



HAL
open science

Guaranteed Simulation of Dynamical Systems with Integral Constraints and Application on Delayed Dynamical Systems

Paul Rouse, Julien Alexandre Dit Sandretto, Alexandre Chapoutot,
Pierre-Loïc Garoche

► To cite this version:

Paul Rouse, Julien Alexandre Dit Sandretto, Alexandre Chapoutot, Pierre-Loïc Garoche. Guaranteed Simulation of Dynamical Systems with Integral Constraints and Application on Delayed Dynamical Systems. 9th International Workshop on Model-Based Design of Cyber Physical Systems, CyPhy 2019 and 15th International Workshop on Embedded and Cyber-Physical Systems Education, WESE 2019, Oct 2019, New York, United States. pp.89-107, 10.1007/978-3-030-41131-2_5. hal-03199339

HAL Id: hal-03199339

<https://hal.science/hal-03199339>

Submitted on 9 Feb 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Guaranteed Simulation of Dynamical Systems with Integral Constraints and Application on Delayed Dynamical Systems

Paul Rousse^{1(✉)}, Julien Alexandre dit Sandretto^{2(✉)},
Alexandre Chapoutot^{2(✉)}, and Pierre-Loïc Garoche^{1(✉)}

¹ ONERA, 31400 Toulouse, France

{paul.rousse,pierre-loic.garoche}@onera.fr

² U2IS, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France
{julien.alexandre-dit-sandretto,alexandre.chapoutot}@ensta-paristech.fr

Abstract. A reachable set computation method for dynamical systems with an integral constraint over the input set is proposed. These models are typical in robustness analysis when studying the impact of bounded energy noises over a system response and can also model a large family of complex systems. The reachable set is over-approximated using a guaranteed set-based integration method within the interval arithmetic framework.

A Runge-Kutta guaranteed integration scheme with pessimistic bounds over the input provides a first conservative bound over the reachable tube. Then, the integral constraint is used to define a contractor over the reachable tube. This contractor and a propagation step are successively applied on the over-approximation until a fixed point is reached. We evaluated our algorithm with DynIbex library to simulate a delayed system, *i.e.*, an infinite dimensional system that can be modeled as a linear time-invariant system subject to an integral quadratic constraint. Our approach is shown to be tractable and enables the use of interval arithmetic and guaranteed integration for a richer set of dynamical systems.

Keywords: Numerical integration · Dynamical systems with integral constraint · Interval arithmetic

1 Introduction

In this paper, we present a method to compute the flowpipe of a dynamical system with an integral inequality constraint between an unknown input disturbance and the state trajectory. The interval arithmetic and guaranteed simulation frameworks are used. With additional assumptions about the dynamic of the disturbance, the integral constraint gives bounds over the set of disturbances.

A contractor over the set of reachable states is defined out of these bounds. This contractor is then used in a fixed point algorithm with a propagation step (as described in [1]). Our algorithm is implemented using DynIbex library [2] and applied to overapproximate the flowpipe of a dynamical system with an inner delay.

In dynamical system's analysis, two signal norms are frequently used: the ∞ -norm (that corresponds to the maximum vector norm over the time domain) and the 2-norm (that corresponds to the signal's energy). A signal with a 2-norm bound can be equivalently defined with an integral constraint. Disturbances with ∞ -norm bounds are naturally handled by guaranteed integration frameworks. Disturbances with 2-norm have been less studied by the community despite their modeling power. In control theory, many relationships between signals and systems are expressed in terms of 2-norm gains. In Hybrid systems analysis, 2-norm input-output gains have been derived (as in [21]) and can be used to compute overapproximation of the reachable set. [7] proposes the use of model reduction methods to verify large systems. No error bound is used during the verification of the approximated system. In fact, such bounds exist and can be expressed as a 2-norm gain relationship with the input signal. Many complex systems can be, as well, described by a linear time-invariant dynamical system disturbed by a 2-norm bounded signal [17].

Related Works. In the first paragraph, we motivate our choice of model with a challenging application: differential equation with inner delays. The next paragraphs review works in reachability analysis for dynamical systems with integral constraints.

Simulation of differential equations with inner delays is a notoriously complex problem [26, 28]. In [8], the author propose to compute an inner and outer approximation of a delayed system's flowpipe. The solution to the differential delay equation is obtained by integrating ordinary differential equations (ODE) over small steps. The solutions over these time intervals are recursively used until the final time of integration is reached. The infinite dimensional state of the delay (i.e. the memory of the delay) is sampled in time. Taylor series and a classical integration method are used to solve the ODE. In [6], the simulation trace is obtained with a similar approach. Along the simulation trace, a bound over the numerical integration error is derived by solving an optimization problem. In [30], set-boundary based reachability analysis method initially developed for ODE is extended to delay differential equations. A sensitivity analysis is used get an inner and outer approximation of the reachable set. In [6, 8, 26, 28, 30], an outer approximation of past states is used to solve the delay differential equation, local properties (Taylor remainder, local contraction of the flowpipe and sensitivity analysis) are used to get guaranteed bound over the reachable state. In [25], the stability of linear systems with constant delays is studied. The state of the delay operator is expressed as a weighted sum of polynomials functions and a remaining noise signal. It can be shown that these weights are solution of a linear time-invariant system subject to a disturbance. The disturbance satisfies an energetic constraint. Delays modeled as an integral quadratic constraint

have also been used for reachability analysis of delay systems in [20]. The reachable tube is overapproximated using a time-varying ellipsoidal set with time-dependent polynomial radius and center. The overapproximating relationship can be expressed as the positivity of a polynomial over the state and time space. An SDP solver is used to find a solution. The SDP solver provide a certificate of positivity for all positive polynomial and thus the overapproximation relationship can be guaranteed. In this work, the system with delay is as well modeled by an integral quadratic constraint to build a contractor. An initial reachable tube can be roughly overapproximated using guaranteed integration tools. Then, the contractor is used to reduce the pessimism of this reachable tube. To apply this contraction, we use forward propagation of the reachable set as in [22].

Computing the reachable set of dynamical systems with integral constraints can be expressed as an optimal control problem as in [10,16]. A state belongs to the reachable set if the maximum integral value satisfies the positivity constraint along its trajectory. Standard tools from optimal control can then be used. This optimization problem can be locally solved (see, *e.g.*, with the Pontryagin Maximum Principle -PMP-, see [9,10,16,29]) leading to a local description of the reachable set boundary. It also can be solved globally (using Hamilton-Jacobi-Bellman -HJB- viscosity subsolutions, see [27]) leading to global constraints over the reachable set. These methods rely on numerical integration of (partial) differential equations and are often subject to numerical instabilities.

HJB and PMP based methods propagate the constraints along the flow of the dynamical system. Occupation measures and barrier certificates methods aim at finding constraints over the reachable tube of a dynamical system: [21] uses integral constraints for verification purposes using barrier certificates where the positivity of the integral is ensured by using a nonnegative constant multiplier: [11,14] use an occupation measure approach where the integral constraint can be incorporated as a constraint over the moment of the trajectories. A hierarchy of semi-definite conditions is derived for polynomial dynamics. Then, off-the-shelf semi-definite programming solvers are used to solve the feasibility problem. Optimization-based methods do not usually take advantage of the model structure as they consider a large class of systems (convex, Lipschitz or polynomial dynamics for example).

For linear system subject to Integral Quadratic Constraints (IQC), the reachability problem can be expressed as the classical Linear Quadratic Regulator problem [24]. Optimal trajectories belong to a time-varying parabolic surface, whose quadratic coefficients are the solution to a Riccati differential equation. [10,23] describes the reachable set of LTI systems with terminal IQC. [13] formalizes the problem with a game theory approach. Recent works showed that the ellipsoidal method developed in [15] can be extended to a so-called Paraboloid method [22] to get the exact characterization of the reachable set of such systems.

Contributions:

- we developed a framework to analyze systems with integral constraints between an unknown disturbance and the state. We make an additional

assumption about the disturbance dynamic. This assumption asserts that the variation of the disturbance is bounded. We then define a contractor over the set of trajectories. This contractor is used in a fixed point algorithm.

- we use models from robust control theory into guaranteed numerical integration.

Plan: In Sect. 2, we define the system of interest. Guaranteed numerical integration for unconstrained systems is presented in Sect. 3. The main contribution of this work is presented in Sect. 4. Since the integral constraint cannot be directly handled by guaranteed integration software such as DynIbex, we make further assumptions about the disturbance dynamic. These hypotheses are then used to define a narrowing operator out of the integral constraint. In Sect. 5, our approach is used to compute the reachable set of a dynamical system with inner delays. We compare this method to a set-based method.

Notations. \mathbb{IR} is the set of intervals over \mathbb{R} , interval vectors are noted in bold letters. Let the norm of $[\mathbf{x}] \in \mathbb{IR}^n$ be $\|[\mathbf{x}]\| = \max_{x \in [\mathbf{x}]} \|x\|$. For an interval $[\mathbf{x}] \in \mathbb{IR}$, let $\overline{[\mathbf{x}]} = \sup_{x \in [\mathbf{x}]} x$. For $n \in \mathbb{N}$ and an interval I of \mathbb{R} , $\mathbb{L}_{loc}^2(\mathbb{R}^+; \mathbb{R}^n)$ is the set of locally square integrable functions from I to \mathbb{R}^n .

2 System with Integral Constraint over the State

Let the following system:

$$\begin{cases} \dot{x} = f(t, x, w) \\ x(0) \in \mathbf{x}_0 \end{cases} \quad (1)$$

where w is an unknown disturbance in $\mathbb{L}_{loc}^2(\mathbb{R}^+; \mathbb{R}^m)$ that satisfies the integral constraint, for any $\tau \geq 0$:

$$\int_0^\tau \|w(s)\|^2 ds \leq \int_0^\tau g(s, x(s)) ds \quad (2)$$

where $g : \mathbb{R}^+ \times \mathbb{R}^n$ is a given function.

Many systems can be modeled in such way. The robust control community makes frequent use of this model where the integral constraint overapproximates the behavior of complex systems, *e.g.*, saturations, delays and bounded nonlinearities to cite few of them.

Remark 1. The integral constraint does not give any bounds on the disturbance as it can be easily understood from the unit energy disturbed system

$$\begin{cases} \dot{x} = -x + w \\ x(0) = 0 \\ 1 \geq \int_0^1 w^2(\tau) d\tau \end{cases} \quad (3)$$

Let w be defined for any $\epsilon > 0$ by

$$\begin{cases} w(\tau) = \frac{1}{\epsilon} & \text{when } \tau \in [0, \epsilon] \\ w(\tau) = 0 & \text{otherwise.} \end{cases}$$

Since $\int_0^1 w^2(\tau) d\tau = 1$, the inequality in Eq. (3) is verified for every $\epsilon > 0$, however no bounds can be determined for w since $w(0) \rightarrow \infty$ when $\epsilon \rightarrow 0$. Please note that the system defined in Eq. (3) has a bounded reachable set even if the disturbance cannot be bounded at any given time (see [4, Chap. 8.1.2]).

3 Interval Analysis and Guaranteed Numerical Integration

A presentation of the main mathematical tools is given in this section. First, the basics of *interval analysis* is provided in Sect. 3.1. Then, a short introduction of *validated numerical integration* is presented in Sect. 3.2.

3.1 Interval Analysis

The simplest and most common way to represent and manipulate sets of values is *interval arithmetic* (see [18]). An interval $[x_i] = [\underline{x}_i, \overline{x}_i]$ defines the set of reals x_i such that $\underline{x}_i \leq x_i \leq \overline{x}_i$. \mathbb{IR} denotes the set of all intervals over reals. The size (or width) of $[x_i]$ is denoted by $w([x_i]) = \overline{x}_i - \underline{x}_i$.

Interval arithmetic extends to \mathbb{IR} elementary functions over \mathbb{R} . For instance, the interval sum, *i.e.*, $[x_1] + [x_2] = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2]$, encloses the image of the sum function over its arguments.

An interval vector or a *box* $[\mathbf{x}] \in \mathbb{IR}^n$, is a Cartesian product of n intervals. The enclosing property basically defines what is called an *interval extension* or an *inclusion function*.

Definition 1 (Inclusion function). Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $[f] : \mathbb{IR}^n \rightarrow \mathbb{IR}^m$ is said to be an inclusion function of f to intervals if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \quad [f]([\mathbf{x}]) \supseteq \{f(\mathbf{x}), \mathbf{x} \in [\mathbf{x}]\} .$$

It is possible to define inclusion functions for all elementary functions such as \times , \div , \sin , \cos , \exp , etc. The *natural* inclusion function is the simplest to obtain: all occurrences of the real variables are replaced by their interval counterpart and all arithmetic operations are evaluated using interval arithmetic. More sophisticated inclusion functions such as the centered form, or the Taylor inclusion function may also be used (see [12] for more details).

Example 1 (Interval arithmetic). A few examples of arithmetic operations between interval values are given

$$\begin{aligned}
 [-2, 5] + [-8, 12] &= [-10, 17] \\
 [-10, 17] - [-8, 12] &= [-10, 17] + [-12, 8] = [-22, 25] \\
 [-10, 17] - [-2, 5] &= [-15, 19] \\
 \frac{[-2, 5]}{[-8, 12]} &= [-\infty, \infty] \\
 \frac{[3, 5]}{[8, 12]} &= \left[\frac{3}{12}, \frac{5}{8} \right] \\
 \left[\frac{3}{12}, \frac{5}{8} \right] \times [8, 12] &= \left[2, \frac{15}{2} \right]
 \end{aligned}$$

In the first example of division, the result is the interval containing all the real numbers because denominator contains 0.

As an example of inclusion function, we consider a function p defined by

$$p(x, y) = xy + x .$$

The associated natural inclusion function is

$$[p]([x], [y]) = [x][y] + [x],$$

in which variables, constants and arithmetic operations have been replaced by its interval counterpart. And so $p([0, 1], [0, 1]) = [0, 2] \subseteq \{p(x, y) \mid x, y \in [0, 1]\} = [0, 2]$. ■

In the constraint programming community, complex equality and inequality constraints can be handled using so-called *contractors*. A contractor is an operator that associates to a set one of its subset that contains all the points where the constraint is verified (see [5]).

Definition 2. For a constraint f that maps \mathbb{R}^n to a truth value, a contractor Ctc of f associates to a subset of \mathbb{R}^n to a subset of \mathbb{R}^n . For any $[\mathbf{b}], [\mathbf{b}'] \in \mathbb{IR}^n$, Ctc must verifies the following properties:

- the contractance: $Ctc([\mathbf{b}]) \subseteq [\mathbf{b}]$,
- the conservativeness: $\forall x \in [\mathbf{b}] \setminus Ctc([\mathbf{b}]), f(x)$ is not satisfied,
- the monotonicity: $[\mathbf{b}'] \subseteq [\mathbf{b}] \Rightarrow Ctc([\mathbf{b}']) \subseteq Ctc([\mathbf{b}])$

3.2 Validated Numerical Integration Methods

Mathematically, differential equations have no explicit solutions, except for few particular cases. Nevertheless, the solution can be numerically approximated with the help of integration schemes such as Taylor series [19] or Runge-Kutta methods [2, 3].

In the following, we consider a *generic* parametric differential equation as an *interval initial value problem* (IIVP) defined by

$$\begin{cases} \dot{\mathbf{y}} = F(t, \mathbf{y}, \mathbf{x}, \mathbf{p}, \mathbf{u}) \\ 0 = G(t, \mathbf{y}, \mathbf{x}, \mathbf{p}, \mathbf{u}) \\ \mathbf{y}(0) \in \mathcal{Y}_0, \mathbf{x}(0) \in \mathcal{X}_0, \mathbf{p} \in \mathcal{P}, \mathbf{u} \in \mathcal{U}, t \in [0, t_{\text{end}}] \end{cases}, \quad (4)$$

with $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^s \mapsto \mathbb{R}^n$ and $G : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^s \mapsto \mathbb{R}^m$. The variable \mathbf{y} of dimension n is the differential variable while the variable \mathbf{x} is an algebraic variable of dimension m with an initial condition $\mathbf{y}(0) \in \mathcal{Y}_0 \subseteq \mathbb{R}^n$ and $\mathbf{x}(0) \in \mathcal{X}_0 \subseteq \mathbb{R}^m$. In other words, differential-algebraic equations (DAE) of index 1 are considered, and in the case of $m = 0$, this differential equation simplifies to an ordinary differential equation (ODE). Note that usually, the initial values of algebraic variable \mathbf{x} are computed by numerical algorithms used to solve DAE but we consider it fixed here for simplicity. Variable $\mathbf{p} \in \mathcal{P} \subseteq \mathbb{R}^r$ stands for parameters of dimension r and variable $\mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^s$ stands for a control vector of dimension s . We assume standard hypotheses on F and G to guarantee the existence and uniqueness of the solution to such problem.

A validated simulation of a differential equation consists in a discretization of time, such that $t_0 \leq \dots \leq t_{\text{end}}$, and a computation of enclosures of the set of states of the system $\mathbf{y}_0, \dots, \mathbf{y}_{\text{end}}$, by the help of a guaranteed integration scheme. In details, a guaranteed integration scheme is made of

- an integration method $\Phi(F, G, \mathbf{y}_j, t_j, h)$, starting from an initial value \mathbf{y}_j at time t_j and a finite time horizon h (the step-size), producing an approximation \mathbf{y}_{j+1} at time $t_{j+1} = t_j + h$, of the exact solution $\mathbf{y}(t_{j+1}; \mathbf{y}_j)$, *i.e.*, $\mathbf{y}(t_{j+1}; \mathbf{y}_j) \approx \Phi(F, G, \mathbf{y}_j, t_j, h)$;
- a truncation error function $\text{lte}_\Phi(F, G, \mathbf{y}_j, t_j, h)$, such that

$$\mathbf{y}(t_{j+1}; \mathbf{y}_j) = \Phi(F, G, \mathbf{y}_j, t_j, h) + \text{lte}_\Phi(F, G, \mathbf{y}_j, t_j, h).$$

Basically, a validated numerical integration method is based on a numerical integration scheme such as Taylor series [19] or Runge-Kutta methods [2, 3] which is extended with interval analysis tools to bound the *local truncation error*, *i.e.*, the distance between the exact and the numerical solutions. Mainly, such methods work in two stages at each integration step, starting from an enclosure $[\mathbf{y}_j] \ni \mathbf{y}(t_j; \mathbf{y}_0)$ at time t_j of the exact solution, we proceed by:

- i. a computation of an *a priori* enclosure $[\tilde{\mathbf{y}}_{j+1}]$ of the solution $\mathbf{y}(t; \mathbf{y}_0)$ for all t in the time interval $[t_j, t_{j+1}]$. This stage allows one to prove the existence and the uniqueness of the solution.
- ii. a computation of a tightening of state variable $[\mathbf{y}_{j+1}] \ni \mathbf{y}(t_{j+1}; \mathbf{y}_0)$ at time t_{j+1} using $[\tilde{\mathbf{y}}_{j+1}]$ to bound the local truncation error term $\text{lte}_\Phi(F, G, \mathbf{y}_j, t_j, h)$.

A validated simulation starts with the interval enclosures $[\mathbf{y}(0)]$, $[\mathbf{x}(0)]$, $[\mathbf{p}]$ and $[\mathbf{u}]$ of respectively, \mathcal{Y}_0 , \mathcal{X}_0 , \mathcal{P} , and \mathcal{U} . It produces two lists of boxes:

- the list of discretization time steps: $\{t_0, \dots, t_{\text{end}}\}$;
- the list of state enclosures at the discretization time steps: $\{[\mathbf{y}_0], \dots, [\mathbf{y}_{\text{end}}]\}$;
- the list of *a priori* enclosures: $\{[\tilde{\mathbf{y}}_0], \dots, [\tilde{\mathbf{y}}_{\text{end}}]\}$.

Figure 1 represents the enclosures $[\tilde{\mathbf{y}}_i]$ and $[\mathbf{y}_i]$ and their membership properties with the trajectories of the dynamical system.

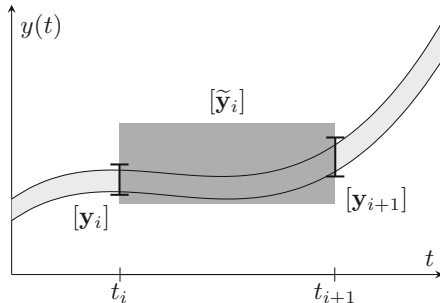


Fig. 1. The trajectories (in light gray) are overapproximated by $[\mathbf{y}_i]$ (thick line segment) at time step t_i . The *a priori* enclosure $[\tilde{\mathbf{y}}_i]$ (in gray) contains the trajectories over the time interval $[t_i, t_{i+1}]$.

4 Dynamical Systems with Integral Constraints

This section presents the main contribution of our work. For system described by Eq. (1) subject to the integral constraint defined by Eq. (2), we compute an overapproximation of its flowpipe over the time domain $[0, T]$, where the time horizon $T > 0$ is given. A first overapproximation of the flowpipe is computed using pessimistic bounds over the disturbances. The integral constraint in Eq. (2) is used to derive contractor. This contractor and a propagation step are applied in a fixed point algorithm until a contraction factor is reached. We run the algorithm over a simple example.

4.1 Extended System

We extend the system's state with the integral value corresponding to the integral constraint in Eq. (2):

$$\begin{cases} \dot{z}(t) = g(t, x(t)) - \|w(t)\|^2 \\ z(0) = 0 \end{cases} \quad (5)$$

Then, Eq. (2) can be equivalently expressed for z :

$$\forall t \in \mathbb{R}^+, z(t) \geq 0. \quad (6)$$

As mentioned in Remark 1, no L_∞ bounds can be derived for L_2 bounded signals. To study such systems, we make further assumptions about the disturbance:

Assumption 1. w is continuous, differentiable and of continuous derivative over \mathbb{R}^+ .

This assumption seems reasonable in the case of real systems modeling since disturbances modeled by integral constraints correspond to physical quantities. Since the continuity of a function over a closed interval implies its boundedness, Assumption 1 implies that the signal w is bounded and of bounded variation over $[0, T]$. Therefore, there exists $[\mathbf{w}] \in \mathbb{I}\mathbb{R}^m$ and $[\mathbf{w}'] \in \mathbb{I}\mathbb{R}^m$ such that for all $t \in [0, T]$:

$$\begin{cases} w(t) \in [\mathbf{w}] \\ \dot{w}(t) \in [\mathbf{w}'] \end{cases} \quad (7)$$

Using Assumption 1 and Eq. (5), the following system will be studied:

$$\mathcal{S} : \begin{cases} \dot{x}(t) = f(t, x(t), w(t)) \\ \dot{z}(t) = g(t, x(t), w(t)) - \|w(t)\|^2 \\ \dot{w}(t) \in [\mathbf{w}'] \\ x(0) \in [\mathbf{x}_0] \\ z(0) = 0 \\ 0 \leq z(t) \\ w(t) \in [\mathbf{w}] \end{cases} \quad (8)$$

where $[\mathbf{x}_0] \in \mathbb{I}\mathbb{R}^n$ is the set of initial states. We use the following notation $(x, z, w) \in \mathcal{S}$ iff $(x, w) \in \mathbb{L}_{loc}^2([0, T]; \mathbb{R}^n) \times \mathbb{L}_{loc}^2([0, T]; \mathbb{R}) \times \mathbb{L}_{loc}^2([0, T]; \mathbb{R}^m)$ is a trajectory of \mathcal{S} .

Equation (7) gives prior bounds over the disturbance w . They can be used to propagate the trajectories using standard guaranteed integration frameworks. Thanks to this, we get a first a priori overapproximation of the reachable set. In the next section, we use this first overapproximation and a contractor (defined out of the integral inequality) in a fixed point algorithm in order to get a tighter overapproximation of the reachable set.

4.2 Bounds over w

In this section, Eq. (7) and the integral constraint in Eq. 6 are used to derive bounds over the disturbance w . These bounds are then used to define a contractor over the *a priori* enclosure of the trajectories.

We present a preliminary result to Property 2:

Property 1. For $[\mathbf{v}] \in \mathbb{I}\mathbb{R}^p$, $p \in \mathbb{N}$ and $r > 0$, if $[\mathbf{v}] \leq r$ then $[\mathbf{v}] \subset [-r, r]^p$.

Proof. In an Euclidean space, the norm 1 and norm 2 satisfies $\sqrt{v_1^2 + \dots + v_p^2} \leq |v_1| + \dots + |v_p|$ for any $(v_1, \dots, v_p) \in \mathbb{R}^p$. ■

When w satisfies Eq. (7) and a given integral constraint, hard bounds (meaning in ∞ -norm) can be derived over w :

Property 2. For a $w \in \mathbb{L}_{loc}^2([0, h]; \mathbb{R}^m)$ defined over an interval of length $h > 0$. If w satisfies Eq. (7) (with given bounds $[\mathbf{w}], [\mathbf{w}'] \in \mathbb{I}\mathbb{R}^m$), then for any $r > 0$:

$$\int_0^h \|w(\tau)\|^2 d\tau \leq r \Rightarrow \forall \tau \in [0, h], w(\tau) \in [\mathbf{W}_r],$$

where $[\mathbf{W}_r] = [-k, k]^n$ with $k = \sqrt{\frac{r}{h}} + \frac{h}{2} \|\mathbf{w}'\|$ (where $\|\mathbf{w}'\|$ is the maximum Euclidean norm over the elements of $[\mathbf{w}']$).

Proof. By applying the Cauchy-Schwartz inequality between the signal w and $t \mapsto 1$ for the inner product of square integrable function, we have:

$$\left\| \int_0^h w(\tau) d\tau \right\|^2 \leq h \int_0^h \|w(\tau)\|^2 d\tau \leq hr.$$

By Eq. (7), $w(\tau) = w_0 + \int_0^\tau w_1(\kappa) d\kappa$ with $w_0 \in [\mathbf{w}]$ and $w_1(\cdot) \in [\mathbf{w}']$. Using the reverse triangular inequality, we have:

$$\left\| \int_0^h w_0 d\tau \right\| \leq \sqrt{rh} + \left\| \int_0^h \int_0^\tau w_1(\kappa) d\kappa \right\|.$$

Then, we get:

$$\|hw_0\| \leq \sqrt{hr} + \frac{h^2}{2} \|\mathbf{w}'\|. \quad (9)$$

This relationship is derived over $[0, h]$ but is also valid for any time interval $[t, t+h]$ of width h , $t > 0$. Therefore, by using Property 1 and Eq. (9), we have: $\forall \tau \in [0, h], w(\tau) \in [\mathbf{W}_r]$. \blacksquare

We then use Property 2 to derive bounds in the specific case of Eq. (5). Let a system trajectory $(x, z, w) \in \mathcal{S}$, such that at a given $t \in [0, T]$ and $h > 0$ s.t. $t+h \in [0, T]$, and for all $\tau \in [t, t+h]$:

$$\begin{cases} (x(t), z(t), w(t)) \in [\mathbf{y}_t] \\ (x(\tau), z(\tau), w(\tau)) \in [\tilde{\mathbf{y}}_t] \end{cases} \quad \text{where} \quad \begin{cases} [\mathbf{y}_t] = [\mathbf{x}_t] \times [\mathbf{z}_t] \times [\mathbf{w}_t] \\ [\tilde{\mathbf{y}}_t] = [\tilde{\mathbf{x}}_t] \times [\tilde{\mathbf{z}}_t] \times [\tilde{\mathbf{w}}_t] \end{cases}. \quad (10)$$

The trajectories belong to $[\mathbf{y}_t]$ at t and are in $[\tilde{\mathbf{y}}_t]$ between $[t, t+h]$. At $t+h$, for a given $t \geq 0$ and a given $h \geq 0$, Eq. (8) implies that z satisfies:

$$z(t+h) = z(t) + \int_t^{t+h} g(t, x(t)) d\tau - \int_t^{t+h} \|w(\tau)\|^2 d\tau.$$

By applying Eq. 6 at $t+h$ implies that $z(t+h) \geq 0$, we have the following relationship:

$$\int_t^{t+h} \|w(\tau)\|^2 d\tau \leq z(t) + \int_t^{t+h} g(\tau, x(\tau)) d\tau. \quad (11)$$

Let the function

$$q(z, x) = z + \int_t^{t+h} g(\tau, x(\tau)) d\tau. \quad (12)$$

By using an interval evaluation $[q]$ of q , the upperbound of $q(z, x)$ can be evaluated for $z \in [\mathbf{z}_t]$ and $x \in [\tilde{\mathbf{x}}_t]$. We denote by $[q]([\mathbf{z}_t], [\tilde{\mathbf{x}}_t])$ this upperbound. For any $w \in \mathbb{L}_{loc}^2([t, t+h], [\tilde{\mathbf{w}}_t])$, Eq. 11 implies:

$$\int_t^{t+h} \|w(\tau)\|^2 d\tau \leq \overline{[q]([\mathbf{z}_t], [\tilde{\mathbf{x}}_t])}.$$

Then, Property 2 can be used to derive bounds over the disturbance w :

Property 3. For a $w \in \mathbb{L}_{loc}^2([t, t+h]; \mathbb{R}^m)$ defined over an interval of length $h > 0$, $t > 0$. If w satisfies Eq. (7) (with given bounds $[\mathbf{w}], [\mathbf{w}'] \in \mathbb{I}\mathbb{R}^m$), then for any $\tau \in [t, t+h]$:

$$w(\tau) \in [\mathbf{W}_q], \quad (13)$$

where $[\mathbf{W}_q]([\tilde{\mathbf{x}}_t], [\mathbf{z}_t]) = [-r, r]^m$ with $r = \sqrt{\frac{[q]([\mathbf{z}_t], [\tilde{\mathbf{x}}_t])}{h}}$ and q defined in Eq. (12).

Proof. This is a direct application of Property 2. ■

We then define the operator over $[\mathbf{y}_t]$ and $[\tilde{\mathbf{y}}_t]$

$$\mathcal{C}([\mathbf{y}_t], [\tilde{\mathbf{y}}_t]) = ([\mathbf{y}_t] \cap [\mathbf{Y}_g]([\tilde{\mathbf{x}}_t], [\mathbf{z}_t]), [\tilde{\mathbf{y}}_t] \cap [\mathbf{Y}_g]([\tilde{\mathbf{x}}_t], [\mathbf{z}_t])) \quad (14)$$

where $[\mathbf{y}_t]$ and $[\tilde{\mathbf{y}}_t]$ are defined in Eq. (10),

$$[\mathbf{Y}_g] = [-\infty, \infty]^n \times [0, \infty] \times [\mathbf{W}_q],$$

with $[\mathbf{W}_q]$ defined in Property 3.

Proposition 1. \mathcal{C} defined in Eq. (14) is a contractor.

Proof. By Property 3, we have, for $\tau \in [t, t+h]$,

$$w(\tau) \in [\mathbf{W}_q],$$

i.e., all the disturbance signals of \mathcal{S} belongs to $[\mathbf{W}_q]$, so the contractor is conservative. Since the contractor is defined as an intersection with $[\mathbf{y}_t]$ and $[\tilde{\mathbf{y}}_t]$ respectively, we have

$$([\mathbf{y}_t], [\tilde{\mathbf{y}}_t]) \subseteq \mathcal{C}([\mathbf{y}_t], [\tilde{\mathbf{y}}_t]),$$

\mathcal{C} is contractive. For any $([\mathbf{y}'_t], [\tilde{\mathbf{y}}'_t])$ such that $[\mathbf{y}'_t] \subseteq [\mathbf{y}_t]$ and $[\tilde{\mathbf{y}}'_t] \subseteq [\tilde{\mathbf{y}}_t]$,

$$\mathcal{C}([\mathbf{y}'_t], [\tilde{\mathbf{y}}'_t]) \subseteq \mathcal{C}([\mathbf{y}_t], [\tilde{\mathbf{y}}_t]),$$

i.e. \mathcal{C} is monotone. ■

4.3 Integral Constraint Propagation

The contractor defined by Eq. (14) is used in a fixed point algorithm as in [1]. A priori enclosure of the trajectory is computed using bounds Eq. (7) over w . The integration algorithm gives

- the discretization time steps: $\{t_0, \dots, t_{end}\}$;
- the state enclosure at the discretization time steps: $\mathcal{Y}^0 = \{[y_0^0], \dots, [y_{end}^0]\}$;
- the *a priori* enclosures: $\tilde{\mathcal{Y}}^0 = \{[\tilde{y}_0^0], \dots, [\tilde{y}_{end}^0]\}$.

We then apply the contractor over each couple of discretized time-step boxes $[y_i^0] \in \mathcal{Y}^0$ and their associated *a priori* enclosures $[\tilde{y}_i^0] \in \tilde{\mathcal{Y}}^0$. These 2 steps are repeated in a fixed point algorithm until the contraction factor is lower than a given value. In this approach, time steps are computed at the first iteration of the algorithm and are not updated.

Example 2. We study the following linear time-invariant system disturbed by an unknown signal w constrained by a 2-norm inequality:

$$\begin{cases} \dot{x}(t) = -x(t) + w(t) \\ \int_0^t w(\tau)^2 d\tau \leq \int_0^t 0.01x(\tau)^2 d\tau \\ x(0) \in [-1, 1] \end{cases} \quad (15)$$

with $[\mathbf{w}] = [-1, 1]$ and $[\mathbf{w}'] = [-1, 1]$ in Eq. (7) for $t \in [0, 2.5]$. Figure 2 shows the reachable set of this dynamical system computed with the method described in this section.

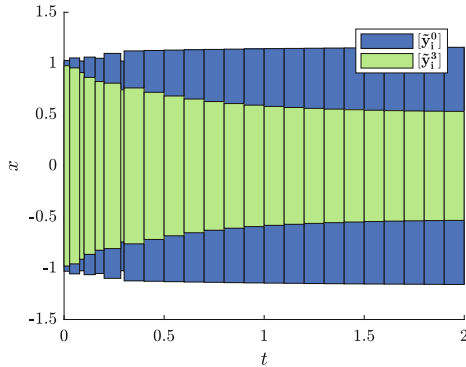


Fig. 2. Computation of the overapproximation of the reachable set of Example 2 using the algorithm presented in Sect. 4. Blue boxes corresponds to the *a priori* enclosures at the first iteration of the algorithm $\tilde{\mathcal{Y}}^0$, green boxes are the *a priori* enclosure at the 3rd iteration $\tilde{\mathcal{Y}}^3$ of the algorithm. (Color figure online)

5 Examples

In this part, we present applications of the method described in Sect.4 for a tank system (Subsect. 5.1) and a delayed system (Subsect. 5.2). The results are discussed in the Subsect. 5.3.

5.1 Tank System

We consider a 2-tanks system (see Fig. 3) described by the following dynamical equation

$$\begin{cases} \dot{h}_1 = -r_1\sqrt{h_1} + w \\ \dot{h}_2 = -r_2\sqrt{h_2} + r_1\sqrt{h_1} \\ \int_0^\tau w(\tau)^2 d\tau \leq \int_0^t \left(r_2\sqrt{h_2(\tau)}\right)^2 d\tau \end{cases}, \quad (16)$$

with initial conditions:

$$\begin{cases} h_1(0) = 1 \\ h_2(0) = 0 \end{cases}.$$

$r_1 = 0.1$ and $r_2 = 0.001$ are given constants depending on the hole diameter, h_1 and h_2 are the respective level of water of tanks 1 and 2. The pump is operated externally and is considered as an unknown disturbance, we model it as the set of signals w that verify the integral constraint in Eq. (16).

The plot in Fig. 3 corresponds to the reachable set overapproximation computed using the algorithm described in Sect. 4. The overapproximation of the reachable set over h_1 is larger than that over h_2 . This is a consequence that the disturbance is directly added into the Tank 1 (to h_1) and is filtered by Tank 1 before influencing Tank 2. At $t = 0.6$, the reachable set over h_1 computed with the integral constraint is 9 times smaller than the reachable set computed with only the prior bound over the unknown disturbance w (see Figs. 3a and b).

5.2 Delayed System with Integral Quadratic Constraint

For $u, v \in \mathbb{L}_{loc}^2(\mathbb{R}^+; \mathbb{R})$, the delay operator D_h over an input signal u is defined by the following relationship:

$$v = D_h(u) \Leftrightarrow \begin{cases} v(t) = u(t-h) & \text{for all } t \geq h \\ v(t) = 0 & \text{otherwise.} \end{cases} \quad (17)$$

Guaranteed integration of differential equation with delays is challenging. Since they act as a memory of the past input signal over an interval of width h , the state of the delay belongs to $\mathbb{L}_{loc}^2([0, h], \mathbb{R})$. The dimension of the system state space is therefore non finite.

The stability of linear time-invariant (LTI) systems with internal delays is studied in [25]. The state of the delay is projected over finite Legendre polynomial basis. These projections are time-dependent values since the state of the

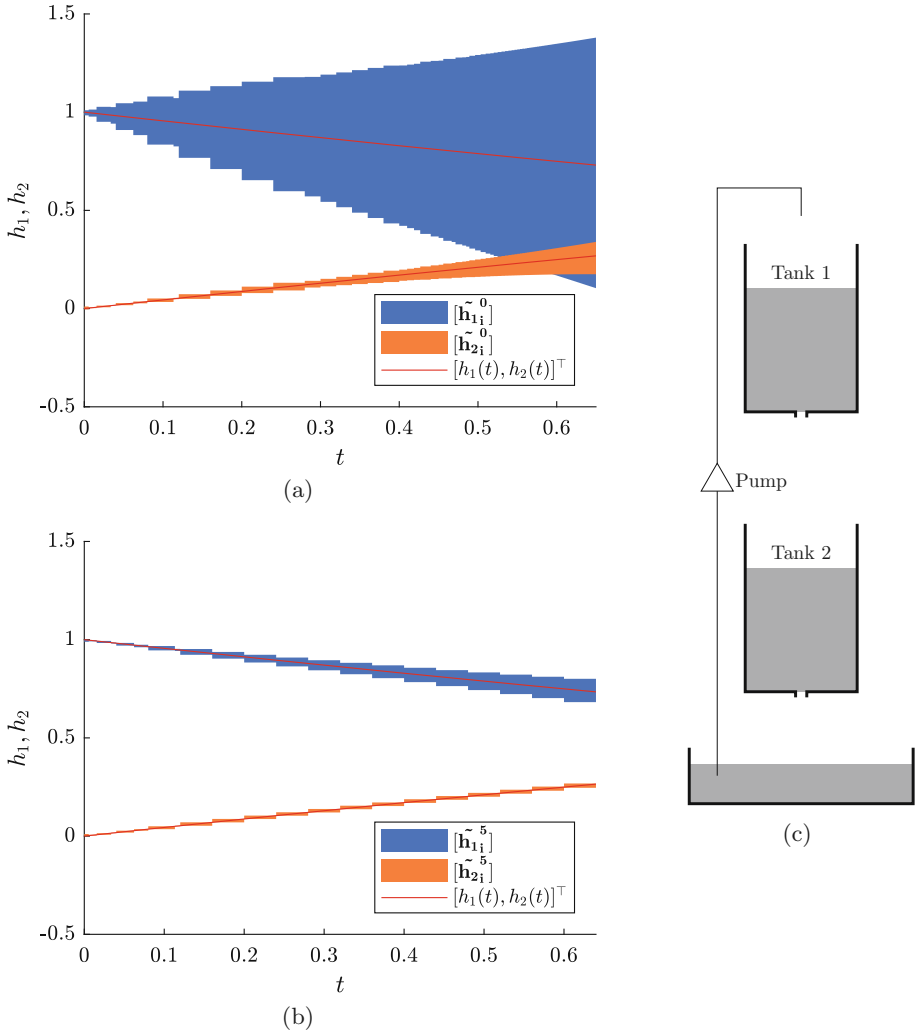


Fig. 3. The tank system described by Eq. (16) is represented in Fig. 3c. The algorithm presented in Sect. 4 is used to compute the overapproximation of h_1 (in blue) and h_2 (in orange). In Fig. 3a, the prior overapproximation of the reachable set $\tilde{\mathcal{Y}}^0$ is shown. Figure 3b shows the *a priori* enclosures $\tilde{\mathcal{Y}}^5$ at the 5th iteration of the algorithm. (Color figure online)

delay is also time-varying. The time derivative of these projections only depends on the input of the delay operator. Then the norm of the state is overapproximated using a Bessel inequality. By integrating this inequality, we get an Integral Quadratic Constraint (IQC) between the output of the delay operator, its input, the derivative of its inputs, the projections over the truncated basis of Legendre polynomial and an error signal. The IQC models the energy of the Legendre

expansion's remainder (i.e. the error signal). In [25], the stability of the delayed LTI system is assessed for all possible error signal which verifies the derived IQC. We use this IQC to overapproximate the reachable set of such system.

In what follows, we use the first order of the IQC relationship described in [25, Theorem 5]. The state ξ corresponds to the average value of the delay's state. The remaining energy of the state is bounded by an integral quadratic constraint.

$$\left\{ \begin{array}{l} \dot{\xi}(t) = -15\xi(t) + 1.5v(t) - w(t) \quad \text{with} \quad \xi(0) = 0 \\ \text{under the IQC} \quad \int_0^t w(s)^2 ds \leq \int_0^t \left[0.0025\dot{v}(s)^2 - 0.75(v(s) - \xi(s))^2 \right] ds \end{array} \right. \quad (18)$$

The IQC system Eq. (18) is used to overapproximate the delay in the following system:

$$\left\{ \begin{array}{l} \dot{x} = -x - k_c D_h(x) \\ x(0) = 0 \end{array} \right. \quad (19)$$

where $k_c = 4$ and $h = 0.01$. Equations (17, 18 and 19) are then combined in a unique linear time-invariant system with an integral quadratic constraint.

$$\left\{ \begin{array}{l} \dot{X}(t) = AX + B_w w(t) + B_u u(t) \\ X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \int_0^t w(\tau)^2 d\tau \leq \int_0^t \begin{bmatrix} X(\tau) \\ u(\tau) \end{bmatrix}^\top M \begin{bmatrix} X(\tau) \\ u(\tau) \end{bmatrix} \end{array} \right. \quad (20)$$

where the matrices are defined by

$$A = \begin{bmatrix} 1.0417 & 15.6250 \\ -6.0417 & -15.6250 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1.0000 \\ -1.0000 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1.0417 \\ -0.0417 \end{bmatrix}$$

and

$$M = \begin{bmatrix} -12.4566 & -30.5990 & 0.0434 \\ -30.5990 & -68.3594 & 0.6510 \\ 0.0434 & 0.6510 & 0.0434 \end{bmatrix}.$$

The bounds in Eq. (13) are $[\mathbf{w}] = [-10, 10]$ and $[\mathbf{w}'] = [-1, 1]$. The initial noise set is defined such that $[\mathbf{w}_0] = [\mathbf{w}]$.

Figure 4 corresponds to the flowpipe of the delayed system modeled with the integral quadratic constraint. Y_{IQC} is the reachable tube of the corresponding system.

5.3 Discussion

The main motivation of this work is to use Integral quadratic constraint (IQC) models in a guaranteed integration framework. IQC models are widely used in

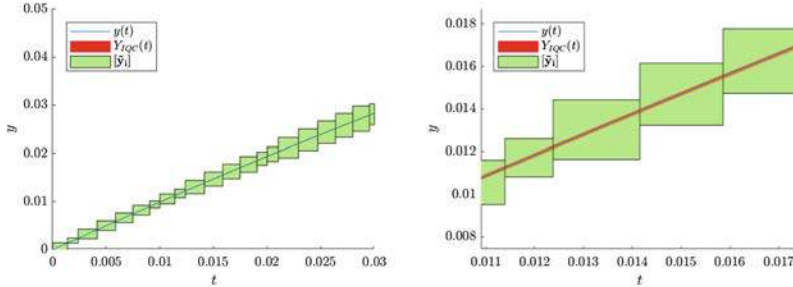


Fig. 4. Computation of the flowpipe of the system Eq. (20) using guaranteed numerical integration framework described in Sect. 3 and the contractor \mathcal{C} introduced in Sect. 4. y (the blue line) corresponds to the response of the delayed system. Y_{IQC} is the exact flowpipe of system computed using the paraboloid method presented in [22]. (Color figure online)

the robust control community for stability analysis of dynamical systems. When the IQC system is stable, there exists an invariant over the set of states (x, z) and the maximal reachable z value (i.e. the maximal integral value reachable) is bounded for any trajectory.

In our approach, such an invariant does not exist. The overapproximation of the maximal reachable z is constantly increasing. Consequently, bounds provided by the fixed point algorithm are also strictly increasing. When these bounds reach the prior bounds given by Eq. (7) over the disturbance, the reachable set tends to the reachable set computed without the integral constraint. Figure 5 corresponds to the reachable set of Example 2 for a larger horizon of integration. The integral constraints provide bounds over w . However, when the energy level is too high, these bounds are strictly included in bounds given by Eq. (7). At $t = 15$ s, the reachable set converges to the reachable set of the system with no integral constraint between the disturbance and the state.

The bounds of the noise input depends on the result of the used guaranteed set integration method. Therefore, if the later are too pessimistic, the proposed contraction method will only rely on the bounds $[\mathbf{w}]$ and $[\mathbf{w}']$ of Eq. (7).

In our approach, a larger class of systems is considered compared to the linear case treated in [22]. Contrary to IQC models, only the dependence in the disturbance needs to be quadratic for the integral constraint.

In term of scalability, our approach needs the state of the original dynamical system to be extended from n variables to $p = n + m + 1$ variables (m states for w , 1 state for z). Since the noise signal span in a subspace of R^n , m is always smaller than n . Since m is often close to 1 (the delay modeled as an integral quadratic constraint introduce a 1 dimensional noise signal), p is close to n (or $2n$ in the worst case). However only the integration part can suffer from the dimension of the system. Based on the advantage of our approach, a less expansive integration method can be used for large systems for a similar result.

We presented 2 examples. A non-linear system in Sect. 5.1 and a linear system with delay in Sect. 5.2. A similar approach could be applied to non-linear system with delay since the integration method can handle non-linear differential equations.

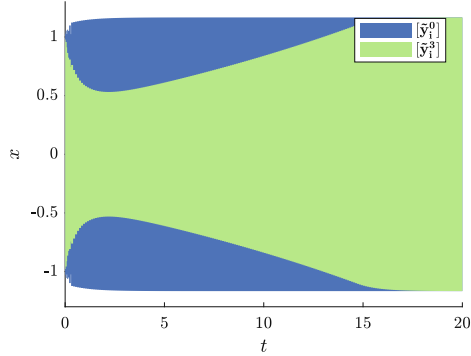


Fig. 5. Computation of the flowpipe of the system Eq. (3) in Example 2 over $[0, 20]$ using guaranteed numerical integration framework described in Sect. 3 and the contractor \mathcal{C} introduced in Sect. 4. In blue, the reachable set when only Eq. (7) is used (*i.e.*, when the integral constraint is not used). In green, the reachable set of the system when the integral constraint is taken into account. (Color figure online)

6 Conclusion

We presented a method to compute an overapproximation of the reachable tube for dynamical systems with integral constraints over the input set. The integral constraint is expressed as a contractor over the set of trajectories and used in a fixed point algorithm together with a propagation process.

The method developed in this work is guaranteed (we compute an overapproximation of the reachable tube). However, our overapproximations tend to constantly grow in size, even when the reachable set is known to be bounded. At each time instant, the integral term of the integral constraint is overapproximated by an interval, its upper bound is used to compute bounds over disturbances. For a trajectory, the worst case disturbance consumes all the integral constraint and the best case disturbance consumes none of it. So the integral level is always growing and at the same time, the reachable set is overapproximated with the worst case disturbance level. In future works, a template based scheme will be used to overapproximate the maximum reachable integral value of the constraint.

Models with integral constraints are a classical tool from the robust control community. In this field, they represent energy gains between signals of the system and a disturbance signal. Many complex systems can be analyzed in

this way. In future works, more applications will be discussed. More specifically, error bounds over reduced models can be expressed as a 2-norm constraints with the input signal of the system. Simplification of models is very appealing for guaranteed integration since the computational time is mainly dependent on the system dimension. Being able to reduce the order of the system and to bound the error with a 2-norm gain would lead to more efficient algorithm.

Our method provides ways to verify physical systems where the sensors are subject to energy bounded disturbances. Currently, most of these noise models are bounds over the signal. Such models are problematic when they disturb an integrating dynamic. Energy bounded noises might lead to more realistic noise models and therefore to better overapproximation of dynamical system reachable set. Future work will include verification of robotic systems subject to sensor noises.

References

1. Alexandre dit Sandretto, J., Chapoutot, A.: Contraction, propagation and bisection on a validated simulation of ODE. In: Summer Workshop on Interval Methods (2016)
2. Alexandre dit Sandretto, J., Chapoutot, A.: Validated explicit and implicit Runge-Kutta methods. *Reliab. Comput.* **22**, 79–103 (2016)
3. Alexandre dit Sandretto, J., Chapoutot, A.: Validated simulation of differential algebraic equations with Runge-Kutta methods. *Reliab. Comput.* **22** (2016)
4. Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V.: *Linear Matrix Inequalities in System and Control Theory*, vol. 15. SIAM, Philadelphia (1994)
5. Chabert, G., Jaulin, L.: Contractor programming. *Artif. Intell.* **173**, 1079–1100 (2009)
6. Chen, M., Fränzle, M., Li, Y., Mosaad, P.N., Zhan, N.: Validated simulation-based verification of delayed differential dynamics. In: Fitzgerald, J., Heitmeyer, C., Gnesi, S., Philippou, A. (eds.) *FM 2016*. LNCS, vol. 9995, pp. 137–154. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-48989-6_9
7. Chou, Y., Chen, X., Sankaranarayanan, S.: A study of model-order reduction techniques for verification. In: Abate, A., Boldo, S. (eds.) *NSV 2017*. LNCS, vol. 10381, pp. 98–113. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-63501-9_8
8. Goubault, E., Putot, S., Sahlmann, L.: Inner and outer approximating flowpipes for delay differential equations. In: Chockler, H., Weissenbacher, G. (eds.) *CAV 2018*. LNCS, vol. 10982, pp. 523–541. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-96142-2_31
9. Graettinger, T.J., Krogh, B.H.: Hyperplane method for reachable state estimation for linear time-invariant systems. *J. Optim. Theory Appl.* **69**(3), 555–588 (1991)
10. Gusev, M.I., Zykov, I.V.: On extremal properties of boundary points of reachable sets for a system with integrally constrained control. In: *Proceedings of 20th World Congress International Federation of Automatic Control*, vol. 50, pp. 4082–4087. Elsevier (2017)
11. Henrion, D., Korda, M.: Convex computation of the region of attraction of polynomial control systems. *IEEE Trans. Autom. Control* **59**(2), 297–312 (2014)
12. Jaulin, L., Kieffer, M., Didrit, O., Walter, É.: *Applied Interval Analysis*. Springer, London (2001). <https://doi.org/10.1007/978-1-4471-0249-6>

13. Jönsson, U.: Robustness of trajectories with finite time extent. *Automatica* **38**(9), 1485–1497 (2002)
14. Korda, M.: Moment-sum-of-squares hierarchies for set approximation and optimal control. Ph.D. thesis, EPFL, Switzerland (2016)
15. Kurzhanski, A.B., Varaiya, P.: On ellipsoidal techniques for reachability analysis. Part I: external approximations. *Optim. Methods Softw.* **17**(2), 177–206 (2002)
16. Lee, E.B., Markus, L.: *Foundations of Optimal Control Theory*. Wiley, New York (1976)
17. Megretski, A., Rantzer, A.: System analysis via integral quadratic constraints. *IEEE Trans. Autom. Control* **42**(6), 819–830 (1997)
18. Moore, R.E.: *Interval Analysis*. Prentice Hall, Upper Saddle River (1966)
19. Nedialkov, N.S., Jackson, K., Corliss, G.: Validated solutions of initial value problems for ordinary differential equations. *Appl. Math. Comput.* **105**(1), 21–68 (1999)
20. Pfifer, H., Seiler, P.: Integral quadratic constraints for delayed nonlinear and parameter-varying systems. *Automatica* **56**, 36–43 (2015)
21. Prajna, S., Jadbabaie, A.: Safety verification of hybrid systems using barrier certificates. In: Alur, R., Pappas, G.J. (eds.) *HSCC 2004*. LNCS, vol. 2993, pp. 477–492. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-540-24743-2_32
22. Rousse, P., Garoche, P.-L., Henrion, D.: Parabolic set simulation for reachability analysis of linear time invariant systems with integral quadratic constraint. In: *Proceedings of European Control Conference, Naples* (2019)
23. Savkin, A.V., Petersen, I.R.: Recursive state estimation for uncertain systems with an integral quadratic constraint. *IEEE Trans. Autom. Control* **40**(6), 1080–1083 (1995)
24. Scherer, C.W., Veenman, J.: Stability analysis by dynamic dissipation inequalities: on merging frequency-domain techniques with time-domain conditions. *Syst. Control Lett.* **121**, 7–15 (2018)
25. Seuret, A., Gouaisbaut, F.: Hierarchy of LMI conditions for the stability analysis of time-delay systems. *Syst. Control Lett.* **81**, 1–7 (2015)
26. Shampine, L.F., Thompson, S.: Numerical solution of delay differential equations. In: Gilsinn, D.E., Kalmár-Nagy, T., Balachandran, B. (eds.) *Delay Differential Equations*, pp. 1–27. Springer, Boston (2009). https://doi.org/10.1007/978-0-387-85595-0_9
27. Soravia, P.: Viscosity solutions and optimal control problems with integral constraints. *Syst. Control Lett.* **40**(5), 325–335 (2000)
28. Szczelina, R.: Rigorous integration of delay differential equations. Ph.D. thesis (2015)
29. Varaiya, P.: Reach set computation using optimal control. In: Inan, M.K., Kurshan, R.P. (eds.) *Verification of Digital and Hybrid Systems*. NATO ASI Series, vol. 170, pp. 323–331. Springer, Heidelberg (2000). https://doi.org/10.1007/978-3-642-59615-5_15
30. Xue, B., Mosaad, P.N., Fränzle, M., Chen, M., Li, Y., Zhan, N.: Safe over- and under-approximation of reachable sets for delay differential equations. In: Abate, A., Geeraerts, G. (eds.) *FORMATS 2017*. LNCS, vol. 10419, pp. 281–299. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-65765-3_16