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# Connectivity of Kautz networks

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We show that undirected Kautz graphs (Theorem 3.1) and modified Kautz graphs (Theorem 4.2) have their connectivities equal to their minimum degrees. In view of their other properties, these results show that Kautz graphs are very good fault-tolerant networks.

## 1. Introduction

An important consideration in the design of communication networks as well as distributed computer systems is the interconnection network. This network is usually modeled by a graph or a digraph in which the vertices represent the switching elements or processors. Communication links are represented by edges if they are bidirectional or by arcs if they are unidirectional. Several factors have to be taken into account in the design of interconnection networks (see, for instance, [4, 9]):

- Communication delays between processors must be short: the graph must have a small diameter or mean distance.
- The number of processors directly connected to a given processor is limited: the graph has a given maximum degree.

– Finally, an interconnection network must be fault-tolerant. Indeed, in a system consisting of a large number of processors, the probability that a processor or a link happens to be faulty becomes important. One minimal requirement is that the system must work even in the case of node or link failures. This means that the associated graph is sufficiently connected.

Different networks that are good from the view point of first two criteria have been proposed in the literature. Among them are the Kautz networks (defined later). The aim of this paper is to show that they have also the best connectivity and are, therefore, highly reliable.

Other criteria of reliability have been considered and are surveyed in [3], where the theorems proved here are stated without proofs.

## 2. Definitions and notations

We represent the nodes of an interconnection network by the vertices and the links by the edges (or arcs) of an undirected (or directed) graph  $G=(V, E)$ . The definitions not given here can be found in [1]. We make precise some notation in the undirected case. Similar notation will be used for digraphs.

Let  $\Gamma(x)$  denote the set of neighbors of a vertex  $x$ .

The *degree*  $d(x)$  of  $x$  is the cardinality of  $\Gamma(x)$ .

The *maximum degree*  $\Delta = \Delta(G)$  of  $G$  is the maximum over all the degrees of the vertices of  $G$ .

The *minimum degree*  $\delta = \delta(G)$  of  $G$  is the minimum over all the degrees of the vertices of  $G$ .

The *distance*  $d(x, y)$  between  $x$  and  $y$  is the length of a shortest path between  $x$  and  $y$ .

The *diameter*  $D = D(G)$  of  $G$  is the maximum distance over all the pairs of vertices.

We call  $(\Delta, D)$ -graph a graph with maximum degree  $\Delta$  and diameter  $D$ .

The number of vertices of a  $(\Delta, D)$ -graph is bounded by the *Moore bound*:

$$n(\Delta, D) = 1 + \Delta \frac{(\Delta - 1)^D - 1}{\Delta - 2} \quad \text{if } \Delta > 2.$$

The  $(\Delta, D)$ -problem, which consists in finding a  $(\Delta, D)$ -graph with the maximum number of vertices, has been extensively studied. See [2] for further details.

A graph  $G$  is  $k$ -connected if there exist  $k$  internally vertex-disjoint paths between any pair of vertices. The *connectivity*  $\kappa(G)$  of  $G$  is the greatest integer  $k$  such that  $G$  is  $k$ -connected. Similarly, a graph is  $l$ -edge-connected if there exist  $l$  edge-disjoint paths between any pair of vertices. The *edge-connectivity*  $\lambda(G)$  of  $G$  is the greatest integer  $l$  such that  $G$  is  $l$ -edge-connected.

It is well known that  $\kappa(G) \leq \lambda(G) \leq \delta(G)$ .

### 2.1. Definitions of Kautz graphs

First we give three equivalent definitions of Kautz digraphs.

### 2.1.1. From an alphabet

The Kautz digraph  $K(d, D)$  (defined in [13]) with in- and out-degree  $d$  and diameter  $D$  is the digraph whose vertices are labeled with words  $(x_1, \dots, x_D)$ , where  $x_i$  belongs to an alphabet of  $d+1$  letters, and  $x_i \neq x_{i+1}$ , for  $1 \leq i \leq D-1$ . The vertex  $(x_1, \dots, x_D)$  is joined to the  $d$  vertices  $(x_2, \dots, x_D, \alpha)$ , where  $\alpha$  is any letter different from  $x_D$ . This digraph has  $d^D + d^{D-1}$  vertices.

### 2.1.2. From line digraph iterations

A definition using line digraph iterations has been given by Fiol et al. [8]. Recall that the *line digraph* of a digraph  $G$  is the digraph  $L(G)$  whose vertices represent the arcs of  $G$ . There is an arc in  $L(G)$  from  $x$  to  $y$  if and only if  $x$  represents the arc  $(u, v)$  in  $G$  and  $y$  represents the arc  $(v, w)$  in  $G$  for some vertices  $u, v$  and  $w$  in  $G$ . If  $G$  is  $d$ -regular and has diameter  $D$ , then  $L(G)$  is  $d$ -regular and has diameter  $D+1$  (except when  $G$  is a circuit).

Let  $K_{d+1}^*$  be the complete symmetric digraph on  $d+1$  vertices (with no loop); then  $L^{D-1}(K_{d+1}^*)$  is the Kautz digraph  $K(d, D)$ .

### 2.1.3. Generalization

The third definition is arithmetic and gives rise to a generalization of these digraphs for every integer  $n$ . This generalization of the Kautz digraphs was first studied by Imase and Itoh [11]. The  $\Pi_{d,n}$  digraph has the set of integers modulo  $n$  as vertex set. Its arc set  $A$  is defined as follows:

$$A = \{(x, y) / y \equiv -dx - a, 1 \leq a \leq d\}.$$

If  $n = d^D + d^{D-1}$ , then  $\Pi_{d,n} = K(d, D)$ .

$\Pi_{d,n}$  has  $n$  vertices and is  $d$ -regular (provided that  $n > d$ ). Its diameter is at most  $\lceil \log_d n \rceil$ . If  $n$  is  $d^p + d^{p-q}$  (where  $p$  and  $q$  are two integers, with  $q$  odd and  $q \leq p$ ), then its diameter is  $\lceil \log_d n \rceil - 1$  (see [11] for a proof).

### 2.1.4. Undirected case

Undirected Kautz graphs are obtained from the associated digraphs by removing the orientation and the resulting parallel edges. In what follows, we will denote by  $UK(d, D)$  the undirected Kautz graph of diameter  $D$  and maximum degree  $\Delta = 2d$ , minimum degree  $2d-1$  (indeed,  $K(d, D)$  contains some parallel edges but no loop). So,  $UK(d, D)$  is the graph whose vertices are the words of length  $D$  from an alphabet of  $d+1$  letters, with no two consecutive identical letters, in which the vertex  $(x_1, \dots, x_D)$  is joined to the vertices  $(\alpha, x_1, \dots, x_{D-1})$  and  $(x_2, \dots, x_D, \beta)$ , with  $\alpha \neq x_1$ , and  $\beta \neq x_D$ .

Although the Kautz graph  $UK(d, D)$  has only  $(\Delta/2)^D + (\Delta/2)^{D-1}$  vertices, which is somewhat far from the Moore bound, there is, at present, no known family of graphs with more vertices defined for any maximum degree  $\Delta$  and any diameter  $D$ . In any case, they have considerably more vertices than the classical hypercubes, used as interconnection networks of current parallel computers.

Another good family of networks is that of de Bruijn networks, whose definition is identical, except that the condition  $a_i \neq a_{i+1}$  is released. The connectivity of de Bruijn networks has been determined by Esfahanian and Hakimi [6] and Schlumberger [17]. The proof given below for Kautz graphs can be easily adapted to prove shortly that an undirected de Bruijn network has its connectivity equal to its minimal degree  $2d-2$ .

### 3. Connectivity of Kautz networks

#### 3.1. Directed case

From the second definition (line digraph iterations), it can be shown that the connectivity of  $K(d, D)$  is  $d$ . Indeed, the connectivity of  $L(G)$  is at least the arc connectivity of  $G$ . This was noted by different authors [7, 12, 15, 16].

#### 3.2. Undirected case

We did not find any result concerning the connectivities of Kautz graphs in the literature. In the present paper we show that undirected Kautz graphs are also maximally connected.

**Theorem 3.1.** *The connectivity of the Kautz graph  $UK(d, D)$  is  $2d-1$  if  $D > 1$ .*

**Proof (preliminary remarks).** Let  $x = (x_1, \dots, x_D)$  be any vertex of a Kautz graph. The set of *left neighbors* of  $x$  is the set of vertices  $(x_2, \dots, x_D, *)$ . It will be denoted by  $\Gamma^+(x)$ . Similarly, the set of vertices  $(*, x_1, \dots, x_{D-1})$  called the *right neighbors* of  $x$  will be denoted by  $\Gamma^-(x)$ . We call *left path (right path)* from  $x$  to  $y$  the shortest path from  $x$  to  $y$  (from  $y$  to  $x$ ) in the associated directed graph. Note that if  $v$  follows  $u$  in a left path (right path) then  $v \in \Gamma^+(u)$  ( $v \in \Gamma^-(u)$ ). We define the '*left distance*'  $d_L(x, y)$  ('*right distance*'  $d_R(x, y)$ ) from  $x$  to  $y$  as the length of a shortest left path (right path) from  $x$  to  $y$ . Note that it is a nonsymmetric function and that  $d_L(x, y) \neq d_R(y, x)$ . If  $D \geq 3$ , a vertex  $x = (x_1, \dots, x_D)$  is said to be *binary* if and only if  $x_i = x_{i+2}$ ,  $1 \leq i \leq D-2$ . Such a vertex will be denoted by  $\overline{x_1 x_2}$ . A vertex  $x = (x_1, \dots, x_D)$  is said to be *ternary* if and only if  $x_i = x_{i+3}$ ,  $1 \leq i \leq D-3$  and  $D \geq 4$  or  $x_1 \neq x_3$  of  $D=3$ . Such a vertex will be denoted by  $\overline{x_1 x_2 x_3}$ . Let us first give simple useful lemmas concerning the sets of neighbors of vertices or left and right paths in a Kautz graph.

**Lemma 3.2.** *Let  $x$  and  $y$  be adjacent vertices in a Kautz graph. Then  $|\Gamma(x) \cap \Gamma(y)| \leq 1$ . Moreover,  $\Gamma(x) \cap \Gamma(y) = \{y'\}$  if and only if  $x = \overline{x_1 x_2 x_3}$  and  $\{y, y'\} = \{\overline{x_2 x_3 x_1}, \overline{x_3 x_1 x_2}\}$ .*

**Proof of Lemma 3.2.** Let  $x = (x_1, \dots, x_D)$ . Then

$$\Gamma(x) = \{(x_2, \dots, x_D, *)\} \cup \{(*, x_1, \dots, x_{D-1})\}.$$

Suppose  $y = (x_2, \dots, x_D, \lambda)$ . Then

$$\Gamma(y) = \{(x_3, \dots, x_D, \lambda, *)\} \cup \{(*, x_2, \dots, x_D)\}.$$

Recall that in a Kautz graph two consecutive letters in a vertex are different. Then, if  $y' \in \Gamma(x) \cap \Gamma(y)$ , the only possibility is  $y = \overline{x_2 x_3 x_1}$  and  $y' = \overline{x_3 x_1 x_2}$ .  $\square$

**Lemma 3.3.** *If  $\Gamma^+(x) \cap \Gamma^-(x) \neq \emptyset$ , then  $x$  is a binary vertex  $\overline{x_1 x_2}$  and  $\Gamma^+(x) \cap \Gamma^-(x) = \{\overline{x_2 x_1}\}$ .*

**Lemma 3.4.** *Let  $x$  be a nonbinary vertex such that  $u \in \Gamma^+(x)$  and  $v \in \Gamma^-(x)$ . Then  $\Gamma^+(v) \cap \Gamma(u) = \Gamma^-(u) \cap \Gamma(v) = \{x\}$ .*

**Definitions.** Let  $S^+(x, t)$  be the set of vertices  $v$  such that the left path from  $x$  to  $v$  contains  $t$ . Similarly, let  $S^-(x, t)$  be the set of vertices  $v$  such that the right path from  $x$  to  $v$  contains  $t$ .

**Lemma 3.5.** *Let  $x$  and  $t$  be any two vertices in  $G$  such that  $d_L(x, t) = k$ . Then*

$$|S^+(x, t)| \leq \frac{d^{D+1-k} - 1}{d-1}.$$

Similarly, if  $d_R(x, t) = k$ , then

$$|S^-(x, t)| \leq \frac{d^{D+1-k} - 1}{d-1}.$$

**Proof of Lemma 3.5.** The number of vertices  $v$  such that  $d_L(t, v) = i$  is at most  $d^i$ . Furthermore, if the left path from  $x$  to  $v$  contains  $t$ , then  $d_L(t, v) \leq D - k$  as  $d_L(x, t) = k$ . Therefore,  $|S^+(x, t)|$  is at most

$$\sum_{i=0}^{D-k} d^i = \frac{d^{D+1-k} - 1}{d-1}. \quad \square$$

In what follows,  $F$  will be a set of vertices (the set of faulty vertices) such that  $|F| \leq 2d - 2$ .

For any vertex  $x$  in  $V(G) - F$ , we define

$$S^+(x) = \bigcup_{t \in F} S^+(x, t).$$

$S^+(x)$  represents the set of vertices which cannot be reached from  $x$  by the shortest left path in  $G - F$  (including those of  $F$ ). Similarly,

$$S^-(x) = \bigcup_{t \in F} S^-(x, t).$$

Let  $s(x) = \inf(|S^+(x)|, |S^-(x)|)$ .

**Lemma 3.6.** Let  $x$  be any vertex in  $G$  such that  $\{\{x\} \cup \Gamma^+(x)\} \cap F = \emptyset$ . Then  $s(x) \leq 2(d^{D-1} - 1)$ .

**Proof of Lemma 3.6.** Let  $t$  be any vertex in  $F$ . Then  $d_L(x, t) > 1$ . By Lemma 3.5, we have

$$|S^+(x, t)| \leq \frac{d^{D-1} - 1}{d - 1}.$$

Therefore,

$$s(x) \leq |S^+(x, t)| |F| \leq 2(d^{D-1} - 1). \quad \square$$

**Proof of Theorem 3.1 (continued).** We consider three cases.

*Case 1:*  $D \geq 3, d \geq 3$ .

Let us first choose a vertex  $z$  such that  $\{\{z\} \cup \Gamma^+(z)\} \cap F = \emptyset$ . Such a vertex always exists. Indeed, we only need to show that the number of vertices  $n$  is greater than  $(d+1)|F|$ . But, as  $D \geq 3$ , we clearly have

$$n = d^D + d^{D-1} \geq d^3 + d^2 > (d+1)(2d-2) \geq (d+1)|F|.$$

To prove the theorem, we only need to show that there exists a path between  $z$  and any other vertex in  $V(G) - F$ . For that purpose, we will show that, for each vertex  $x$ , there exists a vertex  $x'$  such that at least one of the two routes (left and right) from  $x$  to  $x'$  avoids  $F$  and that at least one of the two routings (left and right) from  $z$  to  $x'$  avoids  $F$ . With the above definitions, it suffices to prove that  $n > s(x) + s(z) - |F|$ .

**Lemma 3.7.** If  $\Gamma^+(x) \cap F = \emptyset$  or  $\Gamma^-(x) \cap F = \emptyset$ , then there exists a path in  $G - F$  between  $x$  and  $z$ .

**Proof of Lemma 3.7.** By Lemma 3.6, we have

$$s(x) \leq 2(d^{D-1} - 1) \quad \text{and} \quad s(z) \leq 2(d^{D-1} - 1).$$

Therefore,

$$s(x) + s(z) \leq 4(d^{D-1} - 1) < d^D + d^{D-1} = n \quad \text{as } d \geq 3. \quad \square$$

**Proof of Theorem 3.1 (conclusion).** Now let us remark that we only need to prove the theorem for nonbinary vertices. Indeed, if  $x$  is a binary vertex, we can find a nonbinary vertex  $u$  joined to  $x$  in  $G - F$ . Let  $u$  be a nonbinary vertex in  $\Gamma(x) - F$ , if any. Otherwise,  $F \subset \Gamma(x)$ . Let  $x'$  be the binary neighbor of  $x$  ( $x' \notin F$ ). Then any neighbor of  $x'$  different from  $x$  is suitable (because  $\Gamma(x) \cap \Gamma(x') = \emptyset$  by Lemma 3.2).

So, let  $x$  be a nonbinary vertex. It follows from Lemma 3.3 that  $\Gamma^+(x) \cap \Gamma^-(x) = \emptyset$ . Let  $l = |\Gamma^+(x) \cap F|$  and  $r = |\Gamma^-(x) \cap F|$ . Without loss of generality, we can suppose that  $l \leq r$ . From  $l + r \leq |F| \leq 2d - 2$ , we obtain  $l \leq d - 1$ .

*Subcase a:*  $1 \leq d - 3$ .

Let  $t$  belong to  $F$ .

If  $t$  belongs to  $\Gamma^+(x)$ , then  $d_L(x, t) = 1$ . By Lemma 3.5

$$|S^+(x, t)| \leq \frac{d^D - 1}{d - 1}.$$

Otherwise, we have  $d_L(x, t) > 1$ . By Lemma 3.5,

$$|S^+(x, t)| \leq \frac{d^{D-1} - 1}{d - 1}.$$

Therefore,

$$s(x) \leq |S^+(x)| \leq l \frac{d^D - 1}{d - 1} + (2d - 2 - l) \frac{d^{D-1} - 1}{d - 1},$$

which is maximal for  $l = d - 3$ . Hence,  $s(x) \leq d^D - d^{D-1} - 2$ . Lemma 3.6 applied to  $z$  gives

$$s(z) \leq 2(d^{D-1} - 1).$$

Hence, we have

$$s(x) + s(z) \leq d^D + d^{D-1} - 4 < n.$$

*Subcase b:  $l = d - 2$ .*

In this case,  $|\Gamma^+(x) - F| = 2$  and  $|F - \Gamma(x)| \leq 2$ .

Suppose that there exists a nonbinary vertex  $u$  in  $\Gamma^+(x) - F$  such that  $\Gamma(u) \cap F \cap \Gamma(x) = \emptyset$ . Then  $|\Gamma(u) \cap F| \leq 2$ . If  $\Gamma^-(u) \cap F = \emptyset$  or  $\Gamma^+(u) \cap F = \emptyset$ , we conclude, by Lemma 3.7, that there exists a path between  $u$  and  $z$  in  $G - F$  and, therefore, between  $x$  and  $z$ . Otherwise,  $|\Gamma^+(u) \cap F| = |\Gamma^-(u) \cap F| = 1$  and  $r = d - 2$ . Let  $v$  belong to  $\Gamma^-(x) - F$ .  $\Gamma^+(v) \cap \Gamma(x) = \emptyset$  and, by Lemma 3.4,  $\Gamma^+(v) \cap \Gamma(u) = \{x\}$ . As  $F \subset \Gamma(x) \cup \Gamma(u)$ ,  $\Gamma^+(v) \cap F = \emptyset$  and we conclude by Lemma 3.7.

Otherwise, as  $\Gamma^+(x)$  contains at most a binary vertex and a ternary vertex,  $\Gamma^+(x) - F$  must contain one of each type, which implies  $D = 3$ ,  $x = (x_1, x_2, x_3)$ ,  $\Gamma^+(x) - F = \{(x_2, x_3, x_1), (x_2, x_3, x_2)\}$  and  $(x_3, x_1, x_2) \in F$ . Let  $u = (x_2, x_3, x_1)$ . If  $\Gamma^-(u) \cap F = \emptyset$ , we conclude by Lemma 3.7. Otherwise, as  $\Gamma^-(u) \cap \Gamma(x) = \emptyset$ ,  $r \leq d - 1$  and there exists a vertex  $v$  in  $\Gamma^-(x) - F$ . If  $\Gamma^+(v) \cap F = \emptyset$ , we conclude by Lemma 3.7. Otherwise, as  $\Gamma^+(v) \cap \Gamma(x) = \emptyset$  and  $\Gamma^+(v) \cap \Gamma^-(u) = \{x\}$ ,  $F$  contains a vertex in  $\Gamma^+(v)$  that is neither in  $\Gamma(x)$  nor in  $\Gamma^-(u)$ . Therefore,  $r = d - 2$  and we can choose  $v$  nonbinary. Hence,  $\Gamma^-(v) \cap \Gamma^+(v) = \emptyset$ . We also have  $\Gamma^-(u) \cap \Gamma^-(v) = \emptyset$  by Lemma 3.4 and  $\Gamma^-(v) \cap \Gamma(x) = \emptyset$  because  $v$  is different from  $(x_3, x_1, x_2)$  (recall that  $(x_3, x_1, x_2) \in F$ ). Hence,  $\Gamma^-(v) \cap F = \emptyset$  and we conclude by Lemma 3.7.

*Subcase c:  $l = d - 1$*

We have  $F \subset \Gamma(x)$ . By Lemma 3.2, we have  $|\Gamma(v) \cap \Gamma(x)| \leq 1$  for any vertex  $v$  in  $\Gamma(x)$ . Therefore,  $|\Gamma(v) \cap F| \leq 1$  for any vertex  $v$  in  $\Gamma(x)$ .

If there is a nonbinary vertex  $u$  in  $\Gamma(x) - F$ , then either  $\Gamma^+(u) \cap F = \emptyset$  or  $\Gamma^-(u) \cap F = \emptyset$ . Therefore, we can conclude by applying Lemma 3.7 to  $u$ .

Otherwise, there is a binary vertex  $u$  in  $\Gamma(x) - F$ . From Lemma 3.2 we have  $\Gamma(u) \cap F = \emptyset$ . Consequently, we can apply Lemma 3.7 to vertex  $u$ .

Case 2:  $D \geq 3$ ,  $d = 2$ .

If  $D \geq 4$ , we can proceed as in the previous case by considering now a vertex  $z$  such that

$$\{\{z\} \cup \Gamma^+(z) \cup \Gamma^+(\Gamma^+(z))\} \cap F = \emptyset.$$

Such a vertex exists because  $|F| \leq 2$ . Indeed,

$$|F| + |\Gamma^+(F)| + |\Gamma^+(\Gamma^+(F))| \leq 14 \leq 2^D + 2^{D-1} \quad \text{as } D \geq 4.$$

Hence,  $s(z) \leq 2(2^{D-2} - 1)$  and Lemma 3.7 is still valid because  $s(x) + s(z) \leq 2(2^{D-1} - 1) + 2(2^{D-2} - 1) < 2^D + 2^{D-1}$ . So, we conclude if  $\Gamma^+(x) \cap F = \emptyset$  or  $\Gamma^-(x) \cap F = \emptyset$ . Otherwise, we finish exactly like in subcase c (here  $l = r = 1 = d - 1$ ). The case  $D = 3$  can be checked easily on the graph itself, which has twelve vertices.

Case 3:  $D = 2$

Let  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  be any two nonadjacent vertices in  $V(G)$ . We will show directly the existence of  $2d - 1$  disjoint paths between  $x$  and  $y$ . If  $x_1, x_2, y_1, y_2$  are all distinct, we can consider the following disjoint paths:

$$\begin{aligned} & [(x_1, x_2), (x_2, y_1), (y_1, y_2)], \\ & [(x_1, x_2), (y_2, x_1), (y_1, y_2)], \\ & [(x_1, x_2), (x_2, y_2), (y_2, x_2), (y_1, y_2)], \\ & [(x_1, x_2), (y_1, x_1), (x_1, y_1), (y_1, y_2)], \\ & [(x_1, x_2), (x_2, x_1), (x_1, y_2), (y_2, y_1), (y_1, y_2)], \\ & [(x_1, x_2), (x_2, *), (*, y_2), (y_2, *), (y_1, y_2)], \\ & [(x_1, x_2), (*, x_1), (y_1, *), (*, y_1), (y_1, y_2)], \end{aligned}$$

where  $*$  is any letter different from  $x_1, x_2, y_1$  and  $y_2$ .

Otherwise, we can assume, without loss of generality, that  $x_1 = y_1$  and then consider the following disjoint paths:

$$\begin{aligned} & [(x_1, x_2), (x_2, y_2), (y_2, x_2), (x_1, y_2)], \\ & [(x_1, x_2), (*, x_1), (x_1, y_2)], \end{aligned}$$

where  $*$  is any letter different from  $x_1$ ,

$$[(x_1, x_2), (x_2, *), (*, y_2), (y_2, *), (x_1, y_2)],$$

where  $*$  is any letter different from  $x_1, x_2$ , and  $y_2$ .  $\square$

**Remark.** This result shows that Kautz graphs are very suitable networks, better in fact than de Bruijn graphs. For the same maximum degree and diameter, they have more vertices and a better connectivity (one less than the best possible one). We will

see later that one can construct a  $2d$ -connected graph by adding some edges to  $UK(d, D)$ .

#### 4. Modified Kautz graphs

Since Kautz graphs are not regular, some authors attempt to modify them in order to get regular, maximally connected graphs (that is, graphs of connectivity equal to the degree).

Kumar and Reddy [14] obtained a  $2d$ -regular graph from the Kautz undirected graph  $UK(d, D)$  by adding a particular matching on the vertices of degree  $2d-1$ , in such a way that the subgraph generated by these vertices is a cycle. They showed that the resulting graph has connectivity  $2d$ . Furthermore, they gave a distributed and fault-tolerant routing which guarantees a path of length at most  $D+4t$  if  $t < d$  nodes are faulty. They presented a routing strategy when  $t < 2d$  nodes are faulty, which results in a maximum path length of  $3D+6$  between any two nonfaulty nodes.

In fact, we can show that the graphs obtained from  $UK(d, D)$  by adding any perfect matching between the vertices of degree  $2d-1$  have connectivity  $2d$  when  $D \geq 4$  and  $d \geq 3$ .

**Theorem 4.1.** *For  $d \geq 3$  and  $D \geq 4$ , any graph obtained from the undirected Kautz graph  $UK(d, D)$  by adding any perfect matching avoiding the existing edges on the vertices of degree  $2d-1$  has connectivity  $2d$ .*

**Proof (preliminary remarks).** We use here the same definitions and notation as in the proof of the previous theorem, except for very few details. In particular, the new neighbor of a binary vertex  $x$  neither belongs to  $\Gamma^+(x)$  nor to  $\Gamma^-(x)$ . Here  $F$  denotes a set of  $2d-1$  vertices.

Let us first choose a vertex  $z$  such that

$$\{\{z\} \cup \Gamma^+(z) \cup \Gamma^+(\Gamma^+(z))\} \cap F = \emptyset.$$

Such a vertex exists as  $n > |F|(1+d+d^2)$  as  $D \geq 4$ .

To prove the theorem, we only need to show that there exists a path between  $z$  and any other vertex in  $V(G) - F$ . In fact, we will show that, for every vertex  $x$ , there exists a vertex  $x'$  such that at least one of the two paths (left and right) from  $x$  to  $x'$  avoids  $F$  and that at least one of the two paths (left and right) from  $z$  to  $x'$  avoids  $F$ . Similarly to Lemma 3.7, we have the following lemma.

**Lemma 4.2.** *Suppose  $\Gamma^+(x) \cap F = \emptyset$  or  $\Gamma^-(x) \cap F = \emptyset$ . Then there exists a path between  $x$  and  $z$  in  $G - F$ .*

**Proof of Theorem 4.1 (conclusion).** We only need to prove the theorem for vertices that are neither binary nor ternary. Indeed, we show that, in both cases, we can find a vertex  $u$  joined to  $x$  in  $G - F$  that is neither binary nor ternary.

First suppose that  $x$  is ternary (recall that a vertex cannot be ternary as well as binary). As  $D \geq 4$ ,  $x$  has no binary neighbor. Either  $\Gamma(x) - F$  contains a nonternary vertex  $u$  or  $|\Gamma(x) \cap F| \geq 2d - 2$ . In this case, let  $v$  be a vertex of  $\Gamma(x) - F$ . We have  $|\Gamma(x) \cap \Gamma(v)| \leq 1$ .  $\Gamma(v)$  contains no binary vertex because  $v$  is ternary and then has at most two vertices of  $F$ . Therefore, as  $d \geq 3$ ,  $\Gamma(v) - F - \{x\}$  contains a vertex, that is neither binary nor ternary.

Suppose now that  $x$  is binary. Vertex  $x$  has no ternary neighbor because  $D \geq 4$ . Either  $\Gamma(x) - F$  contains a nonbinary vertex  $u$  or  $|\Gamma(x) \cap F| \geq 2d - 2$ . In this case, let  $v$  belong to  $\Gamma(x) - F$ .  $\Gamma(v)$  contains no vertex of  $\Gamma(x)$ . Hence,  $\Gamma(v)$  contains at most one vertex of  $F$ . Furthermore,  $\Gamma(v)$  has only two binary vertices. Therefore, there is a vertex  $u$  in  $\Gamma(v) - F$  that is neither binary nor ternary ( $v$  is binary).

Case 1:  $l \leq d - 3$ .

As

$$s(x) \leq (d-3) \binom{d^D - 1}{d-1} + (d+2) \binom{d^{D-1} - 1}{d-1} \quad \text{and}$$

$$s(z) \leq (2d-1) \frac{(d^{D-2} - 1)}{d-1},$$

$$s(x) + s(z) \leq (d-3) \binom{d^D - 1}{d-1} + (d+2) \binom{d^{D-1} - 1}{d-1} \\ + (2d-1) \binom{d^{D-2} - 1}{d-1} < n \quad \text{as } d \geq 3.$$

Case 2:  $l = d - 2$ .

As  $x$  has at most one binary neighbor ( $D \geq 4$ ), we can suppose that there exists a nonbinary vertex  $u$  in  $\Gamma^+(x) - F$ . If  $\Gamma^+(u) \cap F = \emptyset$  or  $\Gamma^-(u) \cap F = \emptyset$ , then we conclude by Lemma 4.2. Otherwise,  $|\Gamma^+(u) \cap F| \geq 1$ ,  $|\Gamma^-(u) \cap F| \geq 1$  and  $r \leq d - 1$ .

If  $r = d - 2$ , we can suppose that both vertices of  $\Gamma^-(x) - F$  are nonbinary (else, interchange  $\Gamma^-(x)$  and  $\Gamma^+(x)$ ). Let us denote them by  $v$  and  $w$ . By Lemmas 3.3 and 3.4, the four sets  $\Gamma^-(u) \cap F$ ,  $\Gamma^+(u) \cap F$ ,  $\Gamma^+(v) \cap F$  and  $\Gamma^+(w) \cap F$  are disjoint and they are also disjoint with  $\Gamma(x)$  (as  $x$  is nonternary). Therefore, as  $|F - \Gamma(x)| = 3$ , one of these sets is empty and we conclude by using Lemma 4.2.

If  $r = d - 1$ , let  $v$  be the vertex of  $\Gamma^-(x) - F$ . As  $\Gamma^+(v) \cap \Gamma(u) = \{x\}$  or  $\Gamma^+(v) \cap (\Gamma(u) \cap F) = \emptyset$  (by Lemma 3.4) and  $\Gamma^+(v) \cap \Gamma(x) = \emptyset$ , we have  $\Gamma^+(v) \cap F = \emptyset$ . Therefore, we conclude by applying Lemma 4.2 to  $v$ .

Case 3:  $l = d - 1$ .

Let  $u$  be the vertex of  $\Gamma^+(x) - F$ . If  $\Gamma^+(u) \cap F = \emptyset$  or  $\Gamma^-(u) \cap F = \emptyset$ , we can apply Lemma 4.2 to  $u$ . Otherwise,  $|\Gamma^+(u) \cap F| = |\Gamma^-(u) \cap F| = 1$ . Therefore,  $u$  is binary and  $r = d - 1$ . As  $\Gamma^+(v) \cap F = \emptyset$ , we conclude by applying Lemma 4.2 to the nonbinary vertex  $v$  of  $\Gamma^-(x) - F$ .  $\square$

## 5. Conclusion and open problems

In this article, we have shown that Kautz networks (or modified ones) are highly reliable and, therefore, well suited for future parallel architectures.

For some applications, it is desirable to have graphs not only with a high connectivity but also such that the diameter does not increase too much after the deletion of edges or vertices, or, in addition, with a large number of vertex-disjoint paths of short length between any two vertices. In [5] Bond and Peyrat proved that, after the deletion of at most  $2d - 2$  vertices in  $UK(d, D)$ , the diameter of the resulting graph is at most  $D + 2$ . A natural question is to ask whether there are  $2d - 1$  vertex-disjoint paths of length at most  $D + 2$ , between any two vertices.

Another question is to consider the same problems for  $UII_{d,n}$ , the generalization of the Kautz graphs which is given for any value of the number of vertices. A first step in that direction has been made in [10], where Homobono proved that, for  $D > 4$  and  $n > d^D$ , the connectivity of  $UII_{d,n}$  is  $2d - 1$  if  $d + 1$  divides  $n$ , and  $2d - 2$  otherwise.

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