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# INDUCED SUBGRAPHS OF THE POWER OF A CYCLE

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**Abstract.** In this article, we show that if  $G$  is an induced subgraph of the  $d$ -th power of a cycle of length  $n$ , and  $G$  has minimum degree  $d+k$ , then  $G$  has at least  $\frac{d+k}{2d} n$  vertices. This answers a problem of A. Kézdy.

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Let  $C_n^d$  be the  $d$ -th power of a cycle on  $n$  vertices, that is, the graph whose vertices are the integers modulo  $n$ , two vertices being joined if, and only if, their difference is less than or equal to  $d$ .

During the problem session at the SIAM Conference in San Francisco in June 1988, A. Kézdy [5] proposed the following problem : Find the minimum number of vertices of any induced subgraph  $G$  of  $C_n^d$ , of minimum degree  $d+1$ . In particular, he asked to prove that such a graph must have at least  $\frac{n}{2}$  vertices.

This question is motivated by the study of the  $d$ -girth of a graph. The  $d$ -girth of a graph  $G$  is defined, according to [6], as the minimum number of vertices of an induced subgraph of  $G$  with minimum degree  $d$  (note that the 2-girth is nothing but the girth). Paul Erdős originally proposed the problem of determining the largest  $d$ -girth of a graph on  $n$  vertices and  $m$  edges. A. Kézdy and M. Markert [6] considered this problem, and in order to find graphs with a large  $(d+1)$ -girth asked to determine the  $(d+1)$ -girth of  $C_n^d$ . In [6], they gave constructions that show that the  $(d+1)$ -girth of  $C_n^d$  is at most  $\frac{n}{2} [1 + \frac{d+1}{d^2+d-\epsilon}]$ , where  $\epsilon=0$  or  $1$  whenever  $d+1 \equiv 0$  or  $1 \pmod{2}$ , respectively. They conjectured that this value is exactly the  $(d+1)$ -girth of  $C_n^d$  and furthermore showed that the  $(d+1)$ -girth of  $C_n^d$  is at least  $n(\sqrt{2}-1)$ .

In this article we show that the  $(d+1)$ -girth of  $C_n^d$  is at least  $\frac{d+1}{2d} n$ , solving the problem proposed in [5] and proving the above conjecture when  $d+1$  is even. In fact, we solve a slightly more general problem by determining the minimum order of an induced subgraph of  $C_n^d$  of degree  $d+k$  (with  $k \geq 1$ ).

Note that a minimum induced subgraph of minimum degree  $d-k$  (with  $k \geq 0$ ) of  $C_n^d$  is a complete graph on  $d-k+1$  vertices.

Note also that these graphs  $C_n^d$ , which are particular circulant graphs, have nice properties. For example, they are the graphs with  $n$  vertices and connectivity  $2d$  with the smallest possible number of edges (see the book of C. Berge [1]). They also have "super line-connectivity properties" (see the article of F.T. Boesch and J.F. Wang [2]). They are also important in the design of fault-tolerant networks. For example G. Chartrand and S.F. Kapoor [3], J.P. Hayes [4] and W.W. Wong and C.K. Wong [7] proved that they are minimum  $2d-2$  hamiltonian graphs (resp.  $2d-2$  edge-hamiltonian). That is, they are the graphs with the minimum number of edges among the graphs on  $n$  vertices that remain hamiltonian after the deletion of up to  $2d-2$  vertices (resp. up to  $2d-2$  edges).

**Theorem.**

Let  $G$  be an induced subgraph of  $C_n^d$  of minimum degree  $d+k$ ,  $1 \leq k \leq d$ , then

$$\frac{|V(G)|}{n} \geq \frac{d+k}{2d}.$$

*Proof.*

Denote by  $V$  the set of the vertices of  $G$  and by  $\bar{V}$  the complement of this set in the interval  $[0, n-1]$ .

Let  $I = [u, v]$  be an interval of cardinality less than or equal to  $d$ , with  $v$  in  $V$ . First observe that if  $I$  contains  $\bar{n}$  elements of  $\bar{V}$ , then the interval  $[v+1, v+d+1]$  must contain at least  $\bar{n}+k$  elements of  $V$  to insure that the degree of  $v$  in  $G$  is at least  $d+k$ .

Now, we can construct an infinite sequence of consecutive intervals of integers modulo  $n$ :  $J_1 = [a_1+1, a_2]$ ,  $J_2 = [a_2+1, a_3]$ , ...,  $J_j = [a_j+1, a_{j+1}]$ , ... satisfying the following properties:

- $a_j$  belongs to  $V$ , for all  $j$ ,  $j \geq 2$
- $|J_j| \leq d$ , for all  $j$ ,  $j \geq 1$
- $n_{j+1} = \bar{n}_j + k$ , for all  $j$ ,  $j \geq 1$ , where  $n_j = |J_j \cap V|$  and  $\bar{n}_j = |J_j \cap \bar{V}|$ .

Indeed, we can define these intervals recursively as follows :

- $J_1 = [a_1+1, a_2]$  is any interval of length at most  $d$  such that  $a_2$  is in  $V$ .
- Suppose we have constructed the sequence up to  $J_j = [a_j+1, a_{j+1}]$ , with  $a_{j+1}$  in  $V$ . Then,  $J_{j+1} = [a_{j+1}+1, a_{j+2}]$ , where  $a_{j+2}$  is the  $\bar{n}_j+k$ -th clockwise neighbor of  $a_{j+1}$ . Note that, by the above observation,  $a_{j+2}$  is well defined as  $a_{j+1}$  has at least  $\bar{n}_j$  counterclockwise neighbors in  $\bar{V}$ . Note also that  $J_{j+1}$  has length at most  $d$ .

As there exist only a finite number of intervals of length at most  $d$ , there exist two positive integers  $m$  and  $p$  such that  $J_m = J_{m+p}$ .

Now, let  $I_1 = J_m$ ,  $I_2 = J_{m+1}$ , ...,  $I_p = J_{m+p-1}$ . This sequence of intervals clearly satisfies the following properties :

- a) for all  $i$ ,  $1 \leq i \leq p$ ,  $|I_i| \leq d$ .
- b) if  $n_i = |I_i \cap V|$  and  $\bar{n}_i = |I_i \cap \bar{V}|$ , then for all  $i$  such that  $2 \leq i \leq p$ ,  $n_i = \bar{n}_{i-1} + k$  and  $n_1 = \bar{n}_p + k$ .
- c) all the integers modulo  $n$  appear in the same number of intervals.

From b) and c), we deduce that

$$\frac{|V|}{n} = \frac{\sum_{i=1}^p n_i}{\sum_{i=1}^p (n_i + \bar{n}_i)}$$



Let  $N = \sum_{i=1}^p n_i$ , then  $\sum_{i=1}^p \bar{n}_i = N - kp$ . Therefore  $\frac{|V|}{n} = \frac{N}{2N - kp} = \frac{1}{2 - \frac{kp}{N}}$ .

However, for all  $i$ ,  $1 \leq i \leq p-1$ ,  $n_i + n_{i+1} = n_i + \bar{n}_i + k \leq d+k$  by a), and similarly  $n_p + n_1 \leq d+k$ . Therefore  $N \leq \frac{p(d+k)}{2}$  and  $\frac{|V|}{n} = \frac{1}{2 - \frac{kp}{N}} \geq \frac{1}{2 - \frac{2k}{d+k}} = \frac{d+k}{2d}$ .

*Remark 1*

If  $d+k$  is even, then the bound is sharp. Indeed the subgraph obtained by alternatively taking  $\frac{d+k}{2}$  vertices in  $V$  and  $\frac{d-k}{2}$  vertices in  $\bar{V}$  has minimum degree  $d+k$ . In fact this subgraph is regular of degree  $d+k$ , and is isomorphic to a power of a cycle.

*Remark 2*

If  $d+k$  is odd, then the bound is not sharp.

For instance, if  $d=2$  and  $k=1$ , then the smallest subgraph of minimum degree 3 of  $C_n^2$  has  $\frac{4}{5}n$  vertices. The extremal graph (already given in [6]) is obtained, when 5 divides  $n$ , by alternatively taking 4 vertices in  $V$  and one in  $\bar{V}$ . To prove that this subgraph is the smallest one, let us show that any vertex in  $\bar{V}$  is followed by at least 4 vertices in  $V$ . First recall that if a vertex is in  $V$ , then at least 3 of its 4 neighbors are also in  $V$ . Then note that there can not be two or more consecutive vertices in  $\bar{V}$ , as the first vertex in  $V$  after the sequence of vertices in  $\bar{V}$  would have at most two neighbors in  $V$ . Let  $\bar{v}$  be a vertex in  $\bar{V}$ . By the preceding remark,  $v+1$  is in  $V$  and its two neighbors  $v+2$  and  $v+3$  are also in  $V$ . Now, as  $v+2$  has already  $\bar{v}$  as neighbor in  $\bar{V}$ ,  $v+4$  is also in  $V$ .

More generally, one can prove that if  $k=d-1$ , then  $\frac{|V(G)|}{n} \geq \frac{2d}{2d+1}$ . An extremal example is obtained by alternatively taking  $2d$  vertices in  $V$  and one in  $\bar{V}$ .

If  $d=4$  and  $k=1$ , we can prove that the smallest subgraph of  $C_n^4$  of minimum degree 5 has  $\frac{12}{19}n$  vertices. The extremal example, when 19 divides  $n$ , corresponds to repetitions of the sequence  $\bar{v} v v \bar{v} v v v \bar{v} v \bar{v} v \bar{v} v v v \bar{v} v v \bar{v}$ , where  $v$  denotes a vertex in  $V$  and  $\bar{v}$  a vertex in  $\bar{V}$ . This example is already given in [6], where it is proved to be unique.

If  $d=5$  and  $k=2$ , we can prove that the smallest subgraph of  $C_n^5$ , with minimum degree 7, has  $\frac{5}{7}n$  vertices. An extremal graph corresponds to repetitions of the sequence  $\bar{v} v v v \bar{v} v v v v \bar{v} v v v \bar{v}$ .

If  $d=6$  and  $k=1$ , A. Kézdy and M. Markert proved that the smallest subgraph of  $C_n^6$ , with minimum degree 7, has  $\frac{24}{41}$  vertices. They also showed that there are two extremal sequences:

$\bar{v} \bar{v} \bar{v} v v v \bar{v} \bar{v} \bar{v} v v v \bar{v} \bar{v} v \bar{v} v v \bar{v} \bar{v} v v v v \bar{v} \bar{v} v v \bar{v} v \bar{v} \bar{v} v v v v \bar{v} \bar{v} v v v$

and  $\bar{v} \bar{v} v \bar{v} v v \bar{v} v v \bar{v} v v \bar{v} \bar{v} \bar{v} v v \bar{v} \bar{v} v v v \bar{v} \bar{v} v \bar{v} v \bar{v} \bar{v} v v \bar{v} \bar{v} v v v \bar{v} \bar{v} v v v$

In view of these results, we offer the following conjecture which contains the conjecture of A. Kézdy and M. Markert [6], when  $k=1$ . They verified it for  $d=2,4,6$ , and 8 and  $k=1$ .

**Conjecture.**

The minimum number of vertices of an induced subgraph of  $C_n^d$ , with odd minimum degree  $d+k$ , is  $\frac{n d (d+3-k)}{2(d^2-(k-2)d-k)}$ .

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