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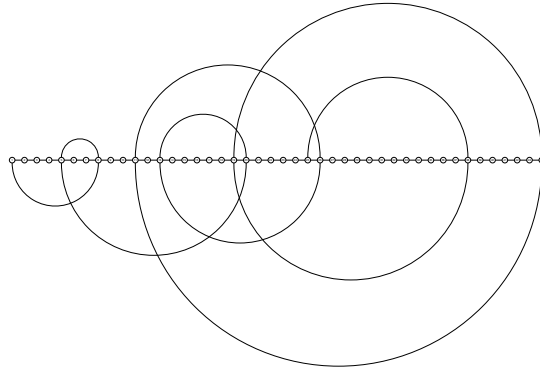
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Clockwise arc diagram of orbits from the parity Collatz function for  $n_0 = 25$  <sup>†</sup>

## 2-adic parity explorations of the $3n + 1$ problem.

Francis Laclé

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### **Abstract**

Starting with a separate 2-adic reformulation of the Collatz function that maps parity orbits, we show that the stopping time  $k$  of this reformulation can be alternatively obtained with the help of some novel ideas as well as hybrid-algorithmic techniques.

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<sup>†</sup>Figure can be zoomed in to read the label on each node when viewing with a PDF viewer.

**Definition 1.** Let  $n \in \mathbb{N}^*$  be the input term and let  $t \in \mathbb{N}$  denote the index of a sequence of  $n$ 's after iteratively applying the Collatz function  $c : \mathbb{N}^* \rightarrow \mathbb{N}^*$ , defined as the assignment

$$c(n_t) := \begin{cases} \frac{n_t}{2} & \text{if } n_t \equiv 0 \pmod{2} \\ 3n_t + 1 & \text{if } n_t \equiv 1 \pmod{2} \end{cases}$$

**Remark 1.** Throughout this paper we will modify  $c(n_t)$  and  $t$ . We have opted to retain the same symbols for clarity purposes.

**Definition 2.** Let  $e$  stand for **even** and  $\mathbb{N}_e = \{2n : n \in \mathbb{N}^*\} = \{2, 4, 6, \dots, 2n\}$  represent the subset of even positive numbers.

**Definition 3.** Let  $o$  stand for **odd** and  $\mathbb{N}_o = \{2n - 1 : n \in \mathbb{N}^*\} = \{1, 3, 5, \dots, 2n - 1\}$  represent the subset of odd positive numbers.

**Corollary 1.** Given the 2-adic valuation of  $n$  where  $\nu$  is the largest possible integer such that  $2^\nu$  divides elements of  $\mathbb{N}_e$  into positive integers, we find the outcome of this division to be always odd, i.e. any outcome  $n \in \mathbb{N}_o$ .

**Corollary 2.** Following Corollary 1 we can substitute  $\nu$  with the 2-adic valuation function  $\nu_2(n)$  (OEIS A007814 [3]) defined as the sequence

$$\nu_2(n) = (0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, \dots).$$

**Remark 2.** A primitive recursive  $\mathbb{N}^*$ -adic valuation function is given in Appendix 2.

**Corollary 3.** With Corollaries 1 and 2 successive orbits of  $\frac{n_t}{2}$  in  $c(n_t)$  given  $n_t \in \mathbb{N}_e$  can be mapped by reformulating the assignment when  $n_t \equiv 0 \pmod{2}$  as follows

$$c(n_t) := \begin{cases} \frac{n_t}{2^{\nu_2(n_t)}} & \text{if } n_t \equiv 0 \pmod{2} \\ 3n_t + 1 & \text{if } n_t \equiv 1 \pmod{2} \end{cases}$$

**Corollary 4.** Any outcome of  $3n_t + 1$  when  $n_t \equiv 1 \pmod{2}$  always equals an  $n_{t+1} \in \mathbb{N}_e$ .

*Proof.* Let  $o$  be any odd integer.  $3 \cdot o$  results in another odd integer. Given the parity present in the natural order of  $\mathbb{N}$ , adding a 1 to this result will therefore always return an element of  $\mathbb{N}_e$ .  $\square$

| $n_0$ | $(c(n_t))_{n_0 \in \mathbb{N}_o}^\infty$   |
|-------|--|
| 1     | (2, 1, ...)                                |
| 3     | (5, 8, 1, ...)                             |
| 5     | (8, 1, ...)                                |
| 7     | (11, 17, 26, 13, 20, 5, 8, 1, ...)         |
| 9     | (14, 7, 11, 17, 26, 13, 20, 5, 8, 1, ...)  |
| 11    | (17, 26, 13, 20, 5, 8, 1, ...)             |
| 13    | (20, 5, 8, 1, ...)                         |
| 15    | (23, 35, 53, 80, 5, 8, 1, ...)             |
| 17    | (26, 13, 20, 5, 8, 1, ...)                 |
| 19    | (29, 44, 11, 17, 26, 13, 20, 5, 8, 1, ...) |
| 21    | (32, 1, ...)                               |
| 23    | (35, 53, 80, 5, 8, 1, ...)                 |

Table 1: Orbits of the first 12 integers in  $\mathbb{N}_o$  of  $c(n_t)$  as given in Corollary 5.

**Corollary 5.** *Following Corollary 4 we can reformulate  $c(n_t)$  by adding one division of 2 when  $n_t \equiv 1 \pmod{2}$  as*

$$c(n_t) := \begin{cases} \frac{n_t}{2^{\nu_2(n_t)}} & \text{if } n_t \equiv 0 \pmod{2} \\ \frac{3n_t + 1}{2} & \text{if } n_t \equiv 1 \pmod{2} \end{cases}$$

*thereby further accelerating the succession. Orbits of the first 12 odd integers are listed for demonstration purposes in Table 1.*

Given Corollary 5 will continue to explore the subset  $n_t \in \mathbb{N}_o$  starting at  $t = 0$ .

**Definition 4.** *Let  $s(n_t)$  denote an expansion of  $\frac{3n_t+1}{2}$  as follows*

$$s(n_t) = \frac{n_t + 2(n_t) + 1}{2}.$$

Observe that after applying  $s(n_t)$  we get an outcome  $n_{t+1}$ , which can be recursively added to and into the expression  $\frac{n_t+2(n_t)+1}{2}$  provided that  $n_t \equiv 1 \pmod{2}$  <sup>†</sup>.

**Remark 3.** *Throughout this paper we will modify both  $s(n_t)$  and the distance between increments of  $t$ . We have opted to retain the same symbols for clarity purposes.*

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<sup>†</sup>Working this out on brown paper results in a shape similar to an upside-down Cantor ternary tree.

**Lemma 1.** *The parity of  $n_{t+1}$  can alter from  $n_t$  after applying the operation  $s(n_t)$  and this alteration depends on whether the outcome  $n_{t+1} = s(n_t)$  contains one or more factors of 2 given  $n_t \in \mathbb{N}_o$ .*

*Proof.* Let  $n_0 \in \mathbb{N}_o$  be the initial input for  $s$ . We have

$$s(n_0) = \frac{n_0 + 2(n_0) + 1}{2} = \frac{3(n_0) + 1}{2} = \frac{3}{2}n_0 + \frac{1}{2} = n_1$$

which can be restated as

$$\begin{aligned} n_1 - \frac{1}{2} &= \frac{3(n_0)}{2} \\ 2(n_1) - 1 &= 3(n_0) \quad (\text{multiply both sides by 2}) \\ 2(n_1) &= 3(n_0) + 1 \quad (\text{add 1 to both sides}) \end{aligned}$$

However,  $n_1$  could or could not be even depending on the number of factors of 2 contained [2]. To complete the proof of existence it is suffice to show one example of each case. For the first case we consider  $n_0 = 11$ . This gives  $n_1 = 17$ , while for the second case we consider  $n_0 = 5$ , which gives  $n_1 = 16$  that contains four factors of two, or  $\nu_2(16) = 4$ , and we are done.  $\square$

**Lemma 2.** *Given  $\mathbb{N}_o$  there exists at least one initial value of  $n_t$  where the outcome of more than one recursive iteration of  $s(n_t)$  results in another  $n_t \in \mathbb{N}_e$  i.e. where  $\nu_2(n_t) > 1$  for a particular  $n_t$ .*

*Proof.* Let  $n_0 \in \mathbb{N}_o$  be the initial value of  $s(n_t)$  and let  $n_1$  be odd, we can insert  $s(n_0)$  given by  $n_1$  into  $s(n_t)$  to obtain

$$s(n_1) = \frac{\frac{n_0 + 2(n_0) + 1}{2} + 2 \left( \frac{n_0 + 2(n_0) + 1}{2} \right) + 1}{2} = n_2$$

which results in the dividend of  $s(n_1)$  being even as  $n_1$  is odd, simplified as

$$s(n_1) = \frac{\frac{n_0 + 2(n_0) + 1}{2} + \frac{2(n_0) + 4(n_0) + 2}{2} + 1}{2} = \frac{\frac{3}{2}n_0 + 3(n_0) + \frac{5}{2}}{2} = \frac{\frac{9}{2}n_0 + \frac{5}{2}}{2} = \frac{9}{4}n_0 + \frac{5}{4} = n_2$$

which can be restated as

$$\begin{aligned} n_2 - \frac{5}{4} &= \frac{9(n_0)}{4} \quad (\text{multiply both sides by 4}) \\ 4(n_2) - 5 &= 9(n_0) \quad (\text{add 5 to both sides}) \\ 4(n_2) &= 9(n_0) + 5. \end{aligned}$$

Similar to the result of Lemma 1 we have that 9 multiplied by an odd integer is an odd integer and an odd integer added with 5 gives an even integer, therefore if  $n_0$  is odd then  $4(n_2)$  given  $s(n_0) \rightarrow s(n_1)$  must be even.

Here  $n_2$  is even or odd following Lemma 1. For example,  $n_0 = 7$  gives  $n_2 = 17$ , while  $n_0 = 19$  gives  $n_2 = 44$  as  $\nu_2(44) = 2$ .

We now solve for the third iteration. Let  $n_2$  equal an odd integer after two iterations of  $s(n_t)$ . We can insert  $n_2$  in  $s(n_t)$  to obtain

$$s(n_2) = \frac{\frac{\frac{3}{2}n_0 + 3(n_0) + \frac{5}{2}}{2} + 2 \left( \frac{\frac{3}{2}n_0 + 3(n_0) + \frac{5}{2}}{2} \right) + 1}{2} = n_3$$

simplified as

$$s(n_2) = \frac{\frac{\frac{3}{2}n_0 + 3(n_0) + \frac{5}{2}}{2} + \frac{3(n_0) + 6(n_0) + 5}{2} + 1}{2} = \frac{\frac{9}{4}n_0 + \frac{9}{2}n_0 + \frac{19}{4}}{2} = \frac{\frac{27}{4}n_0 + \frac{19}{4}}{2} = \frac{27}{8}n_0 + \frac{19}{8} = n_3$$

from which follows that

$$n_3 - \frac{19}{8} = \frac{27(n_0)}{8} \quad (\text{multiply both sides by } 8)$$

$$8(n_3) - 19 = 27(n_0) \quad (\text{add } 19 \text{ to both sides})$$

$$8(n_3) = 27(n_0) + 19$$

As was the case with the previous two results we have that 27 multiplied by an odd integer is an odd integer and an odd integer added with 19 gives an even integer, therefore  $n_0$  must be odd and  $8(n_3)$  must be even.

To complete the proof it is suffice to provide two examples of each case. As a first case we have,  $n_0 = 15$  gives  $n_3 = 53$ , while for the second case we have  $n_0 = 23$  gives  $n_3 = 80$  as  $\nu_2(80) = 4$ .  $\square$

**Theorem 1.** *The coefficients of recursive iterations of  $s(n_t)$  are uniquely defined for some  $k$ , with  $k$  being the exponent having bases 2 and 3.*

*Proof.* Let  $n_t \in \mathbb{N}_o$  denote iterative outcomes of  $s(n_t)$  and let  $k \in \mathbb{N}$  be an iteration index for subsequent outcomes. From Lemma 1 and Lemma 2 we have the following incremental

results up until  $k = 3$

$$\begin{aligned}n_0 &= \frac{1}{1}n_0 + \frac{0}{1} \\n_1 &= \frac{3}{2}n_0 + \frac{1}{2} \\n_2 &= \frac{9}{4}n_0 + \frac{5}{4} \\n_3 &= \frac{27}{8}n_0 + \frac{19}{8}\end{aligned}$$

Then by induction we claim

$$n_4 = \frac{81}{16}n_0 + \frac{65}{16}.$$

for  $k = 4$ . Suppose that this is true, let

$$n_k = \frac{a}{b}n_0 + \frac{z}{b}$$

define all terms where  $a$  is the dividend of the first term,  $b$  be the divisor of both terms respectively, and  $z$  be the dividend of the right term. Then, the claim is valid if we can derive a correspondence with each equation for every  $k$  together with valid examples of  $c(n_t)$ . For  $a$  it is obvious that these are repeated multiplications of base 3, as such  $a = 3^k$ . For  $b$  it is similarly obvious that these are repeated multiplications of base 2, as such  $b = 2^k$ , giving

$$n_k = \frac{3^k}{2^k}n_0 + \frac{z}{2^k}.$$

With  $z$  representing  $(0, 1, 5, 19, \dots, \mathbb{N})$  we claim this to be the enumeration of the dividends of both terms up until  $k - 1$  for  $k > 0$ . For  $1 \leq k \leq 4$  we have

$$\begin{aligned}z_1 &= 1 = 1 + 0 && | k = 1 \\z_2 &= 5 = 1 + 0 + 3 + 1 && | k = 2 \\z_3 &= 19 = 1 + 0 + 3 + 1 + 9 + 5 && | k = 3 \\z_4 &= 65 = 1 + 0 + 3 + 1 + 9 + 5 + 27 + 19 && | k = 4\end{aligned}$$

with  $z_0 = 0$ . This can be rewritten as

$$\begin{aligned}z_0 &= 0 && | k = 0 \\z_1 &= 1 = 3^0 + z_0 && | k = 1 \\z_2 &= 5 = 3^0 + z_0 + 3^1 + z_1 && | k = 2 \\z_3 &= 19 = 3^0 + z_0 + 3^1 + z_1 + 3^2 + z_2 && | k = 3 \\z_4 &= 65 = 3^0 + z_0 + 3^1 + z_1 + 3^2 + z_2 + 3^3 + z_3 && | k = 4\end{aligned}$$

and further expanded as

$$z_0 = 0$$

$$z_1 = 1 = 3^0 + 0$$

$$z_2 = 5 = 3^0 + 0 + 3^1 + 3^0 + 0$$

$$z_3 = 19 = 3^0 + 0 + 3^1 + 3^0 + 0 + 3^2 + 3^0 + 0 + 3^1 + 3^0 + 0$$

$$z_4 = 65 = 3^0 + 0 + 3^1 + 3^0 + 0 + 3^2 + 3^0 + 0 + 3^1 + 3^0 + 0 + 3^3 + 3^0 + 0 + 3^1 + 3^0 + 0 + 3^2 + 3^0 + 0 + 3^1 + 3^0 + 0.$$

The number of times that  $3^0$  or 1 gets included in the enumerative series is always by an additional factor of 2, and so we can simplify to

$$z_0 = 0$$

$$z_1 = 1 = 2^0 + 0$$

$$z_2 = 5 = 2^1 + 0 + 3^1 + 0$$

$$z_3 = 19 = 2^2 + 0 + 3^1 + 0 + 3^2 + 0 + 3^1 + 0$$

$$z_4 = 65 = 2^3 + 0 + 3^1 + 0 + 3^2 + 0 + 3^1 + 0 + 3^3 + 0 + 3^1 + 0 + 3^2 + 0 + 3^1 + 0.$$

Rearranging and cancelling the zeros we get

$$z_1 = 1 = 2^0$$

$$z_2 = 5 = 2^1 + 3^1$$

$$z_3 = 19 = 2^2 + 3^1 + 3^1 + 3^2$$

$$z_4 = 65 = 2^3 + 3^1 + 3^1 + 3^1 + 3^1 + 3^1 + 3^2 + 3^2 + 3^3.$$

which we claim to have a closed form defined as

$$z = a - b = 3^k - 2^k.$$

Let us rewrite outcomes of  $z_k$  as binomial integers, we have

$$z_1 = 1 + 0$$

$$z_2 = 2 + 3$$

$$z_3 = 4 + 15$$

$$z_4 = 8 + 57.$$

For each  $z_k$  we can expand it equally so the right term is equal to  $3^k$  while the left term is



equal to  $-2^k$  such as

$$z_1 = (1 - 3) + (0 + 3)$$

$$z_2 = (2 - 6) + (3 + 6)$$

$$z_3 = (4 - 12) + (15 + 12).$$

If this holds then the additional integers in  $z_4$  should equate 24 as the assertion here is that every pair of additional integers contains a factor of  $2^{k-1}$  times 3. Indeed

$$z_4 = (8 - 24) + (57 + 24) = 65$$

as claimed.

We can now reformulate  $s(n_t)$  explicitly. We assume values of  $n_t \in \mathbb{N}_o$  such that  $s(n_t)$  iterates in  $k + 1$  steps with initial value  $k = 0$  up until an outcome that contains factors of  $2^{n \in \mathbb{N}^*}$ . We have

$$s(n_k) = \frac{3^k(n_0) + 3^k - 2^k}{2^k} = \frac{3^k(n_0) + 3^k}{2^k} - 1$$

up until

$$\max\{k : s(n_k) \equiv 0 \pmod{2}\}.$$

As an example, let us evaluate  $s(n_k)$  for the  $c(n_t)$  function given in Corollary 5 with  $n_0 = 31$  to be the integer that returns an even integer after five consecutive iterations. We have

$$47 \rightarrow 71 \rightarrow 107 \rightarrow 161 \rightarrow 242$$

from which we can work out as

$$s(31_5) = \frac{3^5(31) + 3^5 - 2^5}{2^5} = \frac{243(31) + 243 - 32}{32} = \frac{7533 + 243 - 32}{32} = \frac{7744}{32} = 242.$$

□

**Remark 4.** *It follows from the definition of  $s(n_k)$  that its domain can be expanded to include all  $n \in \mathbb{N}^*$  as initial values  $n_0$  as  $s(n_0) = n_0$ .*

**Corollary 6.** *Following Corollary 5, we can make  $s(n_k)$  part of  $c(n_t)$  such that*

$$c(n_t) := \begin{cases} \frac{n_t}{2^{\nu_2(n_t)}} & \text{if } n_t \equiv 0 \pmod{2} \\ s(n_{t,k}) & \text{if } n_{t,k} \equiv 1 \pmod{2} \end{cases}$$

*The following orbits of initial values of  $n_{0,k} = 1$  till  $n_{0,k} = 12$  for  $n \in \mathbb{N}^*$  with presets  $k = 0$  are listed for demonstration purposes in Table 2.*

| $n_0$ | $(c(n_t))_{n \in \mathbb{N}^*}^\infty$       | $\max\{k : s(n_{t,k}) \equiv 0 \pmod{2}\}$ |
|-------|--|--|
| 1     | (1, 2, 1, ...)                               | (1, 0, ...)                                |
| 2     | (1, ...)                                     | (0, ...)                                   |
| 3     | (3, 5, 8, 1, ...)                            | (2, 0, ...)                                |
| 4     | (1, ...)                                     | (0, ...)                                   |
| 5     | (5, 8, 1, ...)                               | (1, 0, ...)                                |
| 6     | (3, 5, 8, 1, ...)                            | (2, 0, ...)                                |
| 7     | (7, 11, 17, 26, 13, 20, 5, 8, 1, ...)        | (3, 2, 2, 0, ...)                          |
| 8     | (1, ...)                                     | (0, ...)                                   |
| 9     | (9, 17, 7, 11, 17, 26, 13, 20, 5, 8, 1, ...) | (5, 2, 2, 0, ...)                          |
| 10    | (5, 8, 1, ...)                               | (1, 0, ...)                                |
| 11    | (11, 17, 26, 13, 20, 5, 8, 1, ...)           | (2, 2, 2, 0, ...)                          |
| 12    | (3, 5, 8, 1, ...)                            | (2, 0, ...)                                |

Table 2: Orbits of  $c(n_t)$  for initial values of  $n_0 = 1$  till  $n_0 = 12$  for  $n \in \mathbb{N}^*$ .

**Theorem 2.** *From Theorem 1 we derive a substitution for  $k$  of  $s(n)$  where the number of  $k$ -iterations required to reach an  $n \in \mathbb{N}_e$  from  $n = 0$  is the 2-adic valuation function  $v_2(n)$  with an offset of 1, i.e.  $k = v_2(n + 1)$ .*

*Proof.* Let  $n_0$  be an  $n_k \in \mathbb{N}^*$ . We have  $s(n_0) = n_0$  as all  $k = 0$  exponents result in 1 with an even dividend and a divisor of one. Consider  $n_0 = 1$  and  $k = 1$ , evaluated as

$$s(1_1) = \frac{3(1) + 3 - 2}{2} = \frac{3(1) + 1}{2}.$$

Observe that  $\frac{3(1)+1}{2}$  is the expression given in the equivalence relation of Corollary 4 for cases where  $n \equiv 1 \pmod{2}$ . Let us denote this expression in terms of even and odd variables  $e$  and  $o$ , and the dividend and divisor as superscript  $dd$  and  $dr$  respectively. From the rules of modular arithmetic we have

$$\frac{e^{dd}}{e^{dr}} = \frac{o(o) + o}{e} = \frac{o + o}{e}.$$

Outcomes of  $n_0 \equiv 1 \pmod{2}$  for  $e^{dr} > 1$  result in a  $\frac{e^{dd}}{e^{dr}}$  that returns an integer when

$$e^{dd} = e^{dr} \text{ if } e^{dr} > 1 \quad \text{or}$$

$$e^{dd} > e^{dr} \text{ if } e^{dd} = 2^p(e^{dr}) \text{ for } p > 0.$$

Let us now consider the case of  $n \in \mathbb{N}_e$ . We have

$$\frac{o^{dd}}{e^{dr}} = \frac{o(e) + o}{e} = \frac{e + o}{e}.$$

Outcomes of  $n_0 \equiv 0 \pmod{2}$  are therefore rational and will never return an integer, except for the odd case when we have  $2^0$  or 1 as the divisor.

For the odd case  $n_0 = 1$  recall that

$$s(1_1) = \frac{3(1) + 3 - 2}{2} = \frac{3(1) + 1}{2} = \frac{4}{2}$$

continuing the iteration for  $k = 2$  we have

$$s(1_2) = \frac{9(1) + 9 - 4}{4} = \frac{9(1) + 5}{4} = \frac{14}{4}.$$

While both iterations produce  $\frac{e^{dd}}{e^{dr}}$  we observe that solely the first iteration,  $s(1_1)$ , satisfies the constraint  $e^{dd} = 2^p(e^{dr})$  thereby returning the even integer 2. We further observe that  $e^{dd}$  of  $s(1_2)$  contains one less a factor of two than  $e^{dr}$ , namely a factor of  $2^1$  against a factor of  $2^2$ .

Let us consider the subsequent case of  $n_0 = 3$ , of which by definition of  $s(n_k)$  has a maximum of  $0 \leq k \leq 2$ . We have

$$s(3_1) = \frac{3(3) + 3 - 2}{2} = \frac{3(3) + 1}{2} = \frac{10}{2}$$

and

$$s(3_2) = \frac{9(3) + 9 - 4}{4} = \frac{9(3) + 5}{4} = \frac{32}{4}$$

and finally

$$s(3_3) = \frac{27(3) + 27 - 8}{8} = \frac{27(3) + 19}{8} = \frac{100}{8}.$$

The above shows that  $e^{dd}$  of  $s(3_1)$  contains a factor of  $2^1$  against  $2^1$  resulting in the odd integer 5. Continuing the iteration we have  $s(3_2)$  which contains a factor of  $2^5$  against a factor of  $2^2$ , satisfying the  $e^{dd} = 2^p(e^{dr})$  constraint. Finally, we have  $s(3_3)$  which does not satisfy the constraint as it contains a factor of  $2^2$  against a factor of  $2^3$ . Similarly for  $s(5_k)$   $s(7_k)$  we have

$$s(5_1) = \frac{3(5) + 3 - 2}{2} = \frac{3(5) + 1}{2} = \frac{16}{2}$$

and

$$s(7_3) = \frac{27(7) + 27 - 8}{8} = \frac{27(7) + 19}{8} = \frac{208}{8}.$$

Due to the general rational relation between the  $e^{dd}$  and  $e^{dr}$ , an  $n_k \in \mathbb{N}_e$  will result when the rational constraint  $e^{dd} = 2^p(e^{dr})$  is satisfied. To summarise we have for the four examples

$$\begin{aligned} \max\{k = 1 \text{ for } s(1_k) \in \mathbb{N}_e\} & \text{ when } s(1_k) = \frac{2^2}{2^1} \\ \max\{k = 2 \text{ for } s(3_k) \in \mathbb{N}_e\} & \text{ when } s(3_k) = \frac{2^5}{2^2} \\ \max\{k = 1 \text{ for } s(5_k) \in \mathbb{N}_e\} & \text{ when } s(5_k) = \frac{2^3}{2^1} \\ \max\{k = 3 \text{ for } s(7_k) \in \mathbb{N}_e\} & \text{ when } s(7_k) = \frac{2^4}{2^3}. \end{aligned}$$

From the above we observe that each maximum of  $k$  is in fact the 2-adic valuation of  $n_0 + 1 \in \mathbb{N}_e$  from  $n_0$ .

Recall our formulation of  $s(n_k)$  from Theorem 1

$$s(n_k) = \frac{3^k(n_0) + 3^k - 2^k}{2^k} = \frac{3^k(n_0) + 3^k}{2^k} - 1$$

through the negative exponents rule we get

$$s(n_k) = 2^{-k} \left( 3^k(n_0) + 3^k - 2^k \right) = 2^{-k} 3^k(n_0) + 2^{-k} 3^k - 1$$

expressing the above formulation as a rational gives

$$s(n_k) = \frac{1}{2^k} \cdot \frac{3^k}{1} \cdot \frac{n_0}{1} + \frac{1}{2^k} \cdot \frac{3^k}{1} - \frac{2^k}{2^k}.$$

We take the first step of  $k$ -iterations starting at  $s(7_0)$  as the example to further derive the proof. Evaluating the dividend for each  $k$  gives

$$3(7) + 3 - 2 = 22 \quad (\text{divided by } 2 \text{ gives the odd integer } 11)$$

$$9(7) + 9 - 4 = 68 \quad (\text{divided by } 4 \text{ gives the odd integer } 17)$$

$$27(7) + 27 - 19 = 208 \quad (\text{divided by } 8 \text{ gives the } \textit{even} \text{ integer } 26)$$

We can expand each outcome of  $k$  as follows

$$\frac{1}{2} \cdot \frac{3}{1} \cdot \frac{7}{1} + \frac{1}{2} \cdot \frac{3}{1} - \frac{1}{1} = \frac{3}{2} \cdot \frac{7}{1} + \frac{3}{2} - \frac{1}{1} = \frac{21}{2} + \frac{3}{2} - \frac{1}{1} = \frac{24}{2} - \frac{2}{2} = \frac{22}{2} = \frac{11}{1}$$

$$\frac{1}{4} \cdot \frac{9}{1} \cdot \frac{7}{1} + \frac{1}{4} \cdot \frac{9}{1} - \frac{1}{1} = \frac{9}{4} \cdot \frac{7}{1} + \frac{9}{4} - \frac{1}{1} = \frac{63}{4} + \frac{9}{4} - \frac{1}{1} = \frac{72}{4} - \frac{4}{4} = \frac{68}{4} = \frac{34}{2} = \frac{17}{1}$$

$$\frac{1}{8} \cdot \frac{27}{1} \cdot \frac{7}{1} + \frac{1}{8} \cdot \frac{27}{1} - \frac{1}{1} = \frac{27}{8} \cdot \frac{7}{1} + \frac{27}{8} - \frac{1}{1} = \frac{189}{8} + \frac{27}{8} - \frac{1}{1} = \frac{216}{8} - \frac{8}{8} = \frac{208}{8} = \frac{104}{4} = \frac{52}{2} = \frac{26}{1}$$

From the above expansion it can be observed on an elementary level that each  $k$ -iteration adds factors of three to the first two left terms, starting with a multiplication

$$3^k \cdot n_0$$

followed by an addition

$$3^k \cdot n_0 + 3^k$$

and a subtraction of  $-2^k$  for the right dividend term, to finally culminate with a division by  $2^k$ . From this it follows that multiples of 3 with odd numbers result in odd numbers as  $o(o) = o$ . Then, adding these multiples with powers of 3 result in even numbers as  $o(o) + o = e$ . When we consider the formulation of  $s(n_k)$  without  $-\frac{2^k}{2^k} = -1$  we get

$$\frac{3^k(n_0) + 3^k}{2^k}$$

which is equivalent to

$$\frac{3^k(n_0 + 1)}{2^k}$$

This implies that

$$3^k \cdot o + 3^k = 3^k \cdot (o + 1)$$

If  $o$  is odd then  $(o + 1)$  must be even. If  $(o + 1)$  is even then it must contain at least one factor of 2. Illustrated with natural numbers 7 and 8 as

$$7 = 2 \cdot 2 \cdot 2 - 1$$

$$8 = 2 \cdot 2 \cdot 2$$

This leaves us with the following two simplified formulations of  $s(n_{t,k})$

$$s(n_{t,k}) = \frac{3^k(n_t + 1) - 2^k}{2^k} = \frac{3^k(n_t + 1)}{2^k} - 1 \mid n_t \in \mathbb{N}_o$$

and observe for  $s(7_k)$  the following

$$s(7_0) = \frac{8}{1} - \frac{1}{1} = \frac{2 \cdot 2 \cdot 2 - 1}{1} = 2 \cdot 2 \cdot 2 - 1 \in \mathbb{N}_o$$

$$s(7_1) = \frac{24}{2} - \frac{2}{2} = \frac{3 \cdot 2 \cdot 2 \cdot 2 - 2}{2} = 3 \cdot 2 \cdot 2 - 1 \in \mathbb{N}_o$$

$$s(7_2) = \frac{72}{4} - \frac{4}{4} = \frac{3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 - 4}{4} = 3 \cdot 3 \cdot 2 - 1 \in \mathbb{N}_o$$

$$s(7_3) = \frac{216}{8} - \frac{8}{8} = \frac{3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 - 8}{8} = 3 \cdot 3 \cdot 3 - 1 \in \mathbb{N}_e$$

For the case of  $s(7_k)$  we thus have  $k = 3$  as it takes three  $k$ -iterations to factor out all 2's from  $7 + 1$  such that the outcome is again an even integer. More specifically, a subtraction

by  $-\frac{2^k}{2^k}$  in  $s(7_3)$  where its outcome returns an even integer. What remains as a constant term for every  $k$  is  $-\frac{2^k}{2^k}$ . This subtraction retains  $s(n_{t,k})$  to have an even outcome up until the same number of factors of 2's have been iteratively added as the initial  $(n_t + 1)$ , such that divisions by  $2^k$  can factor out enough of the even 2's to return back to an outcome in the form of  $o - 1 = e$ , thus ending the iterative process of  $s(n_{t,k})$ . This process coincides with the outcome of  $\nu_2(7 + 1) = 3$  as  $2^3 = 8 = 7 + 1$ . Hence, through this equivalence we can look ahead at the number of  $k$ -iterations needed for  $n_{t,0} \in \mathbb{N}_o$  to reach an  $n_{t,k} \in \mathbb{N}_e$ , defined as

$$k = \nu_2(n + 1)$$

□

With the demonstration in Theorem 1 and consequently Theorem 2, we exhibit the recursive divisions by factors of two and multiplications by factors of three in elementary form, using  $s(7_{t,k})$  as the exemplar, while highlighting noteworthy terms in bold. We have, with initial values  $t = 0$  and  $k = 0$

$$\begin{aligned} s(7_{0,0}) &= \frac{8}{1} - \frac{1}{1} = \frac{2 \cdot 2 \cdot 2 - 1}{1} = \mathbf{2 \cdot 2 \cdot 2} - 1 \\ s(7_{0,1}) &= \frac{24}{2} - \frac{2}{2} = \frac{22}{2} = \frac{11}{1} = \frac{3 \cdot 2 \cdot 2 \cdot 2 - 2}{2} = \mathbf{3 \cdot 2 \cdot 2} - 1 \\ s(7_{0,2}) &= \frac{72}{4} - \frac{4}{4} = \frac{68}{4} = \frac{34}{2} = \frac{17}{1} = \frac{3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 - 4}{4} = \mathbf{3 \cdot 3 \cdot 2} - 1 \\ s(7_{0,3}) &= \frac{216}{8} - \frac{8}{8} = \frac{208}{8} = \frac{104}{4} = \frac{52}{2} = \frac{26}{1} = \frac{3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 - 8}{8} = \mathbf{3 \cdot 3 \cdot 3} - 1 \\ \frac{26}{2} &= \frac{13}{1} = \frac{3 \cdot 3 \cdot 3 - 1}{2} = \frac{2 \cdot 2(2 \cdot 2 \cdot 2 - 1) - 2}{2} = \mathbf{2(2 \cdot 2 \cdot 2} - 1) - 1 \\ s(13_{1,1}) &= \frac{42}{2} - \frac{2}{2} = \frac{40}{2} = \frac{20}{1} = \frac{3 \cdot 2(2 \cdot 2 \cdot 2 - 1) - 2}{2} = \mathbf{3(2 \cdot 2 \cdot 2} - 1) - 1 \\ \frac{20}{2} &= \frac{10}{1} = \frac{3(2 \cdot 2 \cdot 2 - 1) - 1}{2} = \frac{3(2 \cdot 2 \cdot 2) - 2 - 2}{2} = \mathbf{3(2 \cdot 2)} - 1 - 1 \\ \frac{10}{2} &= \frac{5}{1} = \frac{3(2 \cdot 2) - 1 - 1}{2} = \mathbf{(3 \cdot 2)} - 1 \\ s(5_{2,1}) &= \frac{18}{2} - \frac{2}{2} = \frac{16}{2} = \frac{8}{1} = \frac{(3 \cdot 3 \cdot 2) - 2}{2} = \mathbf{(3 \cdot 3)} - 1 \\ \frac{8}{2} &= \frac{4}{1} = \frac{(3 \cdot 3) - 1}{2} = \frac{2 \cdot 2 \cdot 2}{2} = \mathbf{2 \cdot 2} \\ \frac{4}{2} &= \frac{2}{1} = \frac{2 \cdot 2}{2} = \mathbf{2} \\ \frac{2}{2} &= \frac{1}{1} = 1 \end{aligned}$$

with 1 entering the trivial cycle

$$\begin{aligned} s(1_{3,1}) &= \frac{6}{2} - \frac{2}{2} = \frac{4}{2} = \frac{2}{1} = \frac{2 \cdot 2}{2} = \mathbf{2} \\ \frac{2}{2} &= \frac{1}{1} = 1 \\ &\vdots \end{aligned}$$

| $n_0$ | $n_1 = s(n_0)$ | $n_2 = \frac{n_1}{2^{\nu_2(n_1)}}$ | $n_3 = s(n_2)$ | $n_4 = \frac{n_3}{2^{\nu_2(n_3)}}$ | $n_5 = s(n_4)$ | $n_6 = \frac{n_5}{2^{\nu_2(n_5)}}$ | $n_7 = s(n_6)$ | $n_8 = \frac{n_7}{2^{\nu_2(n_7)}}$ |
|-------|----------------|------------------------------------|----------------|------------------------------------|----------------|------------------------------------|----------------|------------------------------------|
| 1     | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 2     | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 3     | 8              | 1                                  | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 4     | 4              | 1                                  | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 5     | 8              | 1                                  | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 6     | 6              | 3                                  | 8              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 7     | 26             | 13                                 | 20             | 5                                  | 8              | 1                                  | 2              | 1                                  |
| 8     | 8              | 1                                  | 2              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 9     | 14             | 7                                  | 26             | 13                                 | 20             | 5                                  | 8              | 1                                  |
| 10    | 10             | 5                                  | 8              | 1                                  | 2              | 1                                  | 2              | 1                                  |
| 11    | 26             | 13                                 | 20             | 5                                  | 8              | 1                                  | 2              | 1                                  |
| 12    | 12             | 3                                  | 8              | 1                                  | 2              | 1                                  | 2              | 1                                  |

Table 3: Outcomes of the parity function  $c(n_t)$  as defined in Corollary 7 for  $1 \leq n_0 \leq 12 \wedge 1 \leq t \leq 7$ .

**Corollary 7.** *Following previous theorems together with Corollary 6, the 2-adic valuation property for  $n \in \mathbb{N}^*$  can be added to  $s(n_{t,k})$  such that  $k$  becomes a function of  $\nu_2(n_t + 1)$  giving us the reformulation*

$$s(n_t) = \frac{3^{\nu_2(n_t+1)}(n_0) + 3^{\nu_2(n_t+1)}}{2^{\nu_2(n_t+1)}} - 1 = \frac{3^{\nu_2(n_t+1)}(n_0 + 1)}{2^{\nu_2(n_t+1)}} - 1$$

Combining the above with  $\frac{n_t}{2^{\nu_2(n_t)}}$  effectively reformulates  $c(n_t)$  as a parity assignment of (even-)odd-even-odd and so forth orbits  $\forall n_t \in \mathbb{N}^*$ , defined as

$$c(n_t) := \begin{cases} \frac{n_t}{2^{\nu_2(n_t)}} & \text{if } n_t \equiv 0 \pmod{2} \\ s(n_t) & \text{if } n_t \equiv 1 \pmod{2} \end{cases}$$

demonstrated in Table 3 for the following 12 outcomes of initial values of  $n_0 = 1$  till  $n_0 = 12$  for  $n_t \in \mathbb{N}^*$ .

**Corollary 8.** *We follow Corollary 7 with one approach to describe the stopping time  $t$  from  $n_0$  up until  $n_t | c(n_t) = 1$  as a summation series that stems from the parity-alternating formulation of  $c(n_t)$ . Let  $t$  be the iteration index after every operation on  $n$ , we have*

$$t = \begin{cases} \nu_2(n_0) + 2(\nu_2(n_0|_1 + 1)) + \dots + \nu_2(n_t) & \text{if } n_t \neq 2^p \\ \nu_2(n_0) & \text{if } n_t = 2^p \end{cases}$$

where  $p \in \mathbb{N}^*$ . A 2-factor of  $k$  is included as each iteration of  $k$  is an even-multiple of

$\frac{3n+1}{2}$ . Substituting  $i$  for  $\nu_2(n_t)$  and  $k$  for  $\nu_2(n_t + 1)$ , we can simplify to

$$t = \begin{cases} i_0 + 2k_1 + \dots + i_t & \text{if } n_t \neq 2^p \\ i_0 & \text{if } n_t = 2^p \end{cases}$$

Considering Theorem 2 where  $s(n_t)$  returns an  $n \in \mathbb{N}_e$  we note that the above ends with an odd number of terms. 16 examples are shown in Appendix 3 for illustration purposes.

**Definition 5.** We introduce the clockwise arc diagram  $A$  as a visual aid that illustrates a couple of orbital properties of  $c(n_t)$  from Corollary 7 through conjectures 1, 2 and 3. The horizontal graph in the centre represents the number line  $n \in \mathbb{N}^*$ . Arcs on the top represent increments of  $n_t \rightarrow n_{t+1}$  from left to right by  $s(n_t)$ . Arcs on the bottom represent decrements of  $n_t \rightarrow n_{t+1}$  from right to left by  $\frac{n_t}{2^{\nu_2(n_t)}}$ <sup>†</sup>. A few examples are illustrated in Figures 1 and 2<sup>†</sup>.

**Remark 5.** In terms of graph theory, we can restate the main conjecture as follows. Up until reaching a vertex  $n_t = 1$  for a particular  $t \in T$  with the exception of  $n_0 = 1$ , a Hamiltonian or Eulerian path exists in  $A \forall n \in \mathbb{N}^*$ .

**Conjecture 1.** The summation of incremental differences minus the summation of decremental differences is equal to  $n_0 - 1 \forall \mathbb{N}^*$ .

Let  $\mathbf{inc}(n_{t-1}) = s(n_{t-1}) - n_{t-1} \mid t > 0$  and let  $\mathbf{dec}(n_t) = n_t - \frac{n_t}{2^{\nu_2(n_t)}} \mid t > 0$  while iterating  $c(n_t)$ . Through Figures 3 and 4 we can visualise the difference between  $\mathbf{inc}(n_{t-1})$  and  $\mathbf{dec}(n_t)$  to be

$$\left( \sum_{t=0}^{t \mid T \in \mathbb{N}_e} \mathbf{dec}(n_t) \right) - \left( \sum_{t=1}^{t \mid T \in \mathbb{N}_o} \mathbf{inc}(n_{t-1}) \right) = n_0 - 1, \forall n_0 \in \mathbb{N}$$

which can be interpreted as the incremental step from 1 to  $n_0$  when a particular  $n_0$ -orbit converges back to 1.

**Conjecture 2.** Following conjecture 1 it can also be conjectured that

$$\left( \sum_{t=0}^{t \mid T \in \mathbb{N}_e} \mathbf{dec}(n_t) \right) \geq \left( \sum_{t=1}^{t \mid T \in \mathbb{N}_o} \mathbf{inc}(n_{t-1}) \right) \forall n_0 \in \mathbb{N}^*$$

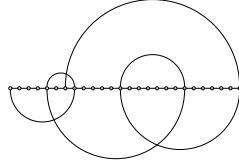
when a particular  $n_0$ -orbit converges back to 1.

Through  $A$  we can visually validate orbits of  $c(n_t)$  as defined in Corollary 7, alternating from  $+\mathbb{N}^* \rightarrow -\mathbb{N}^*$  or vice versa from  $-\mathbb{N}^* \rightarrow +\mathbb{N}^*$ . The following single-step iteration of initial values of  $n_0 = 1$  till  $n_0 = 12$  for  $n \in \mathbb{N}^*$  are listed for demonstration purposes in Table 4.

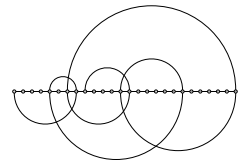
<sup>†</sup>Source code for drawing the clock-wise arc diagrams can be found at <https://github.com/flacle/>.

<sup>†</sup>Figures can be zoomed in to read the label on each node when viewing with a PDF viewer.



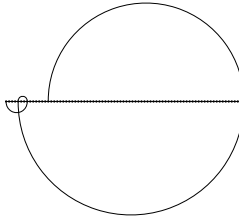


(a)  $n_1 = 26$  is its supremum element.

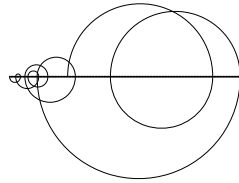


(b)  $n_3 = 26$  is its supremum element.

Figure 1:  $A$  for the parity  $c(n_t)$  function of orbits  $n_0 = 7$  and  $n_0 = 9$ .

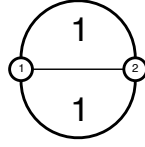


(a)  $n_1 = 80$  is its supremum element.

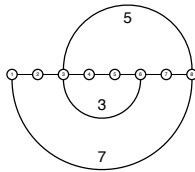


(b)  $n_3 = 152$  is its supremum element.

Figure 2:  $A$  for the parity  $c(n_t)$  function for orbits  $n_0 = 15$  and  $n_0 = 39$ .

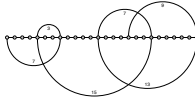


(a)  $n_1 = 2$  is its supremum element and the trivial cycle exemplar.

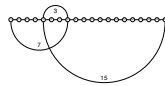


(b)  $n_3 = 8$  is its supremum element.

Figure 3:  $A$  for the parity  $c(n_t)$  function of orbits  $n_0 = 1$  and  $n_0 = 6$ .



(a)  $n_1 = 26$  is its supremum element.



(b)  $n_0 = 20$  is its supremum element.

Figure 4:  $A$  for the parity  $c(n_t)$  function for orbits  $n_0 = 17$  and  $n_0 = 20$ .

| $n_0$ | $c(n_0)$ |
|-------|----------|
| 1     | +1       |
| 2     | -1       |
| 3     | +5       |
| 4     | -3       |
| 5     | +3       |
| 6     | -3       |
| 7     | +19      |
| 8     | -7       |
| 9     | +5       |
| 10    | -5       |
| 11    | +15      |
| 12    | -9       |

Table 4: Alternating operations executed by  $c(n_t)$  as defined in Corollary 7, for initial values of  $n_0 = 1$  till  $n_0 = 12$ .

**Theorem 3.** *We can reformulate  $c(n_t)$  without the classical piecewise notation through the inclusion of divergent subsequences.*

*Proof.* We perceive regularities within pairs of  $n_0$  tuples that are equally distanced from one another, listed in Table 4. For example  $\langle 1, 2 \rangle$ ,  $\langle 5, 6 \rangle$ ,  $\langle 9, 10 \rangle$ , and so on. For the first example we perceive an increase of +1 followed by a decrease of -1. For the second we have +3 followed by -3, similarly for  $n_0 = 9$  we have +5 and -5 and so on, with a distance of  $2^1$ . Suppose that the same rationale can be applied for pairs where  $\nu_2(n_0 + 1) > 1$ . Then, distances between pairs should also exhibit regularities. Indeed, this will be proved by construction to inherit  $z = 3^k - 2^k$  as defined in Theorem 1.

First, let the  $n_0$  terms  $\forall n_0 \in \mathbb{N}^*$  be expanded, illustrated for initial values of  $n_0 = 1$  till  $n_0 = 26$  in Table 5, and forming the sequence

$$n_1 = (2, 1, 8, 1, 8, 3, 26, 1, 14, 5, 26, 3, 20, 7, 80, 1, 26, 9, 44, 5, 32, 11, 80, 3, 38, 13)$$

for  $n_0 = 1$  till  $n_0 = 26$ . This can be further expanded to highlight the recursion within each expression as shown in Table 6. Continuing this approach, expressions are reformulated

| $n_0$ | $n_0 + n_1$  | $n_0$ | $n_0 + n_1$    |
|-------|--------------|-------|----------------|
| 1     | $1 + 1$      | 14    | $14 - 7$       |
| 2     | $2 - 1$      | 15    | $15 + 15 + 50$ |
| 3     | $3 + 3 + 2$  | 16    | $16 - 15$      |
| 4     | $4 - 3$      | 17    | $17 + 9$       |
| 5     | $5 + 3$      | 18    | $18 - 9$       |
| 6     | $6 - 3$      | 19    | $19 + 15 + 10$ |
| 7     | $7 + 7 + 12$ | 20    | $20 - 15$      |
| 8     | $8 - 7$      | 21    | $21 + 11$      |
| 9     | $9 + 5$      | 22    | $22 - 11$      |
| 10    | $10 - 5$     | 23    | $23 + 21 + 36$ |
| 11    | $11 + 9 + 6$ | 24    | $24 - 21$      |
| 12    | $12 - 9$     | 25    | $25 + 13$      |
| 13    | $13 + 7$     | 26    | $26 - 13$      |

Table 5: Expanded  $n_0$  terms. For example,  $7 + 19 = 14 + 7 = 7 + 7 + 12$ .

| $n_0$ | $n_0 + n_1$                  | $n_0$ | $n_0 + n_1$                    |
|-------|------------------------------|-------|--------------------------------|
| 1     | $1 + 1$                      | 14    | $14 - 7$                       |
| 2     | $2 - 1$                      | 15    | $15 + 15 \cdot 1 + 50 \cdot 1$ |
| 3     | $3 + 3 \cdot 1 + 2 \cdot 1$  | 16    | $16 - 15 \cdot 1$              |
| 4     | $4 - 3 \cdot 1$              | 17    | $17 + 9$                       |
| 5     | $5 + 3$                      | 18    | $18 - 9$                       |
| 6     | $6 - 3$                      | 19    | $19 + 3 \cdot 5 + 2 \cdot 5$   |
| 7     | $7 + 7 \cdot 1 + 12 \cdot 1$ | 20    | $20 - 3 \cdot 5$               |
| 8     | $8 - 7 \cdot 1$              | 21    | $21 + 11$                      |
| 9     | $9 + 5$                      | 22    | $22 - 11$                      |
| 10    | $10 - 5$                     | 23    | $23 + 7 \cdot 3 + 12 \cdot 3$  |
| 11    | $11 + 3 \cdot 3 + 2 \cdot 3$ | 24    | $24 - 7 \cdot 3$               |
| 12    | $12 - 3 \cdot 3$             | 25    | $25 + 13$                      |
| 13    | $13 + 7$                     | 26    | $26 - 13$                      |

Table 6: Demonstration of recurrence within each expression.

| $n_0$ | $n_0 + n_1$  | $n_0$ | $n_0 + n_1$  |
|-------|--|-------|--|
| 1     | $1 + (2^1 - 1) \cdot 1 + (3^1 - 2^1 - (2^1 - 1)) \cdot 1$  | 14    | $14 - 1 \cdot 7$   |
| 2     | $2 - 1 \cdot 1$  | 15    | $15 + (2^4 - 1) \cdot 1 + (3^4 - 2^4 - (2^4 - 1)) \cdot 1$   |
| 3     | $3 + (2^2 - 1) \cdot 1 + (3^2 - 2^2 - (2^2 - 1)) \cdot 1$  | 16    | $16 - 15 \cdot 1$  |
| 4     | $4 - 3 \cdot 1$  | 17    | $17 + (2^1 - 1) \cdot 9 + (3^1 - 2^1 - (2^1 - 1)) \cdot 9$   |
| 5     | $5 + (2^1 - 1) \cdot 3 + (3^1 - 2^1 - (2^1 - 1)) \cdot 3$  | 18    | $18 - 9$   |
| 6     | $6 - 1 \cdot 3$  | 19    | $19 + (2^2 - 1) \cdot 5 + (3^2 - 2^2 - (2^2 - 1)) \cdot 5$   |
| 7     | $7 + (2^3 - 1) \cdot 1 + (3^3 - 2^3 - (2^3 - 1)) \cdot 1$  | 20    | $20 - 3 \cdot 5$   |
| 8     | $8 - 7 \cdot 1$  | 21    | $21 + (2^1 - 1) \cdot 11 + (3^1 - 2^1 - (2^1 - 1)) \cdot 11$ |
| 9     | $9 + (2^1 - 1) \cdot 5 + (3^1 - 2^1 - (2^1 - 1)) \cdot 5$  | 22    | $22 - 11$  |
| 10    | $10 - 1 \cdot 5$   | 23    | $23 + (2^3 - 1) \cdot 3 + (3^3 - 2^3 - (2^3 - 1)) \cdot 3$   |
| 11    | $11 + (2^2 - 1) \cdot 3 + (3^2 - 2^2 - (2^2 - 1)) \cdot 3$ | 24    | $24 - 7 \cdot 3$   |
| 12    | $12 - 3 \cdot 3$   | 25    | $25 + (2^1 - 1) \cdot 13 + (3^1 - 2^1 - (2^1 - 1)) \cdot 13$ |
| 13    | $13 + (2^1 - 1) \cdot 7 + (3^1 - 2^1 - (2^1 - 1)) \cdot 7$ | 26    | $26 - 13$  |

Table 7: Reformulation as expressions that contain power operations for every  $n_0 \in \mathbb{N}_o$ .

with power operations such as

$$\begin{aligned}
3 + 3 \cdot 1 + 2 \cdot 1 &= 3 + (2^2 - 1) \cdot 1 + (3^2 - 2^2 - 3) \cdot 1 = 3 + (2^2 - 1) \cdot 1 + (3^2 - 2^2 - (2^2 - 1)) \cdot 1 \\
7 + 7 \cdot 1 + 12 \cdot 1 &= 7 + (2^3 - 1) \cdot 1 + (3^3 - 2^3 - 7) \cdot 1 = 7 + (2^3 - 1) \cdot 1 + (3^3 - 2^3 - (2^3 - 1)) \cdot 1 \\
15 + 15 \cdot 1 + 50 \cdot 1 &= 15 + (2^4 - 1) \cdot 1 + (3^4 - 2^4 - 15) \cdot 1 = 15 + (2^4 - 1) \cdot 1 + (3^4 - 2^4 - (2^4 - 1)) \cdot 1
\end{aligned}$$

and so on, listed in Table 7 for every  $n_0 \in \mathbb{N}_o$  up until  $n_0 = 26$ . We further observe that a similar approach can be derived for each For  $n_0 \in \mathbb{N}_e$  subsequence giving

$$2 - (2^1 - 1) \cdot 1,$$

$$4 - (2^2 - 1) \cdot 1,$$

$$8 - (2^3 - 1) \cdot 1,$$

and so on. Table 8 combines both approaches, completing the expansion for  $n_0 \in \mathbb{N}^*$  up until  $n_0 = 26$ .

As such,  $s(n_t)$  from Theorem 2 applies an expansion of two terms during increments and an expansion of one term during decrements. Given this expanded form we can symbolically derive a separate piecewise function for iterates of  $c(n_t)$  namely

$$F(n_t, p, d) = \begin{cases} -d \cdot (2^p - 1) & \text{if } n_t \equiv 0 \pmod{2} \\ +d \cdot (2^p - 1) + d \cdot (3^p - 2^p - (2^p - 1)) & \text{if } n_t \equiv 1 \pmod{2} \end{cases}$$

| $n_0$ | $n_0 + n_1$  | $n_0$ | $n_0 + n_1$  |
|-------|--|-------|--|
| 1     | $1 + (2^1 - 1) \cdot 1 + (3^1 - 2^1 - (2^1 - 1)) \cdot 1$  | 14    | $14 - (2^1 - 1) \cdot 7$                                     |
| 2     | $2 - (2^1 - 1) \cdot 1$                                    | 15    | $15 + (2^4 - 1) \cdot 1 + (3^4 - 2^4 - (2^4 - 1)) \cdot 1$   |
| 3     | $3 + (2^2 - 1) \cdot 1 + (3^2 - 2^2 - (2^2 - 1)) \cdot 1$  | 16    | $16 - (2^4 - 1) \cdot 1$                                     |
| 4     | $4 - (2^2 - 1) \cdot 1$                                    | 17    | $17 + (2^1 - 1) \cdot 9 + (3^1 - 2^1 - (2^1 - 1)) \cdot 9$   |
| 5     | $5 + (2^1 - 1) \cdot 3 + (3^1 - 2^1 - (2^1 - 1)) \cdot 3$  | 18    | $18 - (2^1 - 1) \cdot 9$                                     |
| 6     | $6 - (2^1 - 1) \cdot 3$                                    | 19    | $19 + (2^2 - 1) \cdot 5 + (3^2 - 2^2 - (2^2 - 1)) \cdot 5$   |
| 7     | $7 + (2^3 - 1) \cdot 1 + (3^3 - 2^3 - (2^3 - 1)) \cdot 1$  | 20    | $20 - (2^2 - 1) \cdot 5$                                     |
| 8     | $8 - (2^3 - 1) \cdot 1$                                    | 21    | $21 + (2^1 - 1) \cdot 11 + (3^1 - 2^1 - (2^1 - 1)) \cdot 11$ |
| 9     | $9 + (2^1 - 1) \cdot 5 + (3^1 - 2^1 - (2^1 - 1)) \cdot 5$  | 22    | $22 - (2^1 - 1) \cdot 11$                                    |
| 10    | $10 - (2^1 - 1) \cdot 5$                                   | 23    | $23 + (2^3 - 1) \cdot 3 + (3^3 - 2^3 - (2^3 - 1)) \cdot 3$   |
| 11    | $11 + (2^2 - 1) \cdot 3 + (3^2 - 2^2 - (2^2 - 1)) \cdot 3$ | 24    | $24 - (2^3 - 1) \cdot 3$                                     |
| 12    | $12 - (2^2 - 1) \cdot 3$                                   | 25    | $25 + (2^1 - 1) \cdot 13 + (3^1 - 2^1 - (2^1 - 1)) \cdot 13$ |
| 13    | $13 + (2^1 - 1) \cdot 7 + (3^1 - 2^1 - (2^1 - 1)) \cdot 7$ | 26    | $26 - (2^1 - 1) \cdot 13$                                    |

Table 8: Combination of both approaches  $n_0 \in \mathbb{N}^*$ .

where  $p > 0$  denotes power and  $d > 0$  the divergence index for  $F$ , i.e. the first occurrence of an increment of  $d$  in  $n$  starting at 1. The additional  $2^p - 1 \mid n_t \equiv 1 \pmod{2}$  can be factored out to obtain

$$F(n_t, p, d) = (n_t \equiv 1) d \cdot (3^p - 2^p) - (n_t \equiv 0) d \cdot (2^p - 1) \pmod{2}$$

that emphasises a modular symmetry between increments and decrements from the parity-alternating formulation of  $c(n_t)$ . As an example, the orbit for  $n_t = 7$  converges to 1 as follows

$$7 + F(7, 3, 1) + F(26, 1, 13) + F(13, 1, 7) + F(20, 2, 5) + F(5, 1, 3) + F(8, 3, 1) = 1.$$

Table 9 shows divergence subsequences for  $1 \leq p \leq 4$  for  $n \in \mathbb{N}^*$  up until  $n = 26$ , while Figure 5 combines subsequences into one chart up until  $n = 100$  for illustration purposes.

Suppose  $F(n_t, p, d)$  can be merged back into the metric space of  $c(n_t)$ , then it should be possible to make both  $p$  and  $d$  functions of  $n_t$ . For  $p$  we observe that it satisfies the sequence

$$(1, 1, 2, 2, 1, 1, 3, 3, 1, 1, 2, 2, 1, 1, 4, 4, \dots).$$

which is OEIS A050603 [4] <sup>†</sup>. For  $p$  as a function of  $n_t$  we thus obtain

$$p(n_t) = \nu_2(n_t + 1) + \nu_2(n_t) = \nu_2(n_t^2 + n_t) = \nu_2(n_t + (n_t \equiv 1)) \pmod{2}$$

---

<sup>†</sup>Also OEIS A136480 [5] minus one.

| $n$ | $F(n, 1, n)$ | $F(n, 2, n)$ | $F(n, 3, n)$ | $F(n, 4, n)$ |
|-----|--------------|--------------|--------------|--------------|
| 1   | 1            | 5            | 19           | 65           |
| 2   | -2           | -6           | -14          | -30          |
| 3   | 3            | 15           | 57           | 195          |
| 4   | -4           | -12          | -28          | -60          |
| 5   | 5            | 25           | 95           | 325          |
| 6   | -6           | -18          | -42          | -90          |
| 7   | 7            | 35           | 133          | 455          |
| 8   | -8           | -24          | -56          | -120         |
| 9   | 9            | 45           | 171          | 585          |
| 10  | -10          | -30          | -70          | -150         |
| 11  | 11           | 55           | 209          | 715          |
| 12  | -12          | -36          | -84          | -180         |
| 13  | 13           | 65           | 247          | 845          |
| 14  | -14          | -42          | -98          | -210         |
| 15  | 15           | 75           | 285          | 975          |
| 16  | -16          | -48          | -112         | -240         |
| 17  | 17           | 85           | 323          | 1105         |
| 18  | -18          | -54          | -126         | -270         |
| 19  | 19           | 95           | 361          | 1235         |
| 20  | -20          | -60          | -140         | -300         |
| 21  | 21           | 105          | 399          | 1365         |
| 22  | -22          | -66          | -154         | -330         |
| 23  | 23           | 115          | 437          | 1495         |
| 24  | -23          | -72          | -168         | -360         |
| 25  | 25           | 125          | 475          | 1625         |
| 26  | -26          | -78          | -182         | -390         |

Table 9: Divergent subsequences for powers  $1 \leq p \leq 4$  for  $n \in \mathbb{N}^*$  up until  $n = 26$ .

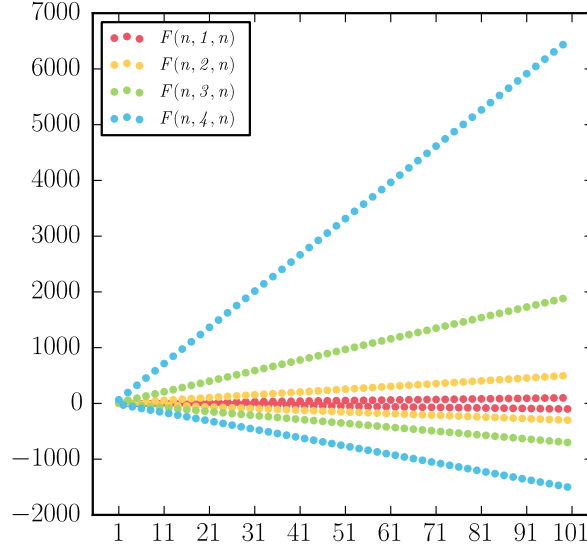


Figure 5: Chart showing the metric spaces of  $1 \leq p \leq 4$  for  $1 \leq n \leq 100$ .

for the exponential  $p$ -terms of  $F$  where  $p(n_t) = p(n_t + 1)$  for all consecutive pairs of  $\langle n_t, n_t + 1 \rangle \in \mathbb{N}^* \mid n_t \in \mathbb{N}_o \wedge n_t + 1 \in \mathbb{N}_e$ . As proven in Theorem 2, the equivalence  $k = \nu_2(n_t + 1)$  within  $s(n_t, k)$  is the lookahead power when  $n_t \in \mathbb{N}_o$  while  $\nu_2(n_t)$  is the power that denotes the maximum number of factors of two contained in every  $n_t \in \mathbb{N}_e$ , as such  $p(n_t)$  combines both powers for all  $n_t \in \mathbb{N}^*$ , equating the sequence above.

For the divergence index  $d$ , we also define a function that satisfies the following sequence

$$(1, 1, 1, 1, 3, 3, 1, 1, 5, 5, 3, 3, 7, 7, 1, 1, \dots)$$

for  $n_t \in \mathbb{N}^*$ . We observe this to be similar to the “change point” sequence as given in OEIS A163575 [6] albeit obtained with the ceiling instead of the flooring function. This yields the following

$$d^{\lceil}(n_t) = \left\lceil \frac{n_t}{2^{p(n_t)}} \right\rceil = \frac{n_t + (n_t \equiv 1)}{2^{p(n_t)}} \pmod{2}$$

where  $\lceil$  denotes the ceiling variety. Following Corollary 5,  $d^{\lceil}(n_t) \mid n_t \in \mathbb{N}_e$  is  $\frac{n_t}{2^{\nu_2(n_t)}}$  that is equivalent to  $c(n_t) \mid n_t \in \mathbb{N}_e$ . While for  $n_t \in \mathbb{N}_o$  it is  $\frac{n_t+1}{2^{\nu_2(n_t+1)}}$  that is equivalent to the denominator in  $s(n_t)$  from Theorem 2 and the equivalence

$$\frac{3^{\nu_2(n_t+1)}(n_t) + 3^{\nu_2(n_t+1)}}{2^{\nu_2(n_t+1)}} = \frac{3^{\nu_2(n_t+1)}(n_t + 1)}{2^{\nu_2(n_t+1)}}.$$



Both  $p(n_t)$  and  $d^\lceil(n_t)$  exhibit cycles for occurrences due to Corollary 5. For  $p(n_t)$  its cycles are of equal 2-adic valuation distances, where the metric is defined as the following function

$$D(n) = n + 2^{p(n)+1}$$

that would reach the same subsequence of  $p(n)$ , mapping a metric to the next occurrence of a subsequence, i.e. to a proceeding pair  $\langle n, n+1 \rangle$  equal with respect to  $p(n)$ . Thus arriving at the next pair that contains the same 2-adic valuation outcome as the current pair. E.g.

$$D(7) = 23, D(9) = 13, D(11) = 19$$

$$p(7) = 3, p(23) = 3, p(9) = 1, p(13) = 1, p(11) = 2, p(19) = 2$$

for natural odd numbers, or

$$D(8) = 24, D(10) = 14, D(12) = 20$$

$$p(8) = 3, p(24) = 3, p(10) = 1, p(14) = 1, p(12) = 2, p(20) = 2$$

for natural even numbers. For  $d^\lceil(n_t)$  we designate it the label of a divergence index as it embeds a counter for  $D(n)$  with a fixed metric of 2 starting at 1, i.e. for the first encounter of a pair in the sequence of  $p(n)$ . E.g.

$$p(1) = 1, p(5) = 1, p(9) = 1, p(13) = 1$$

$$d^\lceil(1) = 1, d^\lceil(5) = 3, d^\lceil(9) = 5, d^\lceil(13) = 7$$

at the onset of  $2^1 - 1$ , or

$$p(3) = 2, p(11) = 2, p(19) = 2, p(27) = 2$$

$$d^\lceil(3) = 1, d^\lceil(11) = 3, d^\lceil(19) = 5, d^\lceil(27) = 7$$

at the onset of  $2^2 - 1$ .

Finally, we can combine and simplify it all together with  $n_t$ , which gives

$$c_1(n_t) := n_t + (n_t \equiv 1) d^\lceil(n_t) \left( 3^{p(n_t)} - 2^{p(n_t)} \right) - (n_t \equiv 0) d^\lceil(n_t) \left( 2^{p(n_t)} - 1 \right) \pmod{2}$$

that embeds orbitals  $\forall n_t \in \mathbb{N}^*$ . E.g., for  $n_0 = 7$  we have the iterations

$$\underbrace{c_1(c_1(c_1(c_1(c_1(c_1(7))))))}_{6 \text{ times}} = 1$$

□

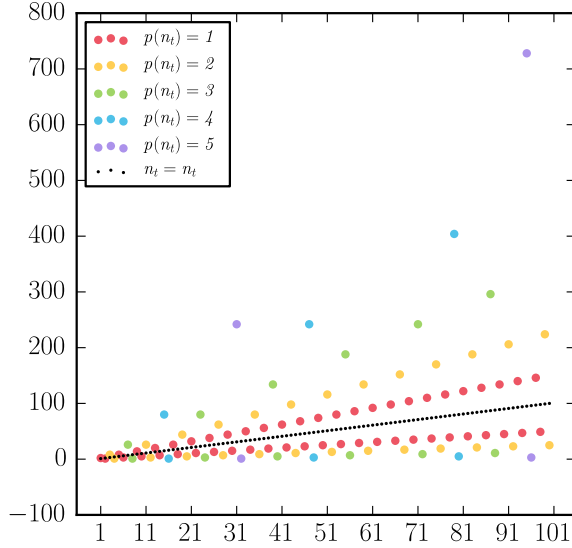


Figure 6: Subsequences  $1 \leq p(n_t) \leq 5$  of  $F$  contained within  $c(n_t)$  for  $1 \leq n_t \leq 100$ .

Figure 6 shows divergence subsequences  $F(n_t, p, d)$  when mapped back into  $c_1(n_t)$  together with an identity sequence  $n_t = n_t$  (1-slope). Figure 7 shows the same result but excluding the addition term  $n_t$  that is given in  $c_1(n_t)$ , thereby isolating iterative operations, and including a constant sequence  $n_t = 0$  (0-slope).

**Remark 6.** Subscript 1 applied on  $c(n_t)$  as  $c_1(n_t)$  denotes single-step iterations in the parity-alternating formulation of  $c(n_t)$ , also seen in A. E.g.,  $c_1(7) = 26$  followed by  $c_1(26) = 13$ .

**Corollary 9.** Given  $c_1(n_t)$  where  $n_{t-1} \in \mathbb{N}_o$  and  $n_t \in \mathbb{N}_e$  for  $t > 0$ , we illustrate the embedding of seemingly continuous 2-adic subspaces that exclude 0, or  $\mathbf{Z}_2 \setminus 0$  in Table 10 where rows represent  $3^{p(n_{t-1})} - 2^{p(n_{t-1})}$ , columns represent  $d(n_{t-1}) \in \mathbb{N}_o$ , and each cell the 2-adic valuation of odd operations within  $c_1(n_t)$  namely  $\nu_2(n_{t-1} + d(n_{t-1})(3^{p(n_{t-1})} - 2^{p(n_{t-1})}))$ . Followed by Table 11 that lists the same results but without the  $\nu_2$  operation. Both tables are limited to  $p(n_{t-1}) \leq 17$  and  $d(n_{t-1}) \leq 33$  for illustrative purposes.

**Lemma 3.** Through Corollary 9 we derive a reformulation of the orbit-transforming function  $c_1(n_t)$  as defined in Theorem 3 with a correspondence to Corollary 7.

*Proof.* Let us first consider  $c_1(n_t) \forall n_t \in \mathbb{N}_o$ . Table 11 illustrates ever-expanding "jumps" within  $\mathbb{N}^*$ . To illustrate this further, indices are listed in Table 12 for  $1 \leq n \leq 32$  and for

| $3^{p(n_t-1)} - 2^{p(n_t-1)}$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | 29 | 31 | 33 |
|-------------------------------|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| $3^1 - 2^1$                   | 1 | 3 | 1 | 2 | 1 | 5  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  |
| $3^2 - 2^2$                   | 3 | 1 | 2 | 1 | 4 | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 5  | 1  | 2  | 1  | 3  |
| $3^3 - 2^3$                   | 1 | 4 | 1 | 2 | 1 | 3  | 1  | 2  | 1  | 9  | 1  | 2  | 1  | 3  | 1  | 2  | 1  |
| $3^4 - 2^4$                   | 4 | 1 | 2 | 1 | 3 | 1  | 2  | 1  | 5  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  |
| $3^5 - 2^5$                   | 1 | 3 | 1 | 2 | 1 | 4  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 5  | 1  | 2  | 1  |
| $3^6 - 2^6$                   | 3 | 1 | 2 | 1 | 5 | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  | 3  |
| $3^7 - 2^7$                   | 1 | 5 | 1 | 2 | 1 | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  | 3  | 1  | 2  | 1  |
| $3^8 - 2^8$                   | 5 | 1 | 2 | 1 | 3 | 1  | 2  | 1  | 4  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 6  |
| $3^9 - 2^9$                   | 1 | 3 | 1 | 2 | 1 | 6  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  |
| $3^{10} - 2^{10}$             | 3 | 1 | 2 | 1 | 4 | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 7  | 1  | 2  | 1  | 3  |
| $3^{11} - 2^{11}$             | 1 | 4 | 1 | 2 | 1 | 3  | 1  | 2  | 1  | 5  | 1  | 2  | 1  | 3  | 1  | 2  | 1  |
| $3^{12} - 2^{12}$             | 4 | 1 | 2 | 1 | 3 | 1  | 2  | 1  | 8  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  |
| $3^{13} - 2^{13}$             | 1 | 3 | 1 | 2 | 1 | 4  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 6  | 1  | 2  | 1  |
| $3^{14} - 2^{14}$             | 3 | 1 | 2 | 1 | 6 | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  | 3  |
| $3^{15} - 2^{15}$             | 1 | 6 | 1 | 2 | 1 | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  | 3  | 1  | 2  | 1  |
| $3^{16} - 2^{16}$             | 6 | 1 | 2 | 1 | 3 | 1  | 2  | 1  | 4  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 5  |
| $3^{17} - 2^{17}$             | 1 | 3 | 1 | 2 | 1 | 5  | 1  | 2  | 1  | 3  | 1  | 2  | 1  | 4  | 1  | 2  | 1  |

Table 10: Outcomes of  $\nu_2 \left( n_{t-1} + d(n_{t-1}) \left( 3^{p(n_{t-1})} - 2^{p(n_{t-1})} \right) \right)$  for  $1 \leq p(n_{t-1}) \leq 17$  and  $1 \leq d(n_{t-1}) \leq 33$ .

| $3^{p(n_t-1)} - 2^{p(n_t-1)}$ | 1         | 3         | 5         | 7         | 9          | 11         | 13         | 15         | 17         | 19         | 21         | 23         | 25         | 27         | 29         | 31         | 33         |
|-------------------------------|-----------|-----------|-----------|-----------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| $3^1 - 2^1$                   | 2         | 8         | 14        | 20        | 26         | 32         | 38         | 44         | 50         | 56         | 62         | 68         | 74         | 80         | 86         | 92         | 98         |
| $3^2 - 2^2$                   | 8         | 26        | 44        | 62        | 80         | 98         | 116        | 134        | 152        | 170        | 188        | 206        | 224        | 242        | 260        | 278        | 296        |
| $3^3 - 2^3$                   | 26        | 80        | 134       | 188       | 242        | 296        | 350        | 404        | 458        | 512        | 566        | 620        | 674        | 728        | 782        | 836        | 890        |
| $3^4 - 2^4$                   | 80        | 242       | 404       | 566       | 728        | 890        | 1052       | 1214       | 1376       | 1538       | 1700       | 1862       | 2024       | 2186       | 2348       | 2510       | 2672       |
| $3^5 - 2^5$                   | 242       | 728       | 1214      | 1700      | 2186       | 2672       | 3158       | 3644       | 4130       | 4616       | 5102       | 5588       | 6074       | 6560       | 7046       | 7532       | 8018       |
| $3^6 - 2^6$                   | 728       | 2186      | 3644      | 5102      | 6560       | 8018       | 9476       | 10934      | 12392      | 13850      | 15308      | 16766      | 18224      | 19682      | 21140      | 22598      | 24056      |
| $3^7 - 2^7$                   | 2186      | 6560      | 10934     | 15308     | 19682      | 24056      | 28430      | 32804      | 37178      | 41552      | 45926      | 50300      | 54674      | 59048      | 63422      | 67796      | 72170      |
| $3^8 - 2^8$                   | 6560      | 19682     | 32804     | 45926     | 59048      | 72170      | 85292      | 98414      | 111536     | 124658     | 137780     | 150902     | 164024     | 177146     | 190268     | 203390     | 216512     |
| $3^9 - 2^9$                   | 19682     | 59048     | 98414     | 137780    | 177146     | 216512     | 255878     | 295244     | 334610     | 373976     | 413342     | 452708     | 492074     | 531440     | 570806     | 610172     | 649538     |
| $3^{10} - 2^{10}$             | 59048     | 177146    | 295244    | 413342    | 531440     | 649538     | 767636     | 885734     | 1003832    | 1121930    | 1240028    | 1358126    | 1476224    | 1594322    | 1712420    | 1830518    | 1948616    |
| $3^{11} - 2^{11}$             | 177146    | 531440    | 885734    | 1240028   | 1594322    | 1948616    | 2302910    | 2657204    | 3011498    | 3365792    | 3720086    | 4074380    | 4428674    | 4782968    | 5137262    | 5491556    | 5845850    |
| $3^{12} - 2^{12}$             | 531440    | 1594322   | 2657204   | 3720086   | 4782968    | 5845850    | 6908732    | 7971614    | 9034496    | 10097378   | 11160260   | 12223142   | 13286024   | 14348906   | 15411788   | 16474670   | 17537552   |
| $3^{13} - 2^{13}$             | 1594322   | 4782968   | 7971614   | 11160260  | 14348906   | 17537552   | 20726198   | 23914844   | 27103490   | 30292136   | 33480782   | 36669428   | 39858074   | 43046720   | 46235366   | 49424012   | 52612658   |
| $3^{14} - 2^{14}$             | 4782968   | 14348906  | 23914844  | 33480782  | 43046720   | 52612658   | 62178506   | 71744354   | 81310472   | 90876410   | 100442348  | 110008286  | 119574224  | 129140162  | 138706100  | 148272038  | 157837976  |
| $3^{15} - 2^{15}$             | 14348906  | 43046720  | 71744354  | 100442348 | 129140162  | 157837576  | 186535790  | 215233604  | 243931418  | 272629232  | 301327046  | 330024860  | 358722674  | 387420488  | 416118302  | 444816116  | 473513930  |
| $3^{16} - 2^{16}$             | 43046720  | 129140162 | 215233604 | 301327046 | 387420488  | 473513930  | 559607372  | 645700814  | 731794256  | 817887698  | 903981140  | 990074582  | 1076168024 | 1162261466 | 1248354908 | 1334448350 | 1420541792 |
| $3^{17} - 2^{17}$             | 129140162 | 387420488 | 645700814 | 903981140 | 1162261466 | 1420541792 | 1678822118 | 1937102444 | 2195382770 | 2453663086 | 2711943422 | 2970223748 | 3228504074 | 3486784400 | 3745064726 | 4003345052 | 4261625378 |

Table 11: Outcomes of  $n_{t-1} + d(n_{t-1}) \left( 3^{p(n_{t-1})} - 2^{p(n_{t-1})} \right)$  for  $1 \leq p(n_{t-1}) \leq 17$  and  $1 \leq d(n_{t-1}) \leq 33$ . Equal colours emphasise the distance expansions that take place on  $\mathbf{Z}_2$  from lower powers  $p(n_{t-1})$ .

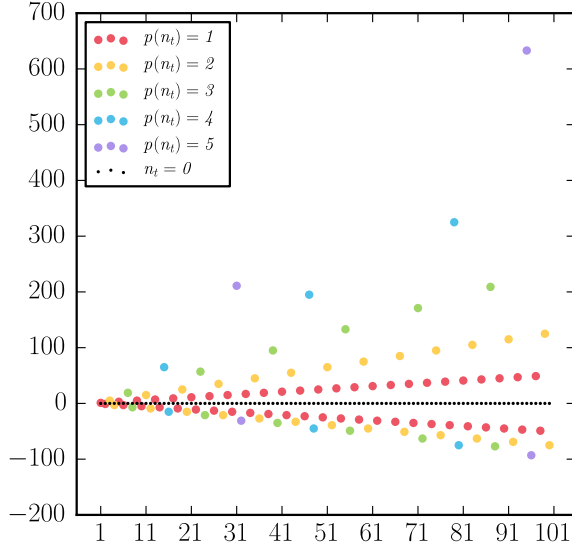


Figure 7: Subsequences  $1 \leq p(n_t) \leq 5$  of  $F$  contained within  $c(n_t)$  but now without the addition term  $n_t$ , or  $(n \equiv 1) d^I(n) (3^{p(n)} - 2^{p(n)}) - (n \equiv 0) d^I(n) (2^{p(n)} - 1) \pmod{2}$ .

| $3^{p(n-1)} - 2^{p(n-1)}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
|---------------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $3^1 - 2^1$               | - | 1 | - | - | - | - | - | 2 | - | -  | -  | -  | -  | 3  | -  | -  | -  | -  | -  | 4  | -  | -  | -  | -  | -  | 5  | -  | -  | -  | -  | -  | 6  |
| $3^2 - 2^2$               | - | - | - | - | - | - | - | 1 | - | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 2  | -  | -  | -  | -  | -  | -  |
| $3^3 - 2^3$               | - | - | - | - | - | - | - | - | - | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 1  | -  | -  | -  | -  | -  | -  |

Table 12: Indices per power expressed in  $c_1(n_t)|n_t \in \mathbb{N}_o$  for  $1 \leq p \leq 3$ , illustrating ever-expanding "jumps" in  $\mathbb{N}^*$ .

the exponential order of  $1 \leq p \leq 3$ . Suppose that the expansion holds  $\forall n_t \in \mathbb{N}_o$ , then there should be a correspondence with  $s(n_t)$  following previous conclusions brought forth in this paper. For  $p = 1$  we have

$$1 \rightarrow 2, 2 \rightarrow 8, 3 \rightarrow 14, 4 \rightarrow 20, 5 \rightarrow 26, 6 \rightarrow 32, \dots$$

with a metric of 6 starting at table column 2. For  $p = 2$  we have

$$1 \rightarrow 8, 2 \rightarrow 26, \dots$$

with a metric of 18 starting at table column 8, and finally we have  $p = 3$  with a metric of 54 starting at table column 26. For  $p = 1$  we observe that the metric of 6 can be applied on  $\mathbb{N}^*$  with  $3(3^0 \cdot n) - 1|n \in \mathbb{N}_o$ . For example  $3(3^0 \cdot 1) - 1 = 2$ ,  $3(3^0 \cdot 3) - 1 = 8$ , and so on. For  $p = 2$  we similarly can apply the metric of 18 on  $\mathbb{N}^*$  with  $3(3^1 \cdot n) - 1|n \in \mathbb{N}_o$ . For example  $3(3^1 \cdot 1) - 1 = 8$ ,  $3(3^1 \cdot 3) - 1 = 26$ , and so on. Considering the increase in power

to  $p = 3$  we expect  $3(3^2n) - 1 \mid n \in \mathbb{N}_o$  to hold. This is true because

$$3(3^2n) - 1 = 3^3n - 1 = 3^p n - 1 \mid n \in \mathbb{N}_o.$$

Considering that the range of  $d^\lceil(n_t)$  is equal to  $\{2n - 1 : n \in \mathbb{N}\}$ , we can substitute  $n$  and  $p$  with their functional counterparts to obtain

$$3^{p(n)} d^\lceil(n) - 1 \mid n \in \mathbb{N}_o.$$

For cases  $n \in \mathbb{N}_e$  of  $c_1(n_t)$  we derive this to be  $d^\lceil(n)$  as follows. From  $c_1(n_t)$  as defined in Theorem 3 we have the following expression

$$n - d^\lceil(n) \left( 2^{p(n)} - 1 \right)$$

for which we can flatten to obtain

$$n - \frac{n(2^{p(n)} - 1)}{2^{p(n)}} = n - \frac{n(2^{\nu_2(n)} - 1)}{2^{\nu_2(n)}} = n - \frac{2^{\nu_2(n)}n - n}{2^{\nu_2(n)}} = n - \frac{n}{2^{\nu_2(n)}}$$

resulting in  $d^\lceil(n_t) \mid n_t \in \mathbb{N}_e$  as a function of  $\frac{n_t}{2^{\nu_2(n_t)}}$  that corresponds with Corollary 7. Both equations can now be combined with an addition operator to obtain

$$c_1(n_t) := (n_t \equiv 1) d^\lceil(n_t) 3^{p(n_t)} - (n_t \equiv 1) + (n_t \equiv 0) d^\lceil(n_t) \pmod{2}$$

To complete the proof we demonstrate the correspondence with  $s(n_t)$  from Corollary 7 for  $n_t \in \mathbb{N}_o$ . Flattening out gives

$$\left\lceil \frac{n_t}{2^{\nu_2(n_t+1)}} \right\rceil 3^{\nu_2(n_t+1)} - 1 = \frac{3^{\nu_2(n_t+1)}(n_t + 1)}{2^{\nu_2(n_t+1)}} - 1 = s(n_t) \mid n_t \in \mathbb{N}_o$$

as claimed. □

**Corollary 10.** *The derivation of  $c_1$  from Lemma 3 can be compacted further.*

*Proof.* Considering sequence

$$n_1 = (2, 1, 8, 1, 8, 3, 26, 1, 14, 5, 26, 3, 20, 7, 80, 1, \dots)$$

resulting from  $c_1(n_0)$ , we have for  $n_t \in \mathbb{N}_o$

$$d^\lceil(n_t) 3^{p(n_t)} - 1 = d^\lceil(n_t) 3^{\nu_2(n_t) + \nu_2(n_t+1)} - 1.$$

Further considering that  $3^{\nu_2(n_t)} = 1 \mid n_t \in \mathbb{N}_o$  and  $3^{\nu_2(n_t+1)} = 1 \mid n_t \in \mathbb{N}_e$ , then

$$c_1(n_t) := d^\lceil(n_t) 3^{\nu_2(n_t+1)} - (n_t \equiv 1) \pmod{2}$$

for the parity-alternating formulation of  $c$  in single-step iterations. □

**Lemma 4.** *Given Lemma 3 and Corollary 10, we can derive  $c_2(n_t)$  by wrapping and embedding the second iteration within  $c_1(n_t)$ .*

*Proof.* If  $n_0 \in \mathbb{N}_o$  then  $c_1(n_0) = n_1 \in \mathbb{N}_e$ . Vice versa, if  $n_0 \in \mathbb{N}_e$  then  $c_1(n_0) = n_1 \in \mathbb{N}_o$ . As such, to iterate in double-steps we can wrap  $d^\lceil(n_t) \mid n_{t-1} \in \mathbb{N}_o$  and, following Lemma 3, embed  $d^\lceil(n_t)$  within itself and within  $p(n_t)$  when  $n_{t-1} \in \mathbb{N}_e$ . This yields

$$c_2(n_t) := (n_t \equiv 1) d_1^\lceil \left( d_1^\lceil(n_t) 3^{p_1(n_t)} - 1 \right) + (n_t \equiv 0) d_1^\lceil \left( d_1^\lceil(n_t) \right) 3^{p_1(d_1^\lceil(n_t))} - (n_t \equiv 0) \pmod{2}$$

that results for example in  $n_0 = 7 \rightarrow n_1 = 13$  crossing over 26, or  $n_0 = 6 \rightarrow n_1 = 8$  crossing under 3 through double-step iterations.

**Remark 7.** *Subscript 2 applied as  $c_2(n_t)$  denotes double-step iterations in the parity-alternating formulation of  $c(n_t)$ , also seen in A. For example,  $c_2(7) = 13$  followed by  $c_2(13) = 5$ . Similarly, subscript 1 applied on  $d$  and  $p$  as  $p_1(n_t)$  and  $d_1^\lceil(n_t)$ , denote the power variant of the original definition  $p(n_t)$  and the divergent index variant of the original definition  $d^\lceil(n_t)$  respectively.*

Let

$$(n_t \equiv 0) d_1^\lceil \left( d_1^\lceil(n_t) \right) 3^{\nu_2(d_1^\lceil(n_t)+1)} - (n_t \equiv 0) \pmod{2}$$

define the operation  $c_2(n_0) = n_2 \mid n_2 \in \mathbb{N}_e$ . First, we define a function for  $\nu_2(d_1^\lceil(n_t) + 1)$  as

$$\begin{aligned} p_2(n_t) &= \nu_2 \left( d_1^\lceil(n_t) + 1 \right) \\ &= \nu_2 \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right) \\ &= \nu_2 \left( \frac{n_t + 2^{\nu_2(n_t)}}{2^{\nu_2(n_t)}} \right) \\ &= \nu_2(n_t + 2^{\nu_2(n_t)}) - \nu_2(2^{\nu_2(n_t)}) \\ &= \nu_2(n_t + 2^{\nu_2(n_t)}) - \nu_2(n_t) \end{aligned}$$

where the last two right-hand expressions follow from an identity extension for  $\mathbb{Q}$  as given in [1] (p. 2), and forming the sequence OEIS A089309 [7]

$$(1, 1, 2, 1, 1, 2, 3, 1, 1, 1, 2, 2, 1, 3, 4, 1, \dots).$$

Intuitively, we recognise  $p_2$  as the 2-adic valuation of  $n$  from  $\mathbb{N}_o \rightarrow \mathbb{N}_e \forall n \in \mathbb{N}^*$ . Furthermore,  $p_2(n) = p_1(n) \mid n \in \mathbb{N}_o$  because

$$\nu_2(n + 1) - 0 = \nu_2(n + 1) + 0.$$

For  $d_1^\lceil (d_1^\lceil (n_t))$  we can define the function  $d_2^\lceil (n_t)$  as

$$d_2^\lceil (n_t) = d_1^\lceil (d_1^\lceil (n_t))$$

and reformulate the embedding by first flattening, followed by a formulation for  $n_t \in \mathbb{N}_e$  then by  $n_t \in \mathbb{N}_o$ , to finally combine both simplifications. To start, let

$$\begin{aligned} d_2^\lceil (n_t) &= d_1^\lceil (d_1^\lceil (n_t)) \\ &= d_1^\lceil \left( \frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}} \right) \\ &= \frac{\frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}}}{2^{\nu_2 \left( \frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}} \right) + \nu_2 \left( \frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}} + 1 \right)}} + \frac{\left( \frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}} \equiv 1 \right)}{2^{\nu_2 \left( \frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}} \right) + \nu_2 \left( \frac{n_t + (n_t \equiv 1)}{2^{\nu_2(n_t) + \nu_2(n_t+1)}} + 1 \right)}} \end{aligned}$$

(mod 2). For the case of  $n_t \in \mathbb{N}_e$  we can cancel out all 0's and obtain

$$\frac{\frac{n_t}{2^{\nu_2(n_t)}}}{2^{\nu_2 \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right)}} + \frac{1}{2^{\nu_2 \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right)}}$$

that through the identity given in  $p_2(n_t)$  can be simplified to

$$\frac{n_t + 2^{\nu_2(n_t)}}{2^{\nu_2(n_t + 2^{\nu_2(n_t)})}}$$

forming the sequence

$$(1, 1, 1, 1, 3, 1, 1, 1, 5, 3, 3, 1, 7, 1, 1, 1, \dots).$$

Similarly, for the case of  $n_t \in \mathbb{N}_o$ , we can cancel out all 0's and obtain

$$\frac{\frac{n_t + 1}{2^{\nu_2(n_t+1)}}}{2^{\nu_2 \left( \frac{n_t + 1}{2^{\nu_2(n_t+1)}} + 1 \right)}} + \frac{1}{2^{\nu_2 \left( \frac{n_t + 1}{2^{\nu_2(n_t+1)}} + 1 \right)}}$$

which is simplified to

$$\frac{n_t + 1 + 2^{\nu_2(n_t+1)}}{2^{\nu_2(n_t+1 + 2^{\nu_2(n_t+1)})}}$$

that forms the sequence

$$(1, 1, 1, 3, 1, 1, 1, 5, 3, 3, 1, 7, 1, 1, 1, 9, \dots).$$

Combining both together we arrive at

$$d_2^\lceil(n_t) = \frac{n_t + (n_t \equiv 1) + 2^{\nu_2(n_t + (n_t \equiv 1))}}{2^{\nu_2(n_t + (n_t \equiv 1) + 2^{\nu_2(n_t + (n_t \equiv 1))})}} \pmod{2}$$

that forms the sequence

$$(1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 1, 1, 1, 1, 1, 1, \dots).$$

For the case of  $c_2(n_0) = n_2 \mid n_2 \in \mathbb{N}_o$  we have

$$(n_t \equiv 1) d_1^\lceil \left( d_1^\lceil(n_t) 3^{p_1(n_t)} - 1 \right) = (n_t \equiv 1) d_1^\lceil(s(n_t)) \pmod{2}.$$

Flattening this out gives

$$\frac{\frac{3^{\nu_2(n_t+1)}(n_t+1)}{2^{\nu_2(n_t+1)}} - 1}{2^{\nu_2\left(\frac{3^{\nu_2(n_t+1)}(n_t+1)}{2^{\nu_2(n_t+1)}} - 1\right)}}$$

that through the negative exponentiation rule and  $p_2(n_t)$  identity we can expand as

$$3^{\nu_2(n_t+1)} 2^{-\nu_2(n_t+1) - \nu_2(3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)}) + \nu_2(2^{\nu_2(n_t+1)})} (n_t + 1) \\ - 2^{-\nu_2(3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)}) + \nu_2(2^{\nu_2(n_t+1)})}.$$

Because  $2^{\nu_2(2^{\nu_2(n_t+1)})} = 2^{\nu_2(n_t+1)}$  we can simplify to

$$\frac{3^{\nu_2(n_t+1)}(n_t+1)}{2^{\nu_2(3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)})}} - \frac{1}{2^{\nu_2(3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)}) - \nu_2(n_t+1)}}$$

and finally combine to obtain

$$\frac{3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)}}{2^{\nu_2(3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)})}}.$$

Adding it all together we arrive at

$$c_2(n_t) := (n_t \equiv 1) \left( \frac{3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)}}{2^{\nu_2(3^{\nu_2(n_t+1)}(n_t+1) - 2^{\nu_2(n_t+1)})}} \right) + (n_t \equiv 0) \left( d_2^\lceil(n_t) 3^{p_2(n_t)} - 1 \right) \pmod{2}$$

which gives the following sequence as its result

$$c_2(n_0) = (1, 2, 1, 2, 1, 8, 13, 2, 7, 8, 13, 8, 5, 26, 5, 2, \dots).$$

□



| $n_0$ | $A_2$ sequences given $c_2(n_t)$  | $n_0$ | $A_2$ sequences given $c_2(n_t)$  |
|-------|---|-------|---|
| 1     | $\ominus$   | 14    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$                     |
| 2     | $\ominus$   | 15    | $\ominus \rightarrow \ominus \rightarrow \ominus$   |
| 3     | $\ominus \rightarrow \ominus$   | 16    | $\ominus \rightarrow \ominus$   |
| 4     | $\ominus \rightarrow \ominus$   | 17    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$   |
| 5     | $\ominus \rightarrow \ominus$   | 18    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$ |
| 6     | $\ominus \rightarrow \ominus \rightarrow \ominus$   | 19    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$                     |
| 7     | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$                     | 20    | $\ominus \rightarrow \ominus \rightarrow \ominus$   |
| 8     | $\ominus \rightarrow \ominus$   | 21    | $\ominus \rightarrow \ominus$   |
| 9     | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$ | 22    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$                     |
| 10    | $\ominus \rightarrow \ominus \rightarrow \ominus$   | 23    | $\ominus \rightarrow \ominus \rightarrow \ominus$   |
| 11    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$                     | 24    | $\ominus \rightarrow \ominus \rightarrow \ominus$   |
| 12    | $\ominus \rightarrow \ominus \rightarrow \ominus$   | 25    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$ |
| 13    | $\ominus \rightarrow \ominus \rightarrow \ominus$   | 26    | $\ominus \rightarrow \ominus \rightarrow \ominus \rightarrow \ominus$   |

Table 13: Sequences of  $A_2$  given  $c_2(n_t)$  for  $1 \leq n_0 \leq 26$ .

**Conjecture 3.** Following Lemma 4 we conjecture nor more nor less than five sub-arc diagrams as part of  $A$  for double-step iterations denoted as  $A_2$  and symbolised as  $\ominus$  a **positive overpass**: an increment crossing the number line and remaining over  $n_{t-2}$ ,  $\ominus$  a **positive underpass**: an increment crossing the number line and under  $n_{t-2}$ ,  $\ominus$  a **negative overpass**: a decrement crossing the number line and remaining over  $n_{t-2}$ ,  $\ominus$  a **negative underpass**: a decrement crossing the number line and under  $n_{t-2}$ , and  $\ominus$  that symbolises a **cycle** circling back to  $n_{t-2}$ .

**Remark 8.** Increments or decrements in  $A_2$  also express the clock or anti-clockwise winding direction. In other words, for  $A_2$  under this conjecture, sequences with  $n_0 \in \mathbb{N}_o$  solely contain positive passes, while sequences with  $n_0 \in \mathbb{N}_e$  solely contain negative passes. Thus, "remaining over" or "under" in the previous statement takes this direction into account.

$A_2$  sequences are listed in Table 13 for  $1 \leq n_0 \leq 26$  for illustration purposes. The orbital length of  $A_2$  of  $c_2(n_t)$  is equal to OEIS A160541 [8] minus one for  $n_0 > 2$ .

**Corollary 11.** Also following Lemma 4 we derive a rational function for  $c_1(n_t)$ .

*Proof.* When  $n_0$  is subtracted from outcomes of  $c_1(n_0)$  we isolate the operation of single-step iteration and observe linear correspondences between elements (visible Figure 6). For

| $n_0$ | $\frac{ax+gf}{b} = y$           | $n_0$ | $\frac{ax+gf}{b} = y$           |
|-------|---------------------------------|-------|---------------------------------|
| 1     | $\frac{2x+1\cdot 2}{4} = y$     | 14    | $\frac{-2x+0\cdot 4}{4} = y$    |
| 2     | $\frac{-2x+0\cdot 4}{4} = y$    | 15    | $\frac{130x+65\cdot 2}{32} = y$ |
| 3     | $\frac{10x+5\cdot 2}{8} = y$    | 16    | $\frac{-30x+0\cdot 32}{32} = y$ |
| 4     | $\frac{-6x+0\cdot 8}{8} = y$    | 17    | $\frac{2x+1\cdot 2}{4} = y$     |
| 5     | $\frac{2x+1\cdot 2}{4} = y$     | 18    | $\frac{-2x+0\cdot 4}{4} = y$    |
| 6     | $\frac{-2x+0\cdot 4}{4} = y$    | 19    | $\frac{10x+5\cdot 2}{8}$        |
| 7     | $\frac{38x+19\cdot 2}{16} = y$  | 20    | $\frac{-6x+0\cdot 8}{8} = y$    |
| 8     | $\frac{-14x+0\cdot 16}{16} = y$ | 21    | $\frac{2x+1\cdot 2}{4} = y$     |
| 9     | $\frac{2x+1\cdot 2}{4} = y$     | 22    | $\frac{-2x+0\cdot 4}{4} = y$    |
| 10    | $\frac{-2x+0\cdot 4}{4} = y$    | 23    | $\frac{38x+19\cdot 2}{16} = y$  |
| 11    | $\frac{10x+5\cdot 2}{8} = y$    | 24    | $\frac{-14x+0\cdot 16}{16} = y$ |
| 12    | $\frac{-6x+0\cdot 8}{8} = y$    | 25    | $\frac{2x+1\cdot 2}{4} = y$     |
| 13    | $\frac{2x+1\cdot 2}{4} = y$     | 26    | $\frac{-2x+0\cdot 4}{4} = y$    |

Table 14: Linear fractions within element pairs as seen in Figure 6, for  $1 \leq n_0 \leq 26 \mid c_1(n_t)$ .

each correspondence we can identify a linear fraction in the form of

$$\frac{ax + gf}{b}$$

with outcomes listed in Table 14. As  $n_0 \in \mathbb{N}_o$  of  $c_1(n_t)$  results in an  $n_t \in \mathbb{N}_e$  we can remove factors of two for each as listed in Table 15. For  $f$  we identify this in reduced form to be

$$f(n_0) = 2^{\nu_2(n_0)}$$

or  $f(n_0) = 2^{\nu_2(n_0)+1}$  in unreduced form. For  $g$  we have

$$g(n_0) = 3^{\nu_2(n_0+1)} - 2^{\nu_2(n_0+1)}$$

where  $g(n_0) = 0 \mid n_0 \in \mathbb{N}_e$ . For  $b$  we observe these to be powers of two forming the sequence

$$(2, 2, 4, 4, 2, 2, 8, 8, 2, 2, 4, 4, 2, 2, 16, 16, \dots)$$

or

$$(2^1, 2^1, 2^2, 2^2, 2^1, 2^1, 2^3, 2^3, 2^1, 2^1, 2^2, 2^2, 2^1, 2^1, 2^4, 2^4, \dots)$$

where we denote the exponent as  $\delta_1(n_0)$  and in fact

$$\delta_1(n_0) = p_1(n_0).$$

| $n_0$ | $\frac{ax+gf}{b} = y$        | $n_0$ | $\frac{ax+gf}{b} = y$          |
|-------|------------------------------|-------|--------------------------------|
| 1     | $\frac{1x+1\cdot1}{2} = y$   | 14    | $\frac{-1x+0\cdot2}{2} = y$    |
| 2     | $\frac{-1x+0\cdot2}{2} = y$  | 15    | $\frac{65x+65\cdot1}{16} = y$  |
| 3     | $\frac{5x+5\cdot1}{4} = y$   | 16    | $\frac{-15x+0\cdot16}{16} = y$ |
| 4     | $\frac{-3x+0\cdot4}{4} = y$  | 17    | $\frac{1x+1\cdot1}{2} = y$     |
| 5     | $\frac{1x+1\cdot1}{2} = y$   | 18    | $\frac{-1x+0\cdot2}{2} = y$    |
| 6     | $\frac{-1x+0\cdot2}{2} = y$  | 19    | $\frac{5x+5\cdot1}{4}$         |
| 7     | $\frac{19x+19\cdot1}{8} = y$ | 20    | $\frac{-3x+0\cdot4}{4} = y$    |
| 8     | $\frac{-7x+0\cdot8}{8} = y$  | 21    | $\frac{1x+1\cdot1}{2} = y$     |
| 9     | $\frac{1x+1\cdot1}{2} = y$   | 22    | $\frac{-1x+0\cdot2}{2} = y$    |
| 10    | $\frac{-1x+0\cdot2}{2} = y$  | 23    | $\frac{19x+19\cdot1}{8} = y$   |
| 11    | $\frac{5x+5\cdot1}{4} = y$   | 24    | $\frac{-7x+0\cdot8}{8} = y$    |
| 12    | $\frac{-3x+0\cdot4}{4} = y$  | 25    | $\frac{1x+1\cdot1}{2} = y$     |
| 13    | $\frac{1x+1\cdot1}{2} = y$   | 26    | $\frac{-1x+0\cdot2}{2} = y$    |

Table 15: Simplified linear fractions within element pairs of  $c_1(n_t)$  as seen in Figure 6.

Finally for  $a$  we identify this to be

$$a(n_0) = 3^{\nu_2(n_0+1)} - 2^{\delta_1(n_0)}.$$

Bringing it all together we arrive at

$$\frac{(3^{\nu_2(n+1)} - 2^{\delta_1(n)})n + (3^{\nu_2(n+1)} - 2^{\nu_2(n+1)})2^{\nu_2(n)}}{2^{\delta_1(n)}}$$

where removing the  $2^{\delta_1(n_t)}$  term from  $a(n_t)$  gives

$$c_1(n_t) := \frac{3^{\nu_2(n_t+1)}n + (3^{\nu_2(n_t+1)} - 2^{\nu_2(n_t+1)})2^{\nu_2(n_t)}}{2^{\delta_1(n_t)}}.$$

as claimed.  $\square$

**Corollary 12.** *Following Theorem 3, Lemma's 3, and 4, and Corollaries 10 and 11 we now derive a rational function for  $c_2(n_t) \forall n_t \in \mathbb{N}^*$ .*

*Proof.* Similar to Corollary 11, we start by subtracting  $n_0$  from  $c_2(n_0)$  to isolate the operation of double-step iterations. Outcomes are listed in Table 16 and Figure 8. Similarly evident from Figure 8 are apparent 2-tuple outcomes, e.g.  $\langle 9, -2 \rangle$  with  $\langle 10, -2 \rangle$  or  $\langle 17, -4 \rangle$

| $n_0$ | $c_2(n_0) - n_0$ | $n_0$ | $c_2(n_0) - n_0$ |
|-------|------------------|-------|------------------|
| 1     | $1 - 1 = 0$      | 14    | $26 - 14 = 12$   |
| 2     | $2 - 2 = 0$      | 15    | $5 - 15 = -10$   |
| 3     | $1 - 3 = -2$     | 16    | $2 - 16 = -14$   |
| 4     | $2 - 4 = -2$     | 17    | $13 - 17 = -4$   |
| 5     | $1 - 5 = -4$     | 18    | $14 - 18 = -4$   |
| 6     | $8 - 6 = 2$      | 19    | $11 - 19 = -8$   |
| 7     | $13 - 7 = 6$     | 20    | $8 - 20 = -12$   |
| 8     | $2 - 8 = -6$     | 21    | $1 - 21 = -20$   |
| 9     | $7 - 9 = -2$     | 22    | $26 - 22 = 4$    |
| 10    | $8 - 10 = -2$    | 23    | $5 - 23 = -18$   |
| 11    | $13 - 11 = 2$    | 24    | $8 - 24 = -16$   |
| 12    | $8 - 12 = -4$    | 25    | $19 - 25 = -6$   |
| 13    | $5 - 13 = -8$    | 26    | $20 - 26 = -6$   |

Table 16: Subtraction of  $n_0$  terms from  $c_2(n_0)$  as given in Lemma 3 for  $1 \leq n_0 \leq 26$ .

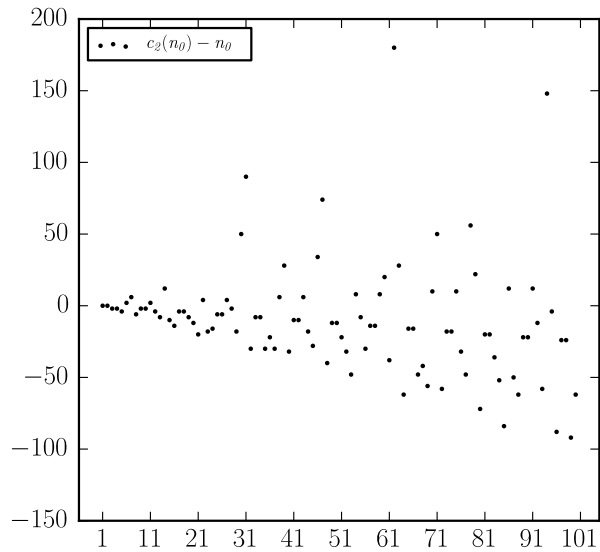


Figure 8: Outcomes of  $c_2(n_0) - n_0$  for  $1 \leq n_0 \leq 100$ .

| $n_0$ | $\frac{a}{b}x + \frac{g}{h} = y$     | $n_0$ | $\frac{a}{b}x + \frac{g}{h} = y$         |
|-------|--------------------------------------|-------|--|
| 1     | $-\frac{2}{8}x + \frac{1}{4} = y$    | 14    | $\frac{22}{32}x + \frac{19}{8} = y$      |
| 2     | $-\frac{2}{8}x + \frac{1}{2} = y$    | 15    | $-\frac{350}{512}x + \frac{65}{256} = y$ |
| 3     | $-\frac{46}{64}x + \frac{5}{32} = y$ | 16    | $-\frac{58}{64}x + \frac{1}{2} = y$      |
| 4     | $-\frac{10}{16}x + \frac{1}{2} = y$  | 17    | $-\frac{2}{8}x + \frac{1}{4} = y$        |
| 5     | $-\frac{26}{32}x + \frac{1}{16} = y$ | 18    | $-\frac{2}{8}x + \frac{1}{2} = y$        |
| 6     | $\frac{2}{16}x + \frac{5}{4} = y$    | 19    | $-\frac{14}{32}x + \frac{5}{16}$         |
| 7     | $\frac{22}{32}x + \frac{19}{16} = y$ | 20    | $-\frac{10}{16}x + \frac{1}{2} = y$      |
| 8     | $-\frac{52}{64}x + \frac{1}{2} = y$  | 21    | $-\frac{122}{128}x + \frac{1}{64} = y$   |
| 9     | $-\frac{2}{8}x + \frac{1}{4} = y$    | 22    | $\frac{2}{16}x + \frac{5}{4} = y$        |
| 10    | $-\frac{2}{8}x + \frac{1}{2} = y$    | 23    | $-\frac{202}{256}x + \frac{19}{128} = y$ |
| 11    | $\frac{2}{16}x + \frac{5}{8} = y$    | 24    | $-\frac{46}{64}x + \frac{5}{4} = y$      |
| 12    | $-\frac{14}{32}x + \frac{5}{4} = y$  | 25    | $-\frac{2}{8}x + \frac{1}{4} = y$        |
| 13    | $-\frac{10}{16}x + \frac{1}{8} = y$  | 26    | $-\frac{2}{8}x + \frac{1}{2} = y$        |

Table 17: Linear fractions within element pairs as seen in Figure 8, for  $1 \leq n_0 \leq 26 \mid c_2(n_t)$ .

with  $\langle 18, -4 \rangle$  with linear correspondences between outcomes of the first element and consequently of the second element of every pair. Solving for each correspondence symbolically we arrive at a set of partial fractions, listed in Table 17. These can be further simplified and combined in the form of

$$\frac{a \cdot n_0 + g \cdot f}{b}$$

, listed in Table 18. For  $f$  we observe this to remain  $f(n_0) = 2^{\nu_2(n_0)}$ .

For  $g$  we have the sequence

$$(1, 1, 5, 1, 1, 5, 19, 1, 1, 1, 5, 5, 1, 19, 65, \dots)$$

which we identify to be  $g(n_0) = 3^{p_2(n_0)} - 2^{p_2(n_0)}$  with  $p_2(n_0)$  as derived in Lemma 4. For  $b$  we observe these to be powers of two forming the sequence

$$(4, 4, 32, 8, 16, 8, 16, 16, 4, 4, 8, 16, 8, 16, 256, 32, \dots)$$

or

$$(2^2, 2^2, 2^5, 2^3, 2^4, 2^3, 2^4, 2^4, 2^2, 2^2, 2^3, 2^4, 2^3, 2^4, 2^8, 2^5, \dots)$$

| $n_0$ | $\frac{ax+gf}{b} = y$           | $n_0$ | $\frac{ax+gf}{b} = y$              |
|-------|---------------------------------|-------|------------------------------------|
| 1     | $\frac{-1x+1 \cdot 1}{4} = y$   | 14    | $\frac{11x+19 \cdot 2}{16} = y$    |
| 2     | $\frac{-1x+1 \cdot 2}{4} = y$   | 15    | $\frac{-175x+65 \cdot 1}{256} = y$ |
| 3     | $\frac{-23x+5 \cdot 1}{32} = y$ | 16    | $\frac{-29x+1 \cdot 16}{32} = y$   |
| 4     | $\frac{-5x+1 \cdot 4}{8} = y$   | 17    | $\frac{-1x+1 \cdot 1}{4} = y$      |
| 5     | $\frac{-13x+1 \cdot 1}{16} = y$ | 18    | $\frac{-1x+1 \cdot 2}{4} = y$      |
| 6     | $\frac{1x+5 \cdot 2}{8} = y$    | 19    | $\frac{-7x+5 \cdot 1}{16}$         |
| 7     | $\frac{11x+19 \cdot 1}{16} = y$ | 20    | $\frac{-5x+1 \cdot 4}{8} = y$      |
| 8     | $\frac{-13x+1 \cdot 8}{16} = y$ | 21    | $\frac{-61x+1 \cdot 1}{64} = y$    |
| 9     | $\frac{-1x+1 \cdot 1}{4} = y$   | 22    | $\frac{1x+1 \cdot 2}{8} = y$       |
| 10    | $\frac{-1x+1 \cdot 2}{4} = y$   | 23    | $\frac{-101x+19 \cdot 1}{128} = y$ |
| 11    | $\frac{1x+5 \cdot 1}{8} = y$    | 24    | $\frac{-23x+5 \cdot 8}{32} = y$    |
| 12    | $\frac{-7x+5 \cdot 4}{16} = y$  | 25    | $\frac{-1x+1 \cdot 1}{4} = y$      |
| 13    | $\frac{-5x+1 \cdot 1}{8} = y$   | 26    | $\frac{-1x+1 \cdot 2}{4} = y$      |

Table 18: Simplified linear fractions within element pairs as seen in Figure 8, for  $1 \leq n_0 \leq 26 \mid c_2(n_t)$ .

where the exponent we have identified as the function

$$\begin{aligned}
\delta_2(n_0) &= p_1(c_1(n_0)) + p_1(n_0) \\
&= \nu_2(c_1(n_0)^2 + c_1(n_0)) + \nu_2(n_0^2 + n_0) \\
&= \nu_2(c_1(n_0) + (c_1(n_0) \equiv 1)) + \nu_2(n_0 + (n_0 \equiv 1)) \pmod{2}
\end{aligned}$$

embedding  $c_1(n_t)$  as a function of  $p_1$  and where  $n_t + 2^{\delta_2(n_t)+1}$  represents the  $\Delta x$  metric to reach a pair  $\langle n_t, n_t + 1 \rangle$  of equal 2-adic valuation in  $\mathbb{N}^* \mid n_t \in \mathbb{N}_o \wedge n_t + 1 \in \mathbb{N}_e$  during double-step iterations. Removing  $\nu_2$  from  $\nu_2(c_1(n_0) + (c_1(n_0) \equiv 1)) \pmod{2}$  in  $p_1(c_1(n_0))$  yields the following sequence

$$(2, 2, 8, 2, 8, 4, 26, 2, 14, 6, 26, 4, 20, 8, 80, 2, \dots)$$

while removing  $\nu_2$  from  $\nu_2(c_1(n_0)^2 + c_1(n_0))$  in  $p_1(c_1(n_0))$  yields the following sequence

$$(6, 2, 72, 2, 72, 12, 702, 2, 210, 30, 702, 12, 420, 56, 6480, 2, \dots).$$

Because of the following  $p$ -adic property

$$\nu_p(xy) = \nu_p(x) + \nu_p(y)$$

where  $x \wedge y \in \mathbb{N}$  we have that

$$\nu_p(x+1) + \nu_p(y) = \nu_p((x+1)y) = \nu_p((xy+y)).$$

As such, the result of the 2-adic valuation function holds even though the two integer sequences above are not equal.

For  $\delta_2(n_0)$  as the exponent of powers of two we thus complete the denominator as

$$b(n_0) = 2^{\delta_2(n_0)}.$$

Finally for  $a$  we identify this to be

$$a(n_0) = 3^{p_2(n_0)} - 2^{\delta_2(n_0)}.$$

**Remark 9.** *Subscript 2 applied on  $\delta(n_0)$  as  $\delta_2(n_0)$  denotes the denominator-exponent variety of double-step iterations in the parity-alternating formulation of  $c(n_t)$ .*

Adding it all together results in

$$\frac{(3^{p_2(n)} - 2^{\delta_2(n)})n + (3^{p_2(n)} - 2^{p_2(n)})2^{\nu_2(n)}}{2^{\delta_2(n)}}$$

Subtracting  $2^{\delta_2(n)}$  term from  $a(n_0)$  allows for the reformulation with  $n_t$  to obtain

$$c_2(n_t) := \frac{3^{p_2(n_t)}n_t + (3^{p_2(n_t)} - 2^{p_2(n_t)})2^{\nu_2(n_t)}}{2^{\delta_2(n_t)}}$$

which allows us to apply double-steps iteratively. For example for  $n_0 = 7$  we have

$$\underbrace{c_2(c_2(c_2(7)))}_{3 \text{ times}} = 1$$

that maps

$$7 \rightarrow 13 \rightarrow 5 \rightarrow 1$$

as claimed. □

**Corollary 13.** *We can remove a factor of  $2^{\nu_2(n_t)}$  from  $\frac{a \cdot n_0 + g \cdot f}{b}$  as given in Corollary 11 and Corollary 12 thereby reducing the number of unknowns from three to two fully.*

*Proof.* Let

$$c_1(n_t) := \frac{3^{\nu_2(n_t+1)}n + (3^{\nu_2(n_t+1)} - 2^{\nu_2(n_t+1)})2^{\nu_2(n_t)}}{2^{\delta_1(n_t)}}.$$

We can include  $\frac{n_t}{2^{\nu_2(n_t)}}$  (sequence OEIS A000265 [9]) in  $c_1(n_t)$  as

$$c_1(n_t) := \frac{3^{\nu_2(n_t+1)} \frac{n_t}{2^{\nu_2(n_t)}} + 3^{\nu_2(n_t+1)} - 2^{\nu_2(n_t+1)}}{2^{\nu_2(n_t+1)}}$$

where the denominator is the result of  $2^{\delta_1(n_t) - \nu_2(n_t)}$ . Furthermore, from the simplifications that were identified in Theorem 2 we also obtain

$$c_1(n_t) := \frac{3^{\nu_2(n_t+1)} \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right)}{2^{\nu_2(n_t+1)}} - 1$$

Similarly, for  $c_2(n_t)$  we also include  $\frac{n_t}{2^{\nu_2(n_t)}}$  and obtain

$$c_2(n_t) := \frac{3^{p_2(n_t)} \frac{n_t}{2^{\nu_2(n_t)}} + 3^{p_2(n_t)} - 2^{p_2(n_t)}}{2^{p_1(c_1(n_t)) + \nu_2(n_t+1)}}$$

where the denominator is the result of  $2^{\delta_2(n_t) - \nu_2(n_t)}$ . Here, we also have the identity

$$p_1(c_1(n_t)) + \nu_2(n_t + 1) = \nu_2(c_1(n_t)) + \nu_2(c_1(n_t) + 1) + \nu_2(n_t + 1) = \nu_2(c_1(n_t)) + p_2(n_t)$$

with the last term being equal to the function  $p_2(n)$  from Lemma 4 because

$$\begin{aligned} (p_1(c_1(n_t)) + \nu_2(n_t + 1)) &= (\nu_2(c_1(n_t)) + 0 + \nu_2(n_t + 1)) \mid (n_t \equiv 1) \pmod{2} \\ &= \left( \nu_2(c_1(n_t)) + \nu_2\left(\frac{n_t}{2^0} + 1\right) \right) \mid (n_t \equiv 1) \pmod{2} \end{aligned}$$

and

$$\begin{aligned} (p_1(c_1(n_t)) + \nu_2(n_t + 1)) &= (0 + \nu_2(c_1(n_t) + 1) + 0) \mid (n_t \equiv 0) \pmod{2} \\ &= \nu_2\left(\frac{n_t}{2^{\nu_2(n_t)}} + 1\right) \mid (n_t \equiv 0) \pmod{2}. \end{aligned}$$

As such, we can similarly simplify  $c_2(n_t)$  to

$$c_2(n_t) := \frac{3^{p_2(n_t)} \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right)}{2^{\nu_2(c_1(n_t)) + p_2(n_t)}} - \frac{1}{2^{\nu_2(c_1(n_t))}}.$$

□

**Lemma 5.** *Following previous Corollaries, a rational function for  $c_3(n_t)$  exists, effectively mapping  $n_0 \rightarrow n_3$ .*



*Proof.* By induction, suppose that a rational function for  $c_3(n_t)$  can be deduced from the previously derived rational functions  $c_2(n_t)$  and  $c_1(n_t)$  with three unknowns. We consider once again

$$\frac{a(n) \cdot n + g(n) \cdot f(n)}{b(n)} + n.$$

As  $b(n)$  is present both in the numerator (as part of  $a(n)$ ) and the denominator we will deduce  $b(n)$  first. Considering the result from previous Corollaries we define  $\delta_3(n)$  as

$$\delta_3(n) = p_1(c_2(n)) + p_1(c_1(n)) + p_1(n)$$

that embeds previous  $c$ -step functions and where

$$b(n) = 2^{\delta_3(n)}.$$

Then we can assert

$$D_3(n) = n + 2^{\delta_3(n)+1}$$

that would reach the same subsequence of  $\delta_3(n)$ , mapping a metric to the next occurrence of a subsequence, i.e. to a proceeding pair equal with respect to  $\delta_3(n)$ . Thus arriving at another pair that contains the same 2-adic valuation outcome as the current pair. E.g.

$$D_3(7) = 71, D_3(9) = 73, D_3(11) = 43$$

$$\delta_3(7) = 5, \delta_3(71) = 5, \delta_3(9) = 5, \delta_3(73) = 5, \delta_3(11) = 4, \delta_3(43) = 4$$

for natural odd numbers, or

$$D_3(8) = 72, D_3(10) = 74, D_3(12) = 268$$

$$\delta_3(8) = 5, \delta_3(72) = 5, \delta_3(10) = 5, \delta_3(74) = 5, \delta_3(12) = 7, \delta_3(268) = 7$$

for natural even numbers. Indeed, for  $\delta_3(n)$  we have the sequence

$$\delta_3(n) = (3, 3, 6, 4, 5, 6, 5, 5, 5, 5, 4, 7, 4, 5, 9, 6, \dots)$$

or

$$2^{\delta_3(n)} = (2^3, 2^3, 2^6, 2^4, 2^5, 2^6, 2^5, 2^5, 2^5, 2^5, 2^4, 2^7, 2^4, 2^5, 2^9, 2^6, \dots)$$

while for  $D_3(n)$  the sequence

$$D_3(n) = (17, 18, 131, 36, 69, 134, 71, 72, 73, 74, 43, 268, 45, 78, 1039, 144, \dots).$$

For  $f(n)$  we claim this to remain

$$f(n) = 2^{\nu_2(n)}$$

equal to previous  $c$ -step functions. For the exponent in the numerator, which we shall now denote as  $r$  that is contained within  $a(n)$  and  $g(n)$ , we claim the following. If

$$\begin{aligned} r_1(n_t) &= \nu_2(n_t + 1) \mid c_1(n_t) \\ r_2(n_t) &= p_2(n_t) = \nu_2\left(\frac{n_t}{2^{\nu_2(n_t)}} + 1\right) \mid c_2(n_t) \end{aligned}$$

then

$$\begin{aligned} r_3(n_t) &= \nu_2(c_2(n_t) + 1) + \nu_2\left(\frac{n_t}{2^{\nu_2(n_t)}} + 1\right) \mid c_3(n_t) \\ &= \nu_2(c_2(n_t) + 1) + r_2(n_t) \mid c_3(n_t) \end{aligned}$$

resulting in the sequence

$$r_3(n_t) = (2, 1, 3, 1, 2, 2, 4, 1, 4, 1, 3, 2, 2, 3, 5, 1, \dots).$$

It's worth highlighting that this is a continuation that recurrently follows from

$$r_2(n_t) = p_2(n_t) = \nu_2(c_1(n_t) + 1) + r_1(n_t) \mid c_2(n_t)$$

because

$$\nu_2(c_1(n_t) + 1) + \nu_2(n_t + 1) = 0 + \nu_2(n_t + 1) \mid (n_t \equiv 1) \pmod{2}$$

and

$$\nu_2(c_1(n_t) + 1) + \nu_2(n_t + 1) = \nu_2\left(\frac{n_t}{2^{\nu_2(n_t)}} + 1\right) + 0 \mid (n_t \equiv 0) \pmod{2}.$$

Finalising  $a(n)$  we thus have

$$a(n_0) = 3^{r_3(n_0)} - 2^{\delta_3(n_0)}.$$

For  $g(n)$  we experimentally obtain

$$\begin{aligned} g(n_0) &= \left( \left( 3^{\nu_2(c_2(n_0)+1)} - 2^{\nu_2(c_2(n_0)+1)} \right) \left( 2^{\nu_2(c_1(n_0))} \right) - \left( 3^{\nu_2(c_2(n_0)+1)} - 2^{\nu_2(c_2(n_0)+1)} \right) \right) 2^{\nu_2(n_0+1)} \\ &\quad + 3^{r_3(n_0)} - 2^{r_3(n_0)} \\ &= \left( \left( 3^{r_3(n_0)-r_2(n_0)} - 2^{r_3(n_0)-r_2(n_0)} \right) \left( 2^{\nu_2(c_1(n_0))} \right) - \left( 3^{r_3(n_0)-r_2(n_0)} - 2^{r_3(n_0)-r_2(n_0)} \right) \right) 2^{\nu_2(n_0+1)} \\ &\quad + 3^{r_3(n_0)} - 2^{r_3(n_0)} \\ &= \left( 3^{r_3(n_0)-r_2(n_0)} - 2^{r_3(n_0)-r_2(n_0)} \right) \left( 2^{\nu_2(c_1(n_0))} - 1 \right) 2^{\nu_2(n_0+1)} + 3^{r_3(n_0)} - 2^{r_3(n_0)} \end{aligned}$$

Bringing it all together in non-simplified or original form we have

$$\frac{\left( 3^{r_3(n)} 2 - 2^{\delta_3(n)+1} \right) n + \left( \left( 3^{r_3(n)-r_2(n)} - 2^{r_3(n)-r_2(n)} \right) \left( 2^{\nu_2(c_1(n))} - 1 \right) 2^{\nu_2(n+1)} + 3^{r_3(n)} - 2^{r_3(n)} \right) 2^{\nu_2(n)+1}}{2^{\delta_3(n)+1}}$$

| $n_0$ | $\frac{a}{b}x + \frac{g}{h} = y$ | $n_0$ | $\frac{a}{b}x + \frac{g}{h} = y$   |
|-------|----------------------------------|-------|------------------------------------|
| 1     | $\frac{2x+7\cdot2}{16} = y$      | 14    | $\frac{-10x+19\cdot4}{64} = y$     |
| 2     | $\frac{-10x+1\cdot4}{16} = y$    | 15    | $\frac{-538x+451\cdot2}{1024} = y$ |
| 3     | $\frac{-74x+47\cdot2}{128} = y$  | 16    | $\frac{-122x+1\cdot32}{128} = y$   |
| 4     | $\frac{-26x+1\cdot8}{32} = y$    | 17    | $\frac{2x+7\cdot2}{16} = y$        |
| 5     | $\frac{-46x+19\cdot2}{64} = y$   | 18    | $\frac{-10x+1\cdot4}{16} = y$      |
| 6     | $\frac{-110x+5\cdot4}{128} = y$  | 19    | $\frac{34x+125\cdot2}{128}$        |
| 7     | $\frac{98x+73\cdot2}{64} = y$    | 20    | $\frac{-122x+1\cdot8}{128} = y$    |
| 8     | $\frac{-58x+1\cdot16}{64} = y$   | 21    | $\frac{-238x+67\cdot2}{256} = y$   |
| 9     | $\frac{98x+103\cdot2}{64} = y$   | 22    | $\frac{-14x+5\cdot4}{32} = y$      |
| 10    | $\frac{-58x+1\cdot4}{64} = y$    | 23    | $\frac{-350x+185\cdot2}{512} = y$  |
| 11    | $\frac{22x+23\cdot2}{32} = y$    | 24    | $\frac{-494x+5\cdot16}{512} = y$   |
| 12    | $\frac{-238x+5\cdot8}{256} = y$  | 25    | $\frac{22x+29\cdot2}{32} = y$      |
| 13    | $\frac{-14x+11\cdot2}{32} = y$   | 26    | $\frac{-26x+1\cdot4}{32} = y$      |

Table 19: Non-simplified or original linear fractions within element pairs for  $1 \leq n_0 \leq 26 \mid c_3(n_t)$ .

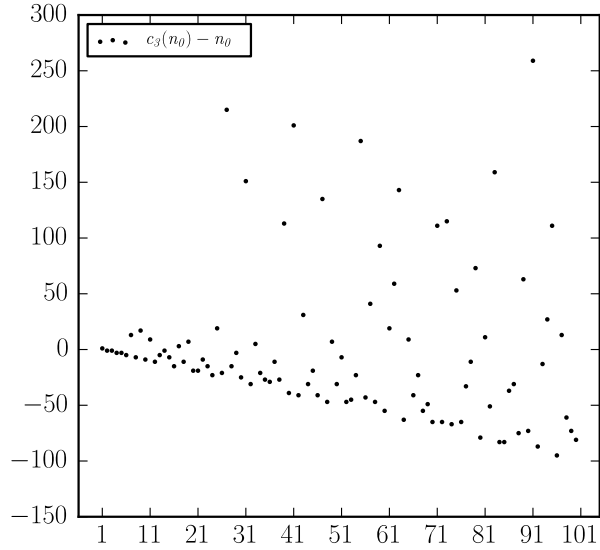


Figure 9: Outcomes of  $c_3(n_0) - n_0$  for  $1 \leq n_0 \leq 100$ .

of which its results are listed in Table 19 and Figure 9. From here we can reformulate to  $c_3(n_t)$  by subtracting  $2^{\delta_3(n)+1}$  from  $a(n)$ . Additionally, removing a factor of  $2^{\nu_2(n_t)+1}$  from all terms to obtain

$$c_3(n_t) := \frac{3^{r_3(n_t)} \frac{n_t}{2^{\nu_2(n_t)}} + (3^{r_3(n_t)-r_2(n_t)} - 2^{r_3(n_t)-r_2(n_t)}) (2^{\nu_2(c_1(n_t))} - 1) 2^{\nu_2(n_t+1)} + 3^{r_3(n_t)} - 2^{r_3(n_t)}}{2^{\delta_3(n_t)-\nu_2(n_t)}},$$

which can be reduced to

$$c_3(n_t) := \frac{3^{r_3(n_t)} \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right) + (3^{r_3(n_t)-r_2(n_t)} - 2^{r_3(n_t)-r_2(n_t)}) (2^{\nu_2(c_1(n_t))} - 1) 2^{\nu_2(n_t+1)} - 2^{r_3(n_t)}}{2^{\delta_3(n_t)-\nu_2(n_t)}}.$$

From Corollary 13 we also have the identity

$$2^{\delta_3(n_t)-\nu_2(n_t)} = 2^{\nu_2(c_2(n_t))+\nu_2(c_1(n_t))+r_3(n_t)}$$

thereby allowing the further reduction of  $c_3(n_t)$  to

$$c_3(n_t) := \frac{3^{r_3(n_t)} \left( \frac{n_t}{2^{\nu_2(n_t)}} + 1 \right)}{2^{\nu_2(c_2(n_t))+\nu_2(c_1(n_t))+r_3(n_t)}} - \frac{2^{\nu_2(c_2(n_t)+1)} - (3^{\nu_2(c_2(n_t)+1)} - 2^{\nu_2(c_2(n_t)+1)}) (2^{\nu_2(c_1(n_t))} - 1)}{2^{p_1(c_2(n_t))+\nu_2(c_1(n_t))}}$$

yielding for  $n_0 \rightarrow n_3$  the sequence

$$c_3(n_0) = (2, 1, 2, 1, 2, 1, 20, 1, 26, 1, 20, 1, 8, 13, 8, 1, \dots).$$

For example for  $n_0 = 7$  we have

$$\underbrace{c_3(c_3(7))}_{2 \text{ times}} = 1$$

that maps

$$7 \rightarrow 20 \rightarrow 1$$

as claimed. □

**Corollary 14.** *As  $g(n_t)$  gets more complex to deduce for every iteration jump, we propose  $a_k(n_t)$  and  $b_k(n_t)$ , where  $k$  denotes jumps in iteration steps, to first obtain successive results of  $g(n_t)$  computationally, denoted as  $g_k(n)$  with  $t = 0$ .*



and let  $k \in \mathbb{N}$  denote the iteration index for every number of  $k$ -iteration steps. Then, we can substitute as

$$C_k(n) = \frac{a_k(n) \cdot \frac{n}{2^{\nu_2(n)}} + g_k(n)}{b_k(n)}$$

that contains the expansion

$$C_{\langle 1, \dots, k \rangle}(n) = \frac{3^{r_{\langle 1, \dots, k \rangle}(n)} \cdot \frac{n}{2^{\nu_2(n)}} + g_{\langle 1, \dots, k \rangle}(n)}{2^{\delta_{\langle 1, \dots, k \rangle}(n)}}$$

up until  $k = 1$  that forms the basis with  $c_0(n) = n$ . The zipper function thus symbolically encodes continuous inclusions of previous iterations from  $k \dots 1$  where outcomes of  $n$  trend to converge around the trivial parity cycle  $(2; 1)$  giving the visual impression of closing a zipper for increments  $k$ . As a result we highlight the example of  $C_7(n)$  that gets defined as

$$C_7(n) = \frac{3^{\nu_2(c_6(n)+1)+\nu_2(c_5(n)+1)+\nu_2(c_4(n)+1)+\dots+\nu_2(c_0(n)+1)} \cdot \frac{n}{2^{\nu_2(n)}} + g_{\langle 1, \dots, 7 \rangle}(n)}{2^{p_1(c_6(n))+p_1(c_5(n))+p_1(c_4(n))+\dots+p_1(c_0(n))}}$$

which through the results of computational verification as listed in Table 21 yields the sequence

$$C_7(n) = (2, 1, 2, 1, 2, 1, 2, 1, 8, 1, 2, 1, 2, 1, 2, 1, \dots).$$

**Remark 12.**  $g_k(n)$  encapsulates all terms as previously stated and is by Corollary 14 the only term that is (at this moment) reliant on the outcome of  $c_k(n)$ .

**Theorem 4.** Following Definition 6 we can induce and validate identifications of subsequent results of  $g_k(n) \mid k > 3$  computationally.

*Proof.* Let

$$\begin{aligned} g_1(n) &= \left( 3^{\nu_2(I_0)} - 2^{\nu_2(I_0)} \right) \\ g_2(n) &= \left( 3^{\nu_2(I_1 I_0)} - 2^{\nu_2(I_1 I_0)} \right) \\ g_3(n) &= \left( 3^{\nu_2(I_2 I_1 I_0)} - 2^{\nu_2(I_2 I_1 I_0)} \right) \\ &\quad + 2^{\nu_2(I_0)} \left( 3^{\nu_2(I_2)} - 2^{\nu_2(I_2)} \right) \left( 2^{\nu_2(V_1)} - 1 \right). \end{aligned}$$

The  $p$ -adic property for multiplication as stated in Corollary 12 is used here as a way to shorten the exponents of each identified result from a series to a product. We also further denote  $I_{k-1, \dots, 1}$  as the variety for iterates  $(c_{k-1, \dots, 1}(n_t) + 1)$  and  $V_{k-1, \dots, 1}$  as the variety for

iterates  $(c_{k-1, \dots, 1}(n_t) + 0)$ . For  $4 \leq k \leq 6$  of  $g_k(n)$  we have experimentally identified the following results

$$\begin{aligned}
g_4(n) &= \left(3^{\nu_2(I_3 I_2 I_1 I_0)} - 2^{\nu_2(I_3 I_2 I_1 I_0)}\right) \\
&\quad + 2^{\nu_2(I_1 I_0)} \left(3^{\nu_2(I_3 I_2)} - 2^{\nu_2(I_3 I_2)}\right) \left(2^{\nu_2(V_2 V_1)} - 1\right) \\
g_5(n) &= \left(3^{\nu_2(I_4 I_3 I_2 I_1 I_0)} - 2^{\nu_2(I_4 I_3 I_2 I_1 I_0)}\right) \\
&\quad + 2^{\nu_2(I_2 I_1 I_0)} \left(3^{\nu_2(I_4)} - 2^{\nu_2(I_4)}\right) \left(2^{\nu_2(V_3 V_2 V_1)} - 1\right) \\
&\quad + 3^{\nu_2(I_4)} 2^{\nu_2(I_1 I_0)} \left(3^{\nu_2(I_3 I_2)} - 2^{\nu_2(I_3 I_2)}\right) \left(2^{\nu_2(V_2 V_1)} - 1\right) \\
g_6(n) &= \left(3^{\nu_2(I_5 I_4 I_3 I_2 I_1 I_0)} - 2^{\nu_2(I_5 I_4 I_3 I_2 I_1 I_0)}\right) \\
&\quad + 2^{\nu_2(I_3 I_2 I_1 I_0)} \left(3^{\nu_2(I_5 I_4)} - 2^{\nu_2(I_5 I_4)}\right) \left(2^{\nu_2(V_4 V_3 V_2 V_1)} - 1\right) \\
&\quad + 3^{\nu_2(I_5 I_4)} 2^{\nu_2(I_1 I_0)} \left(3^{\nu_2(I_3 I_2)} - 2^{\nu_2(I_3 I_2)}\right) \left(2^{\nu_2(V_2 V_1)} - 1\right).
\end{aligned}$$

Then, by induction we claim

$$\begin{aligned}
g_7(n) &= \left(3^{\nu_2(I_6 I_5 I_4 I_3 I_2 I_1 I_0)} - 2^{\nu_2(I_6 I_5 I_4 I_3 I_2 I_1 I_0)}\right) \\
&\quad + 2^{\nu_2(I_4 I_3 I_2 I_1 I_0)} \left(3^{\nu_2(I_6)} - 2^{\nu_2(I_6)}\right) \left(2^{\nu_2(V_5 V_4 V_3 V_2 V_1)} - 1\right) \\
&\quad + 3^{\nu_2(I_6 I_5 I_4)} 2^{\nu_2(I_1 I_0)} \left(3^{\nu_2(I_3 I_2)} - 2^{\nu_2(I_3 I_2)}\right) \left(2^{\nu_2(V_2 V_1)} - 1\right) \\
&\quad + 3^{\nu_2(I_6)} 2^{\nu_2(I_4 I_3 I_2 I_1)} \left(3^{\nu_2(I_5 I_4)} - 2^{\nu_2(I_5 I_4)}\right) \left(2^{\nu_2(V_4 V_3 V_2 V_1)} - 1\right)
\end{aligned}$$

resulting in the sequence

$$g_7(n) = (175, 37, 1319, 37, 619, 269, 3089, 37, 1631, 121, 1423, 269, 995, 347, 32731, 37, \dots).$$

and validated computationally in Appendix 5 for successive  $k$  integers, making Definition 6 thereby fulfilled. We provide a result of Appendix 6 of a first-time convergence to the minimum element 1, which is  $n_0 = 99999999999999999999999999999999$  and a stopping time of  $k = 314$  (for the parity-alternating reformulation of the Collatz function).  $\square$

**Conjecture 4.** We conjecture the following three constraints  $\forall n \in \mathbb{N}^*$  from Theorem 4. First, an upper bound for the total number or series of expressions for each  $k$  of  $g$ , given as the function

$$B(k) = \left\lfloor \frac{k+1}{2} \right\rfloor \mid k > 0$$

that calculates a growth of one extra expression for every other  $g_k$  by embedding the last expression of  $g_{k-2}$  multiplied with additive factors of  $3^{\nu_2(I_{k-1})}$ . The second is a sequence of  $I_{k-1}$ 's in the second expression that coincides with subsequences of OEIS A004523 [10].

For example,  $((2, 3), (4), (4, 5), (6), \dots)$ . The third constraint is a combination of at most four types of expressions, which are  $(3^{\nu_2(\langle I \rangle)} - 2^{\nu_2(\langle I \rangle)})$ ,  $2^{\nu_2(\langle I \rangle)}$ ,  $(2^{\nu_2(\langle V \rangle)} - 1)$ , and lastly  $3^{\nu_2(\langle I \rangle)}$ . All three constraints are demonstrated by Appendix 5.



## Appendix

### 1 Binary opposite modulo

For definition in Appendix 2 we employ a binary opposite modulo, defined as

$$\mathbf{bomod}(x, y) = \begin{cases} 1 & \text{if } x - y \lfloor \frac{x}{y} \rfloor = 0 \\ 0 & \text{if } x - y \lfloor \frac{x}{y} \rfloor > 0 \end{cases}$$

using floored division. For example if  $y = 2$  then  $\mathbf{bomod}(2, 2) = 1$ , conversely  $\mathbf{bomod}(1, 2) = 0$  for  $y \in \mathbb{N}^*$ . This provides versatility in testing membership of integers.

**Remark 13.** *This concept was developed to assist in computing  $p$ -adic valuations on some graphing calculators such as Desmos to accelerate the research brought forth herein.*

### 2 Primitive recursive $\mathbb{N}^*$ -adic valuation function

Given Appendix 1 we can define a primitive recursive function of the  $\mathbb{N}^*$ -adic valuation function  $\nu_n(x)$  described in Corollary 2 other than the common algorithmic approach that uses a **while-loop** as

$$\nu_y(x) = \sum_{i=1}^{\lfloor \log_y(x) \rfloor} \mathbf{bomod}(x, y^i)$$

for  $x > 0 \wedge y > 0$  with the accompanying code for the Python 3 language

---

```
from mpmath import mp
mp.dps = 300 # adjust precision as needed

# Binary opposite modulo function, returns 1 if 0 or 0 if 1
def bomod(x, y):
    if mp.fsub(x, mp.fmul(y, mp.floor(mp.fdiv(x, y)))) > 0:
        return 0
    else:
        return 1

# General n-adic valuation function in primitive form (a = n-adic number, e.g. 2 prime)
def v(n, a):
    k = 0
    # in Python, end of range function is exclusive, hence + 1
    for i in range(1, int(mp.floor(mp.fadd(mp.log(n, a), 1)))):
        k = k + bomod(n, mp.power(a, i))
    return k
```

---

### 3 One approach to derive $t$ following Corollary 7

$$n_0 = 1, t = 0 + 2(1) + 1 = 3$$

$$n_0 = 2, t = 1$$

$$n_0 = 3, t = 0 + 2(2) + 3 = 7$$

$$n_0 = 4, t = 2$$

$$n_0 = 5, t = 0 + 2(1) + 3 = 5$$

$$n_0 = 6, t = 1 + 2(2) + 3 = 8$$

$$n_0 = 7, t = 0 + 2(3) + 1 + 2(1) + 2 + 2(1) + 3 = 16$$

$$n_0 = 8, t = 3$$

$$n_0 = 9, t = 0 + 2(1) + 1 + 2(3) + 1 + 2(1) + 2 + 2(1) + 3 = 19$$

$$n_0 = 10, t = 1 + 2(1) + 3 = 6$$

$$n_0 = 11, t = 0 + 2(2) + 1 + 2(1) + 2 + 2(1) + 3 = 14$$

$$n_0 = 12, t = 2 + 2(2) + 3 = 9$$

$$n_0 = 13, t = 0 + 2(1) + 2 + 2(1) + 3 = 9$$

$$n_0 = 14, t = 1 + 2(3) + 1 + 2(1) + 2 + 2(1) + 3 = 17$$

$$n_0 = 15, t = 0 + 2(4) + 1 + 2(1) + 6 = 17$$

$$n_0 = 16, t = 4$$

## 4 Computational verification of $g_k(n_t)$

---

```
from mpmath import mp
mp.dps = 300 #adjust precision as needed
import functools

# general algorithmic way to write a n-adic valuation function
@functools.lru_cache(maxsize=None)
def v(n, a):
    i = 0
    while mp.fmod(n, a) == 0:
        n = mp.fdiv(n, a)
        i = i + 1
    return i

# p_1(n) as defined in the paper
@functools.lru_cache(maxsize=None)
def p(n):
    return v(mp.fadd(n, mp.fmod(n, 2)), 2)

# parity alternating Collatz function
def c(n):
    nm = mp.fadd(n, mp.fmod(n, 2))
    d = mp.fdiv(nm, mp.power(2, v(nm, 2)))
    three = mp.power(3, v(mp.fadd(n, 1), 2))
    return int(mp.fsub(mp.fmul(d, three), mp.fmod(n, 2)))

# demonstration of the zipper function (to compute g(n))
def gFromZipper(nMax, k):
    nt = []
    b = []
    r = []
    a = []
    g = []
    # integers of c(n)
    cint = []
    for n in range(1, nMax+1):
        # temp store for results of c(n)
        nk = []
        nk.append(n)
        delta = v(mp.fadd(n, 1), 2)
```

```

rk = delta
# temp store to recall last result of c(n)
ck = n
for ki in range(1, k):
    # iterate c(n), a.k.a. c_1, c_2, etc.
    ck = c(nk[ki-1])
    nk.append(ck)
    delta = mp.fadd(delta, p(ck))
    rk = mp.fadd(v(mp.fadd(ck, 1), 2), rk)
bn = mp.power(2, delta)
b.append(int(bn))
r.append(int(rk))
ak = mp.power(3, rk)
a.append(int(ak))
xdiv = mp.fdiv(n, mp.power(2, v(n, 2)))
akx = mp.fmul(ak, xdiv)
akxb = mp.fdiv(akx, bn)
gkb = mp.fsub(c(ck), akxb)
cint.append(c(ck))
gk = mp.fmul(gkb, bn)
g.append(int(gk))

#return(a)
#return(b)
return(g)
#return(cint)

# test range
for n in range(1, 33 + 1):
    derived = gFromZipper(33, n)
    print(derived)

```

---

## 5 Computational validation of $g_k(n_t)$

---

```
# see Appendix 3 for c, p, v and gFromZipper functions
def ckMul(ck, s, e):
    exponent = 1
    for k in range(s, e):
        exponent = mp.fmul(exponent, ck[k])
    return exponent

def express1(ckOdd, s, e):
    exponent = ckMul(ckOdd, s, e)
    return mp.fsub(mp.power(3, v(exponent, 2)), mp.power(2, v(exponent, 2)))

def express2(ckOdd, s, e):
    exponent = ckMul(ckOdd, s, e)
    return mp.power(2, v(exponent, 2))

def express3(ckEven, s, e):
    exponent = ckMul(ckEven, s, e)
    return mp.fsub(mp.power(2, v(exponent, 2)), 1)

def express4(ckOdd, s, e):
    exponent = ckMul(ckOdd, s, e)
    return mp.power(3, v(exponent, 2))

def t1t2t3(ckOdd, ckEven, e):
    t1 = express2(ckOdd, 0, e - 2)
    if (mp.fmod(e, 2) == 0):
        t2 = express1(ckOdd, e - 2, e)
    else:
        t2 = express1(ckOdd, e - 1, e)
    # start at k = 1 as we can remove v_2(c_0) from all
    t3 = express3(ckEven, 1, e - 1)
    return mp.fprod([t1, t2, t3])

def gkV(nMax, nMin, kMax):
    # storage for sum of k-expressions for every n
    nExpressionSums = []
    # go through n-columns (we start at n = 1)
    for n in range(nMin, nMax + 1):
        ckOdd = []
```

```

ckEven = []
ckOdd.append(int(mp.fadd(n, 1))) # precompute all c_k iterates
ckEven.append(n)
for ck in range(0, kMax - 1):
    ckEven.append(int(c(ckEven[ck])))
    ckOdd.append(int(mp.fadd(ckEven[ck + 1], 1)))
kExpresss = [] # storage of expressions per k
# max number of expressions for every k
maxExpressions = int(mp.floor(mp.fdiv(mp.fadd(kMax, 1), 2)))
# keep track of self-expressions
expressRemainder = maxExpressions
for express in range(maxExpressions):
    if (express == 0):
        kExpresss.append(express1(ckOdd, 0, kMax))
        expressRemainder = expressRemainder - 1
    elif (express == 1):
        kExpresss.append(t1t2t3(ckOdd, ckEven, kMax))
        expressRemainder = expressRemainder - 1
    else: # express >= 2
        # for each additional expression we look back
        for iter in range(expressRemainder, expressRemainder + 1):
            if (mp.fmod(kMax, 2) == 0):
                lookB4 = expressRemainder * 2
                t4 = express4(ckOdd, kMax - lookB4, kMax)
                t4 = mp.fmul(t4, t1t2t3(ckOdd, ckEven, kMax - lookB4))
                expressRemainder = expressRemainder - 1
            else:
                lookB4 = expressRemainder * 2 - 1
                t4 = express4(ckOdd, kMax - lookB4, kMax)
                t4 = mp.fmul(t4, t1t2t3(ckOdd, ckEven, kMax - lookB4))
                expressRemainder = expressRemainder - 1
            kExpresss.append(t4)
        nExpressionSums.append(int(mp.fsum(kExpresss)))
return(nExpressionSums)

for k in range(1, 33 + 1):
    obtained = gFromZipper(33, k)
    validated = gkV(33, 1, k)
    difference = list(map(int.__sub__, obtained, validated))
    print(difference) # expectation is a matrix with only zeros

```

---



## Postface

I would first of all like to extend my apologies as I, the author, does not have a formal education/training in pure mathematics but has instead a degree in computer science. Hence, the information presented in this paper could be described differently through more formal/advanced mathematical concepts. Unfortunately, time is a luxury in these uncertain times and the exploration would not have been possible if the author were to undertake a formal education.

This information should be best considered as meta extracts from an external observer that took a submarine into the number theory ocean, to find a rock that is lurking at the depth of the sea floor, and start chipping it away to see what lies inside. Communicating these ideas are meant to catch the attention of the experts and professionals to help validate them and determine whether it regresses or progresses our understanding of this truth seeking problem.

The author wishes to thank especially his amazing wife, brilliant kids, extended family, and friends for their consistent support throughout this pattern seeking quest, followed by Quanta Magazine with Prof. Terrence Tao for the introduction to the  $\sim 84$  years old conjecture of Lothar Collatz. Last but not least, the vibrant communities that maintain the Encyclopaedia of Integer Sequences, Wikipedia, Math StackExchange, Python, and the folks at Desmos for making this useful graphing calculator free.



## References

- [1] Amdeberhan, T.; Manna, D.; Moll, V.H. *The 2-adic valuations of Stirling numbers*, 2007.
- [2] Pólya, G.; Tarjan, R.E.; Woods, D.R. *Notes on Introductory Combinatorics*, Springer, pp. 21–22, ISBN 9780817649524 2009.
- [3] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A007814> 2021.
- [4] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A050603> 2021.
- [5] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A136480> 2021.
- [6] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A163575> 2021.
- [7] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A089309> 2021.
- [8] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A160541> 2021.
- [9] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A000265> 2021.
- [10] Sloane, N.J.A. *An On-Line Version of the Encyclopedia of Integer Sequences*, <https://oeis.org/A004523> 2021.