



HAL
open science

A Sharp Discrepancy Bound for Jittered Sampling

Benjamin Doerr

► **To cite this version:**

| Benjamin Doerr. A Sharp Discrepancy Bound for Jittered Sampling. 2021. hal-03203695v1

HAL Id: hal-03203695

<https://hal.science/hal-03203695v1>

Preprint submitted on 21 Apr 2021 (v1), last revised 30 Dec 2021 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Sharp Discrepancy Bound for Jittered Sampling

Benjamin Doerr
Laboratoire d'Informatique (LIX)
CNRS
École Polytechnique
Institut Polytechnique de Paris
Palaiseau
France

April 19, 2021

Abstract

For $m, d \in \mathbb{N}$, a jittered sampling point set P having $N = m^d$ points in $[0, 1)^d$ is constructed by partitioning the unit cube $[0, 1)^d$ into m^d axis-aligned cubes of equal size and then placing one point independently and uniformly at random in each cube. We show that there are constants $c \geq 0$ and C such that for all d and all $m \geq d$ the expected non-normalized star discrepancy of a jittered sampling point set satisfies

$$c dm^{\frac{d-1}{2}} \sqrt{1 + \log\left(\frac{m}{d}\right)} \leq \mathbb{E}D^*(P) \leq C dm^{\frac{d-1}{2}} \sqrt{1 + \log\left(\frac{m}{d}\right)}.$$

This discrepancy is thus smaller by a factor of $\Theta\left(\sqrt{\frac{1 + \log(m/d)}{m/d}}\right)$ than the one of a uniformly distributed random point set of m^d points. This result improves both the upper and the lower bound for the discrepancy of jittered sampling given by Pausinger and Steinerberger (Journal of Complexity (2016)). It also removes the asymptotic requirement that m is sufficiently large compared to d .

1 Introduction

1.1 Star Discrepancy

The *star discrepancy* $D^*(P)$ of a set of N points in the d -dimensional unit cube $[0, 1]^d$ is a measure for the uniformity of the distribution of these points. It is defined by

$$D^*(P) := \sup_{B \in \mathcal{B}} | |P \cap B| - N\lambda(B) |,$$

where $\lambda(\cdot)$ denotes the Lebesgue measure and \mathcal{B} is the set of all axis-parallel rectangles $[0, x) := \prod_{i=1}^d [0, x_i)$, $x = (x_1, \dots, x_d) \in [0, 1]^d$. We call these rectangles anchored in the origin also *boxes*. The star discrepancy thus is a worst-case measure for how well P satisfies the target of having the fair number $N\lambda(B)$ of points in each box B .

Evenly distributed points sets and the star discrepancy have found applications in various areas such as machine learning [ASYM16], heuristic search [KM05, OT05, TG07], statistics [FW93], and computer graphics [Owe03]. Most prominent is its role in numerical integration, where the *Koksma-Hlawka inequality* [Kok43, Hla61] bounds the integration error in terms of the star discrepancy: For all functions $f : [0, 1]^d \rightarrow \mathbb{R}$ having variation in the sense of Hardy and Krause bounded by 1, we have

$$\left| \int_{[0,1]^d} f(x) dx - \frac{1}{|P|} \sum_{p \in P} f(p) \right| \leq \frac{1}{|P|} D^*(P).$$

We remark that in this work we use the non-normalized version of the star discrepancy as defined above, which is more common in those areas of discrepancy theory that aim at a unified view on discrepancies and exploit connections between different discrepancy notions, e.g., between geometric discrepancies like the star discrepancy and combinatorial discrepancies like hypergraph discrepancies. See the books of Matoušek [Mat99] and Chazelle [Cha00] for introductions to this field. In contexts closer related to numerical integration, motivated by results like the Koksma-Hlawka inequality, a normalized version of the star discrepancy is more common. The normalized star discrepancy simply is $\frac{1}{|P|}$ times the notion we use. There is little risk of confusion since the normalized version always is at most one and usually far less than one, whereas our notion is at least $1/2$ and usually much larger than 1.

1.2 Estimates for the Star Discrepancy

The interest in discrepancies from various research communities has led to a huge body of research (see, e.g., [Cha00, DP10, DT97, Mat99, Nie92]), which we cannot

fully review here. The classic view on geometric discrepancies is to treat the dimension d as a constant and investigate the asymptotic behavior for growing numbers N of points. In this view, a large number of constructions of point sets have been exhibited that have a discrepancy of $D^*(P) = O(\log(N)^{d-1})$. It is also known that such a polylogarithmic discrepancy cannot be avoided, though the correct exponent of the optimal behavior is not known and is a famous open problem for decades, see, e.g., [BL13]. As a side remark, note that axis-parallel regular grids or random point sets have discrepancies of order $\Theta(N^{(d-1)/d})$ and $\Theta(\sqrt{n})$ and thus are not overly interesting in this context.

From the viewpoint of numerical iteration in high dimension, a behavior exponential in d like $(\log N)^{d-1}$ is not very desirable, since such bounds become interesting often only for numbers N of points that are far beyond any practical meaning. For this reason, Heinrich, Novak, Wasilkowski, and Woźniakowski [HNWW01] started the quest for discrepancy bounds that both make the dependence on d fully explicit and that give reasonable discrepancy guarantees also when N is only of moderate size compared to d . Interestingly, this brought random constructions back on stage, and in fact, they are at the moment the best constructions in the regime where N is not very large (say exponential) compared to d .

In this first paper opening this modern view on discrepancies, Heinrich et al. prove that random point sets have an expected discrepancy of order \sqrt{dN} , that is, there is a constant C such that for all d and N a set P of N points chosen independently and uniformly at random in $[0, 1]^d$ satisfies $\mathbb{E}D^*(P) \leq C\sqrt{dN}$. This is asymptotically tight [Doe14] in the sense that there is a constant $c > 0$ such that for all d and all $N \geq d$ the corresponding random point set satisfies $\mathbb{E}D^*(P) \geq c\sqrt{dN}$.

Determining the leading constant C remains a major open problem. For the upper bound, the original proof of Heinrich et al. does not easily reveal information on C . Aistleitner [Ais11] gave an alternative, more direct proof that also shows that with positive probability, $D^*(P) \leq 10\sqrt{dN}$. The currently strongest estimate, lowering the 10 to 2.525, is due to Gnewuch and Hebbinghaus [GH21]. For the lower bound, the elementary proof of [Doe14] clearly can be made more precise and then give a reasonable constant, but this has not been done so far.

1.3 Jittered Sampling

Given the success of random point sets, it is natural to think of constructions that employ randomness in a more clever way than just by taking all decisions independently and uniformly at random. The two most prominent such dependent randomized constructions are *Latin hypercube samplings* [MBC79] and *jittered sampling* (also called *stratified sampling*) [Bel81, CPC84]. The first analysis of the discrepancy of such a construction in the modern paradigm of making the influence

on d explicit was conducted by Pausinger and Steinerberger [PS16], who showed the following bounds (we note that Latin hypercube samples were later analyzed in [DDG18, GH21]).

Assume that we can write $N = m^d$ for some integer m . To obtain a random N -point set P via jittered sampling, we partition the unit cube $[0, 1)^d$ into m^d axis-parallel cubes of identical size and, independently, place a uniformly distributed random point in each cube.

We formulate all results in the following in terms of m and d , and recall that the number N of points is $N = m^d$. To ease comparing the different results, we write all bounds with an explicit term $\sqrt{dm^d}$ (or $\sqrt{m^d}$ when d is treated as a constant), which is the order of magnitude of the discrepancy of a uniform random point set. For constant d , Beck [Bec87] showed that the expected discrepancy of such a point set P satisfies

$$\mathbb{E}D^*(P) = O\left(\sqrt{m^d}\sqrt{\frac{\log m}{m}}\right).$$

Pausinger and Steinerberger proved that for all $d \in \mathbb{N}$ and m sufficiently large relative to d ,

$$\frac{1}{10}\sqrt{dm^d}\sqrt{\frac{1}{m/d}} \leq \mathbb{E}D^*(P) \leq \sqrt{dm^d}\sqrt{\frac{\log m}{m/d}}.$$

It is not stated in the paper for which values of m and d this bound is valid, that is, what “ m sufficiently large compared to d ” precisely means, and the authors state it as an open problem to overcome this asymptotic requirement.

Concerning the lower bound, Pausinger and Steinerberger conjecture that it is not tight and speculate that a lower bound of

$$\Omega\left(\sqrt{dm^d}\frac{1 + \sqrt{\log(m)/d}}{\sqrt{m/d}}\right)$$

“might actually be very close to the truth.” Note that this bound is asymptotically stronger than the previous lower bound only for m superexponential in d , that is, $m = 2^{\omega(d)}$.

1.4 Our Result: A Tight Discrepancy Estimate for Jittered Sampling

We show that both this conjecture (and thus also the proven lower bound) and the proven upper bound are not tight, but that instead the true order of magnitude is

$$\Theta\left(\sqrt{dm^d}\sqrt{\frac{1 + \log(m/d)}{m/d}}\right)$$

for all m and d such that $m \geq d$.

Our result shows that it is the ratio of m and d that describes by how much better jittered sampling is compared to uniform random sampling. For $m = \Theta(d)$, the expected star discrepancies are asymptotically the same, namely $\Theta(\sqrt{dm^d})$. When $m = \omega(d)$, jittered sampling is superior, leading to discrepancies smaller by a factor of $\Theta\left(\sqrt{\frac{\log(m/d)}{m/d}}\right)$. Note that the upper bound of [PS16] shows an advantage of jittered sampling only for $m = \omega(d \log d)$ and “ m large enough compared to d ”.

In this first work determining the asymptotic order of magnitude of jittered sampling point sets, we did not aim at making the leading constant or any lower order terms precise, though in principle we do not see any obstacles for obtaining reasonable absolute bounds for both the lower bound (mainly relying on an estimate on the maximum of independent binomial random variables) and the upper bound (mainly relying on Aistleitner’s dyadic chaining method).

2 Notation and Preliminaries

Throughout this work, we use the following notation: Given an N -point set $P \subseteq [0, 1]^d$, we denote for each Lebesgue-measurable set A by $\lambda(A)$ its Lebesgue measure and by

$$\text{disc}(A) := \text{disc}_P(A) := |P \cap A| - N\lambda(A)$$

the *signed non-normalized discrepancy* of the set A .

The (non-normalized) *star discrepancy* of the point set P is $D^*(P) := \sup_B |\text{disc}(B)|$, where B runs over all *axis-aligned rectangles* (boxes) with lower left corner in the origin, that is, all sets $[0, x) := \prod_{i=1}^d [0, x_i)$, $x \in [0, 1]^d$.

For a positive integer n , we use $[n] := \{1, \dots, n\}$ as shorthand for the set of the first n positive integers. We also write $[a..b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

For given m and (usually suppressed) d , we call $G_m := \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}^d$ the *m-grid* (in dimension d). For $x \in G_m$, we call $C_x := [x, x + \frac{1}{m}\mathbf{1}_m) = \prod_{i=1}^d [x_i, x_i + \frac{1}{m})$ an *m-cube* (or *cube*) in dimension d . A *jittered sampling point set* of $N = m^d$ points is obtained from taking independently and uniformly at random one point from each m -cube.

We now prove an elementary lower bound for tail probabilities of the binomial distribution in the special case $p = 1/2$. Such estimates can be proven via normal approximations and the Berry-Esseen theorem, see, e.g., [Fel71, XVI.5]. To avoid such deep methods in this otherwise elementary combinatorial paper, we now show the following estimate.

Lemma 1. Let $k \in \mathbb{N}$. Let X_1, \dots, X_k be independent random variables each having a binomial distribution with parameters n and $\frac{1}{2}$. We denote the maximum of these by $X_{\max} := \max\{X_i \mid i \in [k]\}$. If $c \in \mathbb{R}$ is such that

$$\alpha(c) := \sqrt{\frac{n(\ln k - \frac{1}{2} \ln \ln k - c)}{2(1 + \sqrt{2 \ln(k)/n})}}$$

is at least \sqrt{n} and small enough to satisfy $\alpha(c) + \frac{n}{\alpha(c)} \leq \frac{n}{2}$, then

$$\Pr[X_{\max} \geq \frac{n}{2} + \alpha(c)] \geq 1 - \exp\left(-\frac{1}{1.5e^{169/6}\sqrt{2\pi}}e^c\right).$$

Consequently, for $e^6 \leq k \leq e^{n/2}$, we have

$$\mathbb{E}[\max\{0, X_{\max} - \frac{n}{2}\}] \geq \sqrt{\frac{n(\ln k - \ln \ln k)}{2(1 + \sqrt{2 \ln(k)/n})}} \left(1 - \exp\left(-\frac{\sqrt{\ln k}}{1.5e^{169/6}\sqrt{2\pi}}\right)\right).$$

Proof. Let X be a random variable having a binomial distribution with parameters n and $\frac{1}{2}$. Let $\alpha \geq 0$ such that $\frac{n}{2} + \alpha \in \mathbb{N}$. We first give a lower bound for the probability that X exceeds its expectation by exactly α . Using Stirling's approximation in the version

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n}}$$

for all $n \geq 1$ due to Robbins [Rob55] and the elementary estimate $1 + x \leq e^x$ valid for all $x \in \mathbb{R}$, we compute for $\alpha < \frac{n}{2}$ that

$$\begin{aligned} \Pr[X = \frac{n}{2} + \alpha] &= 2^{-n} \binom{n}{\frac{n}{2} + \alpha} \\ &= 2^{-n} \frac{n!}{(\frac{n}{2} + \alpha)! (\frac{n}{2} - \alpha)!} \\ &\geq \frac{1}{\sqrt{2\pi}\sqrt{n}(1 + \frac{2\alpha}{n})^{n/2+\alpha+1/2}(1 - \frac{2\alpha}{n})^{n/2-\alpha+1/2}} e^{-1/6} \\ &= \frac{1}{e^{1/6}\sqrt{2\pi}} \frac{1}{\sqrt{n}(1 + \frac{2\alpha}{n})^{2\alpha}(1 - (\frac{2\alpha}{n})^2)^{n/2-\alpha+1/2}} \\ &\geq \frac{1}{e^{1/6}\sqrt{2\pi}} \frac{1}{\sqrt{n} \exp((2\alpha)^2/n) \exp(-(2\alpha/n)^2(n/2 - \alpha + 1/2))} \\ &\geq \frac{1}{e^{1/6}\sqrt{2\pi}} \frac{1}{\sqrt{n}} e^{-2\alpha^2/n - 4\alpha^3/n^2}. \end{aligned}$$

Note that the lower bound in the last line is also valid for $\alpha = \frac{n}{2}$, simply because it is less than 2^{-n} .

Let $\alpha \geq \sqrt{n}$ with $\alpha + \frac{n}{\alpha} \leq \frac{n}{2}$. We now estimate the probability that X exceeds its expectation by at least α via the probability that $X \in [\mathbb{E}X + \alpha, \mathbb{E}X + \alpha + \frac{n}{\alpha}]$. Using that $\binom{n}{\frac{n}{2} + \alpha}$ and thus $\Pr[X = \frac{n}{2} + \alpha]$ is decreasing in α for integral $\alpha \geq 0$, we compute

$$\begin{aligned} \Pr[X \geq \frac{n}{2} + \alpha] &\geq \Pr[\frac{n}{2} + \alpha \leq X \leq \frac{n}{2} + \alpha + \frac{n}{\alpha}] \\ &\geq \lfloor \frac{n}{\alpha} \rfloor \Pr[X = \lfloor \frac{n}{2} + \alpha + \frac{n}{\alpha} \rfloor] \\ &\geq \frac{1}{e^{1/6} \sqrt{2\pi}} \frac{\lfloor \frac{n}{\alpha} \rfloor}{\sqrt{n}} e^{-2(\alpha+n/\alpha)^2/n - 4(\alpha+n/\alpha)^3/n^2} \\ &\geq \frac{1}{1.5e^{169/6} \sqrt{2\pi}} \frac{\sqrt{n}}{\alpha} e^{-2\alpha^2/n - 4\alpha^3/n^2}, \end{aligned} \tag{1}$$

where the last estimate uses $\alpha \geq \sqrt{n}$, $\alpha \leq \frac{n}{2}$, and $\alpha \geq 1$.

Consequently, the probability that all of X_1, \dots, X_k are below $\frac{n}{2} + \alpha$ is

$$\begin{aligned} \Pr[X_{\max} < \frac{n}{2} + \alpha] &= \Pr[\forall i \in [k] : X_i < \frac{n}{2} + \alpha] \\ &\leq \left(1 - \frac{1}{1.5e^{169/6} \sqrt{2\pi}} \frac{\sqrt{n}}{\alpha} e^{-2\alpha^2/n - 4\alpha^3/n^2} \right)^k \\ &\leq \exp \left(- \frac{k}{1.5e^{169/6} \sqrt{2\pi}} \frac{\sqrt{n}}{\alpha} e^{-2\alpha^2/n - 4\alpha^3/n^2} \right). \end{aligned}$$

Let $c \in \mathbb{R}$ such that $\alpha = \alpha(c) = \sqrt{\frac{1}{2}n(\ln k - \frac{1}{2} \ln \ln k - c)(1 + \sqrt{2 \ln(k)/n})^{-1}}$ satisfies $\alpha \geq \sqrt{n}$ and $\alpha + \frac{n}{\alpha} \leq \frac{n}{2}$. With $\alpha \leq \sqrt{\frac{1}{2}n \ln k}$, we continue the previous estimate and compute

$$\begin{aligned} &\exp \left(- \frac{k}{1.5e^{169/6} \sqrt{2\pi}} \frac{\sqrt{n}}{\alpha} e^{-(2\alpha^2/n)(1+2\alpha/n)} \right) \\ &\leq \exp \left(- \frac{k}{1.5e^{169/6} \sqrt{2\pi}} \frac{\sqrt{n}}{\sqrt{\frac{1}{2}n \ln(k)}} \exp \left(- \frac{(\ln k - \frac{1}{2} \ln \ln k - c)(1 + 2\alpha/n)}{1 + \sqrt{2 \ln(k)/n}} \right) \right) \\ &\leq \exp \left(- \frac{k}{1.5e^{169/6} \sqrt{\pi}} \frac{1}{\sqrt{\ln(k)}} \exp(-(\ln k - \frac{1}{2} \ln \ln k - c)) \right) \\ &\leq \exp \left(- \frac{1}{1.5e^{169/6} \sqrt{\pi}} e^c \right). \end{aligned}$$

To prove the second claim, where $e^6 \leq k \leq e^{n/2}$, let $c_k = \frac{1}{2} \ln \ln k$ and $\alpha = \alpha_k = \alpha(c_k)$. Note that α_k is increasing in k in the range $k \in [e^6, e^{n/2}]$. Hence when $n \geq 10$, we have $\alpha_k \geq \sqrt{n}$ for all $k \in [e^6, e^{n/2}]$. For $\alpha \geq \sqrt{n}$, the expression $\alpha + \frac{n}{\alpha}$ is increasing in α . Hence $\alpha_k + \frac{n}{\alpha_k} \leq \alpha_{e^{n/2}} + \frac{n}{\alpha_{e^{n/2}}} \leq \frac{n}{4} + \frac{n}{n/4} \leq \frac{n}{4} + 4$, which is at

most $\frac{n}{2}$ when $n \geq 16$. Hence for $n \geq 16$ and any $k \in [e^6, e^{n/2}]$, we can use the first claim of this lemma and compute

$$\begin{aligned} \mathbb{E}[\max\{0, X_{\max} - \frac{n}{2}\}] &\geq \alpha_k \cdot \Pr[X_{\max} \geq \frac{n}{2} + \alpha_k] \\ &\geq \sqrt{\frac{n(\ln k - \ln \ln k)}{2(1 + \sqrt{2 \ln(k)/n})}} \left(1 - \exp\left(-\frac{\sqrt{\ln k}}{1.5e^{169/6}\sqrt{2\pi}}\right)\right). \end{aligned}$$

For $n < 16$, the second claim is trivially fulfilled since $E[\max\{0, X_{\max} - \frac{n}{2}\}] \geq E[\max\{0, X_1 - \frac{n}{2}\}] = \frac{n}{2} \Pr[X_1 = n] = 2^{-n} \frac{n}{2} \geq \alpha_{e^{n/2}} (1 - \exp(-\sqrt{\ln(e^6)}(1.5)^{-1}e^{-169/6}(2\pi)^{-1/2})) \geq \alpha_k (1 - \exp(-\sqrt{\ln(k)}(1.5)^{-1}e^{-169/6}(2\pi)^{-1/2}))$. \square

3 Proof of the Lower Bound

In this section, we prove that for all $m \geq d$ the discrepancy of a jittered sampling point set having m^d point in $[0, 1]^d$ is at least of order $\sqrt{dm^d} \sqrt{\frac{1+\log(m/d)}{m/d}}$.

Theorem 2. *There is a constant $C > 0$ such that for all $m, d \in \mathbb{N}_{\geq 2}$ with $m \geq d$, the expected discrepancy of a jittered sampling point set $P \subset [0, 1]^d$, $|P| = m^d$, is at least*

$$\mathbb{E}D^*(P) \geq C\sqrt{dm^d} \sqrt{\frac{1 + \log(m/d)}{m/d}}.$$

To ease the presentation, we treat the ‘‘small’’ case that m is at most a constant factor larger than d separately in Lemma 6. The more interesting case, naturally, is that m is of larger order than d . For this, we prove the following result, which we state in a non-asymptotic fashion, noting again that in this work we did not optimize the leading constant or the lower order terms.

Lemma 3. *Let $m, d \in \mathbb{N}_{\geq 2}$ with $\lfloor \frac{m}{d} \rfloor \geq e^6$, and $N = m^d$. Let P be a random N -point set in $[0, 1]^d$ obtained from jittered sampling. Then*

$$\begin{aligned} \mathbb{E}D^*(P) &\geq (2e)^{-\frac{1}{2}} dm^{\frac{d-1}{2}} \sqrt{\ln(\lfloor \frac{m}{d} \rfloor) - \ln \ln(\lfloor \frac{m}{d} \rfloor)} \\ &\quad \left(1 + \sqrt{\frac{2 \ln(\lfloor \frac{m}{d} \rfloor)}{(m - \lfloor \frac{m}{d} \rfloor)^{d-1}}}\right)^{-\frac{1}{2}} \left(1 - \exp\left(-\frac{\sqrt{\ln \lfloor \frac{m}{d} \rfloor}}{1.5e^{169/6}\sqrt{2\pi}}\right)\right). \end{aligned}$$

In particular, $\mathbb{E}D^*(P) \geq C\sqrt{dm^d} \sqrt{\frac{1+\log(m/d)}{m/d}}$ with some constant C that does not depend on m and d .

While we did not optimize for the leading constant, our result can be written as $\mathbb{E}D^*(P) \geq (1 - f(\frac{m}{d}))(2e)^{-\frac{1}{2}}\sqrt{dm^d}\sqrt{\frac{\ln(m/d)}{m/d}}$, where f is a function tending to zero when the argument tends to infinity. Consequently, our leading constant of $(2e)^{-\frac{1}{2}} \geq 0.4288$ is not too bad, and in fact, larger than the $1/10$ of [PS16].

In the proof of Lemma 3, we use the elementary observation that any measurable set has expected signed discrepancy zero when all cubes intersecting it contain exactly one random point distributed uniformly in the cube.

Lemma 4. *Let $A \subseteq [0, 1]^d$ be a measurable set. Let P be a set of $N = m^d$ points such that each m -cube having non-empty intersection with A contains exactly one point of P and this point is uniformly distributed in this cube. Then regardless of the position of the other points of P , we have $\mathbb{E} \text{disc}(A) = 0$.*

Proof. Let \mathcal{Q} be the set of all cubes and \mathcal{Q}_A the set of cubes having non-empty intersection with A . Then

$$\text{disc}(A) = \sum_{Q \in \mathcal{Q}} \text{disc}(A \cap Q) = \sum_{Q \in \mathcal{Q}_A} \text{disc}(A \cap Q).$$

Let $Q \in \mathcal{Q}_A$. With probability $\lambda(A \cap Q)/\lambda(Q)$, the random point in Q lies also in A and we have $\text{disc}(A \cap Q) = 1 - N\lambda(A \cap Q)$. Otherwise, $P \cap (A \cap Q)$ is empty, giving $\text{disc}(A \cap Q) = -N\lambda(A \cap Q)$. Consequently,

$$\begin{aligned} \mathbb{E} \text{disc}(A \cap Q) &= \frac{\lambda(A \cap Q)}{\lambda(Q)}(1 - N\lambda(A \cap Q)) - \left(1 - \frac{\lambda(A \cap Q)}{\lambda(Q)}\right)N\lambda(A \cap Q) \\ &= \frac{\lambda(A \cap Q)}{\lambda(Q)} - N\lambda(A \cap Q) = 0, \end{aligned}$$

where the last equality follows from $\lambda(Q) = 1/N$. By linearity of expectation, $\mathbb{E} \text{disc}(A) = \sum_{Q \in \mathcal{Q}_A} \mathbb{E} \text{disc}(A \cap Q) = 0$. \square

The main strength of the above observation is that it allows to combine certain rectangles with large discrepancy into an anchored box with expected discrepancy (in the probability space conditional on the discrepancies of these rectangles) equal to the sum of the discrepancies of these rectangles. In simple words, we do not need to care about what happens outside these rectangles if we are only interested in a statement on the expected discrepancy. Here is the precise formulation of this principle, which might be helpful for other lower bound proofs as well.

Lemma 5. *Let $r_1, \dots, r_d \in [0, 1)$ be integer multiples of $1/m$. Let $B_0 := [0, r)$, where $r = (r_1, \dots, r_d)$. Let P be a random set of m^d points in $[0, 1]^d$ obtained from jittered sampling. Let $R_1, \dots, R_d \subseteq [0, 1]^d$ be half-open rectangles, possibly depending on P , such that, for each $i \in [d]$, $B_0 \dot{\cup} R_i$ is an anchored box extending*

B_0 in the i -th direction. In other words, for each $i \in [d]$ there is an $s_i \in [r_i, 1)$ such that $R_i = \prod_{j=1}^{i-1} [0, r_j) \times [r_i, s_i) \times \prod_{j=i+1}^d [0, r_j)$. Assume that each R_i , resp. the s_i , is independent of the position of the points outside $\bar{R}_i := \prod_{j=1}^{i-1} [0, r_j) \times [r_i, 1) \times \prod_{j=i+1}^d [0, r_j)$. Let $B = \prod_{i=1}^d [0, s_i)$ be the smallest anchored box containing the R_i . Then

$$\mathbb{E}D^*(P) \geq \mathbb{E}[\text{disc}(B)] = \sum_{i=1}^d \text{disc}(R_i).$$

Note that the lemma in particular covers the case that the s_i are chosen as to maximize the discrepancy of the R_i . Consequently, with this lemma we can construct a box with large expected discrepancy by finding R_i with large signed discrepancy. The idea of this *construction principle* is to partially reveal the random point set, namely where we expect to find a large discrepancy, but to leave the uninteresting parts random and be happy with the expected contribution of zero in these areas.

Proof of Lemma 5. Conditional on the s_i , by construction, all points in cubes not contained in $\bar{R} := \bigcup_{i \in [d]} \bar{R}_i$ are uniformly distributed in their cube. Also, no cube intersects both $A := B \setminus \bigcup_{i \in [d]} R_i$ and \bar{R} . Hence all points in cubes with non-empty intersection with A are distributed uniformly in their cube. By Lemma 4 we have $\mathbb{E} \text{disc}(A) = 0$. We thus have

$$\begin{aligned} \mathbb{E}D^*(P) &\geq \mathbb{E}|\text{disc}(B)| \geq \mathbb{E} \text{disc}(B) \\ &= \mathbb{E} \text{disc}(A) + \sum_{i=1}^d \mathbb{E} \text{disc}(R_i) = \sum_{i=1}^d \mathbb{E} \text{disc}(R_i). \quad \square \end{aligned}$$

Before giving the precise proof of Lemma 3, let us give a brief outline of the main ideas and compare them to the proof of Pausinger and Steinerberger [PS16]. The main argument of [PS16] is the following. Let $x_1, \dots, x_d \in [\frac{m-1}{m}, 1)$ and $x = (x_1, \dots, x_d)$. When assuming that m is sufficiently large compared to d , then the discrepancy of the box $B = [0, x)$ is very close to the discrepancy of the union $R := R_1 \cup \dots \cup R_d$ of the slices $R_i := R_i(x_i) := [0, \frac{m-1}{m})^{i-1} \times [\frac{m-1}{m}, x_i) \times [0, \frac{m-1}{m})^{d-i}$. Note that here the above lemma would have directly shown that the expected star discrepancy of P is at least the discrepancy of R . By construction, the discrepancy of $R_i(x_i)$ is, as a random variable, just the discrepancy of the interval $[0, m(x_i - \frac{m-1}{m}))$ in a one-dimensional random point set of $(m-1)^{d-1}$ points in $[0, 1)$. This one-dimensional discrepancy problem can be analyzed quite well, in particular, the expected maximum discrepancy (over all choices of x_i) can be determined. Consequently, the expected maximum discrepancy of a suitable choice of B is at least $d/2$ times this number (the factor of $1/2$ stems from the fact that we need the discrepancies of the R_i to have the same sign).

From a broader perspective, the main idea of [PS16] is to all regard boxes $B = [0, x)$ with x lying in the upper right sub-cube $C^+ := [\frac{m-1}{m}, 1)^d$ – and only these – and to then exploit that for these the discrepancy is well described by one-dimensional discrepancies, which can be analyzed very precisely. While the reduction to the one-dimensional discrepancy problem allow a very precise analysis of the maximum discrepancy of a box $[0, x)$, $x \in C^+$, this approach carries the risk that the restricted choice of boxes underestimates the star discrepancy significantly.

For this reason, we follow a different road. We do not restrict ourselves to boxes $[0, x)$, $x \in C^+$, but overcome the increased complexity of the larger range for x by restricting ourselves to a suitable *discrete* set of choices for x . Taking $r = 1 - \frac{1}{m} \lfloor \frac{m}{d} \rfloor \approx 1 - \frac{1}{d}$, our x will be such that all x_i are in $[r, 1)$ and are integral multiples of $\frac{1}{2m}$. Clearly, with this relatively small discrete set of boxes, our bounds will necessarily be off the truth by constant factors. However, the more diverse set of boxes together with the right (not very difficult) combinatorial way of selecting a large-discrepancy box among them will enable us to prove the stronger (and in fact asymptotically tight) lower bound.

The main combinatorial observation is that for all $j = 0, \dots, \lfloor \frac{m}{d} \rfloor - 1$, the rectangle $R'_1 = [r + \frac{j}{m}, r + \frac{j}{m} + \frac{1}{2m}) \times [0, r)^{d-1}$ has the same discrepancy distribution, which is the deviation of a binomial random variable with parameters $N' \geq \frac{N}{em}$ and $p = \frac{1}{2}$ from its expectation. Consequently, by elementary properties of the maximum of $\lfloor \frac{m}{d} \rfloor$ independent binomial random variables (Lemma 1), with good probability there is a choice for j such that the signed discrepancy of R'_1 satisfies $\text{disc}(R'_1) \geq C\sqrt{n \log(\frac{m}{d})}$ for some absolute constant C . By construction, the discrepancy of R'_1 is identical to the one of $R_1 := [r, r + \frac{j}{m} + \frac{1}{2m}) \times [0, r)^{d-1}$. Repeating this argument in each dimension and taking as B the smallest anchored box that contains all R_i , with Lemma 5 again we obtain a box with $\mathbb{E} \text{disc}(B) \geq Cd\sqrt{n \log(\frac{m}{d})}$ as desired.

Proof of Lemma 3. Let $k := \lfloor \frac{m}{d} \rfloor$ and $r = \frac{m-k}{m} \geq 1 - \frac{1}{d}$. For each $i \in [d]$ choose x_i such that $R_i := [0, r)^{i-1} \times [r, x_i) \times [0, r)^{d-i}$ has maximum signed discrepancy $\text{disc}(R_i) := |P \cap R_i| - N\lambda(R_i)$. Note that $\text{disc}(R_i) \geq 0$ since taking $x_i = 1$ would give a rectangle containing only full cubes, and hence, having discrepancy zero.

Let $x = (x_1, \dots, x_d)$ and $B := \prod_{i \in [d]} [0, x_i)$ be the smallest box containing all R_i . Note that the choice of the x_i depends only on the position of the points in the cubes in $\bar{R}_i := [0, r)^{i-1} \times [r, 1) \times [0, r)^{d-i}$. By Lemma 5, we have

$$\mathbb{E}D^*(P) \geq \sum_{i=1}^d \mathbb{E} \text{disc}(R_i) = d \mathbb{E} \text{disc}(R_1), \quad (2)$$

where the last equality exploits the symmetry between the R_i .

So it suffices to analyze $\text{disc}(R_1)$. For $j = 0, \dots, k-1$, let $y_j := r + \frac{j}{m}$ and $z_j := y_j + \frac{1}{2m}$. Let $S_j := [r, z_j) \times [0, r)^{d-1}$ and $T_j = [y_j, z_j) \times [0, r)^{d-1}$. Since S_j and T_j differ by full cubes only, we have $\text{disc}(S_j) = \text{disc}(T_j)$. Now each T_j is composed of the “left” half of exactly $N' = (m-k)^{d-1} \geq (1 - \frac{1}{d})^{d-1} m^{d-1} \geq \frac{1}{e} m^{d-1}$ cubes. Consequently, $|P \cap T_j|$ follows a binomial distribution with parameters N' and $p = \frac{1}{2}$. Since no cube intersects non-trivially two different T_j , the discrepancies of the T_j (and thus of the S_j) are independent. Let X_j , $j = 0, \dots, k-1$, be independent random variables with binomial distribution with parameters N' and $p = \frac{1}{2}$. Then $\text{disc}(R_1) \geq \max\{0, \text{disc}(S_j) \mid j = 0, \dots, k-1\}$ and the latter is distributed as $\max\{0, X_j - \mathbb{E}X_j \mid j = 0, \dots, k-1\}$.

Since $k \geq e^6$ by assumption and further $k \leq m-k \leq N' \leq e^{N'/2}$, by the second part of Lemma 1 we have

$$\begin{aligned} \mathbb{E} \text{disc}(R_1) &\geq \sqrt{\frac{N'(\ln k - \ln \ln k)}{2(1 + \sqrt{2 \ln(k)/N'})}} \left(1 - \exp\left(-\frac{\sqrt{\ln k}}{1.5e^{169/6}\sqrt{2\pi}}\right)\right) \\ &\geq (2e)^{-\frac{1}{2}} m^{\frac{d-1}{2}} \sqrt{\ln(\lfloor \frac{m}{d} \rfloor) - \ln \ln(\lfloor \frac{m}{d} \rfloor)} \left(1 + \sqrt{\frac{2 \ln(\lfloor \frac{m}{d} \rfloor)}{(m - \lfloor \frac{m}{d} \rfloor)^{d-1}}}\right)^{-\frac{1}{2}} \\ &\quad \left(1 - \exp\left(-\frac{\sqrt{\ln(\lfloor \frac{m}{d} \rfloor)}}{1.5e^{169/6}\sqrt{2\pi}}\right)\right). \end{aligned}$$

Together with (2), this proves the claim. \square

We now discuss the case that m is of similar order as d . The following result, in particular, extends the $\Omega(dm^{\frac{d-1}{2}})$ lower bound of Lemma 3 to arbitrary $m = \Theta(d)$ and shows that in this case, apart from constant factors, jittered sampling point sets have a discrepancy not smaller than fully random point sets.

Lemma 6. *Let $m, d \in \mathbb{N}_{\geq 2}$. Let $N = m^d$ and P be a random N -point set obtained from jittered sampling. Then*

$$\mathbb{E}D^*(P) \geq \frac{4}{5e^{8+1/6}\sqrt{2\pi}} \exp\left(-\frac{32}{(m-1)^{\frac{d-1}{2}}}\right) d(m-1)^{\frac{d-1}{2}}.$$

In particular, if $m \geq \gamma d$ for some constant $\gamma > 0$, then $\mathbb{E}D^(P) \geq Cdm^{\frac{d-1}{2}}$ for some constant $C > 0$ which depends only on γ .*

If $\Gamma d \geq m \geq \gamma d$ for some constants $\Gamma \geq \gamma > 0$, then $\mathbb{E}D^(P) \geq C'\sqrt{dN}$ for a constant $C' > 0$ which depends only on γ and Γ .*

Proof. We use a simplified version of the proof of Lemma 3. Let $r = \frac{m-1}{m}$. For each $i \in [d]$, choose $x_i \in [r, 1)$ such that $R_i := [0, r)^{i-1} \times [r, x_i) \times [0, r)^{d-i}$ has maximum signed discrepancy $\text{disc}(R_i) = |P \cap R_i| - N\lambda(R_i)$. Let $x = (x_1, \dots, x_d)$ and $B :=$

$\prod_{i \in [d]} [0, x_i)$. As in the proof of Lemma 3, we have $\mathbb{E}D^*(P) \geq d \mathbb{E} \text{disc}(R_1)$, so again it only remains to estimate $\mathbb{E} \text{disc}(R_1)$.

Clearly, we have $\mathbb{E} \text{disc}(R_1) \geq \mathbb{E} \max\{0, \text{disc}(R'_1)\}$ with $R'_1 := [r, r + \frac{1}{2m}) \times [0, r)^{d-1}$. Now $\text{disc}(R'_1)$ follows a binomial distribution with parameters $N' = (m-1)^{d-1}$ and $p = \frac{1}{2}$. If $N' \geq 16$, then for $\alpha = \sqrt{N'}$ we have $\alpha + N'/\alpha \leq N'/2$ and thus equation (1) shows that with probability at least $\frac{4}{5e^{8+1/6}\sqrt{2\pi}}e^{-32/\sqrt{N'}}$ we have a discrepancy of $\sqrt{N'}$ or more. Consequently, $\mathbb{E} \text{disc}(R_1) \geq \mathbb{E} \max\{0, \text{disc}(R'_1)\} \geq \frac{4}{5e^{8+1/6}\sqrt{2\pi}}e^{-32/\sqrt{N'}}\sqrt{N'}$.

If $N' < 16$, then we note that $R'_1 := [r, r + \frac{1}{2N'm}) \times [0, r)^{d-1}$ with probability $1 - (1 - \frac{1}{2N'})^{N'} \geq 1 - \exp(-\frac{1}{2}) \geq 0.39$ contains at least one point. Hence $\mathbb{E} \text{disc}(R_1) \geq \mathbb{E} \max\{0, \text{disc}(R'_1)\} \geq 0.39(1-0.5) \geq 0.19 \geq \frac{4}{5e^{8+1/6}\sqrt{2\pi}} \cdot 4 \geq \frac{4}{5e^{8+1/6}\sqrt{2\pi}}e^{-32/\sqrt{N'}}\sqrt{N'}$.

Hence in either case, $\mathbb{E}D^*(P) \geq d \mathbb{E} \text{disc}(R_1) \geq d \frac{4}{5e^{8+1/6}\sqrt{2\pi}}e^{-32/\sqrt{N'}}\sqrt{N'} = \frac{4}{5e^{8+1/6}\sqrt{2\pi}} \exp(-32(m-1)^{-\frac{d-1}{2}})d(m-1)^{\frac{d-1}{2}}$. \square

4 Proof of the Upper Bound

We now show that the lower bound proven above is tight (apart from constant factors independent of m and d).

Theorem 7. *Let $m, d \in \mathbb{N}$ with $m \geq d \geq 2$. Let $N := m^d$. Let P be a random set of N points in $[0, 1)^d$ obtained from jittered sampling. Then*

$$\mathbb{E}D^*(P) \leq 60.9984\sqrt{dm^d} \frac{\sqrt{\ln(4em/d)} + 2.9599}{\sqrt{m/d}}.$$

The main reason why the upper bound proof of Pausinger and Steinerberger [PS16] does not give tight the right order of magnitude is the following. Using a similar reduction to one-dimensional discrepancies as in their lower bound proof, Pausinger and Steinerberger again are able to give a strong bound (including an exponentially decreasing tail) for the maximum discrepancy among all boxes with upper right corner lying in the same m -cube. To obtain an upper bound valid for all boxes, a union bound is employed. Such a union bound, naturally, ignores any positive correlation between the discrepancies of boxes with corner point in different, but close-by m -cubes. So from a broader perspective, the proof again does a very precise analysis inside the m -cubes, but ignores the overall combinatorial structure of the problem.

Two arguments have been used in the past to better exploit the positive correlation of similar boxes. Heinrich et al. [HNWW01] used deep results of Talagrand and Haussler from the theory of empirical processes to give the first

proof of the $O(\sqrt{dN})$ discrepancy bound for N random points in $[0, 1]^d$. With a non-trivial, purely combinatorial decomposition argument called *dyadic chaining*, Aistleitner [Ais11] and later Gnewuch and Hebbinghaus [GH21] reproved this bound and gave explicit (and small) values for the leading constant

In the following, we show that Aistleitner's dyadic chaining technique can also be used for the non-uniformly distributed point sets stemming from jittered sampling. The main difference, and reason for the stronger discrepancy bound, comes from noting that the grid points $\Gamma_0 = \{0, \frac{1}{m}, \frac{2}{m}, \dots, 1\}^d$ form a $\frac{d}{m}$ -cover (see below for a definition) such that each grid point defines a rectangle with discrepancy 0. Consequently, we can start the dyadic chaining construction with these grid points as coarsest cover. Note that this is not a very efficient (that is, small) cover, but this has no influence on the overall efficiency of the construction as one can verify from the proof below.

The stochastic dependencies in the jittered sampling random experiment, interestingly, impose no additional difficulties. Still the number of points in an arbitrary measurable set can be written as sum of independent 0, 1 random variables. These are not anymore identically distributed as in the proof of [Ais11], but this has no influence on the applicability of most Chernoff-type large deviation bounds.

Of course, in addition to these observations, it remains to estimate the expected star discrepancy in an analogous way as done in [Ais11], which requires some care. As for the lower bound in the previous section, in this first work determining the asymptotic order of magnitude of the star discrepancy of jittered sampling point sets, we do not take great care for obtaining a small leading constant, thus simplifying some calculations as compared to [Ais11] and [GH21]. The reader familiar with Aistleitner's proof will also note that we prefer to work with half-open rectangles, but clearly this makes no difference.

We recall the definitions of δ -covers and δ -bracketing covers. Let $\delta > 0$. A set $\Gamma \subseteq [0, 1]^d$ is called δ -cover if for each $y \in [0, 1]^d$ there are $x, z \in \Gamma \cup \{0\}$ such that $x \leq y \leq z$ and $\lambda(\overline{[x, z]}) \leq \delta$. In particular, and this is what we will need only, we have $\lambda(\overline{[x, y]}) \leq \delta$. A set $\Delta \subseteq ([0, 1]^d)^2$ is called δ -bracketing cover if for each $x \in [0, 1]^d$ there is a pair $(v_K^x, w_K^x) \in \Delta$ such that $v_K^x \leq x \leq w_K^x$ and $\lambda(\overline{[v_K^x, w_K^x]}) \leq \delta$.

Proof of Theorem 7. Throughout this proof, we use the notation $\overline{[x, y]} := [0, y] \setminus [0, x]$ for all $x, y \in [0, 1]^d$ with $x \leq y$.

We start by defining a sequence of δ -covers of increasing precision. Let $\delta_0 := \frac{d}{m}$ and $\Gamma_0 := \{0, \frac{1}{m}, \frac{2}{m}, \dots, 1\}^d$. Then Γ_0 is a δ_0 -cover. Let $K := \lfloor \frac{d-1}{2} \log_2 m \rfloor$. For $i = 1, \dots, K-1$, let $\delta_i := 2^{-i} \frac{d}{m}$ and let Γ_i be a δ_i -cover with $|\Gamma_i| \leq (4e/\delta_i)^d = (4e2^i \frac{m}{d})^d =: \gamma_i$. Finally, let $\delta_K := 2^{-K} \frac{d}{m}$ and let Δ_K be a δ_K -bracketing cover with $|\Delta_K| \leq (4e/\delta_K)^d =: \gamma_K$. Such covers exist by Theorem 1.15 of [Gne08], see also Lemma 1 of [Ais11].

By definition of bracketing covers, for each $x \in [0, 1]^d$ there is a pair $(v_K^x, w_K^x) \in \Delta_K$ such that $v_K^x \leq x \leq w_K^x$ and $\lambda(\overline{[v_K^x, w_K^x]}) \leq \delta_K$. By elementary properties of the discrepancy function, we have

$$|\text{disc}([0, x])| \leq \max \{ |\text{disc}([0, v_K^x])|, |\text{disc}([0, w_K^x])| \} + N\delta_K.$$

Consequently,

$$D^*(P) \leq N\delta_K + \max \{ |\text{disc}([0, v_K^x])|, |\text{disc}([0, w_K^x])| \mid x \in [0, 1]^d \}. \quad (3)$$

Note that $N\delta_K \leq \frac{\sqrt{dN}}{\sqrt{m/d}}$ is of asymptotic order not larger than the bound we aim at. Consequently, it suffices in the following to analyze

$$\max \{ |\text{disc}([0, v_K^x])|, |\text{disc}([0, w_K^x])| \mid x \in [0, 1]^d \}.$$

To this aim, note that for each $i \in [0, K-1]$ and each $x \in [0, 1]^d$ there is a $v_i(x) \in \Gamma_i \cup \{0\}$ such that $v_i(x) \leq x$ and $\lambda(\overline{[v_i(x), x]}) \leq \delta_i$.

For each $x \in [0, 1]^d$ we define $p_{K+1}^x := w_K^x$, $p_K^x := v_K^x$, and recursively for $i = K-1, \dots, 0$, we define $p_i^x := v_i(p_{i+1}^x)$. By construction, the sets $B_i^x := \overline{[p_{i-1}^x, p_i^x]}$, $i = 1, \dots, K+1$, are disjoint. By the additivity of $\text{disc}(\cdot)$ and thus the subadditivity of $|\text{disc}(\cdot)|$, we obtain

$$|\text{disc}([0, v_K^x])| \leq \sum_{i=1}^K |\text{disc}(B_i^x)|, \quad (4)$$

$$|\text{disc}([0, w_K^x])| \leq \sum_{i=1}^{K+1} |\text{disc}(B_i^x)|. \quad (5)$$

For this reason, we now proceed by analyzing the discrepancies of the sets B_i^x . To this aim, keep in mind that (i) we have $\lambda(B_i^x) \leq \delta_{i-1}$ for all i and x , and that (ii) for fixed i , the number of different $B_i^x \neq \emptyset$ is at most $\gamma_i = (4e2^i \frac{m}{d})^d$, for $i \in [K]$, and $\gamma_{K+1} := \gamma_K$ when $i = K+1$.

Let us first regard an arbitrary measurable set $S \subseteq [0, 1]^d$. Let \mathcal{Q} be the set of all elementary cubes $\prod_{j=1}^d [\frac{q_j-1}{m}, \frac{q_j}{m})$, $q_1, \dots, q_d \in [m]$. For each $Q \in \mathcal{Q}$, we have

$$\mathbb{E}[|P \cap S \cap Q|] = \Pr[P \cap S \cap Q \neq \emptyset] = \frac{\lambda(S \cap Q)}{\lambda(Q)} = N\lambda(S \cap Q).$$

Consequently, we may write

$$\begin{aligned} \text{disc}(S) &= |P \cap S| - N\lambda(S) \\ &= \sum_{Q \in \mathcal{Q}} (|P \cap S \cap Q| - N\lambda(S \cap Q)) \\ &= \sum_{Q \in \mathcal{Q}} (|P \cap S \cap Q| - \mathbb{E}[|P \cap S \cap Q|]) \end{aligned}$$

as a sum of N independent random variables $Z_Q := |P \cap S \cap Q| - \mathbb{E}[|P \cap S \cap Q|]$, each taking values in the interval $[-1, 1]$, each having expectation $\mathbb{E}[Z_Q] = 0$, and each having variance $\text{Var}[Z_Q] = \text{Var}[|P \cap S \cap Q|] \leq \mathbb{E}[|P \cap S \cap Q|] = N\lambda(S \cap Q)$. Consequently, Bernstein's inequality gives

$$\Pr[|\text{disc}(S)| \geq t] \leq 2 \exp\left(-\frac{t^2}{2 \sum_Q \text{Var}[Z_Q] + 2t/3}\right) \leq 2 \exp\left(-\frac{t^2}{2N\lambda(S) + 2t/3}\right).$$

For all $i \in [K+1]$ and $\ell \in \mathbb{R}_{\geq 1}$ let $t_{i,\ell} := 2C\ell d \sqrt{\frac{N \ln(2^{2i} 4e(m/d))}{m 2^{i-1}}}$ for some $C \geq 1$. Let $x \in [0, 1]^d$ and $i \in [K+1]$. By the above, we have

$$\begin{aligned} \Pr[|\text{disc}(B_i^x)| \geq t_{i,\ell}] &\leq 2 \exp\left(-\frac{t_{i,\ell}^2}{2N\delta_{i-1} + 2t_{i,\ell}/3}\right) \\ &\leq 2 \exp\left(-\frac{t_{i,\ell}^2}{2 \max\{2N\delta_{i-1}, 2t_{i,\ell}/3\}}\right) \\ &\leq 2 \max\left\{\exp\left(-\frac{t_{i,\ell}^2}{4N\delta_{i-1}}\right), \exp\left(-\frac{t_{i,\ell}^2}{4t_{i,\ell}/3}\right)\right\} \\ &\leq 2\left(\exp\left(-\frac{t_{i,\ell}^2}{4N\delta_{i-1}}\right) + \exp\left(-\frac{t_{i,\ell}^2}{4t_{i,\ell}/3}\right)\right) \\ &\leq 2 \exp\left(-\frac{4 \frac{C^2 d^2 \ell^2 N}{m} 2^{-i+1} \ln(2^{2i} 4e(m/d))}{4N 2^{-i+1} (d/m)}\right) \\ &\quad + 2 \exp\left(-\frac{3}{2} C \ell d \sqrt{\frac{N \ln(2^{2i} 4e(m/d))}{m 2^{i-1}}}\right) \\ &\leq 2 \exp(-d\ell^2 \ln(2^{2i} 4e(m/d))) + 2 \exp\left(-\frac{3}{2} C \sqrt{\frac{N \ln(2^{2i})}{m m^{(d-1)/2}}}\right)^{d\ell} \\ &= 2(2^{2i} 4e(m/d))^{-d\ell^2} + 2 \exp\left(\frac{3}{12} C m^{(d-1)/4} \sqrt{2i \ln(2)}\right)^{-6d\ell} \\ &\leq 2(2^{2i} 4e(m/d))^{-d\ell^2} + 2\left(\frac{1}{4} C m^{(d-1)/4} \sqrt{2i \ln(2)}\right)^{-6d\ell} =: q_{i,\ell}. \end{aligned}$$

With a simple union bound, choosing $C = 8^{1/12}(e/4)^{1/6}4/\sqrt{\ln(2)} = 2^{23/12}e^{1/6}/\sqrt{\ln(2)} \leq 5.3573$, and recalling that $d \geq 2$ and $\ell \geq 1$, we estimate

$$\begin{aligned}
\Pr[\exists x \in [0, 1)^d : |\text{disc}(B_i^x)| \geq t_{i,\ell}] &\leq \gamma_i q_{i,\ell} \\
&\leq (4e2^{i\frac{m}{d}})^d \cdot 2(2^{2i}4e\frac{m}{d})^{-d\ell^2} \\
&\quad + (4em^{(d-1)/2}\frac{m}{d})^d \cdot 2(\frac{1}{4}Cm^{(d-1)/4}\sqrt{2i\ln(2)})^{-6d\ell} \\
&\leq 2 \cdot 2^{-id\ell^2} + 2\left(m^{-d+2}i^{-3}\frac{4e4^6}{2C^62^3\ln(2)^3}\right)^{d\ell} \\
&\leq 2 \cdot 2^{-id\ell^2} + 2(\sqrt{1/8}m^{-d+2}i^{-3})^{d\ell} \\
&\leq 2 \cdot 4^{-i\ell^2} + 2 \cdot 8^{-\ell}m^{-d+2}i^{-6} \leq 2(4^{-i\ell} + 8^{-\ell}i^{-6}).
\end{aligned}$$

Another union bound together with $\ell \geq 1$ and $\zeta(6) = \pi^6/945 \leq 1.0174$ gives

$$\begin{aligned}
\Pr[\exists i \in [K+1]\exists x \in [0, 1)^d : |\text{disc}(B_i^x)| \geq t_{i,\ell}] \\
&\leq \sum_{i=1}^{K+1} 2(4^{-i\ell} + 8^{-\ell}i^{-6}) \\
&\leq 2 \cdot 4^{-\ell} \sum_{i=0}^{\infty} 4^{-i} + 2 \cdot 8^{-\ell} \sum_{i=1}^{\infty} i^{-6} \\
&= 4^{-\ell}\frac{8}{3} + 2 \cdot 8^{-\ell}\zeta(6) \leq 4^{-\ell}(\frac{8}{3} + \zeta(6)) < 3.7 \cdot 4^{-\ell}.
\end{aligned}$$

We compute

$$\begin{aligned}
\sum_{i=1}^{K+1} t_{i,\ell} &\leq \sum_{i=1}^{\infty} 2C\ell d \sqrt{\frac{N \ln(2^{2i}4e(m/d))}{m 2^{i-1}}} \\
&\leq 2C\ell d \sqrt{\frac{N}{m}} \left(\sum_{i=1}^{\infty} \sqrt{\frac{2i\ln(2)}{2^{i-1}}} + \sum_{i=1}^{\infty} \sqrt{\frac{\ln(4em/d)}{2^{i-1}}} \right) \\
&\leq 2C\ell d \sqrt{\frac{N}{m}} \left(5(1 + \sqrt{2}/2)\sqrt{2\ln 2} + \sqrt{\ln(4em/d)}\frac{\sqrt{2}}{\sqrt{2}-1} \right) =: \ell D
\end{aligned}$$

using the estimate $\sum_{i=1}^{\infty} \sqrt{i/2^{i-1}} \leq 1 + \sum_{i=2}^{\infty} i/2^{i/2} \leq 1 + \sum_{i=1}^{\infty} (2i)/2^i + \sum_{i=1}^{\infty} (2i+1)/2^{i+1/2} = 1 + 2 \sum_{i=1}^{\infty} i/2^i + \sqrt{2} \sum_{i=1}^{\infty} i/2^i + 2^{-1/2} \sum_{i=1}^{\infty} 2^{-i} = 1 + 4 + 2\sqrt{2} + 2^{-1/2} = 5(1 + \sqrt{2}/2)$.

Consequently, by (3), (4), and (5), we have $\Pr[D^*(P) \geq \ell D + N\delta_K] \leq 3.7 \cdot 4^{-\ell}$ for all $\ell \geq 1$. From this we easily derive the following bound on the expectation of $D^*(P)$.

$$\begin{aligned}
\mathbb{E}D^*(P) &= \int_0^\infty \Pr[D^*(P) \geq x] dx \\
&\leq D + N\delta_K + D \int_1^\infty \Pr[D^*(P) \geq \ell D + N\delta_K] d\ell \\
&\leq D(1 + 3.7 \cdot 4^{-1}/\ln(4)) + N\delta_K \leq 1.6673D + N\delta_K \\
&\leq 1.6673 \cdot 2Cd \sqrt{\frac{N}{m}} \left(5(1 + \sqrt{2}/2)\sqrt{2\ln 2} + \frac{\sqrt{2\ln(4em/d)}}{\sqrt{2}-1} \right) + \sqrt{\frac{dN}{m/d}} \\
&\leq 17.8645 \sqrt{\frac{dN}{m/d}} (10.0499 + 3.4143\sqrt{\ln(4em/d)} + 1/17.8645) \\
&\leq 60.9948 \sqrt{\frac{dN}{m/d}} (\sqrt{\ln(4em/d)} + 2.9599). \quad \square
\end{aligned}$$

5 Conclusion

In this work, we determined (apart from constant factors) the expected star discrepancy of a jittered sampling point set having m^d points, $m \geq d$. This improves both the previous upper and lower bound and also removes the unquantified restriction that “ m is sufficiently large compared to d ”. These improvements are made possible by exhibiting suitable combinatorial structures in the problem which then allow an easier analysis via discrete probability theory. We feel that this might, also for other discrepancy problems on random point sets, be an approach more suitable than trying to describe the discrepancy directly via continuous-domain random variables.

Acknowledgments

This work was triggered by the 2nd International Workshop on Discrepancy Theory and Applications organized by Dmitriy Bilyk, Luca Brandolini, William Chen, Anand Srivastav, Giancarlo Travaglini. The author would like to thank the organizers for inviting him to this fruitful meeting.

References

- [Ais11] Christoph Aistleitner. Covering numbers, dyadic chaining and discrepancy. *Journal of Complexity*, 27:531–540, 2011.
- [ASYM16] Haim Avron, Vikas Sindhwani, Jiyan Yang, and Michael W. Mahoney. Quasi-Monte Carlo feature maps for shift-invariant kernels. *Journal of Machine Learning Research*, 17:120:1–120:38, 2016.
- [Bec87] József Beck. Irregularities of distribution. I. *Acta Mathematica*, 159:1–49, 1987.
- [Bel81] David R. Bellhouse. Area estimation by point-counting techniques. *Biometrics*, 37:303–312, 1981.
- [BL13] Dmitriy Bilyk and Michael Lacey. The supremum norm of the discrepancy function: Recent results and connections. In *Monte Carlo and Quasi-Monte Carlo Methods 2012*, pages 23–38. Springer, 2013.
- [Cha00] Bernard Chazelle. *The Discrepancy Method*. Cambridge University Press, 2000.
- [CPC84] Robert L. Cook, Thomas K. Porter, and Loren C. Carpenter. Distributed ray tracing. In *Conference on Computer Graphics and Interactive Techniques, SIGGRAPH 1984*, pages 137–145. ACM, 1984.
- [DDG18] Benjamin Doerr, Carola Doerr, and Michael Gnewuch. Probabilistic lower bounds for the discrepancy of Latin hypercube samples. In Josef Dick, Frances Y. Kuo, and Henryk Woźniakowski, editors, *Contemporary Computational Mathematics - A Celebration of the 80th Birthday of Ian Sloan*, pages 339–350. Springer, 2018.
- [Doe14] Benjamin Doerr. A lower bound for the discrepancy of a random point set. *Journal of Complexity*, 30:16–20, 2014.
- [DP10] Josef Dick and Friedrich Pillichshammer. *Digital Nets and Sequences*. Cambridge University Press, 2010.
- [DT97] Michael Drmota and Robert F. Tichy. *Sequences, Discrepancies and Applications*, volume 1651 of *Lecture Notes in Mathematics*. Springer, 1997.
- [Fel71] William Feller. *Introduction to Probability Theory and Its Applications*, volume 2. Wiley, 2nd edition, 1971.

- [FW93] Kai-Tai Fang and Yuan Wang. *Number-Theoretic Methods in Statistics*. CRC Press, 1993.
- [GH21] Michael Gnewuch and Nils Hebbinghaus. Discrepancy bounds for a class of negatively dependent random points including Latin hypercube samples. *arXiv*, 2102.04451, 2021.
- [Gne08] Michael Gnewuch. Bracketing numbers for axis-parallel boxes and applications to geometric discrepancy. *Journal of Complexity*, 24:154–172, 2008.
- [Hla61] Edmund Hlawka. Funktionen von beschränkter Variation in der Theorie der Gleichverteilung. *Annali di Matematica Pura ed Applicata*, 54:325–333, 1961.
- [HNWW01] Stefan Heinrich, Erich Novak, Grzegorz W. Wasilkowski, and Henryk Woźniakowski. The inverse of the star-discrepancy depends linearly on the dimension. *Acta Arithmetica*, 96:279–302, 2001.
- [KM05] Shuhei Kimura and Koki Matsumura. Genetic algorithms using low-discrepancy sequences. In *Genetic and Evolutionary Computation Conference, GECCO 2005*, pages 1341–1346. ACM, 2005.
- [Kok43] Jurjen F. Koksma. A general theorem from the uniform distribution modulo 1 (in Dutch). *Mathematica B (Zutphen)*, 1:7–11, 1942/43.
- [Mat99] Jiří Matoušek. *Geometric Discrepancy*. Springer-Verlag, Berlin, 1999.
- [MBC79] Michael D. McKay, Richard J. Beckman, and William J. Conover. Comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21:239–245, 1979.
- [Nie92] Harald Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods*, volume 63 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
- [OT05] Art B. Owen and Seth D. Tribble. A quasi-Monte Carlo Metropolis algorithm. *Proceedings of the National Academy of Sciences*, 102:8844–8849, 2005.
- [Owe03] Art B. Owen. Quasi-monte carlo sampling. In *Monte Carlo Ray Tracing: ACM SIGGRAPH 2003 Course Notes 44*, pages 69–88. ACM, 2003.

- [PS16] Florian Pausinger and Stefan Steinerberger. On the discrepancy of jittered sampling. *Journal of Complexity*, 33:199–216, 2016.
- [Rob55] Herbert Robbins. A remark on Stirling’s formula. *The American Mathematical Monthly*, 62:26–29, 1955.
- [TG07] Olivier Teytaud and Sylvain Gelly. DCMA: yet another derandomization in covariance-matrix-adaptation. In *Genetic and Evolutionary Computation Conference, GECCO 2007*, pages 955–963. ACM, 2007.