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MICROLOCAL ANALYSIS OF SINGULAR MEASURES

VALERIA BANICA AND NICOLAS BURQ

ABSTRACT. The purpose of this article is to investigate the structure of singular measures from a micro-local perspective. We introduce a notion of L^1 -regularity wave front set for scalar and vectorial distributions. Our main result is a proper microlocal characterisation of the support of the singular part of temperate Radon measures and of their polar functions at these points. We deduce a sharp L^1 elliptic regularity result which appears to be new even for scalar measures. We also deduce several consequences including extensions of the results in [9] giving constraints on the polar function at singular points for measures constrained by a PDE. Finally, we also illustrate the interest of this micro-local approach with a result of propagation of singularities for constrained measures.

1. INTRODUCTION

The purpose of this article is the investigation of the singular part of scalar and vectorial measures from a microlocal analysis perspective.

We start by recalling some classical facts about the class $\mathcal{M}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ of locally bounded Radon measures on \mathbb{R}^d with values in \mathbb{R}^m (see for instance [18] or [15]). The Radon-Nikodym Theorem allows to write the polar decomposition $d\mu = \frac{d\mu}{d|\mu|}d|\mu|$, where the non negative measure $|\mu|$ is the total variation of the measure μ and the function $\frac{d\mu}{d|\mu|} \in L^1(\mathbb{R}^d(d|\mu|), \mathbb{S}^{m-1})$ is the Radon-Nikodym derivative of μ with respect to $|\mu|$, called the polar function of μ . Moreover, the Radon-Nikodym Theorem gives the Lebesgue decomposition of $|\mu|$ with respect to the Lebesgue measure, \mathcal{L}^d , so we have

$$d\mu = g d\mathcal{L}^d + \frac{d\mu}{d|\mu|}d|\mu|_s,$$

where $g \in L^1(\mathbb{R}^d, \mathbb{R}^m)$ and the positive measure $|\mu|_s$ satisfies $|\mu|_s \perp \mathcal{L}^d$. Finally, the Radon-Nikodym Theorem applied to each component of μ with respect to \mathcal{L}^d implies that the singular part of μ is $d\mu_s = \frac{d\mu}{d|\mu|}d|\mu|_s$, and in particular $\frac{d\mu}{d|\mu|} = \frac{d\mu_s}{d|\mu|_s}$, μ_s a.e. and $|\mu_s| = |\mu|_s$.

In the following we shall consider pseudo-differential operators and we have gathered in Section 6 the definitions and basic results from the theory we need. We denote by $S_{cl}^k(\mathbb{R}^d)$ the class of symbols of order k and by $\Psi_{cl}^k(\mathbb{R}^d)$ the class of classical pseudo-differential operators of order k . For u a temperate distribution on \mathbb{R}^d with values in \mathbb{R}^m , that is $u = (u_1, \dots, u_m)$ with $u_i \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R})$, if $A = (A_{ij})$ is a $n \times m$ matrix of pseudo-differential

operators then the distribution $Au \in \mathcal{D}'(\mathbb{R}^d, \mathbb{R}^n)$ is defined by

$$(Au)_j = \sum_k A_{jk} u_k, \quad 1 \leq j \leq n.$$

For $A \in \Psi_{cl}^k(\mathbb{R}^d)$ we denote by $a_k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ its homogeneous principal symbol and similarly for matrices of pseudo-differential operators.

We recall that the classical wave front set introduced by Hörmander gives a proper characterisation of the singular support of a distribution, i.e. the complementary of points where the distribution is \mathcal{C}^∞ . Moreover, for vector-valued distributions a generalisation of this concept is also available through the refined notion of polarisation wave front set introduced by Denker in [8] (see also [10]).

1.1. Main result. In this article we introduce an L^1 -regularity wave front set for vector valued temperate distributions.

Definition 1.1. For $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m)$ we define the set

$$WF_{L^1}(u) \subseteq (T^*\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^m = \mathbb{R}^d \times \mathbb{R}^{d*} \times \mathbb{R}^m$$

by the following: $(x_0, \xi_0, \omega_0) \notin WF_{L^1}(u)$ if and only if there exist $A \in M_{1 \times m}(\Psi_{cl}^0(\mathbb{R}^d))$, $N \in \mathbb{N}^*$, $Q_i \in \Psi_{cl}^0(\mathbb{R}^d)$ and $f_i \in L^1(\mathbb{R}^d, \mathbb{R})$ for $1 \leq i \leq N$, satisfying

$$(1.1) \quad Au = \sum_{i=1}^N Q_i f_i,$$

with A elliptic at (x_0, ξ_0, ω_0) in the sense that

$$a_0(x_0, \xi_0) \omega_0 \neq 0.$$

Among the class $\mathcal{M}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ we shall consider the temperate Radon measures

$$\mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m) := \cup_{\kappa \in \mathbb{R}^+} \mathcal{M}_\kappa(\mathbb{R}^d, \mathbb{R}^m),$$

$$\mathcal{M}_\kappa(\mathbb{R}^d, \mathbb{R}^m) := \{\mu \in \mathcal{M}_{loc}(\mathbb{R}^d, \mathbb{R}^m); \exists C > 0, \forall R > 1, |\mu|(B(0, R)) \leq CR^\kappa\}.$$

Our main result is a connection between the polarisation of the singular part of a measure and its wave front set.

Theorem 1.2. For $\mu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m)$ we have the inclusion:

$$(1.2) \quad \left(x, \frac{d\mu}{d|\mu|}(x)\right) \stackrel{|\mu|_s - a.e.}{\in} \Pi_{13}(WF_{L^1}(\mu) \setminus 0_{pol}).$$

where Π_{13} denotes the projection with respect to the first and third variables, i.e. for $|\mu|_s$ almost all x ,

$$\exists \xi \in \mathbb{S}^{d-1}, \left(x, \xi, \frac{d\mu}{d|\mu|}(x)\right) \in WF_{L^1}(\mu) \setminus 0_{pol}.$$

Remark 1.3. • *Definition 1.1 is equivalent to:*

$$(1.3) \quad WF_{L^1}(u) = \{(x, \xi, \omega) \in \mathbb{R}^d \times \mathbb{R}^{d^*} \times \mathbb{R}^m; \omega \in \cap \ker a_0(x, \xi)\},$$

where the intersection is over all possible $A \in M_{1 \times m}(\Psi_{cl}^0(\mathbb{R}^d))$, $N \in \mathbb{N}^*$, $Q_i \in \Psi_{cl}^0(\mathbb{R}^d)$ and $f_i \in L^1(\mathbb{R}^d, \mathbb{R})$ for $1 \leq i \leq N$, such that $Au = \sum_{i=1}^N Q_i f_i$.

- We can obtain WF_{L^1} by considering in Definition 1.1 the existence of $n, N \in \mathbb{N}^*$, $A \in M_{n \times m}(\Psi_{cl}^0(\mathbb{R}^d))$, $Q^j \in M_{1 \times N}(\Psi_{cl}^0(\mathbb{R}^d))$ and $f^j \in M_{N \times 1}(L^1(\mathbb{R}^d, \mathbb{R}))$ such that $(Au)_j = Q^j f^j$ holds for $1 \leq j \leq n$, and $a_0(x, \xi)\omega \neq 0_{\mathbb{R}^n}$.
- In the case of scalar distributions Definition 1.1 of WF_{L^1} recovers the classical shape of a subset of $T^*\mathbb{R}^d \setminus \{0\}$ and the definition of ellipticity $a_0(x, \xi) \neq 0$ is the usual one.
- The set $WF_{L^1}(u)$ is closed, conical in ξ , linear in ω , and always contains the trivial fiber $0_{pol} := \mathbb{R}^d \times \mathbb{R}^{d^*} \times \{0_{\mathbb{R}^m}\}$.

1.2. Consequences. Starting from Theorem 1.2 we shall deduce a series of results. For simplicity, we chose to work with temperate distributions and temperate Radon measures rather than distributions and locally bounded Radon measures, which would have been possible by simply adding a cut-off in the definition of WF_{L^1} , i.e. replacing Au by $A\chi u$ in (1.1). In the same vein, it is easy to see that for any $\chi \in C_c^\infty(\mathbb{R}^d)$, $\mu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m)$,

$$(1.4) \quad WF_{L^1}(\chi\mu) = WF_{L^1}(\mu) \cap \{(x, \xi, w) \in \mathbb{R}^d \times \mathbb{R}^{d^*} \times \mathbb{R}^m, \chi(x) \neq 0\}.$$

Indeed, let $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ equal to 1 on the support of χ . Then

$$A\mu = Qf \Rightarrow \tilde{\chi}A(\chi\mu) = \chi Qf + \tilde{\chi}[A, \chi]\mu,$$

and $\tilde{\chi}[A, \chi]$ is a pseudo-differential operator of order -1 which sends \mathcal{M}_t to L_{loc}^1 in view of Proposition 6.8. Also, for conciseness, we shall state the results involving hypothesis as equality (1.1) with only one term of type $\Psi_{cl}^0 L^1(\mathbb{R}^d, \mathbb{R})$ in the right-hand side, but of course our results hold for finite sums of such terms.

First we get a proper micro-local characterisation of the singular support of a measure.

Theorem 1.4. *For $\mu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m)$ we have:*

$$\Pi_1(WF_{L^1}(\mu) \setminus 0_{pol}) = \emptyset \iff \mu \in L_{loc}^1(\mathbb{R}^d, \mathbb{R}^m),$$

and moreover

$$\Pi_1(WF_{L^1}(\mu) \setminus 0_{pol}) \subseteq \text{supp } |\mu|_s, \quad |\mu|_s(\text{supp } |\mu|_s \setminus \Pi_1(WF_{L^1}(\mu) \setminus 0_{pol})) = 0.$$

Next, we obtain an elliptic full L^1 -regularity result.

Theorem 1.5. *Let $A \in M_{m \times m}(\Psi_{cl}^0(\mathbb{R}^d))$ be elliptic at $x \in \mathbb{R}^d$ in the sense*

$$a_0(x, \xi)\omega \neq 0_{\mathbb{R}^m}, \quad \forall \xi \in \mathbb{S}^{d-1}, \forall \omega \in \mathbb{R}^{m^*},$$

and let $\mu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m)$. Then the following holds:

$$A\mu \in \Psi_{cl}^0 L^1(\mathbb{R}^d, \mathbb{R})^m \implies \exists V \in \mathcal{V}(x), \mu|_V \in L^1(\mathbb{R}^d, \mathbb{R}^m).$$

Corollary 1.6. *Assume $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m)$, $A \in M_{m \times m}(\Psi_{cl}^k(\mathbb{R}^d))$ is elliptic with $k > 0$ for all $x \in \mathbb{R}^d$, then for $|\alpha| \leq k$ we have*

$$\partial_x^\alpha u \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m), Au \in \Psi_{cl}^0 L^1(\mathbb{R}^d, \mathbb{R})^m \implies \partial_x^\alpha u \in L_{loc}^1(\mathbb{R}^d, \mathbb{R}).$$

Also, if $A \in M_{m \times m}(\Psi_{cl}^0(\mathbb{R}^d))$ is elliptic for all $x \in \mathbb{R}^d$ and if $u \in BV(\mathbb{R}^d, \mathbb{R}^m)$, the set of L^1 -functions with gradient a matrix-valued finite Radon measure, then we recover full L^1 regularity:

$$A \nabla u \in \Psi_{cl}^0 L^1(\mathbb{R}^d, \mathbb{R})^d \implies u \in W_{loc}^{1,1}(\mathbb{R}^d, \mathbb{R}^m).$$

Previously known L^1 elliptic regularity results assuming $u, Pu \in L_{loc}^1$, P elliptic operator of order k involve loss of derivatives or requires the use of Besov spaces $B_\infty^{k,1}$, see [12, Theorem 2.6 and Remark 2.7]. At the exact L^1 level, the following counterexample from [11, Example 7.5] shows that, in general some loss is unavoidable: on the disk of radius one, consider the function

$$u(x) = \log \log(e|x|^{-1}) \in W^{1,1}(B_1),$$

which satisfies

$$\Delta u = -\frac{1}{|x|^2 \log^2(e|x|^{-1})} \in L^1(B_1), \quad D^2 u \notin L^1(B_1).$$

Note that this is (of course!) not in contradiction with Theorem 1.5 as $D^2 u$ is not a Radon measure. Theorem 1.5 eliminates the loss *under the additional assumption that μ is a Radon measure*. To the best of our knowledge, even for scalar equations ($m = 1$), this result is new. We want to emphasize that allowing 0 order pseudo-differential operators is not anecdotal: in fluid mechanics, 0-th order operators are ubiquitous and their lack of L^∞ (or equivalently L^1) boundedness is responsible for many pathological behaviours (see e.g. the double exponential growth of the complexity of solutions to two dimensional Euler equations [14]). Let us now give an example with such a fluid dynamics flavour.

Corollary 1.7. *Let $u \in BV_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Assume that both $\operatorname{div} u$ and $\operatorname{curl} u$ are in $L_{loc}^1(\mathbb{R}^d)$ (or $\Psi^1(L^1)$). Then*

$$\nabla u \in L_{loc}^1(\mathbb{R}^d, \mathbb{R}^{d \times d}).$$

Proof. Replacing u by χu , $\chi \in C_0^\infty$, we can assume that u is compactly supported. Now we have the following equation for any derivative $\partial_i u$, which is a Radon measure by the BV assumption on u ,

$$(1.5) \quad A(\partial_i u) := \frac{1}{(1 + |D_x|^2)^{\frac{1}{2}}} \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & \cdots & \cdots & \partial_d \\ \partial_2 & -\partial_1 & 0 & \cdots & \cdots & 0 \\ 0 & \partial_3 & -\partial_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \partial_d & -\partial_{d-1} \\ -\partial_d & 0 & 0 & \cdots & \cdots & \partial_1 \end{pmatrix} \partial_i u \\ = \frac{\partial_i}{(1 + |D_x|^2)^{\frac{1}{2}}} \begin{pmatrix} \operatorname{div} u \\ \operatorname{curl} u \end{pmatrix},$$

so $A(\partial_i u) \in \Psi_{cl}^0 L^1(\mathbb{R}^d, \mathbb{R})^{d+1}$, and a straightforward calculation shows that the constant coefficients operator $A \in M_{d+1 \times d}(\Psi_{cl}^0(\mathbb{R}^d))$ is elliptic:

$$\forall \xi \neq 0, \ker a_0(\xi) = \{0_{\mathbb{R}^{d+1}}\}.$$

We deduce that $WF_{L^1}(\partial_i u) = 0_{\text{pol}}$ and then Theorem 1.2 yields $\partial_i u \in L_{loc}^1(\mathbb{R}^d, \mathbb{R}^d)$. \square

Remark 1.8. *A by-product of Theorem 1.5 with $A = Id$ is that counter-examples for the lack of continuity on L^1 of zero-order operators can occur only for operators $Q \in \Psi_{cl}^0$ sending an L^1 function outside \mathcal{M}_t . Indeed, if $Qf = \mu \in \mathcal{M}_t$ we deduce from Theorem 1.5 with $A = Id$ that $\mu \in L_{loc}^1$.*

Contrarily to the usual wave fronts defined in microlocal analysis for distributions, in general, the natural implication

$$(1.6) \quad WF_{L^1}(u) = 0_{\text{pol}} \Rightarrow u \in L_{loc}^1,$$

is *not true*. Indeed, recall that general pseudo-differential operators of order 0 are not bounded on L^1 . Taking for example the function $u = D^2 w$ and f from (6.6), we have $f \in L_{comp}^1$, $Q = D^2(-\Delta + 1)^{-1} \in \Psi_{cl}^0$ such that $Qf = u \notin L^1$. On one hand this implies by definition that $WF_{L^1}(u) = 0_{\text{pol}}$, while $u \notin L_{loc}^1$ (because u is a compactly supported distribution but is not in L^1). However, from Theorem 1.4, if we assume in addition $u \in \mathcal{M}_t$, then property (1.6) is true.

We end with the following extension of De Philippis and Rindler's result [9, Theorem 1.1] from the case of constant coefficient differential operators $\mathcal{A} = \sum_{|\alpha| \leq k} A_\alpha \partial_\alpha$, $A_\alpha \in \mathbb{R}^{n \times m}$ of order at least 1 to the case of pseudo-differential operators of nonnegative order, with moreover an extra-degree of freedom on the constraint. This was the starting motivation for us to introduce the L^1 - wave front notion.

Theorem 1.9. *Let $\mu \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^m)$. Let $A \in M_{n \times m}(\Psi_{cl}^0(\mathbb{R}^d))$ such that*

$$A\mu \in \Psi_{cl}^0 L^1(\mathbb{R}^d, \mathbb{R})^n.$$

Then

$$\frac{d\mu}{d|\mu|}(x) \stackrel{|\mu|_s - a.e.}{\in} \bigcup_{|\xi|=1} \ker(a_0(x, \xi)).$$

As a consequence, for A of order $k > 0$ and for Q of order at most k , we can extend the theorem to the more general equation $\mathcal{A}\mu = Qf$. Also, for Q of order strictly less than k we can extend to the more general equation $\mathcal{A}\mu = Q\sigma$ with $\sigma \in \mathcal{M}_t$ that can be treated viewing it as $(A, -Q)(\mu, \sigma) = 0$ and using the fact that the principal symbol of $(A, -Q)$ is $(a_k, 0)$.

The version of Theorem 1.9 in [9] implies several major consequences, and our new version allows to relax the assumptions therein. In particular, Theorem 1.9 shows that Alberti's rank one theorem for gradient of BV functions also holds for measures whose curl

is a combination of derivatives of L^1 functions

$$(1.7) \quad \mu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^{l \times d}), \operatorname{curl} \mu = (\partial_p \mu_{i,q} - \partial_q \mu_{i,p})_{i \leq l, p \neq q \leq d} \in \Psi_{\text{cl}}^1 L^1(\mathbb{R}^d, \mathbb{R})^{d^2 l} \\ \implies \frac{d\mu}{d|\mu|}(x) \stackrel{|\mu|_s\text{-a.e.}}{\in} \{a \otimes \xi; a \in \mathbb{R}^l, \xi \in \mathbb{R}^{d*}\}.$$

The result in [9] has been extended in [4]; it would be interesting to see whether these results of dimensional estimates and rectifiability can be viewed and analyzed from the microlocal analysis point of view.

Finally, the micro-local perspective also allows to extend some invariance properties for constrained measures. We prove some propagation type results in Section 5. However our results in this direction are only preliminary.

1.3. Sketch of the proof of Theorem 1.2. The proof of Theorem 1.2 goes as follows. Arguing by contradiction we consider the set E of positive $|\mu|_s$ -measure on which the inclusion (1.2) fails. We first use Proposition 3.1 to construct at any point $x \in E$ an elliptic operator B_x at $(x, \frac{d\mu}{d|\mu|}(x))$ that smoothens μ in the L^1 -sense of (3.1). Then we use the equation in (3.1) as a starting point to implement an elliptic regularity strategy *à la De Philippis-Rindler* [9] relying on a precise description of tangent measures near μ_s -generic points. The main difficulties in this approach are the following:

- We need some regularity properties of the L^1 functions f_i in (3.1) with respect to the μ_s measure. These properties are a priori valid only $|\mu|_s$ -a.e., while the functions themselves depend on x ; this issue is solved in Lemma 3.3.
- We need to handle the non locality of pseudo-differential operators, which are only *pseudo-local*, and this requires some temperance properties of the scaled functions in the definition of the tangent measures.
- Finally we need to handle the limiting case of 0-th order operators with no gain of regularity.

The article is structured as follows. In Section 2 we give some natural examples illustrating the relevance of WF_{L^1} . Section 3 contains the proof of the main result, namely Theorem 1.2. In Section 4 we give the short proofs that Theorem 1.2 implies Theorems 1.4, 1.5 and 1.9. In Section 5 we consider the propagation of singularities results. In Section 6 are gathered the definitions and basic results on pseudo-differential operators that we use throughout the article. The last section contains general results of temperance properties for measures, obtained from the construction by Preiss in [16], which are important when dealing with pseudo-differential operators.

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2. EXAMPLES

In this section we give some examples of measures for which we can describe the L^1 wave front, WF_{L^1} . We illustrate the relevance of both the cotangent variable ξ and the polarisation variable ω by providing examples for which the dependance with respect to these variables is non trivial.

Proposition 2.1. (1) Let $\mu \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}^m)$. Then $WF_{L^1}(\mu) = 0_{pol}$ (and according to Theorem 1.4 the converse is true if $\mu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R}^m)$).

(2) Let $\mu = (\mu_1, \mu_2)$, $\mu_1 \in L^1_{loc}(\mathbb{R}^d; \mathbb{R}^{m_1})$ and $\mu_2 \in \mathcal{M}_t(\mathbb{R}^d; \mathbb{R}^{m_2})$. Then

$$(x, \xi, \omega) \in WF_{L^1}(\mu) \Leftrightarrow (x, \xi, \omega_2) \in WF_{L^1}(\mu_2) \text{ and } \omega = (0_{\mathbb{R}^{m_1}}, \omega_2).$$

(3) Let $x_0 \in \mathbb{R}^d, \omega_0 \in \mathbb{R}^{m^*}$ and $\mu = \omega_0 \delta_{x_0}$. Then

$$WF_{L^1}(\mu) \setminus 0_{pol} = \{(x_0, \xi, \omega); \xi \in \mathbb{R}^{d^*}, \omega \in \mathbb{R}^* \omega_0\}$$

(4) Let Σ a smooth d_1 dimensional embedded submanifold of \mathbb{R}^d , and μ the scalar surface measure on Σ . Then $WF_{L^1}(\mu)$ is the set of points $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d^*}$ such that $x \in \Sigma$ and ξ is (co)-normal to Σ :

$$\forall X \in T_x \Sigma, \langle \xi, X \rangle = 0.$$

(5) Let $(e_p)_{p=1}^m$ be the canonical basis of \mathbb{R}^m . We consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ of pairwise disjoint points such that $x_n \xrightarrow{n \rightarrow +\infty} x_0$ and

$$\mu = \sum_{n \in \mathbb{N}} 2^{-n} \delta_{x_n} \omega_n, \quad \omega_n = e_p, \text{ if } n \equiv p \pmod{m}.$$

Then

$$WF(\mu) \setminus 0_{pol} = \bigcup_{n \geq 1} \{x_n\} \times \mathbb{R}^{d^*} \times \mathbb{R}^* \omega_n \cup \{x_0\} \times \mathbb{R}^{d^*} \times \mathbb{R}^{m^*}.$$

Proof. Example (1) is trivial, as in the definition of WF_{L^1} , we can choose for any $x_0 \in \mathbb{R}^d$ and $\omega_0 \in \mathbb{R}^{m^*}$, the multiplication operator $A = \chi(x)^t \omega_0$ with $\chi \neq 0$ in a neighborhood of x_0 .

To prove Example (2), we consider first (x, ξ, ω) with $\omega = (\omega_1, \omega_2)$ such that $\omega_1 \neq 0_{\mathbb{R}^{m_1}}$. Assume for example that the first coordinate of ω_1 does not vanish. Let

$$A = (1, 0, \dots, 0) \in M_{1 \times m}(\mathbb{R}),$$

and f be the first component of the vector valued function μ_1 . Then we have

$$A\mu = f \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}),$$

and the constant operator A is elliptic at (x, ξ, ω) because the first coordinate of ω_1 does not vanish. We deduce that $(x, \xi, \omega) \notin WF_{L^1}(\mu)$. Let now $(x, \xi, 0_{\mathbb{R}^{m_1}}, \omega_2) \notin WF_{L^1}(\mu)$. By definition, there exists a A, Q_j, f_j as in Definition 1.1 such that A is elliptic at $(x, \xi, 0, \omega_2)$ and

$$A\mu = \sum_j Q_j f_j.$$

which implies by linearity

$$A \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} = \sum_j Q_j f_j - A \begin{pmatrix} \mu_1 \\ 0 \end{pmatrix}$$

or equivalently

$$A_2 \mu_2 = \sum_j Q_j f_j + A_1 \mu_1, \quad A = (A_1, A_2).$$

Since by assumption $\mu_1 \in L^1$, to conclude that $(x, \xi, \omega_2) \notin WF_{L^1}(\mu_2)$. It remains to check that if A is elliptic at $(x, \xi, 0_{\mathbb{R}^{m_1}}, \omega_2)$ then A_2 is elliptic at (x, ξ, ω_2) , which is clear from

$$a(x, \xi) \begin{pmatrix} 0 \\ \omega_2 \end{pmatrix} = a_2(x, \xi)(\omega_2).$$

Conversely, let now $(x, \xi, \omega_2) \notin WF_{L^1}(\mu_2)$. By definition, there exists a A_2, Q_j, f_j as in Definition 1.1 such that A_2 is elliptic at (x, ξ, ω_2) and

$$A_2 \mu_2 = \sum_j Q_j f_j.$$

With $A = (0_{\mathbb{R}^{m_1}}, A_2)$ we have for any $\omega_1 \in \mathbb{R}^{m_1}$:

$$A \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = A_2 \mu_2, \quad a(x, \xi)(\omega_1, \omega_2) = a_2(x, \xi)\omega_2 \neq 0,$$

and we obtain that $(x, \xi, \omega_1, \omega_2) \notin WF_{L^1}(\mu)$.

For example (3), it is clear from Example (1) that if $(x, \xi, \omega) \in WF_{L^1}(\omega_0 \delta_{x_0}) \setminus 0_{pol}$ then $x = x_0$. Let us assume that $\omega \notin \mathbb{R}\omega_0$. Choosing for A the multiplication by a constant $1 \times m$ vector orthogonal to ω_0 but not to ω , we get $A\omega_0 \delta_{x_0} = 0$ and $a(x_0, \xi)\omega \neq 0$ so $(x_0, \xi, \omega) \notin WF_{L^1}(\omega_0 \delta_{x_0}), \forall \xi \in \mathbb{R}^{d^*}$. As a consequence we have

$$WF_{L^1}(\omega_0 \delta_{x_0}) \setminus 0_{pol} \subseteq \{(x_0, \xi, \omega); \xi \in \mathbb{R}^{d^*}, \omega \in \mathbb{R}^* \omega_0\}.$$

The equality follows from the fact that $WF_{L^1}(\omega_0 \delta_{x_0}) \neq 0_{pol}$ and $WF_{L^1}(\omega_0 \delta_{x_0})$ is invariant under rotations in the ξ variables, as so does $\hat{\delta}_{x_0}$.

For example (4), we perform a change of variables and are reduced to the case when

$$\Sigma = \{(x_1, 0_{\mathbb{R}^{d_2}}); x_1 \in \mathbb{R}^{d_1}\}, \quad \mu = g(x_1) \delta_{x_2=0_{\mathbb{R}^{d_2}}}, \quad g \in C^\infty(\mathbb{R}^{d_1}, \mathbb{R}),$$

where the function g is positive. Let us apply a pseudo-differential operator $\chi(D_x)$ where χ is supported in a conic neighborhood of a point $\xi = (\xi_1, \xi_2)$ with $\xi_1 \neq 0_{\mathbb{R}^{d_1}}$, small enough so that on the support of χ we have $|\xi| \leq C|\xi_1|$. We have

$$\chi(D_x)\mu = \frac{1}{(2\pi)^d} \int e^{i(x_1 - y_1) \cdot \xi_1 + x_2 \cdot \xi_2} \chi(\xi) g(y_1) dy_1 d\xi,$$

and integrations by parts using $\frac{\partial}{\partial y_1} e^{-iy_1 \xi_1} = -i\xi_1 e^{-iy_1 \xi_1}$ gain arbitrary inverse powers of $|\xi_1|$ (hence of $|\xi|$) which shows that $\chi(D_x)\mu \in C^\infty$. We deduce the inclusion

$$WF_{L^1}(\mu) \subset \{(x_1, 0_{\mathbb{R}^{d_2}}, 0_{\mathbb{R}^{d_1}}, \xi_2); x_1 \in \mathbb{R}^{d_1}, \xi_2 \in \mathbb{R}^{d_2}\}.$$

To get the equality, we first remark that it is enough to study the case above when $g = 1$, because we can multiply all pseudo-differential operators by the smooth function $g(x_1)^{-1}$ to the right. According to Theorem 1.4, $\Pi_1 WF_{L^1}$ must have a non trivial intersection with the set $\{(x_1, 0_{\mathbb{R}^{d_2}}); x_1 \in \mathbb{R}^{d_1}\}$ because μ is not L^1 near this point. We get the equality simply because in that particular case $g_1 = 1$, WF_{L^1} is clearly invariant by rotations of the variable ξ_2 as so does $\delta_{0_{\mathbb{R}^{d_2}}}$.

Let us now turn to Example (5). Since the points x_n are pairwise disjoint and converging to x_0 , all the points $x_n, n \geq 1$ are isolated in the sequence and localising μ near each point x_n (i.e. considering $\mu_n = \chi_n \mu$, with $\chi_n = 1$ near x_n and $\chi_n = 0$ in a neighborhood of $x_k, k \neq n$), we get, according to Example (3) that

$$WF(\mu) \setminus 0_{pol} \cap \{(x, \xi, \omega); x \neq x_0\} = \bigcup_{n \geq 1} \{x_n\} \times \mathbb{R}^{d^*} \times \mathbb{R}^* \omega_n.$$

To conclude, it remains to prove that

$$WF(\mu) \setminus 0_{pol} \cap \{(x, \xi, \omega); x = x_0\} = \{x_0\} \times \mathbb{R}^{d^*} \times \mathbb{R}^{m^*}.$$

By linearity with respect to the ω variable, it is actually enough to prove that

$$(x_0, \xi, e_p) \in WF_{L^1}(\mu), \forall \xi \in \mathbb{R}^{d^*}, 1 \leq p \leq m.$$

We argue by contradiction. Assume there exists ξ_0 and p such that $(x_0, \xi_0, e_p) \notin WF_{L^1}(\mu)$. Then there exists $A, Q_j \in \Psi_{cl}^0$ matrices of pseudo-differential operators, $f_j \in L^1$ such that

$$A\mu = \sum_j Q_j f_j, \quad a_0(x_0, \xi_0) e_p \neq 0.$$

We deduce with χ_n as above,

$$A\chi_n \mu = \chi_n Q f + [A, \chi_n] \mu = Q_n f + g, g \in L^1.$$

But for n sufficiently large, x_n is arbitrarily close to x_0 , we get

$$a(x_n, \xi_0) e_p \neq 0.$$

We deduce that

$$(x_n, \xi_0, e_p) \notin WF_{L^1}(\chi_n \mu) = WF_{L^1}(2^{-n} \omega_n \delta_{x_n}) = \{x_n\} \times \mathbb{R}^{d^*} \times \mathbb{R}^* \omega_n,$$

which is a contradiction if we choose $n = p \pmod{m}$ so that $\omega_n = e_p$. \square

In example (5), we have $\mu = \mu_s$ and

$$|\mu|_s = \sum_{n \in \mathbb{N}} 2^{-n} \delta_{x_n}, \quad \frac{d\mu}{d|\mu|}(x) = \sum_{n \in \mathbb{N}} 1_{x=x_n} \omega_n.$$

As a consequence, this example shows that in Theorem 1.2, the inclusion is, in general *not* an equality. This is an indication that the wave front we define in the present work, though sufficient for the applications of our present work, might need to be refined for further applications.

3. PROOF OF THEOREM 1.2

All the results being local, using (1.4), we shall assume without loss of generality that μ is compactly supported. We argue by contradiction, and we suppose that the conclusion of Theorem 1.2 does not hold:

$$|\mu|_s(E) > 0,$$

where

$$E := \{x; (x, \frac{d\mu}{d|\mu|}(x)) \in (\Pi_{13}WF_{L^1}(\mu) \setminus 0_{pol})^c\}.$$

We start by proving a general property of WF_{L^1} that will be the starting point of our approach.

3.1. An elliptic characterisation of WF_{L^1} . In the following Proposition we give a characterisation of the elements in $(\Pi_{13}(WF_{L^1}(u) \setminus 0_{pol}))^c$ for $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m)$.

Proposition 3.1. *For $u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^m)$ the set $(\Pi_{13}(WF_{L^1}(u) \setminus 0_{pol}))^c$ coincides with*

$$(3.1) \quad \left\{ (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^m; \exists B \in M_{1 \times m}(\Psi_{cl}^0(\mathbb{R}^d)) \text{ elliptic at } (x, \omega), \right.$$

$$\left. N \in \mathbb{N}^*, Q_i \in \Psi_{cl}^0(\mathbb{R}^d), f_i \in L^1(\mathbb{R}^d, \mathbb{R}), \forall 1 \leq i \leq N, Bu = \sum_{i=1}^N Q_i f_i \right\}.$$

Here ellipticity at (x, ω) means

$$(3.2) \quad b_0(x, \xi)\omega \neq 0, \quad \forall \xi \in \mathbb{S}^{d-1}.$$

Proof of Proposition 3.1. Let $(x, \omega) \in (\Pi_{13}(WF_{L^1}(u) \setminus 0_{pol}))^c$. Then by definition for all $\xi \in \mathbb{S}^{d-1}$ we have the existence of $N_\xi \in \mathbb{N}^*, B_\xi \in M_{1 \times m}(\Psi_{cl}^0), Q_{\xi,i} \in \Psi_{cl}^0(\mathbb{R}^d)$ and $f_{\xi,i} \in L^1(\mathbb{R}^d, \mathbb{R}), \forall 1 \leq i \leq N_\xi$ such that $B_\xi u = \sum_{i=1}^{N_\xi} Q_{\xi,i} f_{\xi,i}$, with a principal symbol $b_{0,\xi}$ satisfying

$$b_{0,\xi}(x, \xi)\omega \neq 0.$$

By continuity we can find a conical neighborhood $V_\xi \in \mathcal{V}_c(\xi)$ such that

$$b_{0,\xi}(x, \eta)\omega \neq 0, \quad \forall \eta \in V_\xi.$$

and moreover $\delta_\xi \in \{\pm 1\}$ such that

$$(3.3) \quad \delta_\xi b_{0,\xi}(x, \eta)\omega > 0, \quad \forall \eta \in V_\xi.$$

Let χ_ξ be a non-negative smooth cut-off with support in V_ξ , valued one on a conical neighborhood $\tilde{V}_\xi \in \mathcal{V}_c(\xi)$ with $\tilde{V}_\xi \subset V_\xi$. As $\cup_{\xi \in \mathbb{S}^{d-1}} \tilde{V}_\xi$ is a covering of \mathbb{S}^{d-1} we can extract a finite covering

$$(3.4) \quad \exists L \in \mathbb{N}^*, \quad \mathbb{S}^{d-1} \subset \cup_{i=1}^L \tilde{V}_{\xi_i}.$$

Then we define $B := \sum_{l=1}^L Op(\chi_{\xi_l} \delta_{\xi_l}) B_{\xi_l} \in M_{1 \times m}(\Psi_{\text{cl}}^0(\mathbb{R}^d))$ with principal symbol

$$b_0(x, \xi) := \sum_{l=1}^L \chi_{\xi_l} \delta_{\xi_l} b_{0, \xi_l}(x, \xi),$$

satisfying

$$Bu = \sum_{l \in L} \sum_{i=1}^{N_{\xi_l}} Op(\chi_{\xi_l} \delta_{\xi_l}) Q_{\xi_l, i} f_{\xi_l, i}.$$

In particular the right-hand-side is a finite sum of terms Qf with $Q \in \Psi_{\text{cl}}^0(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d, \mathbb{R})$. We consider now

$$b_0(x, \xi) \omega = \sum_{l=1}^L \chi_{\xi_l} \delta_{\xi_l} b_{0, \xi_l}(x, \xi) \omega.$$

On one hand from (3.3) we deduce that all terms in the sum are non negative for all $\xi \in \mathbb{S}^{d-1}$. On the other hand from (3.4) there exists at least a set \tilde{V}_{ξ_j} such that $\xi \in \tilde{V}_{\xi_j}$, thus using again (3.3) we get (3.2). \square

3.2. Preliminaries. We start with a lemma gathering several general measure properties.

Lemma 3.2. *There exists a set \mathcal{N}_0 with $|\mu|_s(\mathcal{N}_0) = 0$ such that for all $x \in \mathcal{N}_0^c$, there exists a sequence $r_j \rightarrow 0$ such that the following three assertions hold.*

$$(i) \quad \lim_{r \rightarrow 0} \frac{\mathcal{L}^d(B(x, r))}{|\mu|_s(B(x, r))} = 0, \quad \lim_{j \rightarrow 0} \frac{|\mu|_a(B(x, r))}{|\mu|_s(B(x, r))} = 0, \quad \lim_{r \rightarrow 0} \int_{B(x, r)} \left| \frac{d\mu}{d|\mu|}(y) - \frac{d\mu}{d|\mu|}(x) \right| d|\mu|_s = 0,$$

There exists $C, c > 0$ such that the family of measures ¹

$$\mu_j := \frac{(T^{x, r_{j_k}})_\# \mu_s}{|\mu|_s(B(x, r_{j_k}))}$$

satisfies the temperate uniform in j bounds:

$$(ii) \quad |\mu_j|(B(x, R)) = \frac{|\mu|_s(B(x, Rr_j))}{|\mu|_s(B(x, r_j))} \leq C \max\{1, R^{d+\frac{1}{2}}\}, \quad |\mu_j|(B_{\frac{1}{3}}) \geq c > 0$$

We can pass to the limit

$$(iii) \quad \mu_j \xrightarrow{j \rightarrow \infty}^* \tilde{\nu} \in Tan(x, \mu) = \frac{d\mu}{d|\mu|}(x) Tan(x, |\mu|),$$

and since $|\mu_j|(B_{\frac{1}{3}}) \geq c > 0$, the restriction of $\tilde{\nu}$ to $B_{\frac{1}{2}}$ is not trivial.

¹by T^{x, r_j} we denote the dilations $\tilde{x} \in \mathbb{R}^d \mapsto x + r_j \tilde{x} \in \mathbb{R}^d$. and by $T_\#^{x, r_j} \sigma$ we denote the push-forward of the measure σ given by $T_\#^{x, r_j} \sigma(A) = \sigma(x + r_j A)$.

Assertion (i) is based on classical characterisation of singular points of a measure (see e.g. [15, Theorem 2.12 (3)], [7, Corollary 1.7.1]). Assertion (ii) is shown in Proposition 7.1. Assertion (iii) is given by the classical properties of tangent measures (see e.g. [17, Section 2.3]) and the lower bound (7.2).

Next we are going to prove the following property on the set $E \setminus \mathcal{N}_0$.

Lemma 3.3. *There exists a set $\tilde{E} \subseteq E \setminus \mathcal{N}_0$ with $|\mu|_s(E \setminus \tilde{E}) = 0$ such that for all $x \in \tilde{E}$, there exists $B \in M_{1 \times m}(\Psi_{cl}^0(\mathbb{R}^d))$, $N \in \mathbb{N}^*$, $Q_i \in \Psi_{cl}^0(\mathbb{R}^d)$, $f_i \in L^1(\mathbb{R}^d, \mathbb{R})$, $\forall 1 \leq i \leq N$, satisfying the identity*

$$Bu = \sum_{i=1}^N Q_i f_i,$$

and the elliptic condition at $(x, \frac{d\mu}{d|\mu|}(x))$:

$$b_0(x, \xi) \frac{d\mu}{d|\mu|}(x) \neq 0, \quad \forall \xi \in \mathbb{S}^{d-1}.$$

Moreover, for the sequence (r_j) constructed in Lemma 3.2 we have the following properties:

$$(iv) \quad \begin{aligned} \forall i \in \{1, \dots, N_x\}, \quad \lim_{r \rightarrow 0} \frac{|f_i \mathcal{L}^d(B(x, r))|}{|\mu|_s(B(x, r))} &= 0, \\ \exists C_{x,i} > 0, \quad \frac{|f_i \mathcal{L}^d(B(x, Rr_j))|}{|\mu|_s(B(x, r_j))} &\leq C_{x,i} \max\{1, R^{d+\frac{1}{2}}\}. \end{aligned}$$

Proof. As $|\mu|_s$ is a finite Radon measure we have $|\mu|_s(E \setminus \mathcal{N}_0) = \sup\{\mu_s(K); K \text{ compact } \subseteq E \setminus \mathcal{N}_0\}$. Thus there exists a compact set $K \subseteq E$ such that $|\mu|_s(K) > \frac{1}{2}|\mu|_s(E) > 0$.

For any $x \in K$ we have $x \in E$, so $(x, \frac{d\mu}{d|\mu|}(x)) \in (\Pi_{13}WF_{L^1}(\mu) \setminus 0_{pol})^c$. Thus we can apply Proposition 3.1 to get the existence of $B_x \in M_{1 \times m}(\Psi_{cl}^0(\mathbb{R}^d))$, $N \in \mathbb{N}^*$, $Q_{x,i} \in M_{m \times 1}(\Psi_{cl}^0(\mathbb{R}^d))$, $f_{x,i} \in L^1(\mathbb{R}^d, \mathbb{R})$, $\forall 1 \leq i \leq N$, such that

$$B_x \mu = \sum_{i=1}^{N_x} Q_{x,i} f_{x,i},$$

with the elliptic property at $(x, \frac{d\mu}{d|\mu|}(x))$:

$$b_{0,x}(x, \xi) \frac{d\mu}{d|\mu|}(x) \neq 0, \quad \forall \xi \in \mathbb{S}^{d-1}.$$

This implies the existence of $r_{x,1} > 0$ and $c_x > 0$ such that

$$|b_{0,x}(y, \xi) \frac{d\mu}{d|\mu|}(x)| > c_x > 0, \quad \forall y \in B(x, r_{x,1}), \quad \forall \xi \in \mathbb{S}^{d-1},$$

In particular there exists $\delta_x > 0$ such that

$$(3.5) \quad b_{0,x}(y, \xi) \omega \neq 0, \quad \forall y \in B(x, r_{x,1}), \quad \forall \xi \in \mathbb{S}^{d-1}, \quad \forall \omega \in \mathbb{S}^{m-1}, \quad \left\| \omega - \frac{d\mu}{d|\mu|}(x) \right\| \leq \delta_x.$$

From Lemma 3.2 for all $x \in \mathcal{N}_0^c$, thus for all $x \in K$,

$$(3.6) \quad \lim_{r \rightarrow 0} \int_{B(x,r)} \left| \frac{d\mu}{d|\mu|}(y) - \frac{d\mu}{d|\mu|}(x) \right| d|\mu|_s = 0,$$

with the average integral well-defined, meaning $|\mu|_s(B(x,r)) > 0, \forall r, \forall x \in K$. Thus for all $x \in K$ we get the existence of $r_{x,2}$ such that for all $r \leq r_{x,2}$

$$(3.7) \quad \int_{B(x,r)} \left| \frac{d\mu}{d|\mu|}(y) - \frac{d\mu}{d|\mu|}(x) \right| d|\mu|_s \leq \frac{\delta_x}{2}.$$

For all $x \in K$ and $r \leq r_{x,2}$ we have $|\mu|_s(B(x,r)) > 0$ and the set defined by

$$F_{x,r} := \left\{ y \in B(x,r), \left\| \frac{d\mu}{d|\mu|}(y) - \frac{d\mu}{d|\mu|}(x) \right\| \leq \delta_x \right\},$$

satisfies, in view of (3.7),

$$(3.8) \quad F_{x,r} \subseteq B(x,r), \quad |\mu|_s(F_{x,r}) > \frac{|\mu|_s(B(x,r))}{2}.$$

We denote by \mathcal{N}_1 the set of null $|\mu|_s$ -measure of the zero-density points w.r.t. $|\mu|_s$ of the set K , thus

$$(3.9) \quad \forall x \in K \setminus \mathcal{N}_1, \quad \exists r_{x,3}, \quad \forall r \leq r_{x,3} \quad \frac{3}{4} |\mu|_s(B(x,r)) \leq |\mu|_s(K \cap B(x,r)).$$

The closed balls $\overline{B}(x, \frac{r_x}{2}), x \in K \setminus \mathcal{N}_1$ with $r_x = \min\{r_{x,1}, r_{x,2}, r_{x,3}\}$ form a covering of the bounded set $K \setminus \mathcal{N}_1$, so by Besicovitch's covering theorem ([15, Theorem 2.7]) we can extract a countable covering: there exists $\{x_n\}_{n \in \mathbb{N}} \subseteq K \setminus \mathcal{N}_1$ such that

$$K \setminus \mathcal{N}_1 \subseteq \cup_{n \in \mathbb{N}} \overline{B}(x_n, \frac{r_{x_n}}{2}) \subseteq \cup_{n \in \mathbb{N}} B(x_n, r_{x_n}).$$

For all $n \in \mathbb{N}$ and $1 \leq i \leq N_{x_n}$ there exists a set $\mathcal{N}_{n,i}$ of null $|\mu|_s$ -measure such that

$$\lim_{r \rightarrow 0} \frac{|f_{x_n,i} \mathcal{L}^d(B(x,r))|}{|\mu|_s(B(x,r))} = 0, \quad \forall x \in (\mathcal{N}_{n,i})^c.$$

This is due to the fact that $|\mu|_s \perp f_{x_n,i} \mathcal{L}^d$ and Theorem 2.12 (3) in [15].

It follows from (3.5) that

$$(3.10) \quad \forall x \in K \setminus \mathcal{N}_1, \quad \exists n \in \mathbb{N}, \quad x \in B(x_n, r_{x_n}), \quad \text{and if } \left\| \frac{d\mu}{d|\mu|}(x) - \frac{d\mu}{d|\mu|}(x_n) \right\| \leq \delta_{x_n},$$

$$\text{then } b_{0,x_n}(x, \xi) \frac{d\mu}{d|\mu|}(x) \neq 0 \quad \forall \xi \in \mathbb{S}^{d-1}.$$

Now we define the sets

$$\tilde{K} := K \setminus (\mathcal{N}_1 \cup_{n \in \mathbb{N}} \cup_{i=1}^{N_{x_n}} \mathcal{N}_{n,i}),$$

and

$$\tilde{E} := (\cup_{n \in \mathbb{N}} F_{x_n, r_{x_n}}) \cap \tilde{K} \subseteq E \setminus \mathcal{N}_0.$$

In view of (3.8)-(3.9) we have

$$(3.11) \quad |\mu|_s(F_{x_n, r_{x_n}} \cap \tilde{K}) = |\mu|_s(F_{x_n, r_{x_n}} \cap K) \\ \geq |\mu|_s(F_{x_n, r_{x_n}}) - |\mu|_s(B(x_n, r_n) \cap K^c) \geq \left(\frac{1}{2} - \frac{1}{4}\right) |\mu|_s(B(x_n, r_x)),$$

and since $|\mu|_s(K) > 0$ we get at least one $n \in \mathbb{N}$ such that $|\mu|_s(B(x_n, r_x)) > 0$ so

$$|\mu|_s(\tilde{E}) > 0.$$

Then (3.10) and the definitions of $F_{x,r}$ and \tilde{E} ensure us that

$$\forall x \in \tilde{E}, \quad \exists n \in \mathbb{N}, \quad b_{0, x_n}(x, \xi) \frac{d\mu}{d|\mu|}(x) \neq 0, \quad \forall \xi \in \mathbb{S}^{d-1}.$$

Therefore for $x \in \tilde{E}$ we define $B := B_{x_n}$, which have the elliptic property in the statement of the Lemma, we define $N := N_{x_n}$, and we have the first part of the Lemma:

$$Bu = \sum_{i=1}^N Q_{x_n, i} f_{x_n, i}.$$

Moreover, $f_{x_n, i}$ satisfies the first assertion of (iv) as by definition $\tilde{E} \subset \mathcal{N}_{x_n, i}^c$. Then it follows that

$$\exists R_{x, x_n, i}, \quad \forall r \in (0, R_{x, x_n, i}), \quad \frac{|f_{x_n, i} \mathcal{L}^d(B(x, r))|}{|\mu|_s(B(x, r))} \leq 1.$$

For $Rr_j \leq R_{x, x_n, i}$ we deduce by combining also with Lemma 3.2 (ii) that

$$(3.12) \quad \frac{|f_{x_n, i} \mathcal{L}^d(B(x, Rr_j))|}{|\mu|_s(B(x, r_j))} = \frac{|f_{x_n, i} \mathcal{L}^d(B(x, Rr_j))|}{|\mu|_s(B(x, Rr_j))} \frac{|\mu|_s(B(x, Rr_j))}{|\mu|_s(B(x, r_j))} \\ \leq \frac{|\mu|_s(B(x, Rr_j))}{|\mu|_s(B(x, r_j))} \leq C \max\{1, R^{d+\frac{1}{2}}\}.$$

On the other hand, by using Lemma 3.2 (i) we have for $Rr_j \geq R_{x, x_n, i}$

$$\frac{|f_{x_n, i} \mathcal{L}^d(B(x, Rr_j))|}{|\mu|_s(B(x, r_j))} \leq C \frac{\|f_{x_n, i}\|_{L^1}}{r_j^{-d}} \leq \frac{C'}{R_{x, x_n, i}^d} R^d.$$

Therefore $f_{x_n, i}$ satisfies also the last assertion of (iv).

Summarizing, we have shown that we can find $\tilde{E} \subset E \setminus \mathcal{N}_0$ of positive $|\mu|_s$ -measure, on which all the conclusion of the Lemma are satisfied, excepted the fact that $|\mu_s|(E \setminus \tilde{E}) = 0$. The argument above is valid also starting from any subset of $E \setminus \mathcal{N}_0$ of positive $|\mu|_s$ -measure. Thus the conclusion of the Lemma are satisfied $|\mu|_s$ -a.e. on $E \setminus \mathcal{N}_0$. Indeed, let us we denote by E_1 the subset of $E \setminus \mathcal{N}_0$ of the points for which the conclusion of the Lemma are not satisfied. If $|\mu_s|(E_1) > 0$ then the construction above would give the existence of a subset $\tilde{E}_1 \subset E_1$ of strictly positive $|\mu_s|$ -measure, thus non-empty, on which the conclusions of the Lemma are both true and false on \tilde{E}_1 . Therefore $|\mu_s|(E_1) = 0$ and the full conclusion of the Lemma follows. \square

3.3. The contradiction argument. We go back to the contradiction argument. From now on, our proof is inspired from the approach in [9], using the elliptic type operator family from Lemma 3.3 instead of an operator constraining the measure to vanish.

We thus apply Lemma 3.3 and as $|\mu|_s(\tilde{E}) > 0$ we can choose now $x_0 \in \tilde{E}$ so that properties in Lemmas 3.2-3.3 are valid with $x = x_0$. We denote

$$\omega_0 := \frac{d\mu}{d|\mu|}(x_0).$$

From Lemma 3.2 (iii) we have that

$$\mu_j \xrightarrow{j \rightarrow \infty} \frac{d\mu}{d|\mu|}(x) \nu,$$

with $\nu \in \text{Tan}(x, |\mu|)$ a positive measure with $\nu_j(B_{\frac{1}{2}}) > 0$. Then we also have (cf Theorem 2.12 (3) in [15] applied with $\mu_s \perp g_R \mathcal{L}^d$, where $g_R(x) := Rg(Rx)$ and $R > 0$)

$$\nu_j := \frac{(T^{x_0, r_j})_{\#} |\mu|_s}{|\mu|_s(B(x_0, r_j))} \xrightarrow{j \rightarrow \infty} \nu.$$

In particular

$$0 < \lim_{j \rightarrow \infty} \nu_j(B_{\frac{1}{3}}) \leq \nu(B_{\frac{1}{2}}),$$

and as $\nu_j \perp \mathcal{L}^d$, there exists sets $E_j \subseteq B_{\frac{1}{2}}$ where $\nu_j \llcorner B_{\frac{1}{2}}$ concentrates and \mathcal{L}^d vanishes:

$$\nu_j(E_j) = \nu_j(B_{\frac{1}{2}}), \quad \mathcal{L}^d(E_j) = 0.$$

By using the full information at (x_0, ω_0) given by Lemma 3.2 we shall get Proposition 3.4 below. Then the two assertions of Proposition 3.4 give us the following contradiction:

$$0 < \nu(B_{\frac{1}{2}}) = \lim_{j \rightarrow \infty} \nu_j(E_j) \leq \lim_{j \rightarrow \infty} |\nu_j - \nu|(E_j) + \nu(E_j) = \lim_{j \rightarrow \infty} |\nu_j - \nu|(B_{\frac{1}{2}}) = 0.$$

Therefore to end the proof of Theorem 1.2 it remains to show Proposition 3.4.

Proposition 3.4. *At least on a subsequence we have*

$$\nu \llcorner B_{\frac{1}{2}} \ll \mathcal{L}^d, \quad \lim_{j \rightarrow \infty} |\nu_j - \nu|(B_{\frac{1}{2}}) = 0.$$

Proof. Let χ be a smooth cut-off function equal to 1 on $B_{\frac{1}{2}}$ and 0 outside $B_{\frac{3}{4}}$, satisfying $\int \chi(x) dx = 1$. The conclusion of the Proposition follows if we show that $\chi \nu \in L^1(\mathbb{R}^d)$ and on a subsequence

$$(3.13) \quad \chi \nu_j \xrightarrow{L^1(\mathbb{R}^d)} \chi \nu.$$

As $\nu_j \xrightarrow{j \rightarrow \infty} \nu$, it is thus enough to prove that $\{\chi \nu_j\}$ is precompact in $L^1_{loc}(\mathbb{R}^d)$.

To get the $L^1_{loc}(\mathbb{R}^d)$ -precompactness of $\{\chi \nu_j\}$ we shall consider the operator B from Lemma 3.3, and use its ellipticity property through an appropriate inversion argument.

Step 1: The inversion formula.

As $x_0 \in \tilde{E}$, the properties in Lemma 3.3 are satisfied for x_0 and in particular we obtain the existence of $B \in M_{1 \times m}(\Psi_{\text{cl}}^0(\mathbb{R}^d))$, $Q \in \Psi_{\text{cl}}^0(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}^d, \mathbb{R})$ satisfying the identity

$$B\mu = Qf,$$

the elliptic condition at (x_0, ω_0) :

$$(3.14) \quad b_0(x_0, \xi)\omega_0 \neq 0, \quad \forall \xi \in \mathbb{S}^{d-1},$$

and moreover f satisfies the properties (iv) in Lemma 3.3. For simplicity we have considered only one term in the right-hand side instead of a finite sum, as finitely many such terms can be treated similarly. According to Proposition 6.2, we can pass from left quantization to right quantization, i.e. we shall work with symbols that depend on y and ξ and are independent of the variable x , and the ellipticity property (3.14) remains valid for them at $y = x_0$. For simplicity we call these symbols again b_l and q . We apply the zoom $\frac{(T^{x_0, r_j})_{\sharp}}{|\mu|_s(B_{r_j}(x_0))}$ and get

$$\tilde{B}^j \mu_j = \tilde{Q}^j f_j,$$

where

$$(3.15) \quad f_j \mathcal{L}^d := \frac{(T^{x_0, r_j})_{\sharp} f \mathcal{L}^d}{|\mu|_s(B(x_0, r_j))}, \quad f_j(x) = \frac{(r_j)^d}{|\mu|_s(B(x_0, r_j))} f(x_0 + r_j x),$$

$$\tilde{b}^j(y, \xi) := b(x_0 + r_j y, \frac{\xi}{r_j}), \quad \tilde{q}^j(y, \xi) := q(x_0 + r_j y, \frac{\xi}{r_j}).$$

We first microlocalize in frequency to suppress the low frequencies by applying to the left $(1 - \chi(D_x))$, and get

$$(3.16) \quad B^j \mu_j := (1 - \chi(D_x)) \tilde{B}^j \mu_j = (1 - \chi(D_x)) \tilde{Q}^j f_j =: Q^j(f_j)$$

with symbols (remark that the following exact formula follows from the independence of \tilde{b}, \tilde{q} with respect to the x variable and (6.2)),

$$b^j(y, \xi) = (1 - \chi(\xi)) b(x_0 + r_j y, \frac{\xi}{r_j}), \quad q^j(y, \xi) := (1 - \chi(\xi)) q(x_0 + r_j y, \frac{\xi}{r_j}).$$

Notice that the symbols b^j and q^j are in $S_{\text{cl}}^0(\mathbb{R}^d)$ with semi-norms in $S_{\text{cl}}^0(\mathbb{R}^d)$ uniformly bounded in j . Indeed, from (3.16) we get

$$(3.17) \quad |\partial_y^\alpha \partial_\xi^\beta b^j(y, \xi)| = \left| \sum_{\delta + \gamma = \beta} \frac{\beta!}{\delta! \gamma!} r_j^{|\alpha| - |\gamma|} \partial_\xi^\delta (1 - \chi(\xi)) \partial_y^\alpha \partial_\xi^\gamma b(x_0 + r_j y, \frac{\xi}{r_j}) \right|$$

$$\leq C r_j^{-|\beta|} \left(1 + \frac{|\xi|}{r_j}\right)^{-|\beta|} \mathbf{1}_{\frac{1}{2} \leq |\xi|} + C \sum_{\substack{\delta \neq 0 \\ \delta + \gamma = \beta}} r_j^{-|\gamma|} \left(1 + \frac{|\xi|}{r_j}\right)^{-|\gamma|} \mathbf{1}_{\frac{1}{2} \leq |\xi| \leq 2} \leq C(1 + |\xi|)^{-|\beta|}$$

Now we localize in space

$$B^j \chi \mu_j = Q^j \chi f_j + [\chi, Q^j] f_j - [\chi, B^j] \mu_j.$$

Next we shall apply a smoothing operator $\zeta_\epsilon(D_x)$ where $\zeta_\epsilon(\xi) = \widehat{\chi}(\epsilon\xi)$. Since

$$\zeta_\epsilon(D_x)\chi\nu_j = \widehat{\chi}(\epsilon D_x)(\chi\nu_j) = \chi_\epsilon \star (\chi\nu_j) \xrightarrow{\epsilon \rightarrow 0} \chi\nu, \quad \chi_\epsilon(x) := \frac{1}{\epsilon^d} \chi\left(\frac{x}{\epsilon}\right),$$

we can choose a sequence $\epsilon_j < \frac{1}{8}$ converging to 0 fast enough so that

$$u_j := \zeta_{\epsilon_j}(D_x)\chi\nu_j = \chi_{\epsilon_j} \star (\chi\nu_j) \xrightarrow{j \rightarrow \infty} \chi\nu.$$

Thus to get $\{\chi\nu_j\}$ precompact in $L^1_{loc}(\mathbb{R}^d)$ it is enough to get $\{u_j\}$ precompact in $L^1_{loc}(\mathbb{R}^d)$. We have by applying $\zeta_{\epsilon_j}(D_x)$:

$$(3.18) \quad \begin{aligned} B^j \omega_0 u_j &= B^j V_j + Q^j g_j \\ -[\zeta_{\epsilon_j}(D_x), B^j] \chi \mu_j + [\zeta_{\epsilon_j}(D_x), Q^j] \chi f_j + \zeta_{\epsilon_j}(D_x) ([\chi, Q^j] f_j - [\chi, B^j] \mu_j), \end{aligned}$$

where

$$V_j := \zeta_{\epsilon_j}(D_x)(\omega_0 \chi \nu_j - \chi \mu_j), \quad g_j = \zeta_{\epsilon_j}(D_x) \chi f_j$$

Next we consider the family of pseudo-differential operators

$$B^j_{\omega_0} := B^j \omega_0,$$

of symbols

$$b^j_{\omega_0}(y, \xi) := b^j(y, \xi) \omega_0 = (1 - \chi(\xi)) b(x_0 + r_j y, \frac{\xi}{r_j}) \omega_0,$$

which are bounded in $S^0_{cl}(\mathbb{R}^d)$ uniformly with respect to j . We split:

$$b^j_{\omega_0}(y, \xi) = (1 - \chi(\xi)) b_0(x_0 + r_j y, \xi) \omega_0 + (1 - \chi(\xi)) (b - b_0)(x_0 + r_j y, \frac{\xi}{r_j}) \omega_0.$$

From (3.14) we get that the principal symbols $b_{0,l}$ satisfy

$$|b_0(y, \xi) \omega_0| \geq c > 0,$$

for all $\xi \in \mathbb{S}^{d-1}$ and y close to x_0 , and following the same lines as in (3.17), the symbol

$$(1 - \chi(\xi)) (b - b_0)(x_0 + r_j y, \frac{\xi}{r_j}) \omega_0$$

is uniformly in S^{-1} and bounded by $C r_j (1 + |\xi|)^{-1}$. So for j large enough, $B^j_{\omega_0}$ is elliptic on B_1 uniformly with respect to j , in the sense:

$$\exists J, c > 0, \quad |b^j_{\omega_0}(y, \xi) \omega_0| \geq c, \quad \forall j \geq J, \forall |\xi| \geq 1, \forall y \in B_1.$$

and we can approximately invert it locally as follows. Let us consider $\tilde{\chi} \in C^\infty_0(B_1)$ equal to 1 on $B_{\frac{7}{8}}$ thus in particular near the support of χ . For $j \geq J$ we define

$$p^j(x, \xi) = \frac{\chi(x)(1 - \chi(\xi))}{b^j_{\omega_0}(x, \xi)},$$

which is in S^0_{cl} uniformly with respect to j , so that, by symbolic calculus of pseudo-differential operators, and by adding the additional cutt-off $\tilde{\chi}$ for later use,

$$P^j \tilde{\chi} B^j_{\omega_0} \tilde{\chi} = \chi(x)(1 - \chi(D_x)) + \tilde{R}^j = \chi(x) + R^j,$$

where $\tilde{\chi}R^j\tilde{\chi} = R^j$ and the family of operators R^j is a family of pseudo-differential operators uniformly bounded in Ψ_{cl}^{-1} , i.e.

$$(3.19) \quad \forall \alpha, \beta, \sup_j \sup_{x, \xi} (1 + |\xi|)^{1+|\beta|} |\partial_x^\alpha \partial_\xi^\beta (r^j(x, \xi))| < +\infty.$$

As $u_j = \chi_{\epsilon_j} \star (\chi \nu_j)$ is supported in an $\epsilon_j < \frac{1}{8}$ neighborhood of $B_{\frac{3}{4}}$ we have $u_j = \tilde{\chi}u_j$. Then we can apply $\tilde{\chi}P^j\tilde{\chi}$ to (3.18) to get:

$$(3.20) \quad u_j = -R^j(u_j) + \tilde{\chi}P^j\tilde{\chi}(B^jV_j + Q^jg_j) \\ - \tilde{\chi}P^j\tilde{\chi}[\zeta_{\epsilon_j}(D_x), B^j]\chi\mu_j + \tilde{\chi}P^j\tilde{\chi}[\zeta_{\epsilon_j}(D_x), Q^j]\chi f_j + \tilde{\chi}P^j\tilde{\chi}\zeta_{\epsilon_j}(D_x)([\chi, Q^j]f_j - [\chi, B^j]\mu_j).$$

Step 2: The compactness arguments.

To get Proposition 3.4 we want to prove that $\{u_j\}$ is precompact in $L_{loc}^1(\mathbb{R}^d)$, and we have to study the contributions of each terms.

We start with the term $R^j(u_j) = \tilde{\chi}R^j\tilde{\chi}u_j$ in (3.20). From (3.19) and Corollary 6.7 applied with $\delta = -1$ the operators R^j are bounded from \mathcal{M}_0 to $W^{1-\epsilon, 1}(\mathbb{R}^d)$ for $0 < \epsilon < 1$, uniformly in j . Also, by the weak convergence of u_j we get that $\tilde{\chi}u_j$ is uniformly bounded in \mathcal{M}_0 . Then Proposition 6.4 with $s = 1 - \epsilon$ implies that $\{R^j(u_j)\}$ is pre-compact in $L_{loc}^1(\mathbb{R}^d)$.

The same argument allows to handle the contribution of $\tilde{\chi}P^j\tilde{\chi}[\zeta_{\epsilon_j}(D_x), Q^j]\chi f_j$, by using this time that χf_j is uniformly bounded in \mathcal{M}_0 due to the first assertion in Lemma 3.3 (iv), and also the fact that the symbol $\zeta(\epsilon_j\xi)$ of the operator $\zeta_{\epsilon_j}(D_x)$ is in $S_{cl}^0(\mathbb{R}^d)$ uniformly with respect to j . The sequence $\{\tilde{\chi}P^j\tilde{\chi}[\zeta_{\epsilon_j}(D_x), B^j]\chi\mu_j\}$ is also precompact in L^1 by the same argument.

Let us now study the sequence $\{\tilde{\chi}P^j\tilde{\chi}\zeta_{\epsilon_j}(D_x)[\chi, B^j]\mu_j\}$. Here the main difference with respect to the previous analysis is that μ_j is not necessarily bounded in \mathcal{M}_0 . However, according to Lemma 3.2 (ii), μ_j is bounded in $\mathcal{M}_{d+1}(\mathbb{R}^d)$ only, i.e. its mass on balls of radius $R > 1$ can grow at most like $R^{d+\frac{1}{2}}$. From Proposition 6.8, we get that the family of operators

$$\tilde{\chi}P^j\tilde{\chi}\zeta_{\epsilon_j}(D_x)[\chi, B^j]\langle x \rangle^{d+\frac{1}{2}}$$

is uniformly in j bounded from \mathcal{M}_0 to $W^{1-\epsilon, 1}$ for an $\epsilon \in (0, 1)$. Then the boundedness of $\langle x \rangle^{-(d+\frac{1}{2})}\mu_j$ in \mathcal{M}_0 from Proposition 3.2 (ii) and Proposition 6.4 give that $\{\tilde{\chi}P^j\tilde{\chi}\zeta_{\epsilon_j}(D_x)[\chi, B^j]\mu_j\}$ is precompact in $L_{loc}^1(\mathbb{R}^d)$.

We get similarly the relative compactness of the sequence $\{\tilde{\chi}P^j\tilde{\chi}\zeta_{\epsilon_j}(D_x)[\chi, Q^j]f_j\}$ by using the boundedness of $\langle x \rangle^{-(d+\frac{1}{2})}f_j$ in \mathcal{M}_0 which follows from the second assertion of Lemma 3.3 (iv).

It remains to study the second term in (3.20):

$$\tilde{\chi}P^j\tilde{\chi}(B^jV_j + Q^jg_j).$$

For this we first note that we have the following property:

$$(3.21) \quad \lim_{j \rightarrow +\infty} \|V_j\|_{L^1} + \|g_j\|_{L^1} = 0.$$

The L^1 -convergence of V_j 's follows from Lemma 3.2 (i)-(ii) exactly as in [9, (2.8)]. For sake of completeness let us recall the short argument. For $\epsilon_j < \frac{1}{8}$, the function V_j is supported in $B_{3/4} + B_{1/8} \subseteq B_{7/8}$, and

$$\begin{aligned} \int_{B_{7/8}} |V_j|(x)dx &\leq \int_{B_1} |\omega_0 \chi \nu_j - \chi \mu_j| \leq \frac{|\omega_0 T_{\#}^{x_0, r_j} |\mu|_s - T_{\#}^{x_0, r_j} \mu|(B_1)}{|\mu|_s(B(x_0, r_j))} \\ &\leq \frac{|\omega_0 |\mu|_s - \mu_s|B(x_0, r_j)}{|\mu|_s(B(x_0, r_j))} + \frac{|\mu_a|B(x_0, r_j)}{|\mu|_s(B(x_0, r_j))} \leq \int_{B(x_0, r_j)} \left| \frac{d\mu}{d|\mu|}(x_0) - \frac{d\mu}{d|\mu|}(x) \right| + \frac{|\mu_a|B(x_0, r_j)}{|\mu|_s(B(x_0, r_j))}. \end{aligned}$$

Then we deduce from Lemma 3.2 (i)-(ii) the L^1 -convergence of V_j to zero. On the other hand, from Lemma 3.3 (iv), we get that χf_j , and hence also $g_j = \chi_j \star \chi f_j$, converges to 0 in L^1 .

Now we can use $L^1 - L^{1,\infty}$ estimates for the 0-order operators $\tilde{\chi} P^j \tilde{\chi} B_l^j$ and $\tilde{\chi} P^j \tilde{\chi} Q^j$ that are Calderon-Zygmund operators, the fact that the bound depends only on a finite number of semi-norms [6], that these semi-norms are uniformly bounded in j , and the convergence (3.21) to get a convergence in measure:

$$\begin{aligned} (3.22) \quad \sup_{\lambda \geq 0} \lambda \mathcal{L}^d(\{|\tilde{\chi} P^j \tilde{\chi}(B^j V_j + Q^j g_j)| > \lambda\}) \\ \leq \sup_{\lambda \geq 0} \lambda \mathcal{L}^d(\{|\tilde{\chi} P^j \tilde{\chi} V_j| > \frac{\lambda}{2}\}) + \sup_{\lambda \geq 0} \lambda \mathcal{L}^d(\{|\tilde{\chi} P^j \tilde{\chi} Q^j g_j| > \frac{\lambda}{2}\}) \\ \leq C \|V_j\|_{L^1} + C \|g_j\|_{L^1} \xrightarrow{j \rightarrow \infty} 0. \end{aligned}$$

To conclude, we are going to use following result from [9].

Lemma 3.5 ([9, Lemma 2.2]). *Consider $\{h_j\}$ a sequence of L^1 -functions supported in B_1 satisfying:*

- a) *The sequence h_j converges weakly to 0, $h_j \xrightarrow{*} 0$ in $\mathcal{D}'(\mathbb{R}^d)$*
- b) *The negative part of h_j tends to 0 in measure*

$$\forall \lambda > 0, \lim_{j \rightarrow +\infty} \mathcal{L}^d(\{h_j^- > \lambda\}) = 0$$

- c) *The sequence of negative parts h_j^- is equi-integrable,*

$$\lim_{\mathcal{L}^d(E) \rightarrow 0} \sup_{j \in \mathbb{N}} \int_E h_j^- dx = 0$$

Then

$$h_j \xrightarrow{L^1_{loc}(B_1)} 0.$$

Proof. Let us recall for completeness the short proof from [9]. Let $\varphi \in C_0^\infty(B_1)$ with $0 \leq \varphi \leq 1$. Since h_j is supported in B_1 , it is enough to prove

$$\lim_{j \rightarrow +\infty} \int_{B_1} \varphi |h_j| dx = 0.$$

We write

$$\int_{B_1} \varphi |h_j| dx = \int_{B_1} \varphi h_j dx + 2 \int_{B_1} \varphi h_j^- dx \leq \int_{B_1} \varphi h_j dx + 2 \int_{B_1} h_j^- dx.$$

As $h_j \xrightarrow{*} 0$ it is enough to show that the last integral converges to 0. Then from the equi-integrability, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{L}^d(E) \leq \delta \Rightarrow \sup_{j \in \mathbb{N}} \int_E h_j^- dx \leq \varepsilon.$$

From the convergence in measure we have

$$\exists J; \forall j \geq J, \mathcal{L}^d(\{h_j^- > \varepsilon\}) \leq \delta.$$

We deduce that for $j \geq J$

$$\int_{B_1} h_j^- dx \leq \int_{\{h_j^- > \varepsilon\} \cap B_1} h_j^- + \int_{\{h_j^- \leq \varepsilon\} \cap B_1} h_j^- \leq \varepsilon(1 + \mathcal{L}^d(B_1)).$$

□

We now check that

$$h_j := \tilde{\chi} P^j \tilde{\chi} (B^j V_j + Q^j g_j),$$

satisfies the assumptions of Lemma 3.5:

- a) from (3.21) we have that V_j and g_j are converging to 0 in L^1 , thus h_j converges weakly to 0,
- b) from (3.22), h_j and hence also h_j^- tends to 0 in measure,
- c) as $u_j \geq 0$, the negative part of h_j is bounded by $(\chi \nu_j - h_j)^+$ which we already proved it is relatively compact in $L^1_{loc}(\mathbb{R}^d)$. As a consequence, the negative part of h_j is equi-integrable.

Thus by applying Lemma 3.5 we deduce that the sequence $\{h_j\}$ converges to 0 in $L^1_{loc}(B_1)$. This concludes the fact that u_j converges in $L^1_{loc}(\mathbb{R}^d)$, and finishes the proof of Proposition 3.4 and hence of Theorem 1.2. □

4. PROOF OF THEOREMS 1.4, 1.5 AND 1.9

Proof of Theorem 1.4. The first assertion of Theorems 1.4 is equivalent to

$$\Pi_{13}(WF_{L^1}(\mu) \setminus 0_{pol}) = \emptyset \iff \mu \in L^1_{loc}.$$

If $\mu \in L^1_{loc}$ then all points $(x, \omega) \in \mathbb{R}^d \times \mathbb{R}^m$ are in the complementary of the projection of the wave front set, $\Pi_{13}(WF_{L^1}(\mu) \setminus 0_{pol})$ since (3.1) is satisfied with $B = \chi \in C_c^\infty(\mathbb{R}^d)$ equal to 1 near x . Conversely, if $\Pi_{13}(WF_{L^1}(\mu) \setminus 0_{pol}) = \emptyset$ then Theorem 1.2 ensures that $\mu \in L^1$.

Concerning the second assertion, let us first consider $x \notin \text{supp } |\mu|_s$. Then there exists $\chi \in C_c^\infty(\mathbb{R}^d)$ equal to 1 near x and such that $\chi \mu \in L^1$. The first assertion ensures that $WF_{L^1}(\chi \mu) = 0_{pol}$, so using (1.4) we get $x \notin \Pi_1(WF_{L^1}(\mu) \setminus 0_{pol})$. So we have the inclusion $\Pi_1(WF_{L^1}(\mu) \setminus 0_{pol}) \subseteq \text{supp } |\mu|_s$. On the other hand Theorem 1.2 gives us the existence of a

set N of null $|\mu|_s$ -measure such that for $x \in N^c$ we have $(x, \frac{d\mu}{d|\mu|}(x)) \in \Pi_{13}(WF_{L^1}(\mu) \setminus 0_{pol})$ so in particular $x \in \Pi_1(WF_{L^1}(\mu) \setminus 0_{pol})$. Therefore we get the last assertion. \square

Proof of Theorem 1.5. Let \tilde{V} a neighborhood of x on which the elliptic property remains valid, and $\chi \in C_c^\infty(\mathbb{R}^d)$ a smooth localisation supported in \tilde{V} , equal to 1 on $V \in \mathcal{V}(x)$. Then Definition 1.1 yields

$$\tilde{V} \times \mathbb{R}^{d^*} \times \mathbb{R}^{m^*} \subseteq WF_{L^1}(\mu)^c,$$

which in view of (1.4) implies $WF_{L^1}(\chi\mu) = 0_{pol}$. Then Theorem 1.4 implies $\chi\mu \in L_{loc}^1$, thus $\mu|_V \in L^1$. \square

Proof of Theorem 1.9. We consider first the case $Q_1 = \dots = Q_N = Id$. Theorem 1.2 gives us the existence for $|\mu|_s$ -almost all x of at least one $\xi_x \in \mathbb{R}^{d^*}$ such that

$$(x, \xi_x, \frac{d\mu}{d|\mu|}(x)) \in WF_{L^1}(\mu).$$

As $A\mu \in L^1$, from definition (1.3) of $WF_{L^1}(\mu)$ we get $a_0(x, \xi_x) \frac{d\mu}{d|\mu|}(x) = 0$, thus the conclusion of Theorem 1.9.

The case $Q_i \neq Id$ for $1 \leq i \leq p$ and $Q_i = Id$ for $p < i \leq N$ follows by considering the equation $(A, -Q_1, \dots, -Q_p)(\mu, f_1, \dots, f_p) = \sum_{i=p+1}^N f_i \in L^1$. Thus we can use the conclusion from the first case with the extended measure $\sigma = (\mu, f_1 d\mathcal{L}^d, \dots, f_p d\mathcal{L}^d)$ as by uniqueness of Radon-Nikodym decomposition we have $\sigma = (g, f_1, \dots, f_p) d\mathcal{L}^d + (\frac{d\mu}{d|\mu|}, 0_{\mathbb{R}^p}) d|\mu|_s$, $d|\sigma|_s = d|\mu|_s$ and $\frac{d\sigma}{d|\sigma|} = (\frac{d\mu}{d|\mu|}, 0_{\mathbb{R}^p})$. \square

5. ON THE SINGULAR PART OF ELEMENTARY CONSTRAINED MEASURES AND MORE

In [9] was considered the question of elementary constrained measures, i.e. measures of the form $\mu_0 = \omega_0 \nu$ with $\omega_0 \in \mathbb{R}^{m^*}$, $\nu \in \mathcal{M}_t(\mathbb{R}^d, \mathbb{R})$, constrained to vanish under the action of a first order linear constant coefficient operator $\sum_{|\alpha|=1} A_\alpha \partial_x^\alpha$, $A_\alpha \in \mathbb{R}^{n \times m}$, satisfying

$$C := \{\xi \in \mathbb{R}^d, \sum_{|\alpha|=1} A_\alpha \omega_0 \xi^\alpha = 0\} \neq \{0_{\mathbb{R}^d}\}.$$

It was noticed in [9] that in this case, passing in Fourier the equation yields $\text{supp } \hat{\nu} \subseteq C$ and ν is invariant in the directions orthogonal to C . Indeed, for $\tilde{x} \in C^\perp$, using that $\hat{\nu}$ is supported in C ,

$$\nu(x + \tilde{x}) = \int e^{i(x+\tilde{x})\xi} d\hat{\nu}(\xi) = \int_C e^{i(x+\tilde{x})\xi} d\hat{\nu}(\xi) = \nu(x).$$

However, this property gives useful informations only for first order operators as in this case, the set C is a vector space. For higher order operators the example of a simple scalar wave equation ($n = m = 1$, $\partial_t^2 - \Delta$) for which the characteristic manifold is

$$C = \{(\tau, \eta) \in \mathbb{R}^{d+1}; \tau^2 = |\eta|^2\},$$

whose orthogonal set C^\perp is reduced to $\{0_{\mathbb{R}^{d+1}}\}$, shows that this invariance property may provide no information. In this section, we give a few elements toward the understanding of more general cases, providing information about the structure of polarisation of the singular part of a constrained measure in non-elementary constrained measure cases. We shall not use our wave front WF_{L^1} , but rely rather on propagation of singularities ideas introduced previously for the study of systems of PDE's, and in particular systems of wave equations (see [5]).

We first notice that the special choice $\mu_0 = \omega_0 \nu$ reduces the study to a system of n scalar equations on the measure ν ,

$$A_j \nu = \sum_{|\alpha|=1} A_\alpha^j \omega_0 \partial_x^\alpha \nu = 0,$$

where A_α^j is the j -th line of the matrix A_α , and the invariance along C^\perp is just the invariance of ν by each of the n transport equations A_j . When studying propagation of singularities for systems, the natural extension of scalar equations (see [8, Sections 3 & 4] and [5] for boundary value problems) is to study systems with diagonal (or at least diagonalisable) *scalar* principal parts, and we start with an elementary result in this simpler case.

Lemma 5.1. *Consider a smooth vector field on \mathbb{R}^d*

$$A := \sum_{i=1}^d b_i(x) \partial_{x_i}, \quad {}^t A := A + \operatorname{div}(b) = \sum_{i=1}^d \partial_{x_i} \circ b_i,$$

a function $H \in C(\mathbb{R}^d, M_{m \times m}(\mathbb{R}))$ and $\mu_0 \in \mathcal{M}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ solution of the system²

$$(5.1) \quad {}^t A \mu_0 + H \mu_0 = \operatorname{div}(b \mu_0) + H \mu = \left(\sum_{i=1}^d \partial_{x_j} (b_i \mu_{0,k}) + \sum_{p=1}^m h_{k,p} \mu_{0,p} \right)_{k=1, \dots, m} = 0_{\mathbb{R}^m}.$$

Then the set

$$Z = \left\{ \left(x, \frac{d\mu_0}{d|\mu_0|}(x) \right), x \in \mathbb{R}^d \right\}$$

is $|\mu_0|$ -a.e. invariant by the flow

$$(x_0, \omega_0) \mapsto \left(x(s), \frac{\omega(s)}{\|\omega(s)\|} \right),$$

where $x(s) = \phi(s, x_0)$ and $\omega(s) = \tilde{\phi}(s, x_0, \omega_0)$ are defined by

$$\dot{x}(s) = b(x(s)), \quad x(0) = x_0, \quad \dot{\omega}(s) = -H(x(s))\omega(s), \quad \omega(0) = \omega_0.$$

Proof. The proof is easy: we just solve the equation! More precisely, we solve the associated equation

$$(5.2) \quad \partial_s \mu + \operatorname{div}(b\mu) + H\mu = 0, \quad \mu|_{s=0} = \mu_0$$

²Abusing notations we still denote by A the vector field $A \operatorname{Id}_{\mathbb{R}^m}$ and ${}^t A = \sum_i \partial_{x_i} \circ b_i \operatorname{Id}_{\mathbb{R}^m}$ its transpose.

Since by the duality method the solution to this equation is unique, if μ_0 solves (5.1), then the unique solution to (5.2) is given by $\mu(s) \equiv \mu_0$. Let us now assume that μ solves (5.2) and define ν by

$$\nu = \phi(-s, \cdot)_{\#} \mu \Leftrightarrow \mu = \phi(s, \cdot)_{\#} \nu.$$

Here we abuse slightly notations and denote by $\phi(s, \cdot)_{\#}$ the push forward of the measure μ by the map

$$(s, x) \mapsto (s, \phi(s, x)),$$

defined by

$$\langle \phi(s, \cdot)_{\#} \mu, \psi \rangle = \langle \mu, \psi(s, \phi(s, x)) \rangle,$$

for any $\psi \in C_0^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m)$. By using the definition of ϕ we get

$$\begin{aligned} \langle \partial_s \mu + \operatorname{div}(b\mu) + H\mu, \psi \rangle &= -\langle \mu, (\partial_1 + b \cdot \nabla_x) \psi \rangle + \langle H\mu, \psi \rangle \\ &= -\langle \nu, (\partial_1 \psi + b \cdot \nabla_x \psi)|_{(s, \phi(s, x))} \rangle + \langle H\mu, \psi \rangle = -\langle \nu, \partial_s(\psi(s, \phi(s, x))) \rangle + \langle H\mu, \psi \rangle \\ &= \langle \partial_s \nu, \psi|_{(s, \phi(s, x))} \rangle + \langle H(\cdot) \phi(s, \cdot)_{\#} \nu, \psi \rangle = \langle \phi(s, \cdot)_{\#} (\partial_s \nu + H(\phi(s, x)) \nu), \psi \rangle. \end{aligned}$$

We deduce that μ solves (5.2) if and only if $\nu = \phi(-s, \cdot)_{\#} \mu(s, \cdot)$ solves

$$\partial_s \nu + H(\phi) \nu = 0, \quad \nu|_{s=0} = \mu_0.$$

To solve this equation we apply the variation of parameter method and compute, with $C(s, x)$ to be defined,

$$\partial_s(C(s, x) \nu)(s, x) = (\partial_s C(s, x) - C(s, x) H(\phi(s, x))) \nu(s, x).$$

Let us now define C as the solution to the differential equation

$$\partial_s C(s, x) = C(s, x) H(\phi(s, x)), \quad C|_{s=0} = \operatorname{Id}.$$

Remark that C is invertible as C^{-1} is the solution to

$$\partial_s C^{-1}(s, x) = -H(\phi(s, x)) C^{-1}(s, x), \quad C^{-1}|_{s=0} = \operatorname{Id}.$$

We get then

$$C(s, x) \nu(t, \cdot) = C(0, x) \nu(0, \cdot) = \mu_0.$$

Summarising we proved

$$\mu_0 = \mu(s, \cdot) = \phi(s, \cdot)_{\#} \nu(s, \cdot) = \phi(s, \cdot)_{\#} (C^{-1}(s, x) \mu_0) \Leftrightarrow \phi(-s, \cdot)_{\#} \mu_0 = C^{-1}(s, x) \mu_0.$$

This implies that the polarisation $\frac{d\mu_0}{d|\mu_0|}$ of μ_0 at $\phi(s, x)$ is colinear to $C^{-1}(s, x)$ applied to the polarisation of μ_0 at x (colinear rather than equal because the Jacobian determinant of the change of variables $x \mapsto \phi(s, x)$ is not necessarily equal to 1 and C^{-1} is not necessarily an isometry). \square

Now, we turn to a more complicated setting involving a first order *non diagonal* equation. We consider two smooth vector fields A_1, A_2 given as previously by functions b_1 and b_2 on \mathbb{R}^2 and $\mu = (\mu_1, \mu_2) \in \mathcal{M}_{loc}(\mathbb{R}^2, \mathbb{R}^{m_1+m_2})$ solution to the coupled system

$$(5.3) \quad \begin{pmatrix} {}^t A_1 & 0 \\ 0 & {}^t A_2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0$$

where we now assume for simplicity that the matrices H_{jk} is C^∞ . We can now state our result which is reminiscent of propagation of singularities type results.

Theorem 5.2. *Assume that the vector fields A_1 and A_2 are linearly independent at each point $x \in \mathbb{R}^2$. Then for $j \in \{1, 2\}$ the sets*

$$Z_j = \left\{ \left(x, \frac{d\mu_j}{d|\mu_j|}(x) \right), x \in \mathbb{R}^d \right\}$$

are $|\mu_{j,s}|$ -a.e. invariant by the flows

$$(x_0, \omega_0) \mapsto \left(x_j(s), \frac{\omega_j(s)}{\|\omega_j(s)\|} \right),$$

where

$$\dot{x}_j(s) = b_j(s), \quad x_j(0) = x_0, \quad \dot{\omega}_j(s) = -H_{jj}(x_j(s))\omega_j(s), \quad \omega_j(0) = \omega_0.$$

In other words, as far as their singular parts are concerned, the propagation formulas for $\mu_{j,s}$ are obtained by forgetting the coupling terms H_{12} and H_{21} in the equation.

Remark 5.3. *For conciseness, we chose to work with smooth vector fields A_j . It is however very likely that this kind of results holds for Lipschitz vector fields. It would be interesting to apply the methods developed in [2] to deal with lower regularity.*

Proof. We reduce the proof to Lemma 5.1 by showing that locally we have the uncoupling property:

$$(5.4) \quad {}^t A_j \mu_{j,s} + H_{jj} \mu_{j,s} = 0.$$

We work near a point $x_0 \in \mathbb{R}^2$ and can replace μ_j by $\chi(x)\mu_j$, $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 near x_0 and after a linear change of variables, we can assume that

$$A_1(x_0) = {}^t A_1(x_0) = \frac{\partial}{\partial x_1}, \quad A_2(x_0) = {}^t A_2(x_0) = \frac{\partial}{\partial x_2}.$$

Consider now a smooth cut off $\zeta_1(\xi)$ vanishing near 0, homogeneous of degree 0 outside $\{\|\xi\| \geq 1\}$ and equal to 1 in a small conical neighborhood of $(0, 1) \cup (0, -1)$. The first step is the following Lemma.

Lemma 5.4. *Near x_0 we have*

$$(1 - \zeta_1)(D_x)\mu_1 \in W^{1-\epsilon, 1}(\mathbb{R}^2).$$

Notice also that this implies $\mu_{1,s} = (\zeta_1(D_x)\mu_1)_s$.

Proof. Let $\zeta_0 \in C_0^\infty(\mathbb{R}^2)$ equal to 1 near 0. Since $\zeta_0(D_x) \in \Psi^{-N}$, we have from Corollary 6.7:

$$(1 - \zeta_1)\zeta_0(D_x)\mu_1 \in W^{1-\epsilon, 1}(\mathbb{R}^2).$$

It remains to study

$$(1 - \zeta_1)(1 - \zeta_0)(D_x)\mu_1.$$

Near x_0 the principal symbol $a_1(x, \xi)$ of the operator ${}^t A_1$ is close to its value at (x_0, ξ) which is $i\xi_1$, hence it is invertible in a neighborhood of the support of $(1 - \zeta_1)(1 - \zeta_0)(\xi)$. As a consequence, we can define

$$p(x, \xi) = \frac{\chi(x)(1 - \zeta_1)(1 - \zeta_0)(\xi)}{a_1(x, \xi)},$$

with $\chi \in C_0^\infty$ smooth, equal to 1 near x_0 . Applying $p(x, D_x)$ to the equation satisfied by μ_1 ,

$$(5.5) \quad {}^t A_1 \mu_1 = -H_{11} \mu_1 - H_{12} \mu_2,$$

we get by symbolic calculus

$$\chi(x)(1 - \zeta_1)(1 - \zeta_0)(D_x)\mu_1 = R_1\mu_1 + R_2\mu_2,$$

where R_j are matrices of pseudo-differential operators of order -1 , which implies by Corollary 6.7 that

$$\chi(x)(1 - \zeta_1)(1 - \zeta_0)(D_x)\mu_1 \in W^{1-\epsilon, 1}(\mathbb{R}^2).$$

□

Let now $\tilde{\zeta}_1$ be a cut-off equal to 1 in a neighborhood of the support of ζ_1 . Applying $\tilde{\zeta}_1(D_x)$ to equation (5.5) we get

$$(5.6) \quad \begin{aligned} & {}^t A_1 \tilde{\zeta}_1(D_x)\mu_1 + H_{11} \tilde{\zeta}_1(D_x)\mu_1 \\ &= -H_{12} \tilde{\zeta}_1(D_x)\mu_2 + [{}^t A_1, \tilde{\zeta}_1(D_x)]\mu_1 + [H_{11}, \tilde{\zeta}_1(D_x)]\mu_1 + [H_{12}, \tilde{\zeta}_1(D_x)]\mu_2, \end{aligned}$$

Now the key point is that the r.h.s. is an L^1 function in a neighborhood of x_0 . Indeed, it is clear for the two last term as they are operators of order -1 applied to measures so Corollary 6.7 can be used³. Let us now study the first term. The function ζ_1 is supported in a small conical neighborhood of $(0, 1) \cup (0, -1)$. We can now choose a smooth function ζ_2 vanishing near 0, homogeneous of degree 0 outside $\{\|\xi\| \geq 1\}$ and equal to 1 in a small conical neighborhood of $(1, 0) \cup (-1, 0)$, and which vanishes on the support of ζ_1 . This is where we use crucially that we are working in \mathbb{R}^2 : in higher dimensions, the cut-off ζ_1 would be required to vanish near the characteristic manifold of X_1 , which at x_0 is

$$C_1 = \{\xi; \xi_1 = 0\}$$

while the cut-off ζ_2 is required to vanish on the characteristic manifold of X_2 which is at x_0 ,

$$C_2 = \{\xi; \xi_2 = 0\}.$$

In dimension 2, these two manifolds intersect at $0_{\mathbb{R}^2}$. In higher dimension they intersect along the plane

$$C_{1,2} = \{\xi; \xi_1 = \xi_2 = 0\},$$

³This is where we use the smoothness of the $A_{j,k}$; this smoothness could be relaxed to Holder continuity or even log continuity

$$\|A(x) - A(y)\| \leq \frac{C}{-\log^\alpha(|x - y|)}, \alpha > 1, \|x - y\| \leq \frac{1}{2}.$$

and consequently such a choice for ζ_2 (equal to 1 near C_1 but vanishing near C_2 , apart from a neighborhood of 0) is possible only in dimension 2. Applying Lemma 5.4 exchanging the

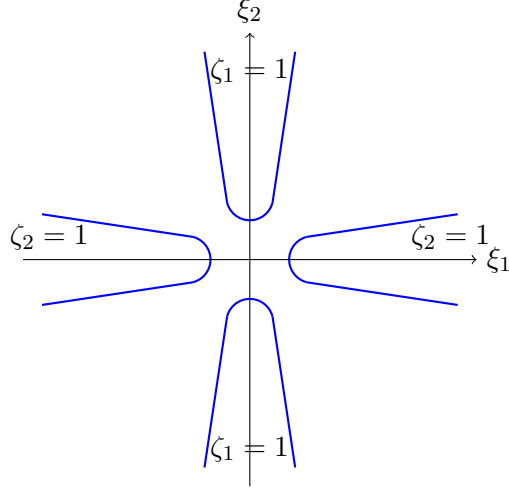


FIGURE 1. The cut-off functions

roles of μ_1 and μ_2 (and the roles of the variables x_1 and x_2), we get by Proposition 6.6 that

$$(1 - \zeta_2)(D_x)\mu_2 \in W^{1-\epsilon,1}(\mathbb{R}^2) \Rightarrow \zeta_1(D_x)\mu_2 = \zeta_1(D_x)(1 - \zeta_2)(D_x)\mu_2 \in W^{1-2\epsilon,1}(\mathbb{R}^2),$$

and consequently near x_0 ,

$$H_{12} \zeta_1(D_x)\mu_2 \in L^1.$$

Finally, to study the second term in the r.h.s. of (5.6), we apply the symbolic calculus formula (6.4) which shows that

$$[{}^t A_1, \zeta_1(D_x)] = -(\nabla_x a_1 \cdot \nabla_\xi \zeta_1)(x, D_x) + R, \quad R \in \Psi^{-1}.$$

Now $R\mu_1 \in W^{1-\epsilon,1}(\mathbb{R}^2)$, and since $\zeta_1 = 1$ on a neighborhood of the support of $\tilde{\zeta}_1$, we get that $\nabla_\xi \zeta_1$ is supported where $(1 - \tilde{\zeta}_1) = 1$ and consequently we have

$$(\nabla_x a_1 \cdot \nabla_\xi \zeta_1)(x, D_x) = (\nabla_x a_1 \cdot \nabla_\xi \zeta_1(1 - \tilde{\zeta}_1))(x, D_x),$$

therefore Lemma 5.4 with $\tilde{\zeta}_1$ and Proposition 6.6 imply

$$(\nabla_x a_1 \cdot \nabla_\xi \zeta_1)(x, D_x)\mu_1 = (\nabla_x a_1 \cdot \nabla_\xi \zeta_1)(x, D_x)(1 - \tilde{\zeta}_1)(D_x)\mu_1 \in W^{1-2\epsilon,1}(\mathbb{R}^2).$$

Summarizing, we have obtained from (5.6) that in a neighborhood of x_0

$${}^t A_1 \zeta_1(D_x)\mu_1 + H_{11} \zeta_1(D_x)\mu_1 \in L^1(\mathbb{R}^d).$$

We can now revisit the proof of Lemma 5.1, with μ_0 replaced by $\zeta_1(D_x)\mu_1$ (wich, according to Lemma 5.4 is also a measure), with the only modification that we now have an L^1 r.h.s. We get

$$\phi_1(-s, \cdot)^\sharp(\zeta_1(D_x)\mu_1) - C_1^{-1}(s, x)(\zeta_1(D_x)\mu_1) \in L_{\text{loc}}^1,$$

Passing to the singular parts (from Lemma 5.4 $\mu_{1,s} = \zeta_1(D_x)\mu_1|_s$), we get near x_0 (and for small s)

$$\phi_1(-s, \cdot)^\sharp\mu_{1,s} = C_1^{-1}(s, x)\mu_{1,s},$$

which implies Theorem 5.2 (and also (5.4)) for μ_1 and small s . The general case is obtained by iterating in s . The proof for μ_2 is similar. \square

6. PSEUDO-DIFFERENTIAL OPERATORS

In this section we have gathered basic facts about pseudo-differential operators. We refer to [13, Chapter XVIII] for a general presentation or [19, Section VI.6] for a presentation closer to our needs of the pseudo-differential calculus (see also [1]). Let us recall the definitions of the symbol classes. Here we adopt the following convention:

$$\forall \alpha \in \mathbb{N}^d, |\alpha| = \sum_{j=1}^d \alpha_j, \alpha! = \prod_{j=1}^d \alpha_j!, \partial_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \circ \cdots \circ \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

We now define the class of symbols of order k by

$$(6.1) \quad S^k(\mathbb{R}^d) = \{a \in C^\infty(\mathbb{R}^{3d});$$

$$\forall \alpha, \beta, \gamma \in \mathbb{N}^d, \sup_{x, y, \xi \in \mathbb{R}^d} |\partial_x^\alpha \partial_y^\gamma \partial_\xi^\beta a(x, y, \xi)| (1 + |\xi|)^{-k + |\beta|} = \|a\|_{k, \alpha, \beta, \gamma} < +\infty\}.$$

The constants $\|a\|_{\alpha, \beta, \gamma}$ are called semi-norms of the symbol a . Most of the time, the symbols we shall consider will not depend on the y variable, but it is convenient to allow this dependence. For simplicity we shall sometimes only consider the subclass S_{cl}^k of symbols admitting homogeneous principal symbol i.e. there exists in addition $\chi \in C_c^\infty(\mathbb{R}^d)$ and $\tilde{a}_k \in C^\infty(\mathbb{R}^d \times \mathbb{S}^d)$ such that:

$$a - (1 - \chi)(\xi) |\xi|^k \tilde{a}_k(x, \frac{\xi}{|\xi|}) \in S^{m-1}.$$

The function $a_k(x, \xi) := |\xi|^k \tilde{a}_k(x, \frac{\xi}{|\xi|})$ is the principal symbol of a .

To any symbol $a \in S_{\text{cl}}^k(\mathbb{R}^d)$ we can associate an operator on $\mathcal{S}'(\mathbb{R}^d)$ by the formula

$$a(x, y, D_x)u(x) = \text{Op}(a)u(x) = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} a(x, y, \xi) u(y) dy d\xi,$$

where this integral is defined as an oscillatory integral. We shall denote $\text{Op}(a) \in \Psi_{\text{cl}}^k(\mathbb{R}^d)$. Remark that if a do not depend on the variable x , we have

$$(6.2) \quad \text{Op}(a)u(x) = \mathcal{F}^{-1} \left(\int e^{-iy\cdot\xi} a(y, \xi) u(y) dy d\xi \right) (x).$$

An operator $Op(a)$ is said to be elliptic at a point (x_0, ξ_0) if the principal symbol $a_k(x, y, \xi)$ is non zero at the point $(x_0, x_0, \frac{\xi_0}{|\xi_0|})$.

Definition 6.1. For any $a \in S^k(\mathbb{R}^d)$ and any sequence $a_i \in S^{k_i}(\mathbb{R}^d)$ with $k_0 = k > k_1 > \dots$, we write

$$a \sim \sum_i a_i \Leftrightarrow \forall M, (a - \sum_{i=0}^M a_i) \in S^{k_{M+1}}.$$

The basic properties of symbolic calculus of pseudo-differential operators are summarized in the following

Proposition 6.2. We have the following symbolic calculus properties

- For any $a \in S_{cl}^k(\mathbb{R}^d)$, there exists $\tilde{a} \in S_{cl}^k(\mathbb{R}^d)$ not depending on the variable y . (resp. $\tilde{\tilde{a}} \in S_{cl}^k(\mathbb{R}^d)$ not depending on the variable x), such that

$$Op(a) = Op(\tilde{a}), \text{ (resp. } Op(a) = Op(\tilde{\tilde{a}}),$$

with

$$\tilde{a}(x, \xi) \sim \sum_N \sum_{|\alpha| \leq N} \frac{i^{|\alpha|}}{\alpha!} (\partial_y^\alpha \partial_\xi^\alpha a(x, y, \xi)) |_{y=x},$$

$$\tilde{\tilde{a}}(y, \xi) \sim \sum_N \sum_{|\alpha| \leq N} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_x^\alpha \partial_\xi^\alpha a(x, y, \xi)) |_{x=y}.$$

In particular,

$$\tilde{a}_k(x, \xi) = a(x, x, \xi), \quad \tilde{\tilde{a}}_k(y, \xi) = a(y, y, \xi),$$

and $Op(a)$ is elliptic at a point (x_0, ξ_0) if and only if $Op(\tilde{a})$ is elliptic at (x_0, ξ_0) in the sense that $\tilde{a}_k(x_0, \xi_0) \neq 0$ (and similarly for the right quantization).

- The formal L^2 adjoint of a pseudo-differential operator, $Op(a)$, is the pseudo-differential operator $Op(a^*)$, with

$$a^*(x, y, \xi) = \overline{a(y, x, \xi)}.$$

- For any $a \in S_{cl}^k(\mathbb{R}^d), b \in S^{\bar{k}}(\mathbb{R}^d)$, there exists $c \in S^{k+\bar{k}}(\mathbb{R}^d)$ such that

$$Op(a) \circ Op(b) = Op(c).$$

Furthermore, if the symbols a, b and c depend only on the x variable (by the previous results we can reduce the analysis to this case), we have

$$(6.3) \quad c = a \# b \sim \sum_N \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha (a(x, \xi)) \partial_x^\alpha (b(x, \xi)).$$

Remark 6.3. *From the explicit formula*

$$a\sharp b(x, \xi) = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot(\eta-\xi)} a(x, \eta) b(y, \xi) dy d\eta,$$

we can actually get a quantitative version of (6.3). Namely, for any N_0 , each semi-norm

$$(6.4) \quad \|a\sharp b - \sum_{N \leq N_0} \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha (a(x, \xi)) \partial_x^\alpha (b(x, \xi))\|_{k-N_0-1, \alpha, \beta}$$

is bounded by a product of a finite number of semi-norms of a and b .

Recal that the Sobolev space $W^{s,p}(\mathbb{R}^d)$ is defined by

$$W^{s,p}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d); (1 - \Delta)^{s/2} u \in L^p(\mathbb{R}^d)\},$$

and that we have the following property.

Proposition 6.4. *For any $1 \leq p \leq +\infty$ and any $s > 0$, $\chi \in C_0^\infty(\mathbb{R}^d)$, the application*

$$u \in W^{s,p}(\mathbb{R}^d) \mapsto \chi u \in L^p(\mathbb{R}^d)$$

is compact.

Pseudo-differential operators are bounded on L^p , and more generally on $W^{s,p}$, for $1 < p < +\infty$. More precisely, we have the following result.

Proposition 6.5 (see e.g. [19, Section VI.5.2]). *Let $A = Op(a) \in \Psi_{cl}^0$. Then, for all $1 < p < +\infty$, $s \in \mathbb{R}$, the operator A is continuous on $W^{s,p}(\mathbb{R}^d)$. Furthermore, its norm is bounded by a finite number of semi-norms*

$$\exists N(d); \|Op(a)\|_{\mathcal{L}(W^{s,p}(\mathbb{R}^d))} \leq C \sup_{|\alpha|+|\gamma|+|\beta| \leq N(d)} \|a\|_{0, \alpha, \beta, \gamma}.$$

In general, pseudo-differential operators are not continuous on L^1 and L^∞ , the basic example being the Hilbert transform (smoothed out near $\xi = 0$), associated to the symbol

$$a(x, y, \xi) = \chi(\xi), \chi \in C^\infty(\mathbb{R}), \chi|_{(-\infty, -1)} = 0, \chi|_{(1, +\infty)} = 1.$$

We also have the following counter example (see the introduction). Let $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 near 0, and

$$(6.5) \quad v(x) := \log \log(e|x|^{-1}) \in W^{1,1}(B_1), w := \chi v \in W_{comp}^{1,1}(\mathbb{R}^2) \\ \Rightarrow (-\Delta + 1)w = \frac{\chi(x)}{|x|^2 \log^2(e|x|^{-1})} + [\chi, \Delta]v + w = f \in L_{comp}^1(B_1), \quad D^2 w \notin L_{loc}^1.$$

We thus get

$$(6.6) \quad \frac{D^2}{(-\Delta + 1)^{-1}} f \notin L_{loc}^1, \quad f \in L_{comp}^1.$$

However, by the dual estimate of the Lemma in [19, Section VI.5.3.1], we have the following weaker result (we shall use only the case $p = 1$).

Proposition 6.6. *Let $\delta \in \mathbb{R}, \epsilon > 0$ and $A = Op(a) \in \Psi^\delta$. Then for all $1 \leq p \leq +\infty, s \in \mathbb{R}$ the operator A is continuous from $W^{s,p}(\mathbb{R}^d)$ to $W^{s-\delta-\epsilon,p}(\mathbb{R}^d)$. Furthermore, its norm is bounded by a finite number of semi-norms*

$$\exists N(d, \epsilon); \|Op(a)\|_{\mathcal{L}(W^{s,p}(\mathbb{R}^d), W^{s-\delta-\epsilon,p}(\mathbb{R}^d))} \leq C \sup_{|\alpha|+|\gamma|+|\beta| \leq N(d)} \|a\|_{-\epsilon, \alpha, \beta, \gamma}.$$

Corollary 6.7. *Let $\delta \in \mathbb{R}, \kappa > 0$ and $A = Op(a) \in \Psi^\delta$. Then, for all $s \in \mathbb{R}$, the operator A is continuous from $\mathcal{M}_0(\mathbb{R}^d)$ to $W^{-\delta-\kappa,1}(\mathbb{R}^d)$. Furthermore, its norm is bounded by a finite number of semi-norms*

$$\exists N(d, \epsilon); \|Op(a)\|_{\mathcal{L}(\mathcal{M}_0(\mathbb{R}^d), W^{-\delta-\kappa,1}(\mathbb{R}^d))} \leq C \sup_{|\alpha|+|\gamma|+|\beta| \leq N(d)} \|a\|_{-\kappa, \alpha, \beta, \gamma}.$$

Indeed, from the continuous inclusion for all $\epsilon > 0$,

$$W^{\kappa/2, \infty}(\mathbb{R}^d) \subset C^0(\mathbb{R}^d),$$

we deduce by duality the continuous inclusion

$$\mathcal{M}_0 \subset W^{-\kappa/2, 1}(\mathbb{R}^d),$$

and Corollary 6.7 follows from Proposition 6.6 with $s = -\kappa/2$ and $\epsilon = \kappa/2$.

We end this section with a result involving weights and bounded measures.

Proposition 6.8. *Let $\delta \in \mathbb{R}, \epsilon > 0$, $A \in \Psi^\delta$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then, for any $k \in \mathbb{R}^+$, the operator*

$$\chi A(1 + |x|^k)$$

is continuous from $\mathcal{M}_0(\mathbb{R}^d)$ to $W^{-\delta-\epsilon, 1}(\mathbb{R}^d)$. Furthermore, its norm is bounded by a finite number of semi-norms of a :

$$(6.7) \quad \exists N(d, \epsilon); \|\chi A(1 + |x|^k)\|_{\mathcal{L}(\mathcal{M}_0; W^{-\delta-\epsilon, 1}(\mathbb{R}^d))} \leq C(1 + \sup_{|\alpha|+|\gamma|+|\beta| \leq N(d)} \|a\|_{-\epsilon, \alpha, \beta, \gamma}).$$

Proof. With respect to Corollary 6.7, the only new point is the presence of the weight and of the cut-off. We use a dyadic partition of unity

$$1 = \sum_{p \geq 0} \phi_p(x), \phi_0 \in C_0^\infty(\mathbb{R}^d), \forall p \geq 1, \phi_p(x) = \phi(2^{-p}x), \phi \in C_0^\infty(\{\frac{1}{2} < \|x\| < 2\}),$$

and write

$$\chi A(1 + |x|^k) = \sum_{p \geq 0} \chi A(1 + |x|^k) \phi_p(x).$$

According to Corollary 6.7 each term is bounded from $\mathcal{M}_0(\mathbb{R}^d)$ to $W^{-\delta-\epsilon, 1}(\mathbb{R}^d)$, and we just have to check that the series of the norms is summable. Consider $K(x, y)$ the kernel of the operator $\chi A(1 + |x|^k) \phi_k(x)$:

$$(6.8) \quad K(x, y) = \frac{1}{(2\pi)^d} \int e^{i(x-y) \cdot \xi} \chi(x) a(x, y, \xi) \phi_p(y) (1 + |y|^k) d\xi.$$

Remark that on the support of this kernel, $\|x\| \leq C$, $\|y\| \geq 2^{p-1}$, and consequently, for k large enough, $\|x - y\| \sim 2^p$. Integrating by parts N times in (6.9) using the identity

$$L(e^{i(x-y)\cdot\xi} = -e^{i(x-y)\cdot\xi}, \quad L = \frac{i(x-y)\cdot\nabla\xi}{\|x-y\|^2},$$

we get

$$(6.9) \quad K(x, y) = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} L^N(a(x, y, \xi))(\chi(x)\phi_p(y)(1 + |y|^k)d\xi,$$

and consequently

$$\chi A(1 + |x|^k)\phi_k(x) = \text{op}(a_N), \quad a_N = L^N(a(x, y, \xi))(\chi(x)\phi_p(y)(1 + |y|^k)) \in S^{-N}(\mathbb{R}^d),$$

with semi-norms in S^{-N} bounded by

$$\|a_N\|_{-N, \alpha, \beta, \gamma} \leq C_{N, \alpha, \beta, \gamma} 2^{-kN} \sum_{|\alpha'| \leq |\alpha|, |\beta'| \leq |\beta|, |\gamma'| \leq |\gamma| + N} \|a\|_{0, \alpha', \beta', \gamma'}.$$

We deduce that $[\chi, B]\phi_k = \chi B\phi_k = \chi_1[\chi, B]\phi_k$ is bounded from $\mathcal{M}(\mathbb{R}^d)$ to $W^{1,1}(\mathbb{R}^d)$ by $C_N 2^{k(p-N)}$ and we conclude by choosing $N > p$. \square

7. TEMPERANCE

To deal with pseudo-differential operators in the passage to the limit when defining tangent measures, we need, in the definition of tangent measures, a temperance property which is actually satisfied on a set of full measure. This property is implicit in the construction by Preiss (see [16, Theorem 2.5]).

Proposition 7.1. *Let ν be a compactly supported bounded non negative Radon measure on \mathbb{R}^d , and let $\{r_j\}_{j \in \mathbb{N}}$ a sequence convergent to zero. Then for ν -a.e. points x_0 and every $\kappa > 0$, there is a subsequence of $\{r_{j_k}\}_{k \in \mathbb{N}}$ and $C, c > 0$ such that*

$$\nu_k := \frac{(T^{x_0, r_{j_k}})_\# \nu}{\nu(B(x, r_{j_k}))}, \quad T^{x_0, r_{j_k}} : x \mapsto x_0 + r_{j_k} x,$$

satisfies the uniform bounds for $k \in \mathbb{N}$ and $R > 1$:

$$(7.1) \quad \nu_k(B(x, R)) = \frac{\nu(B(x, Rr_{j_k}))}{\nu(B(x, r_{j_k}))} \leq CR^{d+\kappa},$$

$$(7.2) \quad \nu_k(B(x, R^{-1})) = \frac{\nu(B(x, R^{-1}r_{j_k}))}{\nu(B(x, r_{j_k}))} \geq cR^{-(d+\kappa)}.$$

Remark 7.2. *The proof gives actually a more precise bound (with a logarithmic loss) $R^d \log^a(1 + R)$, $a > 1$. On the other hand, for absolutely continuous measures $\nu = f d\mathcal{L}^d$, at all Lebesgue points of f , i.e. for \mathcal{L}^d -almost every x_0 we have*

$$\lim_{r \rightarrow 0} \frac{1}{C_d r^d} \int_{B(x_0, r)} f(x) dx = f(x_0).$$

Since $0 < |f(x_0)| < +\infty$ for $f d\mathcal{L}^d$ -almost every x_0 we get easily that in (7.1) -(7.2) we can replace $R^{\pm(d+\kappa)}$ by $R^{\pm d}$ (without the logarithmic loss). We do not know if it is the case for general ν , or even for doubling measures ν .

Proof. We basically follow the construction of tangent measures in [16] with minor differences, but since the temperate bound (7.1) does not seem to appear anywhere explicitly in the literature, we give below the complete proof.

We can use Fubini and get for any set A and radius R

$$(7.3) \quad \nu(A) = \frac{1}{\mathcal{L}^d(B(0, R))} \int \nu(A \cap B(x, R)) dx.$$

We consider for $r > 0, \delta > 0, k \in \mathbb{N}$ the set,

$$E_{r,k,\delta} := \{x \in \mathbb{R}^d, \nu(B(x, 2^k r)) \geq \beta_{k,\delta} \nu(B(x, r))\},$$

where

$$\beta_{k,\delta} := \frac{2^d(2^k + 1)^d \nu(K)}{\delta},$$

K is a compact set where ν is supported, and $\delta > 0$ to be chosen later.

Applying (7.3) to $A = E_{r,k,\delta}$, $R = \frac{r}{2}$, we get

$$\nu(E_{r,k,\delta}) = \frac{1}{\omega_d(\frac{r}{2})^d} \int \nu(E_{r,k,\delta} \cap B(x, \frac{r}{2})) dx,$$

where $\omega_d = \mathcal{L}^d(B(0, 1))$. If $E_{r,k,\delta} \cap B(x, \frac{r}{2}) \neq \emptyset$ then we can use a point z in this intersection to get

$$(7.4) \quad \begin{aligned} \nu(E_{r,k,\delta} \cap B(x, \frac{r}{2})) &\leq \nu(B(x, \frac{r}{2})) \leq \nu(B(z, r)) \\ &\leq \frac{1}{\beta_{r,k,\delta}} \nu(B(z, 2^k r)) \leq \frac{1}{\beta_{r,k,\delta}} \nu(B(x, (2^k + 1)r)). \end{aligned}$$

Therefore for any $r > 0, \delta > 0, k \in \mathbb{N}$, and using again (7.3) with $A = K$ we obtain

$$(7.5) \quad \nu(E_{r,k,\delta}) \leq \delta \frac{1}{\omega_d((2^k + 1)r)^d \nu(K)} \int \nu(B(x, (2^k + 1)r)) dx = \delta,$$

and on the complementary set ${}^c E_{r,k,\delta}$ we have

$$(7.6) \quad \frac{\nu(B(x, 2^k r))}{\nu(B(x, r))} \leq \beta_{k,\delta} = \frac{2^d(2^k + 1)^d \nu(K)}{\delta},$$

while on the complementary set ${}^c E_{r \cdot 2^{-k}, k, \delta}$ we have similarly

$$(7.7) \quad \frac{\nu(B(x, r))}{\nu(B(x, 2^{-k} r))} \leq \beta_{k,\delta} = \frac{2^d(2^k + 1)^d \nu(K)}{\delta}.$$

Let r_j be a sequence convergent to zero and $\epsilon > 0$. Now we consider

$$A_{\{r_j\}, \epsilon} := \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_{r_j, \epsilon},$$

where $A_{r_j, \epsilon} := \cup_{k=1}^{\infty} (E_{r_j, k, a_k \epsilon} \cup E_{r_j 2^{-k}, k, a_k \epsilon})$ and $a_k = k^{-1-\gamma}$, $\gamma > 0$, summable so that in view of (7.5) we have $\nu(A_{r_j, \epsilon}) \leq C\epsilon$. In particular $\nu(A_{\{r_j\}, \epsilon}) \leq C\epsilon$. For any point $x \notin A_{\{r_j\}, \epsilon}$ we get for all $i \in \mathbb{N}$ the existence of $j \geq i$ such that $x \notin A_{r_j, \epsilon}$. This yields a subsequence r_{j_n} such that $x \notin A_{r_{j_n}, \epsilon}$ for all $n \in \mathbb{N}$, thus $x \notin E_{r_{j_n}, k, a_k \epsilon}$ and $x \notin E_{r_{j_n} 2^{-k}, k, a_k \epsilon}$ for all $n, k \in \mathbb{N}$. Then, renaming this sequence r_j for simplicity, from (7.6)-(7.7) we get

$$\frac{\nu(B(x, 2^k r_j))}{\nu(B(x, r_j))} \leq \frac{2^{kd}}{k^{1+\gamma \epsilon}}, \quad \frac{\nu(B(x, 2^{-k} r_j))}{\nu(B(x, r_j))} \geq \frac{k^{1+\gamma \epsilon}}{2^{kd}}, \quad \forall j, k \in \mathbb{N}.$$

As a consequence for all $x \notin A_{\{r_j\}, \epsilon}$ there is a subsequence along which a non-zero tangent measure at x is obtained. By defining

$$A_{\{r_j\}} := \cap_{n=1}^{\infty} A_{\{r_j\}, \frac{1}{n}},$$

we get a set of zero measure such that for all $x \notin A_{\{r_j\}}$, thus in particular $x \notin A_{\{r_j\}, \frac{1}{n_x}}$ for some $n_x \in \mathbb{N}$, there is a subsequence along which a non-zero tangent measure at x is obtained and the bounds (7.1)-(7.2) are satisfied for $R = 2^k$, and hence for all $R \geq 1$. \square

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