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ONE-ENDED SPANNING SUBFORESTS AND TREEABILITY OF GROUPS

CLINTON T. CONLEY, DAMIEN GABORIAU, ANDREW S. MARKS,
AND ROBIN D. TUCKER-DROB

ABSTRACT. We show that several new classes of groups are measure strongly treeable. In particular, finitely generated groups admitting planar Cayley graphs, elementarily free groups, and $\text{Isom}(\mathbf{H}^2)$ and all its closed subgroups. In higher dimensions, we also prove a dichotomy that the fundamental group of a closed aspherical 3-manifold is either amenable or has strong ergodic dimension 2. Our main technical tool is a method for finding measurable treeings of Borel planar graphs by constructing one-ended spanning subforests in their planar dual. Our techniques for constructing one-ended spanning subforests also give a complete classification of the locally finite p.m.p. graphs which admit Borel a.e. one-ended spanning subforests.

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0. INTRODUCTION

This article is a contribution to the study of measured and Borel equivalence relations, in terms of their graphed structures, with applications in the measured group theory of countable and locally compact groups.

Dramatic progress has been realized in the study of discrete groups in relation with topological and geometric ideas over the course of the 20th century, from the early works of Klein, Poincaré, Dehn, Nielsen, Reidemeister and Schreier for instance, to Bass-Serre theory and Thurston's Geometrization program as well as hyperbolic groups and the emergence of geometric group theory as a distinct area of mathematics under the impulse of Gromov. In his monograph [Gro93], Gromov outlined his program of understanding countable discrete groups up to quasi-isometry (e.g., co-compact lattices in the same locally compact second countable group G). In the same text Gromov also introduced the parallel notion of measure equivalence (**ME**) between countable discrete groups [Gro93, 0.5.E], the most emblematic example being lattices in G . Two groups are ME if they admit commuting, free, measure-preserving actions on a nonzero Lebesgue measure space with finite measure fundamental domains. This concept is strongly connected with orbit equivalence (**OE**) in ergodic theory ([Fur99], [Gab02b, Th. 2.3]; see [Gab05, Fur11] for surveys on ME and OE).

The history of orbit equivalence itself can be traced back to the work of Dye [Dye59, Dye63] stemming on the group-measure-space von Neumann algebra of Murray and von Neumann [MvN36]. The abstract and basic objects connecting this turn out to be the standard measure-class preserving equivalence relations, as axiomatized by Feldman-Moore [FM77]. A major milestone is the elucidation of the connections between five properties (see [Con76, CK77, OW80, CFW81]): *the following are equivalent: (1) hyperfiniteness of the group-measure-space von Neumann algebra, (2) hyperfiniteness of the equivalence relation, (3) amenability of the equivalence relation, (4) orbit equivalence with a \mathbf{Z} -action, and (5) when the action is assumed to be probability measure preserving (**p.m.p.**) and free, amenability of the acting group.* As a consequence, the measure equivalence class of \mathbf{Z} consists exactly in all infinite amenable groups. Thus such a useful geometric invariant as the growth becomes apparently irrelevant in measured group theory insofar as amenability is concerned, although our Theorem 2.6 leads us to reconsider this observation.

Much of progress in orbit equivalence has been realized since the 80's following a suggestion of A. Connes at a conference in Santa Barbara in 1978 (see [Ada90])

of studying equivalence relations \mathcal{R} with an additional piece of data: a measurably-varying simplicial complex structure on each equivalence class (aka a **complexing** [AG21]). The 1-dimensional complexings are known as **graphings**. Their acyclic version (**treeings**) were originally studied by S. Adams [Ada90, Ada88]. Both are constitutive of the theory of **cost** [Lev95, Gab00], since this is defined in terms of graphings and treeings allow to compute it [Gab00, Théorème 1]. Graphings and treeings have also played a crucial role in the theory of structurings on countable Borel equivalence relations [JKL02].

Amenability, seen from the perspective of orbit equivalence, can be rephrased as the capability of embellishing almost every orbit (equivalence class) with a measurably-varying oriented line structure [Dye59, OW80, CFW81]. Alternatively, it is easy to equip the classes of any hyperfinite equivalence relations with a **one-ended** tree structure. As a kind of converse, in the p.m.p. context (which will be our context in the introduction through Theorem 9) any treeing of an amenable equivalence relation is (class-wise) at most two-ended ([Ada90]).

Beyond amenability, the simplest groups from the measured theoretic point of view are the **treeable** ones: those admitting a free p.m.p. action whose orbit equivalence relation can be equipped with a treeing¹. This is an extremely rich and still mysterious class of groups (see the survey part of [Gab05]). By a theorem of Hjorth [Hjo06], this is precisely the class of groups Γ that are ME with a free group \mathbf{F}_n . This family splits into four ME-classes: $n = 0$ when Γ is finite, $n = 1$ when Γ is infinite amenable, and $n = 2$ or $n = \infty$ according to their cost belonging to $(1, \infty)$ or $\{\infty\}$ [Gab00].

The first substantial example of a treeable group apart from free products of amenable groups is the fundamental group $\pi_1(\Sigma)$ of a closed hyperbolic surface Σ . Indeed, both $\pi_1(\Sigma)$ and \mathbf{F}_2 share the property of being isomorphic to lattices in $G = \mathrm{SL}(2, \mathbf{R})$. It follows that $\pi_1(\Sigma)$ admits at least one treeable free action, namely the natural action by multiplication $\pi_1(\Sigma) \curvearrowright G/\mathbf{F}_2$ (with Haar measure). It is a longstanding question of [Gab00, Question VI.2] whether treeable groups are **strongly treeable**² (**ST**), i.e., whether **all** their free p.m.p. actions are treeable. This question has been open for twenty years, even for $\pi_1(\Sigma)$ (see [Gab02b, Question p. 176]), and we solve it in this case. This is our first main result:

Theorem 1. Surface groups are strongly treeable. More generally, finitely generated groups admitting a planar Cayley graph are strongly treeable.

A more general statement can be found in Theorem 4.4.

The introduction of the notion of “measurable free factor” in [Gab05], led to the production of some new examples of treeable groups, such as branched surface groups. These are examples from a family that we will discuss now. The **elementarily free groups** are those groups with the same first-order theory as the free

¹For precise definitions of the various notions of treeability, see Appendices A and B

²Observe that unfortunately for consistency of terminology, in [Gab00] the terms “arborable” and “anti-arborable” are used instead of the current better terms “strongly treeable” and “non-treeable” respectively, that we will adopt here.

group \mathbf{F}_2 . We shall use their description (when finitely generated) as fundamental groups of certain tower spaces (according to the results of [Sel06] and [KM98], made utterly complete in [GLS20] – see §5 for more details). A careful analysis of their virtual structure allowed [BTW07] to apply results from [Gab05] to a finite index subgroup in order to achieve their treeability. The question of their strong treeability has remained open since then; we resolve it:

Theorem 2. Finitely generated elementarily free groups are strongly treeable.

Strong treeability has a number of consequences which do not follow from treeability. In particular, if Γ is a strongly treeable group, then by [Gab00, Prop. VI.21] Γ satisfies the fixed price conjecture [Gab00, Question I.8]. The groups having a planar Cayley graph appearing in Theorem 1, such as the cocompact Fuchsian triangle groups, give the first new examples of groups of fixed price greater than 1 since [Gab00]. (The cocompact Fuchsian triangle groups admit finite index subgroups which are surfaces, but both strong treeability and fixed price are not known to pass to finite index super-groups.)

The arguments developed for the above theorem gave us as a by-product the following interesting claim (Corollary 5.11). Let $r \geq 3$ and $\Gamma_1, \Gamma_2, \dots, \Gamma_r$ be countable groups and let $\gamma_i \in \Gamma_i$ be an infinite order element for each $i = 1, 2, \dots, r$. If the Γ_i are all treeable or strongly treeable, then the same holds not only for their free product, but also for its quotient by the normal subgroup generated by the product of the γ_i : $(\Gamma_1 * \Gamma_2 * \dots * \Gamma_r) / \langle\langle \prod_{i=1}^r \gamma_i \rangle\rangle$.

It is worth mentioning that treeability has also had an impact in the theory of von Neumann algebras. Popa’s discovery [Pop06] of the first II_1 factor with trivial fundamental group (namely the group von Neumann algebra $L(\text{SL}(2, \mathbf{Z}) \ltimes \mathbf{Z}^2)$) used his rigidity-deformation theory to establish uniqueness of the HT Cartan subalgebra, thereby reducing the problem to the study of the orbit equivalence relation of the treeable action of $\text{SL}(2, \mathbf{Z})$ on the 2-torus. Later on, Popa and Vaes extended drastically the class of groups whose free p.m.p. actions lead to uniqueness of the Cartan subalgebra [PV14a, PV14b], and thus the study of the group-measure space von Neumann algebra of these actions boils down to that of the action up to OE. This class contains free groups, non-elementary hyperbolic groups, their direct products and all groups that are ME with these groups. The class of countable groups satisfying 2-cohomology vanishing for cocycle actions on II_1 factors is speculated to coincide with the class of treeable groups [Pop18, Remarks 4.5].

However, it is far from the case that all countable groups are treeable. The first examples of non-treeable groups are the infinite Kazhdan property (T) groups [AS90] and the non-amenable cost 1 groups [Gab00, Th. 4], and more generally all non-amenable groups with $\beta_1^{(2)} = 0$ [Gab02a, Proposition 6.10]. The random graph formulation of treeability is the existence of an invariant probability measure supported on the set of spanning trees on the group; in [PP00], Pemantle and Peres prove that a non-amenable direct product of infinite groups can have no such probability measure. The non treeability of non-amenable direct products also follows from the theory of cost [Gab00].

The study of non-treeable groups must involve higher dimensional geometric objects. Recall that the **geometric dimension** of a countable group Γ is the smallest dimension of a contractible complex on which Γ acts freely. Analogously, the second author introduced in [Gab02a, Déf. 3.18] the **geometric dimension** of a p.m.p. standard equivalence relation \mathcal{R} on a probability measure space (X, μ) as the smallest dimension of a measurable bundle of contractible simplicial complexes over X on which \mathcal{R} acts **smoothly**, i.e., for which there exists a Borel fundamental domain for the action of \mathcal{R} . The **ergodic dimension of a group** Γ [Gab02a, Déf. 6.4] is the smallest geometric dimension among all of its free p.m.p. actions (see [Gab21] for more on this notion). Being infinite treeable is thus a synonym of having ergodic dimension 1. The ergodic dimension of a group is an ME-invariant, it is bounded above by its virtual geometric dimension is non-increasing when taking subgroups.

We say that Γ has **strong ergodic dimension** d if all its free p.m.p. actions have geometric dimension d . Honesty and humility force us to admit our ignorance: not a single group is known with two free p.m.p. actions having different geometric dimensions.

By Ornstein-Weiss [OW80], all infinite amenable groups have strong ergodic dimension 1. Recall that non vanishing of ℓ^2 -Betti numbers produces lower bounds for the ergodic dimension (see [Gab02a, Prop. 5.8, Cor. 3.17]). It follows for instance that (with $p_i \geq 2$) $\Gamma = \mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_d}$ or $\Gamma = (\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_d}) * \mathbf{F}_k$ have strong ergodic dimension d [Gab02a, pp. 126-127] while $\Gamma \times \mathbf{Z}$ has strong ergodic dimension $d + 1$ and $\text{Out}(\mathbf{F}_n)$ has strong ergodic dimension $2n - 3$ [GN21, Theorem 1.6, Theorem 1.1].

It is not hard to check that the 3-dimensional manifolds with one of the eight geometric structures of Thurston have ergodic dimension at most 2. For instance the fundamental group of a closed hyperbolic 3-dimensional manifold is a cocompact lattice in $\text{SO}(3, 1)$. It is ME with non-compact lattices in $\text{SO}(3, 1)$, which have geometric dimension 2 and zero first ℓ^2 -Betti number, and thus they have ergodic dimension at most 2. We prove a strong dichotomy theorem for the ergodic dimension of fundamental groups of aspherical manifolds of dimension 3.

Theorem 3 (Theorem 6.2). Suppose Γ is the fundamental group of a closed (i.e., compact without boundary) aspherical (possibly non-orientable) manifold of dimension 3. Then either

- (1) Γ is amenable, or
- (2) Γ has strong ergodic dimension 2.

If one removes the assumption of asphericity, the Kneser-Milnor theorem [Kne29, Mil62] decomposes the fundamental group of an orientable closed 3-dimensional manifold as a free product of amenable groups and groups to which Theorem 3 applies. It follows that

Theorem 4. If M is an orientable closed 3-dimensional manifold, then $\Gamma = \pi_1(M)$ has strong ergodic dimension $\in \{0, 1, 2\}$.

Considering the orientation covering, one deduces that the ergodic dimension is at most 2 if M^3 is non-orientable. However, strongness also holds in this case, but this

shall be treated elsewhere [Gab21]. This touches the delicate open question whether strong ergodic dimension is an invariant of commensurability. If one allows M to have boundary components, then we lose a priori the strongness but we still obtain that every p.m.p. free action of $\Gamma = \pi_1(M)$ has geometric dimension at most 2.

It is worth noting that all our results are proved without appealing to the recent progress in Thurston's geometrization theorem. In higher dimensions, we also obtain a non-trivial bound :

Theorem 5 (Theorem 6.1). Suppose $\Gamma = \pi_1(M)$ is the fundamental group of a compact aspherical manifold M (possibly with boundary) of dimension at least 2. Then all free p.m.p. actions of Γ have ergodic dimension at most $\dim(M) - 1$.

The theory of measure equivalence has been extended beyond countable discrete groups to include all unimodular locally compact second countable (**lcsc**) groups (see the nice survey [Fur11] and see [KKR17] for basic invariance properties). The investigation about their treeability began with a result of Hjorth [Hjo08, Theorem 0.5] stating that the products $G_1 \times G_2$ of infinite lcsc groups are non treeable unless both are amenable. He observed that amenable groups are strongly treeable and asks which other lcsc groups satisfy this property [Hjo08, p. 387]. We produce the first progress in this study since then:

Theorem 6 (See Corollary 4.2). $\text{Isom}(\mathbf{H}^2)$, $\text{PSL}_2(\mathbf{R})$, and $\text{SL}_2(\mathbf{R})$ are all strongly treeable, as are all of their closed subgroups.

While the notion of treeability extends in the natural way to orbit equivalence relations of actions $G \curvearrowright (X, \mu)$ of lcsc groups, an equivalent way of conceiving a treeing in this context is by introducing a cross section $B \subseteq X$ (see section A) to which the restriction $\mathcal{R}|_B$ of the orbit equivalence relation $\mathcal{R}_{G \curvearrowright (X, \mu)}$ has countable classes and is treeable.

This result gives as a by-product the first examples of non trivial fixed price for connected lcsc groups (Definition A.9). In contrast, fixed price 1 for the direct product of some lcsc groups with the integers is obtained in [AM21]. Once a Haar measure is prescribed on G , the quantity $\frac{\text{cost}(\mathcal{R}|_B) - 1}{\text{covolume}(B)}$ does not depend on the cross section B , since the restrictions are pairwise stably orbit equivalent (Proposition A.8). A remarkable consequence of Theorem 6 is that this quantity is also independent of the free p.m.p. action $G \curvearrowright (X, \mu)$, for each of these groups:

Theorem 7 (See Corollary 4.2 and Remark 4.3). The groups $G = \text{Isom}(\mathbf{H}^2)$, $\text{PSL}_2(\mathbf{R})$, and $\text{SL}_2(\mathbf{R})$, and their closed subgroups have fixed price

A central fascination in our study is, given a graphing, the hunt for a subgraphing all of whose connected components are acyclic and have exactly one end (hereafter named a **one-ended spanning subforest**, since several connected components are usually necessary for covering a single class). Besides their intrinsic interest, one-ended spanning subforests prove to be extremely useful in our applications, e.g., Theorems 3.6, 5.1, 6.1, and 6.2.

Much of the technical work in the paper consists of finding new techniques for constructing Borel a.e. one-ended spanning subforests of locally finite Borel graphings.

In particular, in the case of locally finite p.m.p. graphs, we give a complete characterization of what graphs admit Borel a.e. one-ended spanning subforests.

Theorem 8 (Theorem 2.1). Suppose that \mathcal{G} is a measure preserving aperiodic locally finite Borel graph on a standard probability space (X, μ) . Then \mathcal{G} has a Borel μ -a.e. one-ended spanning subforest $\mathcal{T} \subseteq \mathcal{G}$ iff \mathcal{G} is μ -nowhere two-ended.

Here, μ -a.e. **one-ended spanning subforest** means that the set of vertices of the one-ended trees of \mathcal{T} has full μ -measure in X ; while μ -**nowhere two-ended** means that the set of vertices of the two-ended connected components of \mathcal{G} has measure zero in X .

The search for subtrees or subforests has attracted enormous attention in another but related mathematical field: the theory of random graphs and percolation (already alluded to in our introduction to non-treeable groups). Thus, for example Pemantle [Pem91] introduced the spanning forest FUSF for \mathbf{Z}^d , obtained as the limiting measure of the uniform spanning tree on large finite pieces of the lattice. He proved that it is connected if and only if $d \leq 4$. The use of various subforests such as the (wired and free) minimal spanning forests (WMSF, FMSF) or the (wired and free) uniform spanning (WUSF, FUSF) forests are of crucial significance in the theory of percolation on graphs [BLPS01, LPS06]. The WMSF is an instance of a random one-ended spanning subforest. The authors of [LPS06] have shown the equivalence of WMSF \neq FMSF with the famous conjecture of Benjamini-Schramm [BS96] whether $p_c \neq p_u$. The mean valency of the FUSF equals two plus twice the first ℓ^2 -Betti number [Lyo09, Corollary 4.12]. If we knew that adding a random graph of arbitrarily small mean valency could make the FUSF forest connected, then it would solve the cost vs first ℓ^2 -Betti number question of [Gab02a, p. 129]. See also [GL09] and [Tim19] for further connections between treeability and percolation.

Theorem 8 relies on Elek and Kaimanovich's characterization of when a locally finite p.m.p. graph \mathcal{G} is μ -**hyperfinite** (i.e., when there exists a μ -conull subset X_0 of X such that the connectedness equivalence relation $\mathcal{R}_{\mathcal{G}}$ of \mathcal{G} is hyperfinite once restricted to X_0). On the contrary, we say \mathcal{G} is μ -**nowhere hyperfinite** if there does not exist a positive measure subset A of X such that $(\mathcal{R}_{\mathcal{G}})_{\upharpoonright A}$ is hyperfinite.

We also use Theorem 8 to give an interesting dual statement to the well-known part (1) of the following theorem that a graph is μ -hyperfinite if and only if it has complete sections of arbitrarily large measure whose induced subgraphs are finite.

Theorem 9 (Theorem 1.3). Let \mathcal{G} be a locally finite p.m.p graph on a standard probability space (X, μ) .

- (1) \mathcal{G} is μ -hyperfinite if and only if for every $\epsilon > 0$, there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_{\mathcal{G}}$ with $\mu(A) > 1 - \epsilon$ so that $\mathcal{G}_{\upharpoonright A}$ has finite connected components.
- (2) \mathcal{G} is μ -nowhere hyperfinite if and only if for every $\epsilon > 0$ there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_{\mathcal{G}}$ with $\mu(A) < \epsilon$ such that $\mathcal{G}_{\upharpoonright A}$ is $\mu_{\upharpoonright A}$ -nowhere hyperfinite.

By significantly relaxing the measure preserving hypothesis, we arrived at the study of Borel graphings and their behavior with respect to various Borel probability

measures that are not necessarily measure preserving. Although the measure μ is not a priori related to the Borel equivalence relation E under consideration, some properties may hold up to discarding a set of μ -measure 0. In this non p.m.p. context, we obtain one-ended spanning subforests out of Borel planar graphs (see Definition 3.5).

Theorem 10 (See Corollary 3.7). Let \mathcal{G} be a locally finite Borel graph on X whose connected components are planar. Let μ be a Borel probability measure on X . If \mathcal{G} is μ -nowhere two-ended, then \mathcal{G} has a Borel a.e. one-ended spanning subforest.

Without the “ μ -nowhere two-ended” assumption, one still gets the existence of a spanning subforest on a μ -conull set (see Theorem 3.6).

We say that a Borel equivalence relation E on X is **measure treeable** if for each Borel probability measure μ on X , there a μ -conull subset X_0 of X such that the restriction of E to X_0 is Borel treeable (Definition A.1). Observe that when the classes of E are countable then each such μ is dominated by a quasi-invariant measure $\mu' = \sum_{i=1}^{\infty} 2^{-i} g_{i*} \mu$, where $\{g_i\}_{i=1}^{\infty}$ is an enumeration of some countable group G that generates E (see [FM77]). Note that μ and μ' have the same E -invariant null sets. See also Proposition A.4.

A lscg group G is called **measure strongly treeable (MST)** if all orbit equivalence relations generated by all free Borel actions of G are measure treeable (Definition B.1).

In this context, using Theorem 10, our Theorems 1, 2 and 6 take indeed a much stronger non p.m.p. form (see Corollary 4.2, Corollary 4.4, and Theorem 5.1):

Theorem 11. The following groups are measure strongly treeable

- (1) $\text{Isom}(\mathbf{H}^2)$, $\text{PSL}_2(\mathbf{R})$, and $\text{SL}_2(\mathbf{R})$, and all of their closed subgroups.
- (2) Finitely generated groups admitting a planar Cayley graph.
- (3) Finitely generated elementarily free groups.

Our proof follows an idea from [BLPS01] (also used in [Gab05]) of finding an a.e. one-ended spanning subforest in the planar dual (see §3). Suppose \mathcal{G} is a locally finite graph admitting an accumulation-point free planar embedding into \mathbf{R}^2 . Then it is easy to see that subtrees of \mathcal{G} are in one-to-one correspondence with one-ended subforests in the planar dual of \mathcal{G} . We use this correspondence to show the measure treeability of the above groups G by converting the problem of treeing an action of G into finding an a.e. one-ended spanning subforest of the planar dual of graphings of the action which are Borel planar.

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1. ELEK’S REFINEMENT OF KAIMANOVICH’S THEOREM FOR MEASURED GRAPHS

If \mathcal{G} is a locally finite Borel graph on a standard probability space (X, μ) , then the **(vertex) isoperimetric constant of \mathcal{G}** is the infimum of $\mu(\partial_{\mathcal{G}}A)/\mu(A)$ over all Borel subsets $A \subseteq X$ of positive measure such that $\mathcal{G}_{\upharpoonright A}$ has finite connected components. Here, $\partial_{\mathcal{G}}A$ denotes the set of vertices in $X \setminus A$ which are \mathcal{G} -adjacent to a vertex in A . If the measure μ is $\mathcal{R}_{\mathcal{G}}$ -quasi-invariant then the isoperimetric constant of \mathcal{G} can be equivalently phrased in terms of the associated Radon-Nikodym cocycle (see [KM04, §8]).

In [Kai97], Kaimanovich established the equivalence between μ -hyperfiniteness of a measured equivalence relation \mathcal{R} and vanishing of the isoperimetric constant of all bounded graph structures on \mathcal{R} . In [Ele12], Elek sharpened Kaimanovich’s Theorem by establishing the following characterization of hyperfiniteness for a fixed measured graph \mathcal{G} .

Theorem 1.1 (Elek [Ele12]). *Let \mathcal{G} be a locally finite Borel graph on a standard probability space (X, μ) . Then \mathcal{G} is μ -hyperfinite if and only if for every positive measure Borel subset $X_0 \subseteq X$, the isoperimetric constant of $\mathcal{G}_{\upharpoonright X_0}$ is 0.*

While the theorem in [Ele12] is stated for measure preserving bounded degree graphs, it can easily be extended to all locally finite graphs which are not necessarily measure preserving. For the convenience of the reader we indicate the proof.

Proof of Theorem 1.1. Suppose first that \mathcal{G} is μ -hyperfinite. Let $X_0 \subseteq X$ be a Borel set of positive measure and let $\mathcal{H} = \mathcal{G}_{\upharpoonright X_0}$. Then \mathcal{H} is μ -hyperfinite, so after ignoring a null set we can find finite Borel subequivalence relations $\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \dots$ with $\mathcal{R}_{\mathcal{H}} = \bigcup_n \mathcal{R}_n$. Since \mathcal{H} is locally finite, given $\epsilon > 0$, we may find n large enough so that $\mu(A_n) > \mu(X_0)(1 - \epsilon)$, where $A_n = \{x \in X_0 : \mathcal{H}_x \subseteq [x]_{\mathcal{R}_n}\}$ and \mathcal{H}_x denotes the set of \mathcal{H} -neighbors of x . Then $\mathcal{H}_{\upharpoonright A_n} \subseteq \mathcal{R}_n$, so $\mathcal{H}_{\upharpoonright A_n}$ has finite connected components. In addition, $\mu(\partial_{\mathcal{H}}A_n)/\mu(A_n) < \epsilon/(1 - \epsilon)$, so as $\epsilon > 0$ was arbitrary this shows the isoperimetric constant of \mathcal{H} is 0.

Assume now that for every positive measure Borel subset $X_0 \subseteq X$ the isoperimetric constant of $\mathcal{G}_{\upharpoonright X_0}$ is 0. To show that \mathcal{G} is μ -hyperfinite it suffices to show that for any $\epsilon > 0$ there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1 - \epsilon$ such that $\mathcal{G}_{\upharpoonright Y}$ has finite connected components (since then we can find a sequence of such sets Y_n , $n \in \mathbf{N}$, with $\mu(Y_n) \geq 1 - 2^{-n}$, so by Borel-Cantelli $\mathcal{R}_{\mathcal{G}} = \liminf_n \mathcal{R}_{\mathcal{G}_{\upharpoonright Y_n}}$ is μ -hyperfinite). Given $\epsilon > 0$, by Zorn’s Lemma we can find a maximal collection \mathcal{A} of pairwise disjoint nonnull Borel subsets of X subject to

- (i) $\mathcal{G}_{\upharpoonright \bigcup \mathcal{A}}$ has finite connected components;
- (ii) $\mu(\partial_{\mathcal{G}}(\bigcup \mathcal{A})) \leq \epsilon \mu(\bigcup \mathcal{A})$;

(iii) If $A, B \in \mathcal{A}$ are distinct, then no vertex in A is adjacent to a vertex in B .

Let $Y = \bigcup \mathcal{A}$. We now claim that the set $X_0 = X \setminus (Y \cup \partial_{\mathcal{G}} Y)$ is null. Otherwise, by hypothesis we may find a Borel set $A_0 \subseteq X_0$ of positive measure such that $\mathcal{G}_{\upharpoonright A_0}$ has finite connected components and $\mu(\partial_{\mathcal{G}_{\upharpoonright X_0}} A_0) < \epsilon \mu(A_0)$. But then the collection $\mathcal{A}_0 = \mathcal{A} \cup \{A_0\}$ satisfies (i)-(iii) in place of \mathcal{A} , contradicting maximality of \mathcal{A} . Thus, $\mu(Y) = 1 - \mu(\partial_{\mathcal{G}} Y) \geq 1 - \epsilon \mu(Y) \geq 1 - \epsilon$, and $\mathcal{G}_{\upharpoonright Y}$ has finite connected components, which finishes the proof. \square

Remark 1.2. We have used the vertex isoperimetric constant, whereas [Ele12] uses the edge isoperimetric constant. The relationship is as follows. Let \mathcal{G} be a graph on (X, μ) and let M_r be the Borel σ -finite measures on \mathcal{G} given by $M_r(D) = \int_X |D^x| d\mu$. The **edge isoperimetric constant** of \mathcal{G} is the infimum of $M_r(\partial_{\mathcal{G}}^e A)/\mu(A)$ over all Borel subsets $A \subseteq X$ of positive measure such that $\mathcal{G}_{\upharpoonright A}$ has finite connected components. Here, $\partial_{\mathcal{G}}^e A$ is the set of all edges of \mathcal{G} having one endpoint in A and one in $X \setminus A$. (Note that since $\partial_{\mathcal{G}}^e A$ is symmetric, one obtains the same definition if in place of M_r one uses the measure $M_l(D) = \int_X |D_x| d\mu$.) It is then easy to see that if μ is \mathcal{G} -quasi-invariant, and if \mathcal{G} is bounded (meaning that \mathcal{G} is bounded degree and the Radon-Nikodym cocycle $\rho : \mathcal{R}_{\mathcal{G}} \rightarrow \mathbf{R}^+$ associated to μ is essentially bounded on \mathcal{G}) then for any positive measure Borel subset $X_0 \subseteq X$, the edge isoperimetric constant of $\mathcal{G}_{\upharpoonright X_0}$ vanishes if and only if the vertex isoperimetric constant of $\mathcal{G}_{\upharpoonright X_0}$ vanishes.

The combinatorial core of the proof of the forward direction of Theorem 1.1 is the fact that a graph \mathcal{G} is μ -hyperfinite if and only if there are arbitrarily large sets on which its restriction is finite. There is a dual statement for μ -nowhere hyperfiniteness.

Theorem 1.3. *Let \mathcal{G} be a p.m.p. locally finite Borel graph on a standard probability space (X, μ) .*

- (1) \mathcal{G} is μ -hyperfinite if and only if for every $\epsilon > 0$, there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_{\mathcal{G}}$ with $\mu(A) > 1 - \epsilon$ so that $\mathcal{G}_{\upharpoonright A}$ has finite connected components.
- (2) \mathcal{G} is μ -nowhere hyperfinite if and only if for every $\epsilon > 0$ there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_{\mathcal{G}}$ with $\mu(A) < \epsilon$ such that $\mathcal{G}_{\upharpoonright A}$ is $\mu_{\upharpoonright A}$ -nowhere hyperfinite.

Proof. The new content of the theorem is the forward direction of (2). (In our proof of Theorem 1.1 we indicated how to prove part (1)). A key point in our proof is the use of Corollary 2.11 below on the existence of maximal hyperfinite one-ended spanning subforests, which we establish later.

By Corollary 2.11 (whose assumption is satisfied when \mathcal{G} is μ -nowhere hyperfinite) we may find a Borel μ -a.e. one-ended spanning subforest $\mathcal{T} \subseteq \mathcal{G}$ such that $\mathcal{R}_{\mathcal{T}}$ is μ -maximal among the \mathcal{G} -connected, μ -hyperfinite equivalence subrelations of $\mathcal{R}_{\mathcal{G}}$.

Let $A_1 = X$ and for $n \geq 1$ let $A_{n+1} = \{x \in A_n : \deg_{\mathcal{T}_{\upharpoonright A_n}}(x) \geq 2\}$. Then $A_1 \supseteq A_2 \supseteq \dots$ and $\mu(\bigcap_n A_n) = 0$ since \mathcal{T} is one-ended, so after ignoring a null set we may assume that $\bigcap_n A_n = \emptyset$. Observe that $\mathcal{R}_{\mathcal{T}_{\upharpoonright A_n}} = (\mathcal{R}_{\mathcal{T}})_{\upharpoonright A_n}$ for all n . Given $\epsilon > 0$, let n be so large that $\mu(A_n) < \epsilon/2$. Since \mathcal{G} is μ -nowhere hyperfinite and

\mathcal{T} is μ -hyperfinite, the set $\mathcal{G} \setminus \mathcal{R}_{\mathcal{T}}$ meets almost every connected component of \mathcal{T} . We may therefore find a subset $\mathcal{G}_0 \subseteq \mathcal{G} \setminus \mathcal{R}_{\mathcal{T}}$ which is incident with almost every connected component of \mathcal{T} such that the set B , of vertices incident with \mathcal{G}_0 , has measure $\mu(B) < \epsilon/2n$. (Finding \mathcal{G}_0 is easy when \mathcal{T} is ergodic; in general we can simply use the ergodic decomposition of \mathcal{T} .)

Claim. $\mathcal{T} \cup (\mathcal{G}_{\uparrow B})$ is μ -nowhere hyperfinite.

Proof of the claim. Suppose otherwise. Then we may find a non-null $\mathcal{R}_{\mathcal{T} \cup (\mathcal{G}_{\uparrow B})}$ -invariant set D such that $(\mathcal{R}_{\mathcal{T} \cup (\mathcal{G}_{\uparrow B})})_{\uparrow D}$ is hyperfinite. Then the equivalence relation $\mathcal{Q} := (\mathcal{R}_{\mathcal{T} \cup (\mathcal{G}_{\uparrow B})})_{\uparrow D} \sqcup (\mathcal{R}_{\mathcal{T}})_{\uparrow (X \setminus D)}$ is \mathcal{G} -connected and μ -hyperfinite. By our choice of B , each component of $\mathcal{T} \cup (\mathcal{G}_{\uparrow B})$ contains more than one component of \mathcal{T} , so \mathcal{Q} properly contains $\mathcal{R}_{\mathcal{T}}$ since D is non-null. This contradicts the maximality of $\mathcal{R}_{\mathcal{T}}$. \square

For each $x \in B$ let $\pi(x) \in A_n$ denote the unique vertex in A_n which is closest to x with respect to the graph metric in \mathcal{T} , and let p_x denote the unique shortest path through \mathcal{T} from x to $\pi(x)$. The length of each p_x is at most $n-1$, so if we let C denote the set of all vertices which lie along p_x for some $x \in B$, then $\mu(C) \leq n\mu(B) < \epsilon/2$. Let $A = A_n \cup C$. Then $\mu(A) < \epsilon$ and

$$\mathcal{R}_{\mathcal{G}_{\uparrow A}} \supseteq \mathcal{R}_{\mathcal{T}_{\uparrow A}} \vee \mathcal{R}_{\mathcal{G}_{\uparrow B}} = (\mathcal{R}_{\mathcal{T}})_{\uparrow A} \vee \mathcal{R}_{\mathcal{G}_{\uparrow B}} \supseteq (\mathcal{R}_{\mathcal{T} \cup (\mathcal{G}_{\uparrow B})})_{\uparrow A},$$

so that $\mathcal{G}_{\uparrow A}$ is $\mu_{\uparrow A}$ -nowhere hyperfinite since $\mathcal{T} \cup (\mathcal{G}_{\uparrow B})$ is μ -nowhere hyperfinite. \square

2. ONE-ENDED SPANNING SUBFORESTS

In this section, we characterize exactly when a locally finite probability measure preserving Borel graph has a one-ended spanning subforest.

Theorem 2.1 (For p.m.p. graphings). *Suppose that \mathcal{G} is a measure preserving aperiodic locally finite Borel graph on a standard probability space (X, μ) . Then \mathcal{G} has a Borel a.e. one-ended spanning subforest iff \mathcal{G} is μ -nowhere two-ended.*

We further conjecture the following strengthening of this theorem for graphs which are not necessarily measure preserving. In this more general setting, the correct generalization of (μ -a.e.) aperiodicity is μ -nowhere smoothness of \mathcal{G} ; we say that \mathcal{G} is μ -**nowhere smooth** if there is no positive measure Borel subset of X which meets each \mathcal{G} -component in at most one point.

Conjecture 2.2 (Non necessarily p.m.p. graphings). *Suppose that \mathcal{G} is a μ -nowhere smooth locally finite Borel graph on a standard probability space (X, μ) . Then \mathcal{G} has a Borel a.e. one-ended spanning subforest iff \mathcal{G} is μ -nowhere two-ended.*

We know that the forward direction of the above conjecture is true by Lemma 2.4 below. The reverse direction is known to be true in the case when \mathcal{G} is hyperfinite by Lemma 2.10, and when \mathcal{G} is acyclic by the following theorem of [CMTD16].

Theorem 2.3 (Non necessarily p.m.p. treeings [CMTD16, Theorem 1.5]). *Suppose that \mathcal{G} is an acyclic, aperiodic locally finite Borel graph on a standard probability space (X, μ) . If \mathcal{G} is μ -nowhere two-ended, then \mathcal{G} has a Borel a.e. one-ended spanning subforest.*

We will begin by proving the forward direction of Theorem 2.1 (and also Conjecture 2.2). An easy argument shows that the graph associated to a free measure preserving action of \mathbf{Z} cannot have a Borel a.e. one-ended spanning subforest; such a subforest must come from removing a single edge from each connected component of the graph. This set of edges would witness the fact that the graph \mathcal{G} is smooth, contradicting our assumption that the action of \mathbf{Z} is measure-preserving. Our argument is a simple generalization of this idea.

Lemma 2.4. *Suppose that \mathcal{G} is a μ -nowhere smooth locally finite Borel graph on a standard probability space (X, μ) . If there is a set of positive measure on which \mathcal{G} is two-ended, then \mathcal{G} does not admit a Borel a.e. one-ended spanning subforest.*

Proof. By restricting to and renormalizing a Borel \mathcal{G} -invariant subset of positive measure, we may assume that \mathcal{G} is everywhere two-ended and has a Borel a.e. one-ended spanning subforest \mathcal{T} . We will now show \mathcal{G} is smooth. Let Y be the set of connected $C \in [\mathcal{R}_{\mathcal{G}}]^{<\mathbf{N}}$ such that removing C from \mathcal{G} disconnects its connected component into exactly two infinite pieces. Recall that $[\mathcal{R}_{\mathcal{G}}]^{<\mathbf{N}}$ is the Borel set of finite subsets of X made of $\mathcal{R}_{\mathcal{G}}$ -equivalent points. By taking a countable coloring of the intersection graph on Y (see [KM04, Lemma 7.3] and [CM16, Proposition 2]), we may find a Borel set $Z \subseteq Y$ which meets every connected component of \mathcal{G} and so that distinct $C, D \in Z$ are pairwise disjoint and if C and D are in the same connected component, then $|C| = |D|$. By discarding a smooth set, we may assume Z meets each connected component of \mathcal{G} infinitely many times. Let \mathcal{H} be the graph on Z where $C \mathcal{H} C'$ if C and C' are in the same connected component of \mathcal{G} and there is no $D \in Z$ such that removing D from \mathcal{G} places C and C' in different connected components. Note that \mathcal{H} is 2-regular.

Now let $Z' \subseteq Z$ be the set of $C \in Z$ such that there exists a \mathcal{H} -neighbor D of C and a component F of \mathcal{T} , such that C meets F , but D does not meet F .

It is easy to see that Z' meets each connected component of \mathcal{G} and is finite (else \mathcal{T} is not a one-ended spanning subforest), but then \mathcal{G} is smooth. \square

Our proof of the reverse direction of Theorem 2.1 splits into two cases based on Theorem 1.1. In particular, it will suffice to prove Theorem 2.1 for μ -hyperfinite graphs, and graphs having positive isoperimetric constant.

2.1. Measure preserving graphs with superquadratic growth. We begin with a lemma giving a sufficient condition for a graph to possess a one-ended spanning subforest. (In fact, this condition can be shown to be equivalent to the existence of such a subforest)

Let f be a partial function from a set X into itself, and let $y \in X$. The **back-orbit** of y under f is the set of all $x \in \text{dom}(f)$ for which there is some $n \geq 0$ with $f^n(x) = y$.

Lemma 2.5. *Suppose that \mathcal{G} is a locally finite Borel graph on a standard probability space (X, μ) , and there are partial Borel functions $f_0, f_1, \dots \subseteq \mathcal{G}$ such that*

- (1) $\sum_i \mu(\text{dom}(f_i)) < \infty$
- (2) $\bigcup \text{dom}(f_i) = X$