs that may appear in the bottom right corner.

We explain the notation used in the diagram. For $u \in D_n$, we let $\underline{u} \in \{u, \bar{u}\}$ be such that $\underline{u}_1 > 0$. The first step of $\pi^{\underline{u}}$ after crossing the diagonal is an East step and therefore the Galois connection corresponding via the height to $\pi^{\underline{u}}$ is fixed-point free. We let $\text{Pi}_{E,S}^+([n]_0, [n]_0)$ denote the set of East and South step paths from $(0, n)$ to $(n, 0)$ that make an East step after meeting the diagonal, and that are symmetric along the diagonal. We let $\text{GC}_{\text{fpfsa}}([n]_0)$ denote the set of fixed-point free self-adjoint Galois connection of $[n]_0$. Then the height function is a bijection from $\text{Pi}_{E,S}^+([n]_0, [n]_0)$ to $\text{GC}_{\text{fpfsa}}([n]_0)$. For $f \in \text{GC}_{\text{fpfsa}}([n]_0)$, E_f is as in the statement of Lemma 6.5. Finally, if E is a set of edges on the vertices $[n]$ and $\sigma \in S_n$, then we let

$$\sigma \circ E := \{ \{ \sigma(i), \sigma(j) \} \mid \{ i, j \} \in E \} = \{ \{ i, j \} \mid \{ \sigma^{-1}(i), \sigma^{-1}(j) \} \in E \}.$$

We justify now the equality on the bottom line of the diagram. Notice that

$$E^u = \lambda_x^u \circ E_{\pi^u} \quad \text{with} \quad E_{\pi^u} := \{ \{ i, j \} \mid i, j \in [n], i \neq j, (i, j) \text{ lies below } \pi^u \}$$

and that $E_{\pi^u} = E_{\pi^{\underline{u}}} = E_f$ where $f = \text{height}_{\pi^{\underline{u}}}$. Indeed, the condition that (i, j) lies below $\pi^{\underline{u}}$ amounts to saying that i is less of the height of $\pi^{\underline{u}}$ after j East steps. This shows that $E^u = \lambda_x^u \circ E_f$ with $f = \text{height}_{\pi^{\underline{u}}}$.

Finally, the following equivalences are clear: f is a fixed-point free self-adjoint Galois connection of $[n]_0$ if and only if $<$ (the ordering given by the identity permutation) is a degree ordering of the threshold graph $([n], E_f)$ (by Lemma 6.5), if and only if the ordering given by the permutation σ is a degree ordering for the threshold graph $\sigma \circ E$. Thus, in the right bottom corner of the above diagram we have all the pairs (w, E) such that $([n], E)$ is a threshold graph and the linear ordering given by the permutation $w \in S_n$ is among its degree orderings. \square

Let us remark that Theorem 6.6 also yields a natural representation of the weak ordering on D_n as follows: under the bijection, $(w_1, E_1) \leq (w_2, E_2)$ holds if and only if $w_1 \leq w_2$ in the weak ordering of S_n and, moreover, $E_1 \subseteq E_2$. This poset (actually a lattice, since it is isomorphic to D_n) is built out from threshold graphs but is only loosely related to the lattice of threshold graphs of [14] where unlabeled (that is, up to isomorphism) threshold graphs are considered.

That threshold graphs are related to the families B and D in the theory of Coxeter groups has already been observed, see e.g. [8], [22, Exercise 5.25], and [23, Exercise 3.115]. As part of possible future research, it is tempting to investigate further the bijection presented in Theorem 6.6 (which can be further adapted to fit the type B) and try to understand if it plays any important role with respect to the problem, partly solved in [8], of characterizing free sub-arrangements of the Coxeter arrangements B_n .

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