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DISCRETE STICKY COUPLING OF FUNCTIONAL AUTOREGRESSIVE PROCESSES

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In this paper, we provide bounds in various metrics between the successive iterates of two functional autoregressive processes with isotropic Gaussian noise of the form $Y_{k+1} = T_\gamma(Y_k) + \sqrt{\gamma\sigma^2}Z_{k+1}$ and $\tilde{Y}_{k+1} = \tilde{T}_\gamma(\tilde{Y}_k) + \sqrt{\gamma\sigma^2}\tilde{Z}_{k+1}$ in the limit where the parameter $\gamma \rightarrow 0$. More precisely, we give non-asymptotic bounds on $\rho(\mathcal{L}(Y_k), \mathcal{L}(\tilde{Y}_k))$, where ρ is an appropriate weighted Wasserstein distance or a V -distance, uniformly in the parameter γ , and on $\rho(\pi_\gamma, \tilde{\pi}_\gamma)$, where π_γ and $\tilde{\pi}_\gamma$ are the respective stationary measures of the two processes. Of particular interest, this class of processes encompasses the Euler-Maruyama discretization of Langevin diffusions and its variants. To obtain our results, we rely on the construction of a discrete Markov chain $(W_k^{(\gamma)})_{k \in \mathbb{N}}$ for which we are able to bound the moments and show quantitative convergence rates uniform on γ . In addition, we show that this process converges in distribution to the continuous sticky process studied in [20, 18]. Finally, we illustrate our result on two numerical applications.

1. Introduction. We are interested in this paper in Markov chains $(Y_k)_{k \in \mathbb{N}}$ starting from $y \in \mathbb{R}^d$ and defined by recursions of the form

$$(1) \quad Y_{k+1} = T_\gamma(Y_k) + \sigma\sqrt{\gamma}Z_{k+1},$$

where $\sigma > 0$, $\gamma \in (0, \bar{\gamma}]$, for $\bar{\gamma} > 0$, $\{T_\gamma : \gamma \in (0, \bar{\gamma}]\}$ is a family of continuous functions from \mathbb{R}^d to \mathbb{R}^d and $(Z_k)_{k \geq 1}$ is a sequence of i.i.d. d -dimensional standard Gaussian random variables. Note that the Euler-Maruyama discretization of overdamped Langevin

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diffusions or of general Komolgorov processes and its variants belong to this class of processes and in that setting γ corresponds to the discretization step size. Indeed, the Euler scheme consists in taking for any $\gamma \in (0, \bar{\gamma}]$, $T_\gamma(y) = y + \gamma b(y)$ for some $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$. When $b = -\nabla U$ for some potential U , these methods are now popular Markov Chain Monte Carlo algorithms to sample from the target density $x \mapsto e^{-U(x)} / \int_{\mathbb{R}^d} e^{-U(y)} dy$. However, in some applications, explicitly computing ∇U is not an option and further numerical methods must be implemented which come with additional bias since only approximations of ∇U can be used in (1). In this paper, we precisely study this additional source of error. In particular, based on a chain defined by (1), we consider a second Markov chain $(\tilde{Y}_k)_{k \in \mathbb{N}}$ defined by the recursion

$$(2) \quad \tilde{Y}_{k+1} = \tilde{T}_\gamma(\tilde{Y}_k) + \sigma \sqrt{\gamma} \tilde{Z}_{k+1},$$

where $\{\tilde{T}_\gamma : \gamma \in (0, \bar{\gamma}]\}$ is a family of functions from \mathbb{R}^d to \mathbb{R}^d such that for any γ , \tilde{T}_γ is an approximation of T_γ in a sense specified below, and $(\tilde{Z}_k)_{k \geq 1}$ is a sequence of i.i.d. d -dimensional standard Gaussian random variables potentially correlated with $(Z_k)_{k \geq 1}$.

We will enforce below conditions that ensure that both $(Y_k)_{k \in \mathbb{N}}$ and $(\tilde{Y}_k)_{k \geq 1}$ are geometrically ergodic, and denote by π_γ and $\tilde{\pi}_\gamma$ their invariant probability measures respectively. If for any $\gamma > 0$, \tilde{T}_γ is close in some sense to T_γ , the overall process $(\tilde{Y}_k)_{k \in \mathbb{N}}$ can be seen as a perturbed version of $(Y_k)_{k \in \mathbb{N}}$, and $\tilde{\pi}_\gamma$ is expected to be close to π_γ . The main goal of this paper is to establish quantitative bounds between the finite-time laws of the two processes and between their equilibria. We pay particular attention to the dependency of these estimates on γ . Indeed, in the case of the Euler scheme of a continuous-time diffusion, π_γ and the law of $Y_{[t/\gamma]}$ for some $t > 0$ converge to the invariant measure and law at time t of the continuous-time process, and similarly for the perturbed chain. Hence, as $\gamma \rightarrow 0$, our estimates should not degenerate, but rather yield quantitative estimates for the continuous time process. More precisely, the present paper is the discrete-time counterpart of the study conducted by [18] in the continuous-time case, and as γ vanishes we recover estimates that are consistent with those of [18].

As in [18], our results are based on the construction of a suitable coupling of the processes, i.e. a simultaneous construction of a pair $(Y_k, \tilde{Y}_k)_{k \in \mathbb{N}}$ of non-independent chains that marginally follow respectively (1) and (2) and are designed to get and stay close one to the other. We use the maximal reflection coupling for Gaussian laws, namely at each step the two chains are coupled to merge with maximal probability and, otherwise, we use a reflection (see Section 2.2 below). Estimates on the laws of the chains then follow from the study of $(\|Y_k - \tilde{Y}_k\|)_{k \in \mathbb{N}}$, which is itself based on the analysis of a Markov chain $(W_k)_{k \in \mathbb{N}}$ on $[0, +\infty)$ that is such that, by design of the coupling, $\|Y_k - \tilde{Y}_k\| \leq W_k$ for all $k \in \mathbb{N}$. Thus, the question of establishing bounds between the laws of two d -dimensional Markov chains is reduced to the study of a single one-dimensional chain. Besides, together with the Markov property, the auxiliary chain presents some nice features. First, it is stochastically monotonous, i.e. if $(W'_k)_{k \in \mathbb{N}}$ is a Markov chain associated to the same Markov kernel as $(W_k)_{k \in \mathbb{N}}$ and such that $W_0 \leq W'_0$, then for any $k \in \mathbb{N}$, W_k is stochastically greater than W'_k , i.e. for any $t \geq 0$, $\mathbb{P}(W_k \leq t) \geq \mathbb{P}(W'_k \leq t)$. Secondly, $(W_k)_{k \in \mathbb{N}}$ has an atom at 0.

The main results and main steps of this study are the following. First, we prove that $(W_k)_{k \in \mathbb{N}}$ admits a unique invariant measure and that, independently from γ , the moments and mass on $(0, +\infty)$ of this equilibrium are small when the difference between T_γ and \tilde{T}_γ is small. Second, we establish the geometric convergence of the chain towards its equilibrium, at an explicit rate (stable as $\gamma \rightarrow 0$). Finally, we prove that, as $\gamma \rightarrow 0$,

the chain $(W_k)_{k \in \mathbb{N}}$ converges in law to the continuous-time sticky diffusion that played the same role in [18]. This last part is not necessary to get estimates on the finite-time and equilibrium laws of (1) and (2) for a given $\gamma > 0$, but it sheds some new light on the limit sticky process which, in [18], is constructed as the limit of continuous-time diffusions with diffusion coefficients that vanish at zero, rather than discrete-time chains. In some sense, $(W_k)_{k \in \mathbb{N}}$ can be seen as a discretization scheme for the sticky process, see also [2] on this topic.

Besides the obvious continuous/discrete difference between [18] and the present work, let us emphasize a few other distinctions. First, in [18], the one-dimensional sticky process has an explicit invariant measure. This is not the case in our framework, which makes the derivation of the bounds on the moments of the equilibrium a bit more involved. Secondly, in [18], although it is proven that the mass at zero and the first moment of the law of the sticky diffusion converge to their value at equilibrium (which is sufficient to get estimates on the laws of the two initial d -dimensional processes), the question of long-time convergence is not addressed for the sticky diffusion, whereas our long-time convergence results for $(W_k)_{k \geq 0}$ together with its convergence as $\gamma \rightarrow +\infty$ furnish an explicit convergence rate for the sticky diffusion. The proof of the stability of the mass at zero and of the first moment in [18] relies on a concave modification of the distance (such as used e.g. in [16]), which is contracted by the chain before it hits zero. This method does not apply to, say, the second moment of the process. As a consequence, the results of [18] only concern the total variation and \mathcal{W}_1 Wasserstein distances, while we consider a broader class of distances.

Finally, our theoretical results are illustrated through numerical experiments. We study the influence of the discretization scheme generally needed to perform Bayesian inference for parameters of Ordinary Differential Equations (ODEs).

Notation and convention. We denote by $\mathcal{B}(\mathbb{R}^d)$, the Borel σ -field of \mathbb{R}^d endowed with the Euclidean distance and by φ_{σ^2} the density of the one-dimensional Gaussian distribution with zero-mean and variance $\sigma^2 > 0$. In the case $\sigma = 1$, we simply denote this density by φ . $\Delta_{\mathbb{R}^d}$ stands for the subset of $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\}$ and for any $A \subset \mathbb{R}^d$, A^c for its complement. Let μ and ν be two σ -finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. If ν is absolutely continuous with respect to μ , we write $\nu \ll \mu$. We say that ν and μ are equivalent if and only if $\nu \ll \mu$ and $\mu \ll \nu$. We denote by $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ the floor and ceiling function respectively. For $d, n \in \mathbb{N}^*$, $\mathcal{M}_{d,n}(\mathbb{R})$ stands for the set of $d \times n$ real matrices. We denote by $C^k(U, A)$ the set of k -th continuously differentiable functions from an open set $U \subset \mathbb{R}^m$ in $A \subset \mathbb{R}^p$. We use the convention $\sum_{k=n}^p = 0$ and $\prod_{k=n}^p = 1$ for $n < p$, $n, p \in \mathbb{N}$, and $a/0 = +\infty$ for $a > 0$.

2. Sticky reflection coupling.

2.1. Main result. The Markov kernels R_γ associated with $(Y_k)_{k \in \mathbb{N}}$ defined in (1) are given for any $y \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$R_\gamma(y, A) = (2\pi\sigma^2\gamma)^{d/2} \int_{\mathbb{R}^d} \mathbb{1}_A(y') \exp \left\{ -\|y' - T_\gamma(y)\|^2 / (2\sigma^2\gamma) \right\} dy'.$$

Note that \tilde{R}_γ associated with $(\tilde{Y}_k)_{k \in \mathbb{N}}$ is given by the same expression upon replacing T_γ by \tilde{T}_γ . We consider the following assumption on the family $\{T_\gamma : \gamma \in (0, \bar{\gamma}]\}$. This condition will ensure that R_γ is geometrically ergodic (see Proposition 1) and it will be important to derive our main results regarding the distance of R_γ^k and \tilde{R}_γ^k , for $k \in \mathbb{N}$.

H1. *There exist $R_1, L \geq 0$ and $m > 0$ such that for any $\gamma \in (0, \bar{\gamma}]$, there exists a non-decreasing function $\tau_\gamma : [0, +\infty) \rightarrow [0, +\infty)$ satisfying $\tau_\gamma(0) = 0$, $\|T_\gamma(x) - T_\gamma(\tilde{x})\| \leq \tau_\gamma(\|x - \tilde{x}\|)$ for any $x, \tilde{x} \in \mathbb{R}^d$, and*

$$(3) \quad \sup_{r \in (0, +\infty)} \{\tau_\gamma(r)/r\} \leq 1 + \gamma L, \quad \sup_{r \in (R_1, +\infty)} \{\tau_\gamma(r)/r\} \leq 1 - \gamma m.$$

In addition, $\sup_{\gamma \in (0, \bar{\gamma}]} \|T_\gamma(0)\| < +\infty$.

Note that the condition that for any $\gamma \in (0, \bar{\gamma}]$, τ_γ is non-decreasing can be omitted upon replacing in our study τ_γ by the affine majorant

$$(4) \quad \bar{\tau}_\gamma : r \mapsto \begin{cases} (1 + L\gamma)r & \text{if } r \in [0, R_1], \\ (1 + L\gamma)R_1 + (1 - m\gamma)(r - R_1) & \text{otherwise.} \end{cases}$$

Indeed, by definition and (3), for any $r \in [0, +\infty)$, $\tau_\gamma(r) \leq \bar{\tau}_\gamma(r)$, therefore for any $x, \tilde{x} \in \mathbb{R}^d$, $\|T_\gamma(x) - T_\gamma(\tilde{x})\| \leq \bar{\tau}_\gamma(\|x - \tilde{x}\|)$. In addition, an easy computation leads to setting $R_2 = 2R_1(L + m)/m$,

$$\sup_{r \in (0, +\infty)} \{\bar{\tau}_\gamma(r)/r\} \leq 1 + \gamma L, \quad \sup_{r \in (R_2, +\infty)} \{\bar{\tau}_\gamma(r)/r\} \leq 1 - \gamma m/2.$$

Then, $\bar{\tau}_\gamma$ satisfies **H1** and is non-decreasing.

Note that **H1** implies that for any $r \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$, $\tau_\gamma(r) \leq (1 + \gamma L)r$, therefore T_γ is $(1 + \gamma L)$ -Lipschitz. The second condition in (3) ensures that for any $\gamma \in (0, \bar{\gamma}]$, T_γ is a *contraction at infinity*, i.e. for any $x, \tilde{x} \in \mathbb{R}^d$, $\|T_\gamma(x) - T_\gamma(\tilde{x})\| \leq (1 - \gamma m)\|x - \tilde{x}\|$, if $\|x - \tilde{x}\| \geq R_1$.

The assumption **H1** holds for the Euler scheme applied to diffusions with scalar covariance matrices, i.e. (1) with $T_\gamma(x) = x + \gamma b(x)$ and a drift function $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, if, for some $L_b, m_b, R_b > 0$, b is L_b -Lipschitz continuous and satisfies

$$\langle x - y, b(x) - b(y) \rangle \leq -m_b \|x - y\|^2,$$

for all $x, y \in \mathbb{R}^d$ with $\|x - y\| \geq R_b$. Indeed, this implies that for any $x, y \in \mathbb{R}^d$, $\|T_\gamma(x) - T_\gamma(y)\| \leq (1 + L_b\gamma)\|x - y\|$ and, provided $\gamma \in (0, L^2/m_b)$ and $\|x - y\| \geq R_b$, $\|T_\gamma(x) - T_\gamma(y)\|^2 \leq (1 - m_b\gamma)\|x - y\|^2$. Therefore, it suffices to consider τ_γ defined by (4) with $L = L_b$, $m = m_b/2$ and $R_1 = R_b$.

Our results will be stated in term of Wasserstein distances and V -norms, whose definitions are the following. Consider a measurable cost function $\mathbf{c} : \mathbb{R}^{2d} \rightarrow [0, \infty)$. Then the associated Wasserstein distance \mathscr{W}_c is given for two probability measures μ, ν on \mathbb{R}^d by

$$\mathscr{W}_c(\nu, \mu) = \inf_{\pi \in \Pi(\nu, \mu)} \int_{\mathbb{R}^{2d}} \mathbf{c}(x, y) \pi(dx, dy),$$

where $\Pi(\nu, \mu)$ is the set of transference plan between ν and μ , namely the set of probability measures on \mathbb{R}^d whose first and second d -dimensional marginals are respectively ν and μ . In the particular case where $\mathbf{c}(x, y) = \mathbb{1}_{\Delta_{\mathbb{R}^d}^c}(x, y)$, \mathscr{W}_c is simply the total variation distance $\|\cdot\|_{TV}$. For $V : \mathbb{R}^d \rightarrow [1, +\infty)$, the choice $\mathbf{c}(x, y) = \mathbb{1}_{\Delta_{\mathbb{R}^d}^c}(x, y)\{V(x) + V(y)\}$ yields the V -norm (see [11, Theorem 19.1.7]), i.e. $\mathscr{W}_c(\nu, \mu) = \|\nu - \mu\|_V$. Finally, for $\mathbf{c}(x, y) = \|x - y\|^p$ with $p \in [1, +\infty)$, \mathscr{W}_c is the p -th power of the usual Wasserstein distance of order p .

Following the same line as the proof of [9, Theorem 15]¹, we can show that **H1** implies that the Markov kernel R_γ is V_c -uniformly geometrically ergodic where for any $c > 0$ and $x \in \mathbb{R}^d$, $V_c(x) = \exp(c\|x\|^2)$, with a convergence rate that scales linearly with the step size γ .

PROPOSITION 1. *Assume **H1**. Then, setting $\bar{\gamma}_1 = \bar{\gamma} \wedge \{1/\mathfrak{m}\}$, for any $\gamma \in (0, \bar{\gamma}_1]$, R_γ admits a unique stationary distribution π_γ . In addition, there exist $c > 0$, $\rho \in [0, 1)$ and $C \geq 0$ such that for any $x \in \mathbb{R}^d$ and $\gamma \in (0, \bar{\gamma}_1]$, $\|\delta_x R_\gamma^k - \pi_\gamma\|_{V_c} \leq C\rho^{k\gamma} V_c(x)$.*

PROOF. This result is a simple consequence of [9, Corollary 11]. Its proof is given for completeness in Section 5.1.1. \square

Note that this result can be made quantitative, and other convergence results in total variation and Wasserstein distance of order $p \in [1, +\infty)$ can also be established following the same lines as the proof of [9, Corollary 14]. However, these results are out of the scope of the present paper and would be simple adaptations of those in [9] or [17].

We now consider an assumption which quantifies the perturbation associated with \tilde{T}_γ relatively to T_γ , for $\gamma \in (0, \bar{\gamma}]$.

H2. *There exists $c_\infty > 0$ such that $\sup_{x \in \mathbb{R}^d} \|T_\gamma(x) - \tilde{T}_\gamma(x)\| \leq \gamma c_\infty$ for all $\gamma \in (0, \bar{\gamma}]$.*

EXAMPLE 2. *The assumption **H2** holds for the Euler scheme applied to diffusions with scalar covariance matrices, i.e. (1) and (2) with*

$$(5) \quad T_\gamma(x) = x + \gamma b(x) \quad \text{and} \quad \tilde{T}_\gamma(x) = x + \gamma \tilde{b}(x),$$

under the condition that $\sup_{x \in \mathbb{R}^d} \|b(x) - \tilde{b}(x)\| \leq c_\infty$. This setting is exactly the one we introduced to motivate our study. In particular, in the case where $b = -\nabla U$ for some potential U , \tilde{b} may correspond to a numerical approximation of this gradient.

Note that compared to T_γ , $\gamma \in (0, \bar{\gamma}]$, we do not assume any smoothness condition on \tilde{T}_γ . More precisely, we do not assume that \tilde{T}_γ satisfies **H1**. Regarding the ergodicity properties of \tilde{R}_γ associated with \tilde{T}_γ , $\gamma \in (0, \bar{\gamma}]$, we have the following result.

PROPOSITION 3. *Assume **H1** and **H2** and set $\bar{\gamma}_1 = \bar{\gamma} \wedge \{1/\mathfrak{m}\}$. Then, for any $\gamma \in (0, \bar{\gamma}_1]$, \tilde{R}_γ admits a unique stationary distribution $\tilde{\pi}_\gamma$. In addition, there exists $c > 0$ such that for any $\gamma \in (0, \bar{\gamma}_1]$, there exist $\rho_\gamma \in [0, 1)$ and $C_\gamma \geq 0$ such that for any $x \in \mathbb{R}^d$ $\|\delta_x \tilde{R}_\gamma^k - \tilde{\pi}_\gamma\|_{V_c} \leq C_\gamma \rho_\gamma^{k\gamma} V_c(x)$, where $V_c(x) = \exp(c\|x\|^2)$.*

PROOF. The proof is postponed to Section 5.1.2. \square

Similarly to Proposition 1 with respect to R_γ , Proposition 3 implies that \tilde{R}_γ is V_c -uniformly geometrically ergodic. However in contrast to Proposition 1, the dependency of the rate of convergence with respect to the step size γ is not explicit anymore since the results and the method employed in [9] or [17] cannot be applied anymore.

Note that Proposition 1 and Proposition 3 imply that R_γ and \tilde{R}_γ converge to π_γ and $\tilde{\pi}_\gamma$ respectively in total variation and Wasserstein metric of any order $p \in [1, +\infty)$.

¹[9, Theorem 15] consider the case where T_γ comes from the Euler discretization scheme and has form $T_\gamma(x) = x + \gamma b(x)$.

Based on the two assumptions above, we can now state one of our main results. Our goal is to quantify the distance between the laws of the iterates of the two chains $(Y_k)_{k \in \mathbb{N}}$ and (\tilde{Y}_k) , in particular starting from the same initial point $x \in \mathbb{R}^d$ or at equilibrium. Indeed, remark that, in view of Propositions 1 and 3, letting $k \rightarrow +\infty$ in the next statement yields quantitative bounds on $\mathcal{W}_{\mathbf{c}}(\pi_\gamma, \tilde{\pi}_\gamma)$ for any $\gamma \in (0, \bar{\gamma} \wedge \{1/\mathfrak{m}\}]$.

THEOREM 4. *Assume **H1** and **H2** hold. There exist some explicit constants $C, c \geq 0$, $\rho \in [0, 1)$ satisfying for any $k \in \mathbb{N}$, $\gamma \in (0, \bar{\gamma}]$ and $x, \tilde{x} \in \mathbb{R}^d$,*

$$\mathcal{W}_{\mathbf{c}}(\delta_x R_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k) \leq C \rho^{\gamma k} \mathcal{V}(\|x - \tilde{x}\|) + cc_\infty,$$

where $\mathbf{c}(x, \tilde{x}) = \tilde{\mathbf{c}}(\|x - \tilde{x}\|)$, with

$$(\tilde{\mathbf{c}}, \mathcal{V}) \in \{(\mathbb{1}_{(0,+\infty)}, |\cdot|), (|\cdot|, |\cdot|), (\mathbb{1}_{(0,+\infty)} \exp(|\cdot|), \mathbb{1}_{(0,+\infty)} \exp(|\cdot|))\}.$$

REMARK 5. *It is also possible to treat the case of cost functions \mathbf{c} of the form $\mathbf{c}(x, y) = \tilde{\mathbf{c}}(\|x - y\|) (V(x) + V(y))$ with $\tilde{\mathbf{c}}$ as in Theorem 4 and V a positive function, simply by using Hölder's inequality. Indeed, for $p, q > 1$ with $1/p + 1/q = 1$, we can bound*

$$\mathcal{W}_{\mathbf{c}}(\nu, \mu) \leq (\mathcal{W}_{\mathbf{c}_p}(\nu, \mu))^{1/p} \left((\nu(V^q))^{1/q} + (\mu(V^q))^{1/q} \right)$$

with $\mathbf{c}_p(x, y) = \tilde{\mathbf{c}}^p(\|x - y\|)$. Bounds on the $\mathcal{W}_{\mathbf{c}_p}$ distance can then be established as in Theorem 4, while bounds on expected values of V^q , independent from γ , are classically obtained through Lyapunov arguments (see e.g. the proof of Proposition 1 in Section 5.1.1).

The rest of this section is devoted to the proof of this result. In particular, we define in the following the main object of this paper.

2.2. The discrete sticky kernel. We define a Markovian coupling of the two chains $(Y_k)_{k \in \mathbb{N}}$ and $(\tilde{Y}_k)_{k \in \mathbb{N}}$ defined in (1) and (2) by using, at each step, the maximal reflection coupling of the two Gaussian proposals, which is optimal for the total variation distance (i.e that maximizes the probability of coalescence). Let $(U_k)_{k \geq 1}$ be a sequence of i.i.d. uniform random variables on $[0, 1]$ independent from $(Z_k)_{k \geq 1}$ which we recall is a sequence of i.i.d. d -dimensional standard Gaussian random variables. We define the discrete sticky Markov coupling K_γ of R_γ and \tilde{R}_γ as the Markov kernel associated with the Markov chain on \mathbb{R}^{2d} given for $k \in \mathbb{N}$ by

$$(6) \quad \begin{aligned} X_{k+1} &= T_\gamma(X_k) + (\sigma^2 \gamma)^{1/2} Z_{k+1} \\ \tilde{X}_{k+1} &= X_{k+1} B_{k+1} + (1 - B_{k+1}) F_\gamma(X_k, \tilde{X}_k, Z_{k+1}), \end{aligned}$$

where $B_{k+1} = \mathbb{1}_{[0,+\infty)}(p_\gamma(X_k, \tilde{X}_k, Z_{k+1}) - U_{k+1})$ and

$$F_\gamma(x, \tilde{x}, z) = \tilde{T}_\gamma(\tilde{x}) + (\sigma^2 \gamma)^{1/2} \left\{ \text{Id} - 2\mathbf{e}(x, \tilde{x})\mathbf{e}(x, \tilde{x})^\top \right\} z,$$

$$(7) \quad E(x, \tilde{x}) = \tilde{T}_\gamma(\tilde{x}) - T_\gamma(x), \quad \mathbf{e}(x, \tilde{x}) = \begin{cases} \frac{E(x, \tilde{x})}{\|E(x, \tilde{x})\|} & \text{if } E(x, \tilde{x}) \neq 0 \\ \mathbf{e}_0 & \text{otherwise,} \end{cases}$$

$$p_\gamma(x, \tilde{x}, z) = 1 \wedge \left[\frac{\varphi_{\sigma^2 \gamma} \left\{ \|E(x, \tilde{x})\| - (\sigma^2 \gamma)^{1/2} \langle \mathbf{e}(x, \tilde{x}), z \rangle \right\}}{\varphi_{\sigma^2 \gamma} \left\{ (\sigma^2 \gamma)^{1/2} \langle \mathbf{e}(x, \tilde{x}), z \rangle \right\}} \right],$$

where $\mathbf{e}_0 \in \mathbb{R}^d$ is an arbitrary unit-vector, *i.e.* $\|\mathbf{e}_0\| = 1$, and $\varphi_{\sigma^2\gamma}$ is the density of the one-dimensional Gaussian distribution with mean 0 and variance $\sigma^2\gamma$. In other words, K_γ is given for any $\gamma \in (0, \bar{\gamma}]$, $(x, y) \in \mathbb{R}^{2d}$ and $\mathbf{A} \in \mathcal{B}(\mathbb{R}^{2d})$ by

$$\begin{aligned} K_\gamma((x, \tilde{x}), \mathbf{A}) &= \int_{\mathbb{R}^d} \mathbb{1}_{\mathbf{A}}(T_\gamma(x) + (\sigma^2\gamma)^{1/2}z, T_\gamma(x) + (\sigma^2\gamma)^{1/2}z) p_\gamma(x, \tilde{x}, z) \frac{e^{-\|z\|^2/2}}{(2\pi)^{d/2}} dz \\ &\quad + \int_{\mathbb{R}^d} \mathbb{1}_{\mathbf{A}}\left(T_\gamma(x) + (\sigma^2\gamma)^{1/2}z, F_\gamma(x, \tilde{x}, z)\right) (1 - p_\gamma(x, \tilde{x}, z)) \frac{e^{-\|z\|^2/2}}{(2\pi)^{d/2}} dz. \end{aligned}$$

In words, from the initial conditions (x, \tilde{x}) , this coupling works as follows: first, a Gaussian variable Z_{k+1} is drawn for the fluctuation of X_{k+1} . Then, \tilde{X}_{k+1} is made equal to X_k with probability $p(x, \tilde{x}, Z_{k+1})$ and, otherwise, the fluctuation of \tilde{X}_{k+1} with respect to its average $\tilde{T}_\gamma(\tilde{x})$ is given by the orthogonal reflection of Z_{k+1} in the direction $\tilde{T}_\gamma(\tilde{x}) - T_\gamma(x)$.

It is well known that for any $(x, \tilde{x}) \in \mathbb{R}^d$, $K_\gamma((x, \tilde{x}), \mathbf{A} \times \mathbb{R}^d) = R_\gamma(x, \mathbf{A})$ and $K_\gamma((x, \tilde{x}), \mathbb{R}^d \times \mathbf{A}) = \tilde{R}_\gamma(x, \mathbf{A})$, see e.g. [4, Section 3.3], [13, Section 4.1], [17] or [9].

The starting point of our analysis is the next result, which will enable to compare the coupling difference process $\|X_{k+1} - \tilde{X}_{k+1}\|$ with a Markov chain on $[0, +\infty)$. Define $(G_k)_{k \geq 1}$ for any $k \geq 1$ by

$$(8) \quad G_k = \langle \mathbf{e}(X_{k-1}, \tilde{X}_{k-1}), Z_k \rangle,$$

where \mathbf{e} is given by (7). For any $a \geq 0$, $g \in \mathbb{R}$, $u \in [0, 1]$ and $\gamma \in (0, \bar{\gamma}]$ define

$$(9) \quad \mathcal{H}_\gamma(a, g, u) = \mathbb{1}_{[0, +\infty)}(u - \bar{p}_{\sigma^2\gamma}(a, g)) \left(a - 2(\sigma^2\gamma)^{1/2}g \right),$$

where

$$(10) \quad \bar{p}_{\sigma^2\gamma}(a, g) = 1 \wedge \frac{\varphi_{\sigma^2\gamma}\left(a - (\sigma^2\gamma)^{1/2}g\right)}{\varphi_{\sigma^2\gamma}\left((\sigma^2\gamma)^{1/2}g\right)}.$$

PROPOSITION 6. *Assume **H1** and **H2** hold. Then for any $\gamma \in (0, \bar{\gamma}]$, $k \in \mathbb{N}$, almost surely, we have*

$$(11) \quad \|X_{k+1} - \tilde{X}_{k+1}\| \leq \mathcal{G}_\gamma(\|X_k - \tilde{X}_k\|, G_{k+1}, U_{k+1}),$$

where $(X_k, \tilde{X}_k)_{k \in \mathbb{N}}$ are defined by (6), and for any $w \in [0, +\infty)$, $g \in \mathbb{R}$ and $u \in [0, 1]$,

$$\mathcal{G}_\gamma(w, g, u) = \mathcal{H}_\gamma(\tau_\gamma(w) + \gamma c_\infty, g, u).$$

In addition, for any $g \in \mathbb{R}^d$ and $u \in [0, 1]$, $w \rightarrow \mathcal{G}_\gamma(w, g, u)$ is non-decreasing.

PROOF. The proof is postponed to Section 5.1.3. □

Consider now the stochastic process $(W_k)_{k \in \mathbb{N}}$ starting from $\|X_0 - \tilde{X}_0\|$ and defined by induction on k as follows,

$$(12) \quad \begin{aligned} W_{k+1} &= \mathcal{G}_\gamma(W_k, G_{k+1}, U_{k+1}) \\ &= \begin{cases} \tau_\gamma(W_k) + \gamma c_\infty - 2\sigma\sqrt{\gamma}G_k & \text{if } U_{k+1} \geq \bar{p}_{\sigma^2\gamma}(\tau_\gamma(W_k) + \gamma c_\infty, G_k) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By definition (8) and (7), an easy induction implies that $(G_k)_{k \geq 1}$ and $(U_k)_{k \geq 1}$ are independent, $(G_k)_{k \geq 1}$ are i.i.d. standard Gaussian random variables and $(U_k)_{k \geq 1}$ are

i.i.d. uniform random variables on $[0, 1]$. Therefore, $(W_k)_{k \in \mathbb{N}}$ is a Markov chain with Markov kernel Q_γ defined for $w \in [0, +\infty)$ and $A \in \mathcal{B}([0, +\infty))$ by

$$(13) \quad Q_\gamma(w, A) = \delta_0(A) \int_{\mathbb{R}} \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g) \varphi(g) dg \\ + \int_{\mathbb{R}} \mathbb{1}_A(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\gamma^{1/2}g) \{1 - \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g)\} \varphi(g) dg,$$

where φ is the density of the standard Gaussian distribution on \mathbb{R} . By Proposition 6, we have almost surely for any $k \in \mathbb{N}$,

$$(14) \quad \|X_k - \tilde{X}_k\| \leq W_k.$$

Another consequence of Proposition 6 is that Q_γ is stochastically monotonous (see e.g. [23] or [28]), more precisely if $(W_k)_{k \in \mathbb{N}}$ and $(\tilde{W}_k)_{k \in \mathbb{N}}$ are two chains given by (12) with the same variables $(G_k, U_k)_{k \in \mathbb{N}}$ with $W_0 \leq \tilde{W}_0$, then almost surely $W_k \leq \tilde{W}_k$ for all $k \in \mathbb{N}$. This nice property will be used several times in the analysis of this chain.

The main consequence of (14) is the following result.

COROLLARY 7. *Assume **H1** and **H2** hold. Let $\mathbf{c} : \mathbb{R}^{2d} \rightarrow [0, +\infty)$ of the form $\mathbf{c}(x, y) = \tilde{\mathbf{c}}(\|x - y\|)$ for some non-decreasing function $\tilde{\mathbf{c}} : [0, +\infty) \rightarrow [0, +\infty)$, $\tilde{\mathbf{c}}(0) = 0$. For any $x, \tilde{x} \in \mathbb{R}^d$ and $k \in \mathbb{N}$,*

$$\mathcal{W}_{\mathbf{c}}(\delta_x R_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k) \leq \int_{\mathbb{R}^{2d}} \mathbf{c}(y, \tilde{y}) K_\gamma^k((x, \tilde{x}), d(y, \tilde{y})) \leq \int_0^{+\infty} \tilde{\mathbf{c}}(\tilde{w}) Q_\gamma^k(\|x - \tilde{x}\|, d\tilde{w}).$$

PROOF. Let $k \in \mathbb{N}$. By (14) and since $\tilde{\mathbf{c}}$ is non-decreasing, we get almost surely $\tilde{\mathbf{c}}(\|X_k - \tilde{X}_k\|) \leq \tilde{\mathbf{c}}(W_k)$. Taking the expectation concludes the proof. \square

From Corollary 7, the question to get bounds on $\mathcal{W}_{\mathbf{c}}(\delta_x R_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k)$ boils down to the study of the Markov kernel Q_γ on $[0, +\infty)$, which is the main part of our work.

2.3. Analysis of the auxiliary Markov chain. We start with a Lyapunov/drift result.

PROPOSITION 8. *Assume **H1** and **H2** hold. Then for any $w \geq 0$,*

$$Q_\gamma \mathcal{V}_1^*(w) \leq (1 - \gamma \mathfrak{m}) \mathcal{V}_1^*(w) \mathbb{1}_{(R_1, +\infty)}(w) + (1 + \gamma \mathfrak{L}) \mathcal{V}_1^*(w) \mathbb{1}_{[0, R_1]}(w) + \gamma c_\infty,$$

where Q_γ is defined by (13) and for any $w \in \mathbb{R}$, $\mathcal{V}_1^*(w) = |w|$.

PROOF. The proof is postponed to Section 5.1.4. \square

Proposition 8 implies in particular that for any $w \in \mathbb{R}$,

$$Q_\gamma \mathcal{V}_1^*(w) \leq (1 - \gamma \mathfrak{m}) \mathcal{V}_1^*(w) \mathbb{1}_{(R_1, +\infty)}(w) + \gamma[(\mathfrak{L} + \mathfrak{m})R_1 + c_\infty].$$

Then, a straightforward induction shows that for any $k \in \mathbb{N}$,

$$Q_\gamma^k \mathcal{V}_1^*(w) \leq (1 - \gamma \mathfrak{m})^k \mathcal{V}_1^*(w) + [(\mathfrak{L} + \mathfrak{m})R_1 + c_\infty]/\mathfrak{m},$$

and therefore by Corollary 7 taking $\tilde{\mathbf{c}}(t) = t$,

$$(15) \quad \mathcal{W}_1(\delta_x R_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k) \leq (1 - \gamma \mathfrak{m})^k \|x - \tilde{x}\| + [(\mathfrak{L} + \mathfrak{m})R_1 + c_\infty]/\mathfrak{m}.$$

However, this result is not sharp as $k \rightarrow +\infty$. Indeed, in the case $c_\infty = 0$, $R_\gamma = \tilde{R}_\gamma$ and by Proposition 1, it holds that $\mathcal{W}_1(\delta_x R_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k) \rightarrow 0$ as $k \rightarrow +\infty$, while the right-hand side of (15) converges to $(\mathfrak{L} + \mathfrak{m})R_1/\mathfrak{m} \neq 0$. We need to refine our results in order to fill this gap. To this end, we need to analyze more precisely the long-time behavior of Q_γ . A first step is to show that it is ergodic.

PROPOSITION 9. Assume **H 1** and **H 2** hold. For any $\gamma \in (0, \bar{\gamma}]$, Q_γ admits a unique invariant probability measure μ_γ and is geometrically ergodic. In addition, $\mu_\gamma(\{0\}) > 0$ and μ_γ is absolutely continuous with respect to the measure $\delta_0 + \text{Leb}$ on $([0, +\infty), \mathcal{B}([0, +\infty)))$. Finally, in the case $c_\infty \neq 0$, μ_γ and $\delta_0 + \text{Leb}$ are equivalent.

PROOF. The proof is postponed to Section 5.1.5. \square

COROLLARY 10. Assume **H 1** and **H 2** hold. Let $\mathbf{c} : \mathbb{R}^{2d} \rightarrow [0, +\infty)$ of the form $\mathbf{c}(x, y) = \tilde{\mathbf{c}}(\|x - y\|)$ for some non-decreasing function $\tilde{\mathbf{c}} : [0, +\infty) \rightarrow [0, +\infty)$, $\tilde{\mathbf{c}}(0) = 0$. For any $x, \tilde{x} \in \mathbb{R}^d$ and $k \in \mathbb{N}$,

$$(16) \quad \mathcal{W}_{\mathbf{c}}(\delta_x R_\gamma^k, \delta_{\tilde{x}} \tilde{R}_\gamma^k) \leq \int_0^{+\infty} \tilde{\mathbf{c}}(\tilde{w}) \{Q_\gamma^k(\|x - \tilde{x}\|, \cdot) - \mu_\gamma\}(\mathrm{d}\tilde{w}) + \mu_\gamma(\tilde{\mathbf{c}}).$$

where μ_γ is the stationary distribution of Q_γ given by (13). In particular, if $x = \tilde{x}$, $\mathcal{W}_{\mathbf{c}}(\delta_x R_\gamma^k, \delta_x \tilde{R}_\gamma^k) \leq \mu_\gamma(\tilde{\mathbf{c}})$.

PROOF. The proof of (16) is a consequence of Proposition 9 and Corollary 7. The last statement follows from the fact that Q_γ is stochastically monotonous. Indeed, by Proposition 6, for any $w, \tilde{w} \in [0, +\infty)$, $w \leq \tilde{w}$, and $a \in [0, +\infty)$, $Q_\gamma(w, [0, a]) \geq Q_\gamma(\tilde{w}, [0, a])$. Therefore, for any $a \in [0, +\infty)$, $w \mapsto Q_\gamma(w, [0, a])$ is non-increasing on $[0, +\infty)$ and for any non-increasing bounded function f , $Q_\gamma f(w) \geq Q_\gamma f(\tilde{w})$ for any $w, \tilde{w} \in [0, +\infty)$, $w \leq \tilde{w}$. As a result, a straightforward induction shows that for any $k \in \mathbb{N}$, $w, \tilde{w} \in [0, +\infty)$, $w \leq \tilde{w}$, and $a \in [0, +\infty)$, $Q_\gamma^k(w, [0, a]) \geq Q_\gamma^k(\tilde{w}, [0, a])$. Then, we obtain $Q_\gamma^k(0, [0, a]) \geq \int_0^{+\infty} \mu(\mathrm{d}w) Q_\gamma^k(w, [0, a]) = \mu_\gamma([0, a])$. Since $\tilde{\mathbf{c}}$ is non-decreasing on $[0, +\infty)$, we get $Q_\gamma \tilde{\mathbf{c}}(0) \leq \mu_\gamma(\tilde{\mathbf{c}})$, which combined with (16) completes the proof. \square

Corollary 10 then naturally brings us to derive moment bounds for the stationary distribution μ_γ , $\gamma \in (0, \bar{\gamma}]$ and quantitative convergence for Q_γ to μ_γ . Our next results address these two problems.

THEOREM 11. Assume **H 1** and **H 2** hold. For any $\bar{\delta} \in (0, \{\mathbf{L}^{-1} \wedge (\sigma \mathbf{e}^{-1}/c_\infty)^2\}]$ and $\gamma \in (0, \bar{\gamma}]$,

$$(17) \quad \int_{[0, +\infty)} w \mu_\gamma(\mathrm{d}w) \leq c_\infty c_1, \quad \mu_\gamma((0, +\infty)) \leq c_\infty c_2,$$

where μ_γ is the stationary distribution of Q_γ given by (13), and, considering ζ given below in (58),

$$\begin{aligned} c_1 &= \eta_1 R_1 (1 + \mathbf{L}/\mathbf{m}) + 1/\mathbf{m}, \\ c_2 &= \mathbf{e}^{(\bar{\delta} + \bar{\gamma})\mathbf{L}} (c_1 (1 + \bar{\gamma}\mathbf{L})/\bar{\delta}^{1/2} + [\bar{\delta} + \bar{\gamma}]^{1/2})/(\sqrt{2\pi}\sigma) + 2\zeta[\bar{\delta} + \bar{\gamma}]^{1/2} \mathbf{e}^{3(\bar{\delta} + \bar{\gamma})\mathbf{L}}/\sigma^3, \\ \eta_1 &= [\bar{\delta} + \bar{\gamma}]^{1/2} \left[\frac{\zeta \mathbf{e}^{3(\bar{\delta} + \bar{\gamma})\mathbf{L}}}{\sigma^3} + \frac{\mathbf{e}^{(\bar{\delta} + \bar{\gamma})\mathbf{L}}}{2\sqrt{2\pi}\sigma} \right] / \Phi \left(-\frac{(1 + \bar{\gamma}\mathbf{L})R_1 + (\bar{\delta} + \bar{\gamma})c_\infty}{2\bar{\delta}^{1/2}\sigma \mathbf{e}^{-(\bar{\delta} + \bar{\gamma})\mathbf{L}}} \right). \end{aligned}$$

PROOF. The proof is postponed to Section 5.1.6. \square

THEOREM 12. Assume **H 1** and **H 2** hold. For any $a > 0$ and $\gamma \in (0, \bar{\gamma}]$,

$$\int_0^{+\infty} \mathbb{1}_{(0, +\infty)}(w) \exp(aw) \mathrm{d}\mu_\gamma(w) \leq c_\infty c_3,$$

where c_3 is explicitly given in the proof and μ_γ is the stationary distribution of Q_γ given by (13).

PROOF. The proof is postponed to Section 5.1.7. \square

We now specify the convergence of Q_γ to μ_γ for any $\gamma \in (0, \bar{\gamma}]$.

THEOREM 13. Assume **H1** and **H2** hold. There exist explicit constants $\rho \in [0, 1)$ and $C \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}]$, $w \geq 0$,

$$\|\delta_w Q_\gamma^k - \mu_\gamma\|_{\mathcal{V}} \leq C \rho^{\gamma k} \mathcal{V}(w),$$

where $\mathcal{V}(w) = 1 + |w|$ or $\mathcal{V}(w) = \exp(a|w|)$, for $a > 0$.

PROOF. The proof is postponed to Section 5.1.8. \square

Combining the results of Corollary 10, Theorem 11, Theorem 12 and Theorem 13 allows to address the main questions raised in this section and prove Theorem 4.

Discussion on the bounds provided by Theorem 11. In this paragraph, we discuss how the constants c_1, c_2 given in Theorem 11 behaves with respect to the parameters R_1, L, m in the limit $c_\infty \rightarrow 0$ and $\bar{\gamma} \rightarrow 0$. For ease of presentation, we also only consider the case $\sigma = 1$.

- (1) First consider the case $R_1 = 0$. then $c_1 \approx m^{-1}$ and $c_2 \approx [C_1/[m\bar{\delta}^{1/2}] + C_2\bar{\delta}^{1/2}]$ for $\bar{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\})$ and numerical constants $C_1, C_2 \geq 0$. Then, since in the case $R_1 = 0$, we can take $L \rightarrow 0$ and $\bar{\delta}$ can be arbitrary chosen in $(0, +\infty)$. Taking $\bar{\delta} \leftarrow m^{-1}$, we obtain that $c_2 \approx \tilde{C}_1/m^{1/2}$. Note that the dependency of c_1, c_2 with respect to $m^{1/2}$ is then sharp. Indeed, consider the particular example of two auto-regressive processes for which $T_\gamma(y) = (1 - \varrho\gamma)y$ and $\tilde{T}_\gamma(y) = (1 - \varrho\gamma)y + \gamma\varrho a$ for $\gamma \in (0, \varrho^{-1})$ and $a, \varrho > 0$. Then, on the one hand, **H1** and **H2** are satisfied with $R_1 = 0$, $m = \varrho$ and $c_\infty = \varrho a$ which lead to $c_1 c_\infty \sim a$ and $c_2 c_\infty \sim Ca/\varrho^{1/2}$, for some universal constant $C \geq 0$. On the other hand, an easy computation (see e.g. [12]) shows that the stationary distributions π_γ and $\tilde{\pi}_\gamma$ provided by Proposition 1 and Proposition 3 are $\mathbf{N}(0, \varrho^{-1}(2 - \gamma\varrho\gamma)^{-1})$ and $\mathbf{N}(a, \varrho^{-1}(2 - \gamma\varrho\gamma)^{-1})$ respectively. Therefore, we get $\mathcal{W}_1(\pi_\gamma, \tilde{\pi}_\gamma) = a$ and $\|\pi_\gamma - \tilde{\pi}_\gamma\|_{TV} \sim Ca/\varrho^{1/2}$ as $\gamma, a \rightarrow 0$ and we get up to universal constants c_1 and c_2 .
- (2) We now consider the case $R_1 \geq 1$, $L = 0$. Note that in this case $\bar{\delta}$ can be arbitrary taken in $(0, 1)$. Then, for some universal constants C_1, C_2, C_3 , $\eta_1 \sim C_1\bar{\delta}^{1/2}/\Phi\{C_2R_1/\bar{\delta}^{1/2} + C_3c_\infty\bar{\delta}^{1/2}\}$. Therefore, taking $\bar{\delta} \leftarrow m^{-1} \vee R_1^2$, we get that for some universal constants $D_1, D_2, E \geq 0$, $c_1 \approx D_1[(R_1 \vee m^{-1/2}) + m^{-1}]$, $c_2 \approx Em^{-1/2} \vee R_1$. Note that the behaviour of c_2 with respect to R_1 and m is consistent with the results obtained in [18] (see [18, Lemma 1]) for the stationary distributions of continuous sticky processes. Note that it is shown in [18, Example 2] that this bound is sharp with respect to R_1 and m .
- (3) In the case $R_1 \wedge L \geq 1$, taking $\bar{\delta} = L^{-1}$ since we are in the regime $c_\infty \rightarrow 0$, we get that up to logarithmic term and using $\bar{\gamma} \leq L^{-1}$, $c_1, c_2 \approx \exp[e^4(R_1L^{1/2} + c_\infty)^2]$. The estimate for c_2 is also consistent with [18, Lemma 1] which holds for stationary distributions of continuous sticky processes.

3. Continuous-time limit. In the case where T_γ and \tilde{T}_γ are specified by (5), then under appropriate conditions on b and \tilde{b} , it can be shown, see e.g. [9, Proposition 25], that for any $T \geq 0$ and $x \in \mathbb{R}^d$,

$$(18) \quad \lim_{m \rightarrow +\infty} \{\|\delta_x R_{T/m}^m - \delta_x P_T\|_V + \|\delta_x \tilde{R}_{T/m}^m - \delta_x \tilde{P}_T\|_V\} = 0,$$

for some measurable function $V : \mathbb{R}^d \rightarrow [1, +\infty)$ and where $(P_t)_{t \geq 0}$ and $(\tilde{P}_t)_{t \geq 0}$ are the Markov semigroup corresponding to (1) and (2). Then, this naturally implies convergence in total variation and also Wasserstein distance of order p if $\inf_{x \in \mathbb{R}^d} \{V(x)/\|x\|^p\} > 0$. As a consequence, results of Section 2 immediately transfer to the continuous-time processes. More precisely, let $\mathbf{c} : \mathbb{R}^{2d} \rightarrow [0, +\infty)$ of the form $\mathbf{c}(x, y) = \tilde{\mathbf{c}}(\|x - y\|)$ for some non-decreasing function $\tilde{\mathbf{c}} : [0, +\infty) \rightarrow [0, +\infty)$, $\tilde{\mathbf{c}}(0) = 0$. If (18) holds and $\sup_{x, y \in \mathbb{R}^d} \{\mathbf{c}(x, y)/\{V(x) + V(y)\}\} < +\infty$, we get by the triangle inequality that for any $x, \tilde{x} \in \mathbb{R}^d$, $T > 0$, $\mathcal{W}_{\mathbf{c}}(\delta_x P_T, \delta_{\tilde{x}} \tilde{P}_T) \leq \liminf_{m \rightarrow +\infty} \mathcal{W}_{\mathbf{c}}(\delta_x R_{T/m}^m, \delta_{\tilde{x}} \tilde{R}_{T/m}^m)$. Then, results of Section 2 can be applied implying if **H1** and **H2** holds, that for any $x, \tilde{x} \in \mathbb{R}^d$ there exist $C_1, C_2 \geq 0$ such that for any $T \geq 0$, $\mathcal{W}_{\mathbf{c}}(\delta_x P_T, \delta_{\tilde{x}} \tilde{P}_T) \leq C_1 \rho^T + C_2 c_\infty$. We therefore generalize the result provided in [18] which is specific to the total variation distance. We do not give a specific statement for this result which is mainly technical and is not the main subject of this paper. Instead, the goal of this section is to study the continuous-time limit of the coupling (6) (and not only of its marginals) toward some continuous-time sticky diffusion.

More precisely, let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of step sizes such that $\lim_{n \rightarrow +\infty} \gamma_n = 0$ and $w_0 \geq 0$. Then, consider the sequence of Markov chains $\{(W_k^{(n)})_{k \in \mathbb{N}} : n \in \mathbb{N}\}$ for any $n \in \mathbb{N}$, $(W_k^{(n)})_{k \in \mathbb{N}}$ is the Markov chain defined by (12) with $W_0^{(n)} = w_0$, $\gamma = \gamma_n$ and therefore associated with the Markov kernel Q_{γ_n} . Let $\{(\mathbf{W}_t^{(n)})_{t \in (0, +\infty)} : n \in \mathbb{N}\}$ be the continuous linear interpolation of $\{(W_k^{(n)})_{k \in \mathbb{N}} : n \in \mathbb{N}\}$, *i.e.* the sequence of continuous processes defined for any $n \in \mathbb{N}$, $t \in (0, +\infty)$ by

$$(19) \quad \mathbf{W}_t^{(n)} = W_{\lfloor t/\gamma_n \rfloor}^{(n)} + \{W_{\lfloor t/\gamma_n \rfloor + 1}^{(n)} - W_{\lfloor t/\gamma_n \rfloor}^{(n)}\} \{t/\gamma_n - \lfloor t/\gamma_n \rfloor\}.$$

Note that for any $k \in \mathbb{N}$ and $h \in [0, \gamma_n]$, $\mathbf{W}_{k\gamma_n + h}^{(n)} = W_k^{(n)} + (h/\gamma_n)\{W_{k+1}^{(n)} - W_k^{(n)}\}$. We denote by $\mathbb{W} = C([0, +\infty), \mathbb{R})$ endowed with the uniform topology on compact sets, \mathcal{W} its corresponding σ -field and $(W_t)_{t \geq 0}$ the canonical process defined for any $t \in (0, +\infty)$ and $\omega \in \mathbb{W}$ by $W_t(\omega) = \omega_t$. Denote by $(\mathcal{W}_t)_{t \geq 0}$ the filtration associated with $(W_t)_{t \geq 0}$. Note that $\{(\mathbf{W}_t^{(n)})_{t \in (0, +\infty)} : n \in \mathbb{N}\}$ is a sequence of \mathbb{W} -valued random variables. The main result of this section concerns the convergence in distribution of this sequence.

We consider the following assumption on the function τ_γ .

A 1. *There exists a function $\kappa : [0, +\infty) \rightarrow [0, +\infty)$ such that for any $\gamma \in (0, \bar{\gamma}]$, $\tau_\gamma(w) = w + \gamma\kappa(w)$ and $\kappa(0) = 0$. In addition, κ is L_κ -Lipschitz: for any $w_1, w_2 \in (0, +\infty)$, $|\kappa(w_1) - \kappa(w_2)| \leq L_\kappa |w_1 - w_2|$.*

This is not a restrictive condition since, under **H1**, up to a possible modification of τ_γ , it is always possible to ensure **A1**.

Under **A1**, we consider a sticky process [29, 30, 18], which solves the stochastic differential equation

$$(20) \quad d\mathbf{W}_t = \{\kappa(\mathbf{W}_t) + c_\infty\}dt + 2\sigma \mathbb{1}_{(0, +\infty)}(\mathbf{W}_t)dB_t,$$

where $(B_t)_{t \geq 0}$ is a one-dimensional Brownian motion. Note that for any initial distribution μ_0 on $(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$, (20) admits a unique weak solution by [18, Lemma 6].

The main result of this section is the following.

THEOREM 14. *Assume **A1**. Then, the sequence $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ defined by (19) converges in distribution to the solution $(\mathbf{W}_t)_{t \geq 0}$ of the SDE (20).*

The proof of this theorem follows the usual strategy employed to show convergence of a sequence of continuous processes to a Markov process. A first step is to show that under **A 1**, $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ is uniformly bounded in L^q for some $q \geq 2$, on $[0, T]$ for any $T \geq 0$.

PROPOSITION 15. *Assume **A 1**. Then for any $T \geq 0$, there exists $C_T \geq 0$ such that $\sup_{n \in \mathbb{N}} \mathbb{E}[\sup_{t \in [0, T]} \{(\mathbf{W}_t^{(n)})^4\}] \leq C_T$ where $(\mathbf{W}_t^{(n)})_{t \geq 0}$ is defined by (19).*

PROOF. The proof is postponed to Section 5.2.1. \square

Then, we are able to obtain the tightness of the sequence of stochastic processes $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$.

PROPOSITION 16. *Assume **A 1**. Then, $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ is tight in \mathbb{W} .*

PROOF. The proof is postponed to Section 5.2.2. \square

Denote for any $n \in \mathbb{N}$, μ_n the distribution of $(\mathbf{W}_t^{(n)})_{t \geq 0}$ on \mathbb{W} . Then, by Prohorov's Theorem [3, Theorem 5.1, 5.2], $(\mu_n)_{n \in \mathbb{N}}$ admits a limit point. If we now show that every limit point associated with $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ is a solution of the SDE (20) using that $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ is tight again and since (20) admits a unique weak solution, the proof of Theorem 14 will be completed. To establish this result, we use the characterization of solutions of SDEs through martingale problems. More precisely by [6, Theorem 1.27], the distribution μ on \mathbb{W} of $(\mathbf{W}_t)_{t \geq 0}$, solution of (20), is the unique solution to the martingale problem associated with μ_0 , the drift function $w \mapsto \kappa(w) + c_\infty$ and the variance function $2\sigma \mathbb{1}_{(0, +\infty)}$, *i.e.* it is the unique probability measure satisfying on the filtered probability space $(\mathbb{W}, \mathcal{W}, (\mathcal{W}_t)_{t \geq 0}, \mu)$:

- (a) the distribution of \mathbf{W}_0 is μ_0 ;
- (b) the processes $(M_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$ defined for any $t \geq 0$ by

$$(21) \quad M_t = \mathbf{W}_t - \mathbf{W}_0 - \int_0^t \{c_\infty + \kappa(\mathbf{W}_u)\} du, \quad N_t = M_t^2 - 4\sigma^2 \int_0^t \mathbb{1}_{(0, +\infty)}(\mathbf{W}_u) du,$$

are $(\mathcal{W}_t)_{t \geq 0}$ -local martingales.

In other words, it corresponds in showing that $(M_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -local martingales and by [27, Theorem 1.8] identifying its quadratic variation $(\langle M \rangle_t)_{t \geq 0}$ as the process $(4\sigma^2 \int_0^t \mathbb{1}_{(0, +\infty)}(\mathbf{W}_u) du)_{t \geq 0}$. Therefore, Theorem 14 is a direct consequence of the following result.

THEOREM 17. *Assume **A 1**. Let μ_∞ be a limit point of $(\mu_n)_{n \in \mathbb{N}}$. Then, the two processes $(M_t)_{t \geq 0}$ and $(N_t)_{t \geq 0}$ defined by (21) are $(\mathcal{W}_t)_{t \geq 0}$ -martingales on $(\mathbb{W}, \mathcal{W}, (\mathcal{W}_t)_{t \geq 0}, \mu_\infty)$.*

PROOF. The proof is postponed to Section 5.2.4. \square

Consider the differential operators $\mathcal{A}, \tilde{\mathcal{A}}$ defined for any $\psi \in C^2(\mathbb{R})$ by

$$(22) \quad \begin{aligned} \mathcal{A}\psi(w) &= \{\kappa(w) + c_\infty\}\psi'(w) + 2\mathbb{1}_{(0, +\infty)}(w)\sigma^2\psi''(w) \\ \tilde{\mathcal{A}}\psi(w) &= \{\kappa(w) + c_\infty\}\psi'(w) + 2\sigma^2\psi''(w), \end{aligned}$$

where κ is arbitrary extended on \mathbb{R} . Note that \mathcal{A} is the *extended* generator associated with (20). A crucial step in the proof of Theorem 17 is the following.

PROPOSITION 18. *Let $\varphi \in C^3(\mathbb{R})$, satisfying*

$$(23) \quad \sup_{w \in \mathbb{R}} \{ |\varphi|(w)/(1+w^2) + |\varphi'|(w)/(1+|w|) + |\varphi''|(w) + |\varphi^{(3)}|(w) \} < +\infty.$$

Then, for any $N \in \mathbb{N}$, $(t_1, \dots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \dots \leq t_N \leq s < t$, $\psi : [0, +\infty)^N \rightarrow \mathbb{R}$, positive, continuous and bounded, it holds that

$$(24) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\varphi(\mathbf{W}_t^{(n)}) - \varphi(\mathbf{W}_s^{(n)}) - \int_s^t \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) du \right) \psi(\mathbf{W}_{t_1}^{(n)}, \dots, \mathbf{W}_{t_N}^{(n)}) \right] = 0.$$

In addition, if $\varphi''(w) \geq 0$ for any $w \in \mathbb{R}$, it holds that

$$(25) \quad \limsup_{n \rightarrow +\infty} \mathbb{E} \left[\left(\varphi(\mathbf{W}_t^{(n)}) - \varphi(\mathbf{W}_s^{(n)}) - \int_s^t \tilde{\mathcal{A}}\varphi(\mathbf{W}_u^{(n)}) du \right) \psi(\mathbf{W}_{t_1}^{(n)}, \dots, \mathbf{W}_{t_N}^{(n)}) \right] \leq 0.$$

PROOF. The proof is postponed to Section 5.2.3. \square

Note that while Proposition 18-(24) is in general sufficient to conclude on the convergence of the sequence of processes $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ (see e.g. [19]), in our setting, it is not enough to complete the proof of Theorem 14 since the diffusion coefficient associated with \mathcal{A} is discontinuous. To circumvent this issue, we adapt to our sequence $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$ the same strategy employed in [26, Proposition 6].

4. An application in Bayesian statistics: parameter estimation in an ODE.

4.1. *Setting and verifying the assumptions.* Consider an ordinary differential equation ODE on \mathbb{R}^n of the form

$$(26) \quad \dot{x}_\theta(t) = f_\theta(x_\theta(t), t), \quad x_\theta(0) = x_0 \in \mathbb{R}^n,$$

where $\{f_\theta : \theta \in \mathbb{R}^d\}$ is a family of function from $\mathbb{R}^n \times [0, +\infty)$ to \mathbb{R}^n parametrized by some parameter $\theta \in \mathbb{R}^d$. In all this section $x_0 \in \mathbb{R}^n$ is assumed to be fixed and we consider the following assumption.

AO1. *For all $\theta \in \mathbb{R}^d$ there exists a unique solution of (26) defined for all positive times, which we denote by $(x_\theta(t))_{t \geq 0}$. In addition, the functions $(\theta, x, t) \in \mathbb{R}^d \times \mathbb{R}^n \times [0, +\infty) \mapsto f_\theta(x, t)$ and $(\theta, t) \in \mathbb{R}^d \times [0, +\infty) \mapsto x_\theta(t)$ are continuously differentiable.*

In fact the continuous differentiability of $(\theta, t) \mapsto x_\theta(t)$ is a consequence of the one of $(\theta, x, t) \mapsto f_\theta(x, t)$, see e.g. [31, Theorem 4.D].

To fix ideas, throughout this section, we will repeatedly discuss the following case of a logistic equation.

EXAMPLE 19. *For $r \in C^1(\mathbb{R}, \mathbb{R}_+)$, set $f_\theta(x) = x(1 - r(\theta)x)$ for any $\theta, x \in \mathbb{R}$, so that (26) reads*

$$\dot{x}_\theta(t) = x_\theta(t) (1 - r(\theta)x_\theta(t)), \quad x_\theta(0) = x_0,$$

with $x_0 \geq 0$. In this example, AO1 holds and, r and x_0 being positive, for all $\theta \in \mathbb{R}$, the solution of (26) is such that $x_\theta(t) \in [0, e^t x_0]$ for all $t \geq 0$. Indeed, $x \mapsto x(1 - r(\theta))$ is locally Lipschitz continuous, which yields existence and uniqueness of a maximal solution. Since 0 is always an equilibrium, solutions stay positive, from which $x'_\theta(t) \leq x_\theta(t)$ for all $t \geq 0$, implying that $x_\theta(t) \leq e^t x_0$ for all $t \geq 0$. This also implies non-explosion, hence the solution is defined on $[0, +\infty)$.

We consider the problem of estimating θ based on some observation of a trajectory of the ODE. More precisely, for $T > 0$, $N \in \mathbb{N}^*$, $(t_1, \dots, t_N) \in \mathbb{R}^N$, $0 < t_1 < \dots < t_N = T$, the statistical model corresponding to the observation $\mathbf{y} = (y_i)_{i \in \{1, \dots, N\}} \in (\mathbb{R}^n)^N$ is given by

$$(27) \quad y_i = x_\theta(t_i) + \varepsilon_i,$$

for $\theta \in \mathbb{R}^d$ and where $(\varepsilon_i)_{i \in \{1, \dots, N\}}$ are independent and identically random variables on \mathbb{R}^n distributed according to some known positive density φ_ε with respect to the Lebesgue measure. Given a prior distribution with positive density π_0 on \mathbb{R}^d , the a posteriori distribution for this model admits a positive density π with respect to the Lebesgue measure which is characterized by the potential U given by (up to an additive constant)

$$-\log \pi(\theta) = U(\theta) = -\ln \pi_0(\theta) - \sum_{i=1}^N \ln \varphi_\varepsilon(y_i - x_\theta(t_i)).$$

We consider the following assumption on π_0 and φ_ε setting $-\log(\pi_0) = U_0$.

AO2. *The functions π_0 and φ_ε are twice continuously differentiable and there exist $\mathfrak{m}_U > 0$, $L_U, R_U \geq 0$ such that ∇U_0 is L_U -Lipschitz continuous and for any $\theta, \tilde{\theta} \in \mathbb{R}^d$ with $\|\theta - \tilde{\theta}\| \geq R_U$,*

$$\langle \theta - \tilde{\theta}, \nabla U_0(\theta) - \nabla U_0(\tilde{\theta}) \rangle \geq \mathfrak{m}_U \|\theta - \tilde{\theta}\|^2.$$

In practice, expectations with respect to the posterior distribution can be approximated by ergodic means of the Unadjusted Langevin Algorithm (ULA), namely the Markov chain

$$(28) \quad X_{k+1} = X_k - \gamma \nabla U(X_k) + \sqrt{2\gamma} Z_{k+1},$$

where $\gamma > 0$ and $(Z_k)_{k \in \mathbb{N}}$ are independent and identically standard Gaussian variables. The long-time convergence of this algorithm and the numerical bias on the invariant measure due to the time discretization are well understood, see e.g. [7, 14, 15, 8, 10] and references therein. However, in the present case, it is not possible to sample this Markov chain, as the exact computation of

$$(29) \quad \nabla U(\theta) = -\nabla_\theta \ln \pi_0(\theta) + \sum_{i=1}^N \nabla_\theta x_\theta(t_i) \nabla_x \ln \varphi_\varepsilon(y_i - x_\theta(t_i)),$$

is not possible in most cases because of the term involving x_θ and $\nabla_\theta x_\theta$. Here ∇_θ and ∇_x denote the gradient operator with respect to θ and x respectively. Therefore, only approximations of these two functions can be used in place of $(x_\theta(t_i), \nabla_\theta x_\theta(t_i))_{i \in \llbracket 0, N \rrbracket}$, which leads to an additional discretization bias. Our results based on the sticky coupling yields a quantitative bound on this error (with respect to the ideal ULA above). Let us detail this statement.

First, remark that $t \mapsto z_\theta(t) = (x_\theta(t), \nabla_\theta x_\theta(t))$ solves

$$(30) \quad \dot{z}_\theta(t) = F_\theta(z_\theta(t), t) \quad z_\theta(0) = z_0 = (x_0, 0)$$

on $\mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R})$ with for any $x \in \mathbb{R}^n$, $\mathbf{A} \in \mathcal{M}_{d,n}(\mathbb{R})$, $\theta \in \mathbb{R}^d$, $t \geq 0$,

$$(31) \quad F_\theta((x, \mathbf{A}), t) = (f_\theta(x, t), \nabla_\theta f_\theta(x, t) + \mathbf{A} \nabla_x f_\theta(x, t)).$$

Provided f_θ , $\nabla_\theta f_\theta$ and $\nabla_x f$ are computable, in practice this ODE can be approximated by standard numerical schemes. For instance, a basic explicit Euler discretization with time-step $h > 0$ is given by

$$\tilde{z}_\theta^h(0) = z_0, \quad \tilde{z}_\theta^h((k+1)h) = \tilde{z}_\theta^h(kh) + hF_\theta(\tilde{z}_\theta^h(kh), kh) \quad \forall k \in \mathbb{N}$$

and

$$(32) \quad \tilde{z}_\theta^h(t) = \tilde{z}_\theta^h(kh) + (t - kh)F_\theta(\tilde{z}_\theta^h(kh), kh), \quad t \in [kh, (k+1)h).$$

To establish the consistency of this approximation (with some uniformity in θ), we consider the following condition.

AO3. *There exist $L_F, L'_F, C_F, \delta > 0$ and a compact set $K \subset \mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R})$ such that the following holds. For all $t \in [0, T]$ and $\theta \in \mathbb{R}^d$, the ball centered at $z_\theta(t)$ and radius δ is included in K . Moreover, for all $z, \tilde{z} \in K$, $t, s \in [0, T]$ and $\theta, \tilde{\theta} \in \mathbb{R}^d$, $\|F_\theta(z, t)\| \leq C_F$ and*

$$(33) \quad \|F_\theta(z, t) - F_{\tilde{\theta}}(\tilde{z}, s)\| \leq L_F \|\theta - \tilde{\theta}\| + L'_F (\|z - \tilde{z}\| + |t - s|).$$

PROPOSITION 20. *Assume **AO1** and **AO3**. There exist $\bar{h}, C > 0$ such that for all $h \in (0, \bar{h}]$, $\theta \in \mathbb{R}^d$, and $t \in [0, T]$, $\tilde{z}_\theta^h(t) \in K$ and $\sum_{i=1}^N \|z_\theta(t_i) - \tilde{z}_\theta^h(t_i)\| \leq Ch$, where z_θ solves (30) and \tilde{z}_θ^h is given by (32).*

PROOF. The proof is postponed to Section 5.3. □

EXAMPLE 21 (Continuation of Example 19). *Let us check for instance that **AO3** is satisfied for Example 19 provided that r is twice continuously differentiable on $[0, +\infty)$ with, for some $L_r, L'_r, L''_r > 0$,*

$$(34) \quad r, r' \text{ and } r'' \text{ uniformly bounded respectively by } L_r, L'_r \text{ and } L''_r.$$

We may consider for example $r : \theta \mapsto a_1\theta^2/(\theta^2 + a_2)$ for $a_1, a_2 \in (0, +\infty)$. Recall that $f_\theta(x) = x(1 - r(\theta)x)$ and $x_\theta(t) \in [0, e^t x_0]$ for all $\theta \in \mathbb{R}$ and all $t \in [0, T]$, so that for any $t \geq 0$ and $\theta \in \mathbb{R}^d$,

$$|\partial_\theta f_\theta(x_\theta(t))| = |r'(\theta)x_\theta^2(t)| \leq L'_r e^{2t} x_0^2.$$

for all $t \geq 0$. Notice that $1/r(\theta)$ is an equilibrium of the equation, so that it cannot be crossed by other solutions. Hence, on the one hand, if $1 \leq r(\theta)x_0$ then x_θ is non-increasing (in particular $x_\theta(t) \leq x_0$ for all $t \geq 0$) while, on the other hand, if $1 \geq r(\theta)x_0$, then $1 \geq r(\theta)x_\theta(t)$ for all $t \geq 0$. In both cases, we get that for all $t \geq 0$,

$$|\partial_x f_\theta(x_\theta(t))| = |1 - 2r(\theta)x_\theta(t)| \leq 1 + 2L_r x_0.$$

Combining the two previous bounds,

$$\|\partial_\theta x_\theta(t)\| \leq \int_0^t (L'_r e^{2s} x_0^2 + (1 + 2L_r x_0) \|\partial_\theta x_\theta(s)\|) ds,$$

*and thus by Grönwall's inequality, $\|\partial_\theta x_\theta(t)\| \leq M_t = L'_r x_0^2 t e^{(3+2L_r x_0)t}$, for all $t \in [0, T]$ and $\theta \in \mathbb{R}$. Then, for any $\delta > 0$, **AO3** is satisfied with $K = [-\delta, e^T x_0 + \delta] \times [-M_T - \delta, M_T + \delta]$. Since K is compact and as, by (31), for any $x \in \mathbb{R}^n$, $\mathbf{A} \in \mathcal{M}_{d,n}(\mathbb{R})$, $\theta \in \mathbb{R}^d$, $t \geq 0$,*

$$F_\theta((x, \mathbf{A}), t) = \left(x(1 - r(\theta)x), r'(\theta)x^2 + \mathbf{A}(1 - 2r(\theta)x) \right),$$

then (33) easily follows from the condition (34).

Since other schemes may be used, in particular higher-order ones, we consider more generally in the following a solver $\Psi^h : \mathbb{R}^d \rightarrow (\mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R}))^N$ for $h > 0$ satisfying the condition:

AO 4. *There exist $\bar{h}, \mathbf{C}_\Psi, \alpha > 0$ such that, for any $\theta \in \mathbb{R}^d$ and $h \in (0, \bar{h}]$,*

$$\sum_{i=1}^N \|z_\theta(t_i) - \Psi_i^h(\theta)\| \leq \mathbf{C}_\Psi h^\alpha,$$

where z_θ is a solution of (30) and $\Psi_i^h : \mathbb{R}^d \rightarrow \mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R})$ is the i -th component of Ψ^h .

When **AO 3** and **AO 4** are both satisfied, without loss of generality, we assume furthermore that \bar{h} is sufficiently small so that $\mathbf{C}_\Psi \bar{h}^\alpha \leq \delta$. This implies that $\Psi_i^h(\theta) \in \mathbf{K}$ for all $\theta \in \mathbb{R}^d$, $i \in \{1, \dots, N\}$ and $h \in (0, \bar{h}]$.

Writing $\Psi_i^h(\theta) = (\tilde{x}_\theta^h(t_i), G_\theta^h(t_i))$, we can consider for any $\theta \in \mathbb{R}^d$,

$$(35) \quad \tilde{b}_h(\theta) = -\nabla_\theta \ln \pi_0(\theta) + \sum_{i=1}^N G_\theta^h(t_i) \cdot \nabla_x \ln \varphi_\varepsilon(y_i - \tilde{x}_\theta^h(t_i)),$$

as an approximation of ∇U (29). Remark that, now, in contrast to $b(\theta) = \nabla U(\theta)$, it is possible in practice to evaluate $\tilde{b}_h(\theta)$ for $\theta \in \mathbb{R}^d$, provided $\nabla_x \ln \varphi_\varepsilon$ and $\nabla_\theta \ln \pi_0$ can be evaluated. We now assess the error due to the use of \tilde{b}_h in place of the exact gradient in (28) by verifying that the assumption of Section 2 are satisfied. For $\gamma, h > 0$ and $\theta \in \mathbb{R}^d$, denote

$$(36) \quad \mathbf{T}_\gamma(\theta) = \theta - \gamma \nabla U(\theta), \quad \tilde{\mathbf{T}}_{\gamma,h}(\theta) = \theta - \gamma \tilde{b}_h(\theta).$$

When **AO 2** and **AO 3** are both satisfied, there exist $\mathbf{C}_\varphi, \mathbf{L}_\varphi > 0$ such that for all $i \in \{1, \dots, N\}$, the function s_i on the compact set \mathbf{K} given for any $x \in \mathbb{R}^n$ and $\mathbf{A} \in \mathcal{M}_{d,n}(\mathbb{R})$, by

$$(37) \quad \mathbf{s}_i(x, \mathbf{A}) = \mathbf{A} \nabla_x \ln \varphi_\varepsilon(y_i - x)$$

is bounded by \mathbf{C}_s and \mathbf{L}_s -Lipschitz continuous on $\mathbb{R}^n \times \mathcal{M}_{d,n}(\mathbb{R})$.

PROPOSITION 22. *Under **AO 1**, **AO 2**, **AO 3** and **AO 4**, for any $h \in (0, \bar{h})$, the functions \mathbf{T}_γ and $\tilde{\mathbf{T}}_{\gamma,h}$ given by (36) satisfy for any $\bar{\gamma} > 0$, **H1** and **H2**, with*

$$c_\infty = \mathbf{C}_\Psi \mathbf{L}_s h^\alpha, \quad R_1 = \frac{2N\mathbf{C}_s}{\mathbf{m}_U} \vee R_U, \quad \mathbf{m} = \frac{\mathbf{m}_U}{2}, \quad \mathbf{L} = \mathbf{L}_U + \mathbf{L}_s \mathbf{L}_F \sum_{i=1}^N t_i e^{\mathbf{L}'_F t_i}.$$

PROOF. The proof is postponed to Section 5.3. □

Under the conditions of Proposition 22 and using the results of Section 2, we get that the Markov chains $(X_k)_{k \in \mathbb{N}}$ and $(\tilde{X}_k)_{k \in \mathbb{N}}$ associated to \mathbf{T}_γ and $\tilde{\mathbf{T}}_{\gamma,h}$ given by (36) have unique invariant measure π_γ and $\tilde{\pi}_{\gamma,h}$, and that there exist $\bar{\gamma}, \bar{h}, C > 0$ such that for all $\gamma \in (0, \bar{\gamma}]$ and $h \in (0, \bar{h}]$,

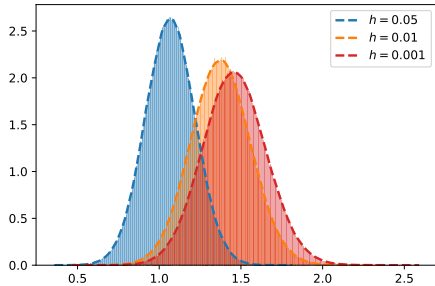
$$(38) \quad \|\pi_\gamma - \tilde{\pi}_{\gamma,h}\|_{\text{TV}} \leq Ch.$$

EXAMPLE 23 (Continuation of Example 19). *As a conclusion, consider the logistic case of Example 19 with the Euler scheme (32), assuming that $r \in \mathbf{C}^2(\mathbb{R}, [0, +\infty))$ satisfies (34). Then, by Example 19 and Example 21, **AO 1**, **AO 3** and **AO 4** hold. Assuming moreover that π_0 and φ_ε are Gaussian, then **AO 2** also holds and we obtain (38).*

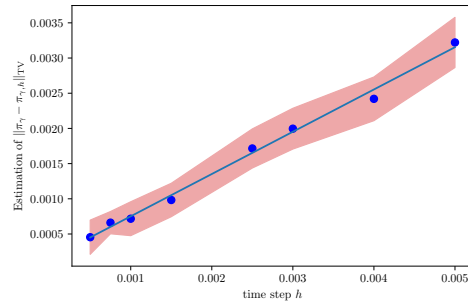
4.2. *Numerical results.* We illustrate our findings on two particular ODEs. First, we consider the ODE associated with the Van der Pol oscillator corresponding to the second order ODE:

$$(39) \quad \ddot{x}_\theta(t) - \theta(1 - x_\theta(t)^2)\dot{x}_\theta(t) + x_\theta(t) = 0,$$

where $\theta \in \mathbb{R}$ is the parameter to infer. It corresponds to (26) with $f_\theta(x_1, x_2) = (x_2, \theta(1 - x_1^2)x_2 - x_1)$. We generate synthetic data solving (39) using the 4th Runge-Kutta method for $T = 10$ and $\theta = 1$. We then select $(x_\theta(t_i))_{i=1}^{25}$ for $(t_i)_{i=1}^{25}$ uniformly chosen in $[0, T]$. The observations $\mathbf{y} = (y_i)_{i=1}^{25}$ are obtained from $(x_\theta(t_i))_{i=1}^{25}$ adding i.i.d. zero-mean Gaussian noise with variance 0.5. We consider the corresponding statistical model (27) where $(\varepsilon_i)_{i=1}^{25}$ are i.i.d. zero-mean Gaussian random variables with variance 0.5. We consider as prior π_0 , the zero-mean Gaussian distribution with variance 0.5. We then use ULA with $\gamma = 10^{-2}$, for which the gradient is estimated using the Euler method with the time steps $h \in \{0.05, 0.01, 0.001\}$. Figure 1a represents the histograms corresponding to the different Markov chains after 10^5 iterations with a burn-in of 10^4 steps. Gaussian kernel density approximation of these histograms are estimated and used as proxy for the density of the invariant distributions $\pi_{\gamma, h}$ of the Markov chain $(\tilde{X}_k)_{k \in \mathbb{N}}$ associated to $\tilde{T}_{\gamma, h}$ given by (36). To obtain a proxy for the density of π_γ , the stationary distribution of $(X_k)_{k \in \mathbb{N}}$ associated to T_γ , we use the same procedure but using the Euler method with $h = 0.0001$. We then estimate the total variation between π_γ and $\pi_{\gamma, h}$ for $h \in \{0.005, 0.004, 0.003, 0.0025, 0.0015, 0.001, 0.00075, 0.0005\}$ using numerical integration. The corresponding results over 10 replications are reported in Figure 1b. We can observe that the total variation distance linearly decreases with h which supports our findings.



(a) Empirical histograms and corresponding KDE for different time steps h



(b) Numerical estimation of $\|\pi_\gamma - \pi_{\gamma, h}\|_{\text{TV}}$

Fig 1: Numerical illustrations for the Van der Pol oscillator (39)

For our second experiment, we consider the Lotka-Volterra model describing the evolution of the population of two interacting biological species denoted by $t \mapsto x_\theta(t) = (u_\theta(t), v_\theta(t))$. The dynamics of these two populations are assumed to be governed by the system of equations given by:

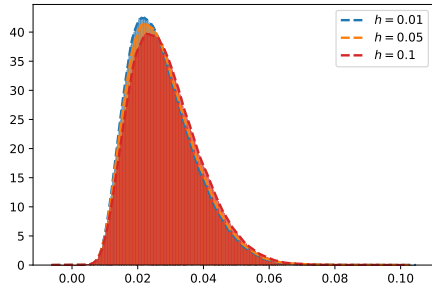
$$(40) \quad \dot{u}_\theta(t) = (\alpha - \beta v_\theta(t))u_\theta(t), \quad \dot{v}_\theta(t) = (-\gamma + \delta u_\theta(t))v_\theta(t),$$

where $\theta = (\alpha, \beta, \gamma, \delta)$ is the parameter to infer. We follow the same methodology presented in [24, Chapter 16]. For this experiment, we consider an other statistical model

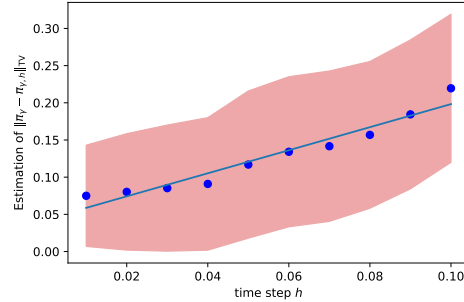
as previously and generate synthetic data $\mathbf{y} = (y_i)_{i=1}^{50}$ accordingly and associated with observation times $(t_i)_{i=1}^{50}$ uniformly spaced in $[0, T]$ for $T = 10$ and the true parameter $\theta_0 = (0.6, 0.025, 0.8, 0.025)$. More precisely, for any $i \in \{1, \dots, 50\}$, $y_i = (u_i^y, v_i^y)$ with $u_i^y = u_\theta(t_i)e^{\varepsilon_{u,i}}$, $v_i^y = v_\theta(t_i)e^{\varepsilon_{v,i}}$ and $(\varepsilon_{u,j}, \varepsilon_{v,j})_{j=1}^{50}$ are i.i.d. one dimensional zero-mean Gaussian random variables with covariance matrix I_2 . The prior π_0 is set to be the Gaussian distribution on \mathbb{R}^4 with means $(1, 0.05, 1, 0.05)$ and standard deviations $(0.5, 0.05, 0.5, 0.05)$. The posterior distribution is then given by $\pi(\theta|\mathbf{y}) \propto \exp(-U(\theta))$, where

$$U(\theta) = -\log \pi_0(\theta) + \sum_{i=1}^{50} \left(\frac{(\log u_i^y - \log u_\theta(t_i))^2 + (\log v_i^y - \log v_\theta(t_i))^2}{2\varsigma^2} \right).$$

We then use ULA with $\gamma = 5 \times 10^{-5}$, for which the gradient is estimated using the Euler method. We focus here on the second component of the chain. The results for the other components are similar. Figure 2a represents the histograms for the second component corresponding to the different Markov chains after 10^7 iterations with a burn-in of 10^3 steps. Gaussian kernel density approximation of these histograms are estimated and used as proxy for the marginal density of the invariant distributions $\pi_{\gamma,h}$ of the Markov chain $(\tilde{X}_k)_{k \in \mathbb{N}}$ associated to $\tilde{T}_{\gamma,h}$ given by (36). To obtain a proxy for the marginal density of π_γ , the stationary distribution of $(X_k)_{k \in \mathbb{N}}$ associated to T_γ , we use the same procedure but using the Euler method with $h = 0.0001$. We then estimate the total variation between π_γ and $\pi_{\gamma,h}$ for $h \in \{k \times 10^{-2} : k \in \{1, \dots, 10\}\}$ using numerical integration. The corresponding results over 10 replications are reported in Figure 2b. We can observe that the total variation distance still linearly decreases with h .



(a) Empirical histograms and corresponding KDE for different time steps h



(b) Numerical estimation of $\|\pi_\gamma - \pi_{\gamma,h}\|_{TV}$

Fig 2: Numerical illustrations for the Lotka-Volterra model (40)

5. Postponed proofs.

5.1. Postponed proofs of Section 2.

5.1.1. *Proof of Proposition 1.* Recall that under **H1**, we have for any $\gamma \in (0, \bar{\gamma}]$,

$$(41) \quad \begin{aligned} \sup_{x \in \mathbb{R}^d, x \neq 0} \{ \|T_\gamma(x) - T_\gamma(0)\| / \|x\| \} &\leq (1 + \gamma L), \\ \sup_{x \in \mathbb{R}^d, \|x\| \geq R_1} \{ \|T_\gamma(x) - T_\gamma(0)\| / \|x\| \} &\leq (1 - \gamma m). \end{aligned}$$

Define for any $x \in \mathbb{R}^d$

$$(42) \quad \bar{V}_{c,\gamma}(x) = \exp(c\|x - T_\gamma(0)\|^2), \text{ for } c = m/(16\sigma^2) .$$

Note that since $\sup_{\gamma \in (0, \bar{\gamma}]} \|T_\gamma(0)\| < +\infty$ by **H1**, $\bar{V}_{c,\gamma}$ goes to $+\infty$ at infinity and

$$\lim_{\|x\| \rightarrow +\infty} \left\{ \sup_{\gamma \in (0, \bar{\gamma}]} \bar{V}_{c,\gamma}/V_c \right\}(x) = 1 = \lim_{\|x\| \rightarrow +\infty} \left\{ \inf_{\gamma \in (0, \bar{\gamma}]} \bar{V}_{c,\gamma}/V_c \right\}(x) = 1 .$$

Therefore, by [9, Corollary 11], it is sufficient to show that there exist $\bar{\gamma}_1 \in (0, \bar{\gamma}]$, $\lambda \in (0, 1)$, $A \geq 0$ such that for any $\gamma \in (0, \bar{\gamma}_1]$ and $x \in \mathbb{R}^d$,

$$(43) \quad R_\gamma \bar{V}_{c,\gamma}(x) \leq \lambda^\gamma \bar{V}_{c,\gamma}(x) + \gamma A .$$

We show that it holds with

$$(44) \quad \begin{aligned} \bar{\gamma}_1 &= \bar{\gamma} \wedge \{1/\mathfrak{m}\} , \quad \lambda = \exp(-c\mathfrak{m}M^2/4) , \quad M = R_1 \vee (8d\sigma^2/\mathfrak{m})^{1/2} , \\ A &= \lambda^{\bar{\gamma}} \exp(cM^2 + \bar{\gamma}\{B_1M^2 + B_2 - \log(\lambda)\})\{B_1M^2 + B_2 - \log(\lambda)\} , \quad B_2 = 2dc\sigma^2 , \\ B_1 &= 4C_1 , \quad C_1 = C_2 \vee C_2^2\bar{\gamma} , \quad C_2 = (2L) \vee (L\bar{\gamma}) \vee (8c\sigma^2) . \end{aligned}$$

Define for any $x \in \mathbb{R}^d$, $\bar{T}_\gamma(x) = T_\gamma(x) - T_\gamma(0)$. Note that for any $\gamma \in (0, \bar{\gamma}_1]$, $2c\sigma^2\gamma \leq 1$ by definition of c (42) and $\bar{\gamma}_1$. Let $x \in \mathbb{R}^d$ and $\gamma \in (0, \bar{\gamma}_1]$. Then, we obtain using that $\int_{\mathbb{R}} e^{az+bz^2-z^2/2} dz = (2\pi(1-2b)^{-1})^{d/2} e^{a^2/(2(1-2b))}$ for any $a \in \mathbb{R}$ and $b \in [0, 1/2)$,

$$\begin{aligned} R_\gamma \bar{V}_{c,\gamma}(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(c\|\bar{T}_\gamma(x) + (\sigma^2\gamma)^{1/2}z\|^2 - \|z\|^2/2) dz \\ &= (1 - 2c\sigma^2\gamma)^{-d/2} \exp\{c(1 - 2c\sigma^2\gamma)^{-1}\|\bar{T}_\gamma(x)\|^2\} . \end{aligned}$$

We now distinguish the case $\|x\| \geq M$ and $\|x\| < M$.

In the first case, we get by (41), $(1 - 2c\sigma^2\gamma)^{-1} \leq 8c\sigma^2\gamma$ and $(1 - \mathfrak{m}\gamma)^2 \leq 1 - \mathfrak{m}\gamma$ since $2c\sigma^2\gamma \leq 1/2$ and $\gamma \leq 1/\mathfrak{m}$, by definition of c and $\bar{\gamma}_1$ (42)-(44),

$$(45) \quad \begin{aligned} R_\gamma \bar{V}_{c,\gamma}(x) &\leq (1 - 2c\gamma\sigma^2)^{-d/2} \exp\left[c(1 - \mathfrak{m}\gamma)(1 + 8c\sigma^2\gamma)\|x\|^2\right] \\ &\leq \exp\left[c(1 - \mathfrak{m}\gamma/4)\|x\|^2 + 2dc\sigma^2\gamma - \mathfrak{m}c\gamma\|x\|^2/4\right] \leq \lambda^\gamma \bar{V}_{c,\gamma}(x) , \end{aligned}$$

where we used for the penultimate inequality that $-\log(1-t) \leq 2t$ for $t \in [0, 1/2]$.

For the case $\|x\| \leq M$, by (41), $(1 - 2c\sigma^2\gamma)^{-1} \leq 8c\sigma^2\gamma$ and $(1 - \mathfrak{m}\gamma)^2 \leq 1 - \mathfrak{m}\gamma$ since $2c\sigma^2\gamma \leq 1/2$ and $\gamma \leq 1/\mathfrak{m}$, by (42)-(44), and using $-\log(1-t) \leq 2t$ for $t \in [0, 1/2]$ again,

$$\begin{aligned} R_\gamma \bar{V}_{c,\gamma}(x) &\leq (1 - 2c\gamma\sigma^2)^{-d/2} \exp\left[c(1 + L\gamma)^2(1 + 8c\sigma^2\gamma)\|x\|^2\right] \\ &\leq \exp\left[c(1 + \gamma B_1)\|x\|^2 + \gamma B_2\right] . \end{aligned}$$

Using that $e^t - 1 \leq te^t$ for $t \geq 0$, we obtain that

$$\begin{aligned} R_\gamma \bar{V}_{c,\gamma}(x) &= \lambda^\gamma \bar{V}_{c,\gamma}(x) + \lambda^\gamma \bar{V}_{c,\gamma}(x) \{\exp\left[c\gamma B_1\|x\|^2 + \gamma B_2 - \gamma \log(\lambda)\right] - 1\} \\ &\leq \lambda^\gamma \bar{V}_{c,\gamma}(x) + \gamma \lambda^{\bar{\gamma}} \bar{V}_{c,\gamma}(x) \{B_1\|x\|^2 + B_2 - \log(\lambda)\} \exp\left[\bar{\gamma}(B_1\|x\|^2 + B_2 - \log(\lambda))\right] , \end{aligned}$$

which combined with (45) completes the proof of (43).

5.1.2. *Proof of Proposition 3.* First note that under **H1** and **H2**, for all $x \in \mathbb{R}^d$, since

$$\|\tilde{T}_\gamma(x) - T_\gamma(0)\| \leq \gamma c_\infty + \|T_\gamma(x) - T_\gamma(0)\|$$

and from (41),

$$(46) \quad \sup_{x \in \mathbb{R}^d, \|x\| \geq R_1} \left[\|\tilde{T}_\gamma(x) - T_\gamma(0)\| / \{(1 - \gamma \mathfrak{m})\|x\| + \gamma c_\infty\} \right] \leq 1.$$

Define for any $x \in \mathbb{R}^d$

$$(47) \quad \bar{V}_{c,\gamma}(x) = \exp(c\|x - T_\gamma(0)\|^2), \text{ for } c = m/(16\sigma^2).$$

Note that $\lim_{\|x\| \rightarrow +\infty} \bar{V}_{c,\gamma}(x) = +\infty$ and $\lim_{\|x\| \rightarrow +\infty} \{\bar{V}_{c,\gamma}/V_c\}(x) = 1$. In addition, by **H1** and **H2**, \tilde{R}_γ is strongly aperiodic, Leb-irreducible and all compact sets are 1-small. It is therefore sufficient to show by [25, Theorem 16.0.1] that there exists $c > 0$, such that for any $\gamma \in (0, \bar{\gamma}_1]$ (recall $\bar{\gamma}_1 = \bar{\gamma} \wedge \{1/\mathfrak{m}\}$), there exists $\lambda_\gamma \in [0, 1)$ and $A_\gamma \geq 0$, such that

$$(48) \quad \tilde{R}_\gamma V_c \leq \lambda_\gamma V_c + A_\gamma.$$

Define for any $x \in \mathbb{R}^d$, $\bar{T}_\gamma(x) = \tilde{T}_\gamma(x) - T_\gamma(0)$. Note that for any $\gamma \in (0, \bar{\gamma}_1]$, $2c\sigma^2\gamma \leq 1$ by definition of c (47) and $\bar{\gamma}_1$. Let $x \in \mathbb{R}^d$ and $\gamma \in (0, \bar{\gamma}_1]$. Then, we obtain using that $\int_{\mathbb{R}} e^{az+bz^2-z^2/2} dz = (2\pi(1-2b)^{-1})^{d/2} e^{a^2/(2(1-2b))}$ for any $a \in \mathbb{R}$ and $b \in [0, 1/2)$,

$$\begin{aligned} R_\gamma \bar{V}_{c,\gamma}(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(c\|\bar{T}_\gamma(x) + (\sigma^2\gamma)^{1/2}z\|^2 - \|z\|^2/2) dz \\ &= (1 - 2c\sigma^2\gamma)^{-d/2} \exp\{c(1 - 2c\sigma^2\gamma)^{-1}\|\bar{T}_\gamma(x)\|^2\}. \end{aligned}$$

If $\|x\| \geq M$, we get by (46), $(1 - 2c\sigma^2\gamma)^{-1} \leq 8c\sigma^2\gamma$ and $(1 - \mathfrak{m}\gamma)^2 \leq 1 - \mathfrak{m}\gamma$ since $2c\sigma^2\gamma \leq 1/2$ and $\gamma \leq 1/\mathfrak{m}$, by (47) and definition of $\bar{\gamma}_1$,

$$\begin{aligned} R_\gamma \bar{V}_{c,\gamma}(x) &\leq (1 - 2c\gamma\sigma^2)^{-d/2} \exp\left[c(1 + 8c\sigma^2\gamma)\{(1 - \mathfrak{m}\gamma)\|x\| + c_\infty\}^2\right] \\ &\leq (1 - 2c\gamma\sigma^2)^{-d/2} \exp\left[c\{(1 - \mathfrak{m}\gamma/2)\|x\| + (1 + 8c\sigma^2\gamma)c_\infty\}^2\right]. \end{aligned}$$

Therefore, we get $\liminf_{\|x\| \rightarrow +\infty} [R_\gamma \bar{V}_{c,\gamma}(x)/\bar{V}_{c,\gamma}(x)] = 0$ which completes the proof of (48).

5.1.3. *Proof of Proposition 6.* The proof is based on this technical lemma.

LEMMA 24. For any $g \in \mathbb{R}$, $u \in [0, 1]$ and $\gamma \in (0, \bar{\gamma}]$, $a, b \in [0, +\infty)$, $a \leq b$,

$$(49) \quad \mathcal{H}_\gamma(a, g, u) \leq \mathcal{H}_\gamma(b, g, u)$$

PROOF. Let $u \in [0, 1]$, $g \in \mathbb{R}$ and $a, b \in [0, +\infty)$, $a \leq b$. We first prove that for any $c \in \mathbb{R}_+$ such that $c - 2(\sigma^2\gamma)^{1/2}g < 0$,

$$(50) \quad \mathcal{H}_\gamma(c, g, u) = 0,$$

which implies that for any $c \in \mathbb{R}_+$, $\mathcal{H}_\gamma(c, g, u) \geq 0$. We need to consider the following two cases.

- (a) If $c - (\sigma^2\gamma)^{1/2}g < 0$. Then, using $-(\sigma^2\gamma)^{1/2}g \leq c - (\sigma^2\gamma)^{1/2}g$, $\varphi_{\sigma^2\gamma}((\sigma^2\gamma)^{1/2}g) = \varphi_{\sigma^2\gamma}(-(\sigma^2\gamma)^{1/2}g)$, and $t \mapsto \varphi_{\sigma^2\gamma}(t)$ is decreasing on $[0, +\infty)$, we get $\bar{p}_\gamma(c, g) = 1$. Therefore (50) holds.

(b) If $0 \leq c - (\sigma^2\gamma)^{1/2}g \leq (\sigma^2\gamma)^{1/2}g$, we obtain similarly to the first case that $\bar{p}_\gamma(c, g) = 1$ and therefore (50) holds.

We now show (49). It is straightforward by (50) if $0 > a - 2(\sigma^2\gamma)^{1/2}g$. If $0 \leq a - 2(\sigma^2\gamma)^{1/2}g$. By using $t \mapsto \varphi_{\sigma^2\gamma}(t)$ is decreasing on $[0, +\infty)$, we obtain

$$(51) \quad \bar{p}_{\sigma^2\gamma}(a, g) \geq \bar{p}_{\sigma^2\gamma}(b, g).$$

Then, (49) follows from (9) and (51). \square

PROOF OF PROPOSITION 6. Let $k \in \mathbb{N}$. By **H1**, **H2** and the triangle inequality, for any $x, \tilde{x} \in \mathbb{R}^d$,

$$(52) \quad \|E(x, \tilde{x})\| \leq \tau_\gamma(\|x - \tilde{x}\|) + \gamma c_\infty.$$

By using (6), and (9) we have,

$$\begin{aligned} \|X_{k+1} - \tilde{X}_{k+1}\| &= B_{k+1}\| - E(X_k, \tilde{X}_k) + 2(\sigma^2\gamma)^{1/2}e(X_k, \tilde{X}_k)e(X_k, \tilde{X}_k)^T Z_{k+1}\| \\ &= B_{k+1}\| - \|E(X_k, \tilde{X}_k)\|e(X_k, \tilde{X}_k) + 2(\sigma^2\gamma)^{1/2}G_{k+1}e(X_k, \tilde{X}_k)\| \\ &= B_{k+1} \left| \|E(X_k, \tilde{X}_k)\| - 2(\sigma^2\gamma)^{1/2}G_{k+1} \right| \\ &= \mathcal{H}_\gamma(\|E(X_k, \tilde{X}_k)\|, G_{k+1}, U_{k+1}). \end{aligned}$$

This gives (11) when combined with (52). Finally, the last statement follows from Lemma 24 and **H1** ensuring that τ_γ is non-decreasing on $[0, +\infty)$. \square

5.1.4. *Proof of Proposition 8.* The proof is an easy consequence of this technical lemma.

LEMMA 25. Assume **H1** and **H2** hold. Then, for any $w \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$, we have that

$$Q_\gamma \mathcal{V}_1^*(w) = \tau_\gamma(w) + \gamma c_\infty.$$

PROOF. For any $w \in \mathbb{R}$, we have,

$$\begin{aligned} Q_\gamma \mathcal{V}_1^*(w) &= \int_{\mathbb{R}} (1 - \bar{p}_{\sigma^2\gamma}(w, g)) \left(\tau_\gamma(w) + \gamma c_\infty - 2(\sigma^2\gamma)^{1/2}g \right) \varphi(g) dg \\ &= \int_{\mathbb{R}} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left(\varphi_{(\sigma^2\gamma)^{1/2}}(g) - \varphi_{(\sigma^2\gamma)^{1/2}}(g) \wedge \varphi_{(\sigma^2\gamma)^{1/2}}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg \\ &= \int_{-\infty}^{(\tau_\gamma(w) + \gamma c_\infty)/2} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left(\varphi_{(\sigma^2\gamma)^{1/2}}(g) - \varphi_{(\sigma^2\gamma)^{1/2}}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg. \end{aligned}$$

By using change of variable $g \rightarrow \mathbf{a} - g$ we have,

$$\begin{aligned} &\int_{-\infty}^{(\tau_\gamma(w) + \gamma c_\infty)/2} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left(\varphi_{(\sigma^2\gamma)^{1/2}}(g) - \varphi_{(\sigma^2\gamma)^{1/2}}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg \\ &= \frac{1}{2} \int_{\mathbb{R}} (\tau_\gamma(w) + \gamma c_\infty - 2g) \left(\varphi_{(\sigma^2\gamma)^{1/2}}(g) - \varphi_{(\sigma^2\gamma)^{1/2}}(\tau_\gamma(w) + \gamma c_\infty - g) \right) dg \\ &= \tau_\gamma(w) + \gamma c_\infty. \end{aligned}$$

\square

PROOF OF PROPOSITION 8. By Lemma 25 and **H1**, for any $w \in [0, +\infty)$,

$$Q_\gamma \mathcal{V}_1^*(w) = \tau_\gamma(w) + \gamma c_\infty \leq (1 - \gamma \mathbf{m}) \mathcal{V}_1^*(w) \mathbb{1}_{(R_1, \infty)}(w) + (1 + \gamma \mathbf{L}) \mathcal{V}_1^*(w) \mathbb{1}_{(0, R_1]}(w) + \gamma c_\infty.$$

This completes the proof. \square

5.1.5. *Proof of Proposition 9.* We first establish that Q_γ admits a unique invariant probability measure μ_γ and is geometrically ergodic. To that end, we show that Q_γ is (a) irreducible and aperiodic, (c) any compact set of $[0, +\infty)$ is small and (d) there exists $\lambda > 0$ and $b \geq 0$ such that $Q_\gamma \mathcal{V}(w) \leq \lambda \mathcal{V}(w) + b$ for any $w \in [0, +\infty)$ with $\mathcal{V}(w) = w + 1$. The proof then follows from [11, Corollary 14.1.6, Theorem 15.2.4].

(a) Let \mathbf{K} be a compact set. Then for any $w \in \mathbf{K}$ we have

$$(53) \quad Q_\gamma(w, \{0\}) \geq \int_{[-1,1]} \bar{p}_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty, g) \varphi(g) dg \geq \eta_{\mathbf{K}},$$

where using **H1**

$$(54) \quad \eta_{\mathbf{K}} = \inf_{(r,g) \in \mathbf{K} \times [-1,1]} \bar{p}_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty, g) \int_{[-1,1]} \varphi(g) dg \\ \geq \inf_{(a,g) \in [0,M] \times [-1,1]} \bar{p}_{\sigma^2\gamma}(a, g) \int_{[-1,1]} \varphi(g) dg,$$

and $M = (1 + \gamma L) \sup(\mathbf{K}) + \gamma c_\infty$. Note that since $(a, g) \rightarrow \bar{p}_{\sigma^2\gamma}(a, g)$ is a continuous positive function, and $[0, M] \times [-1, 1]$ is compact, $\eta_{\mathbf{K}} > 0$. Therefore $\{0\}$ is an accessible $(1, \delta_0)$ -small set and Q_γ is irreducible. In addition, $Q_\gamma(0, \{0\}) > 0$ which implies that Q_γ is strongly aperiodic.

(b) Let now \mathbf{C} be a compact set, we show that \mathbf{C} is small. By (53), for $\mathbf{A} \in \mathcal{B}([0, +\infty))$ and $w \in [0, +\infty)$,

$$Q_\gamma^2(w, \mathbf{A}) \geq \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(\tilde{w}) Q_\gamma(\tilde{w}, \mathbf{A}) Q_\gamma(w, d\tilde{w}) \geq \eta_{\mathbf{C}} Q_\gamma(0, \mathbf{A}).$$

Therefore \mathbf{C} is a $(2, Q_\gamma(0, \cdot))$ -small set.

(c) In addition by **H1** and (13) we have, for any $w \in [0, +\infty)$,

$$\int_{[0, +\infty)} \mathcal{V}(\tilde{w}) Q_\gamma(w, d\tilde{w}) \\ \leq 1 + \int_{\mathbb{R}} (1 - \bar{p}_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty, z)) \left(\tau_\gamma(w) + \gamma c_\infty + 2(\sigma^2\gamma)^{1/2}(g)_- \right) \varphi(g) dg \\ \leq (1 - \gamma \mathfrak{m}) \mathcal{V}(w) + \gamma R_1(\mathfrak{m} + L) + \gamma c_\infty + 2(\sigma^2\gamma)^{1/2}/\sqrt{2\pi}.$$

The proof of the first part of the proposition is complete.

We now establish the second part. Let $\mathbf{A} \in \mathcal{B}(\mathbb{R})$ such that $(\delta_0 + \text{Leb})(\mathbf{A}) = 0$. Then $0 \notin \mathbf{A}$ and $\text{Leb}(\mathbf{A}) = 0$ therefore for any $w \in \mathbb{R}$,

$$Q_\gamma(w, \mathbf{A}) \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{1}_{\mathbf{A}} \left(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\gamma^{1/2}g \right) dg = 0.$$

It follows that $\mu_\gamma(\mathbf{A}) = \mu_\gamma Q_\gamma(\mathbf{A}) = 0$ and $\mu_\gamma \ll (\delta_0 + \text{Leb})$.

Since for any $w \in [0, +\infty)$, $Q_\gamma(w, \{0\}) > 0$, $\delta_{\{0\}}$ is an irreducibility measure, and by [11, Theorem 9.2.15], μ_γ is a maximal irreducibility measure for Q_γ , $\delta_0 \ll \mu_\gamma$ implying that $\mu_\gamma(\{0\}) > 0$.

In the case $c_\infty \neq 0$, by (13) and (10), for any $\mathbf{A} \in \mathcal{B}([0, +\infty))$, $\text{Leb}(\mathbf{A}) > 0$, $Q_\gamma(w, \mathbf{A}) > 0$ for any $w \in [0, +\infty)$ and therefore Leb is an irreducibility measure. Applying [11, Theorem 9.2.15] again, we get that $\delta_0 + \text{Leb} \ll \mu_\gamma$. This completes the proof since we have already shown that $\mu_\gamma \ll (\delta_0 + \text{Leb})$.

5.1.6. *Proof of Theorem 11.*

LEMMA 26. Assume **H1** and **H2** hold. For any $w \in \mathbb{R}$,

$$Q_\gamma(w, \{0\}) = 2\Phi\left(-\frac{\tau_\gamma(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}}\right),$$

where Q_γ is defined by (13) and Φ is the cumulative distribution of the one-dimensional Gaussian distribution with mean 0 and variance 1.

PROOF. Let $w \in \mathbb{R}$. By (12) and the change of variable $g \rightarrow \sigma\sqrt{\gamma}g$, we get

$$\begin{aligned} Q_\gamma(r, \{0\}) &= \int_{\mathbb{R}} \left(1 \wedge \frac{\varphi_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty - \sigma\sqrt{\gamma}g)}{\varphi_{\sigma^2\gamma}(\sigma\sqrt{\gamma}g)}\right) \varphi(g) dg \\ &= \int_{\mathbb{R}} \varphi_{\sigma^2\gamma}(g) \wedge \varphi_{\sigma^2\gamma}(\tau_\gamma(w) + \gamma c_\infty - g) dg = \int_{\mathbb{R}} \varphi_{\sigma^2\gamma}(g) \wedge \varphi_{\sigma^2\gamma}(g - \tau_\gamma(w) + \gamma c_\infty) dg \\ &= \int_{-\infty}^{(\tau_\gamma(w) + \gamma c_\infty)/2} \varphi_{\sigma^2\gamma}(g - \tau_\gamma(w) + \gamma c_\infty) dg + \int_{(\tau_\gamma(w) + \gamma c_\infty)/2}^{+\infty} \varphi_{\sigma^2\gamma}(g) dg \\ &= \int_{-\infty}^{-(\tau_\gamma(w) + \gamma c_\infty)/2} \varphi_{\sigma^2\gamma}(g) dg + \int_{(\tau_\gamma(w) + \gamma c_\infty)/2}^{+\infty} \varphi_{\sigma^2\gamma}(g) dg = 2\Phi\left(-\frac{\tau_\gamma(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}}\right), \end{aligned}$$

and the lemma follows. \square

LEMMA 27. Let $\sigma^2, \gamma > 0$. For any $t, a \geq 0$, we have

$$\int_{\mathbb{R}} \left[1 - 2\Phi\left(-\frac{t - 2\sigma\gamma^{1/2}g}{2a}\right)\right] \bar{p}_{\sigma^2\gamma}(t, g) \varphi(g) dg = 0,$$

where $\bar{p}_{\sigma^2\gamma}$ is defined by (10), φ and Φ are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1 respectively.

PROOF. Using the changes of variable $g \mapsto \sigma\gamma^{1/2}g$, $g \mapsto g - t$ and $g \mapsto -g$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \left[1 - 2\Phi\left(-\frac{t - 2\sigma\gamma^{1/2}g}{2a}\right)\right] \bar{p}_{\sigma^2\gamma}(t, g) \varphi(g) dg \\ &= \int_{\mathbb{R}} \left[1 - 2\Phi\left(-\frac{t - 2\sigma\gamma^{1/2}g}{2a}\right)\right] \left\{1 \wedge \frac{\varphi_{\sigma^2\gamma}(a - \sigma\sqrt{\gamma}g)}{\varphi_{\sigma^2\gamma}(\sigma\sqrt{\gamma}g)}\right\} \varphi(g) dg \\ &= \int_{-\infty}^{t/2} \left[1 - 2\Phi\left(-\frac{t - 2g}{2a}\right)\right] \varphi_{\sigma^2\gamma}(t - g) dg + \int_{t/2}^{+\infty} \left[1 - 2\Phi\left(-\frac{t - 2g}{2a}\right)\right] \varphi_{\sigma^2\gamma}(g) dg \\ &= \int_{-\infty}^{-t/2} \left[1 - 2\Phi\left(-\frac{-t - 2g}{2a}\right)\right] \varphi_{\sigma^2\gamma}(g) dg + \int_{t/2}^{+\infty} \left[1 - 2\Phi\left(-\frac{t - 2g}{2a}\right)\right] \varphi_{\sigma^2\gamma}(g) dg \\ &= \int_{t/2}^{+\infty} \left[1 - 2\Phi\left(-\frac{-t + 2g}{2a}\right)\right] \varphi_{\sigma^2\gamma}(g) dg + \int_{t/2}^{+\infty} \left[1 - 2\Phi\left(-\frac{t - 2g}{2a}\right)\right] \varphi_{\sigma^2\gamma}(g) dg. \end{aligned}$$

Using that $s \in \mathbb{R}$, $1 - 2\Phi(s) = -[1 - 2\Phi(-s)]$ completes the proof. \square

LEMMA 28. Let $\sigma^2, \gamma > 0$. For any $t, s \geq 0$ and $a > 0$,

$$(55) \quad \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{t+s-2\sigma\sqrt{\gamma}g}{2a} \right) \right] \bar{p}_{\sigma^2\gamma}(t, g) \varphi(g) dg \\ = 2\mathbb{P} \left(\sigma\sqrt{\gamma}G \geq t/2, -s-t \leq 2a\tilde{G} - 2\sigma\sqrt{\gamma}G \leq s-t \right),$$

where G, \tilde{G} are two independent one-dimensional standard Gaussian random variables, $\bar{p}_{\sigma^2\gamma}$ is defined by (10), φ and Φ are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1 respectively.

PROOF. Using the changes of variable $g \mapsto \sigma\gamma^{1/2}g$, $g \mapsto g - t$, we get

$$(56) \quad \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{t+s-2\sigma\sqrt{\gamma}g}{2a} \right) \right] \bar{p}_{\sigma^2\gamma}(t, g) \varphi(g) dg \\ = \int_{-\infty}^{-t/2} \left[1 - 2\Phi \left(-\frac{-t+s-2g}{2a} \right) \right] \varphi_{\sigma^2\gamma}(g) dg \\ + \int_{t/2}^{+\infty} \left[1 - 2\Phi \left(-\frac{t+s-2g}{2a} \right) \right] \varphi_{\sigma^2\gamma}(g) dg \\ = 2[\mathbb{P}(\sigma\gamma^{1/2}G \geq t/2) - A - B]$$

$$A = \int_{-\infty}^{-t/2} \Phi \left(-\frac{-t+s-2g}{2a} \right) \varphi_{\sigma^2\gamma}(g) dg, \quad B = \int_{t/2}^{+\infty} \Phi \left(-\frac{t+s-2g}{2a} \right) \varphi_{\sigma^2\gamma}(g) dg.$$

In addition, we have since $(-G, -\tilde{G})$ has the same distribution than (G, \tilde{G}) ,

$$A = \mathbb{P} \left(\sigma\sqrt{\gamma}G \leq -\frac{t}{2}, \tilde{G} \leq -\frac{-t+s-2\sigma\sqrt{\gamma}G}{2a} \right) = \mathbb{P} \left(\sigma\sqrt{\gamma}G \geq \frac{t}{2}, \tilde{G} \geq \frac{-t+s+2\sigma\sqrt{\gamma}G}{2a} \right) \\ B = \mathbb{P} \left(\sigma\sqrt{\gamma}G \geq \frac{t}{2}, \tilde{G} \leq -\frac{t+s-2\sigma\sqrt{\gamma}G}{2a} \right)$$

Therefore, we obtain

$$A + B \\ = \mathbb{P}(\sigma\sqrt{\gamma}G \geq t/2) - \mathbb{P} \left(\sigma\sqrt{\gamma}G \geq t/2, -\frac{t+s-2\sigma\sqrt{\gamma}G}{2a} \leq \tilde{G} \leq \frac{-t+s+2\sigma\sqrt{\gamma}G}{2a} \right) \\ = \mathbb{P}(\sigma\sqrt{\gamma}G \geq t/2) - \mathbb{P} \left(\sigma\sqrt{\gamma}G \geq t/2, -s-t \leq 2a\tilde{G} - 2\sigma\sqrt{\gamma}G \leq s-t \right)$$

Plugging this expression in (56) concludes the proof. \square

LEMMA 29. For any $a \in \mathbb{R}$, $b \in [0, 1]$, it holds

$$\Phi(a+b) - \Phi(a-b) \geq 1 - 2\Phi(-b) - a^2 b \exp(-b^2/2) / \sqrt{2\pi},$$

where Φ are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1.

PROOF. Define $\psi : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ for any $a \in \mathbb{R}$, $b \in [0, 1]$ by

$$\psi(a, b) = \Phi(a+b) - \Phi(a-b) - 1 + 2\Phi(-b) + a^2 b \exp(-b^2/2) / \sqrt{2\pi}.$$

We show that $\psi(a, b) \geq 0$ for any $a \in \mathbb{R}$ and $b \in [0, 1]$. Using that $1 - \Phi(-t) = \Phi(t)$ for any $t \in \mathbb{R}$, we get that $\psi(a, b) = \psi(-a, b)$ and therefore we only need to consider the case $a \leq 0$ and $b \in [0, 1]$. In addition, for any $b \in [0, 1]$, $\psi(0, b) = 0$ and thus, it is sufficient to establish that for any $b \in [0, 1]$, $a \mapsto \psi(a, b)$ is non-increasing on \mathbb{R}_- .

For any $a \leq 0$ and $b \in (0, 1)$, we have using that $\sinh(t) = \int_0^t \cosh(s) ds \leq t \cosh(t)$ and $e^{-t^2/2} \cosh(t) \leq 1$ for any $t \in [0, +\infty)$,

$$\begin{aligned} \sqrt{2\pi} \exp(b^2/2) \frac{\partial \psi}{\partial a}(a, b) &= 2 \exp(-a^2/2) \sinh(-ab) + 2ab \\ &< -2ab[\exp(-a^2b^2/2) \cosh(ab) - 1] \leq 0. \end{aligned}$$

By continuity, it also holds for $a \leq 0$ and $b \in [0, 1]$ which concludes the proof. \square

LEMMA 30. Assume **H1** and **H2** hold. For any $w \in [0, +\infty)$ and $\alpha, \beta \in [0, +\infty)$, such that $\alpha/(2\beta) \leq 1$,

$$\begin{aligned} (57) \quad \int_{[0, +\infty)} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tilde{w}) + \alpha}{2\beta} \right) \right] Q_\gamma(w, d\tilde{w}) \\ \leq 1 - 2\Phi \left(-\frac{\tau_\gamma(w) + \gamma c_\infty + \alpha/(1 + \gamma L)}{2\sqrt{\sigma^2 \gamma + \beta/(1 + \gamma L)^2}} \right) + \zeta \frac{\gamma \alpha}{\beta^3}, \end{aligned}$$

where Φ is the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1,

$$(58) \quad \zeta = (1 + \bar{\gamma} L)^2 \sigma^2 (2\sqrt{2\pi})^{-1} \left[\sup_{t \geq 0} \{t^2 \Phi(-t)\} + 1/8 \right].$$

PROOF. Let $\alpha, \beta \geq 0$ such that $\alpha/(2\beta) \leq 1$. By (13), we have

$$\begin{aligned} &\int_{[0, +\infty)} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tilde{w}) + \alpha}{2\beta} \right) \right] Q_\gamma(w, d\tilde{w}) \\ &= \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \right] (1 - \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g)) \varphi(g) dg \\ &\quad + \left[1 - 2\Phi \left(-\frac{\tau_\gamma(0) + \alpha}{2\beta} \right) \right] \int_{\mathbb{R}} \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g) \varphi(g) dg \\ &= \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \right] (1 - \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g)) \varphi(g) dg \\ (59) \quad &+ \left[1 - 2\Phi \left(-\frac{\tau_\gamma(0) + \alpha}{2\beta} \right) \right] 2\Phi \left(-\frac{\tau_\gamma(w) + \gamma c_\infty}{2\sigma\sqrt{\gamma}} \right). \end{aligned}$$

By **H1**, $t \mapsto 1 - 2\Phi(-t)$ is increasing, we have setting $\psi_\gamma(w) = \tau_\gamma(w) + \gamma c_\infty + \alpha/(1 + \gamma L)$,

$$1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \leq 1 - 2\Phi \left(-\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma}g}{2\beta/(1 + \gamma L)} \right).$$

Using [15, Lemma 20], Lemma 27 and Lemma 28, we get

$$\int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \right] (1 - \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g)) \varphi(g) dg$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\psi_{\gamma}(w) - 2\sigma\sqrt{\gamma}g}{2\beta/(1+\gamma\mathbf{L})} \right) \right] (1 - \bar{p}_{\sigma^2\gamma}(\psi_{\gamma}(w), g)) \varphi(g) dg \\
&\quad - \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\psi_{\gamma}(w) - 2\sigma\sqrt{\gamma}g}{2\beta/(1+\gamma\mathbf{L})} \right) \right] \bar{p}_{\sigma^2\gamma}(\tau_{\gamma}(w) + \gamma c_{\infty}, g) \varphi(g) dg \\
(60) \quad &= 1 - 2\Phi \left(-\frac{\psi_{\gamma}(w)}{2\sqrt{\sigma^2\gamma + \beta/(1+\gamma\mathbf{L})^2}} \right) - 2\mathbb{P}(\{2\sigma\sqrt{\gamma}G \geq \tau_{\gamma}(w) + \gamma c_{\infty}\} \cap \mathbf{A}),
\end{aligned}$$

where

$$\mathbf{A} = \left\{ -\tau_{\gamma}(w) - \gamma c_{\infty} - \frac{\alpha}{1+\gamma\mathbf{L}} \leq \frac{2\beta\tilde{G}}{1+\gamma\mathbf{L}} - 2\sigma\sqrt{\gamma}G \leq -\tau_{\gamma}(w) - \gamma c_{\infty} + \frac{\alpha}{1+\gamma\mathbf{L}} \right\},$$

and G, \tilde{G} are two independent one-dimensional standard Gaussian random variables.

Define $\theta_{\gamma} : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ for any $w \in [0, +\infty)$ and $g \in \mathbb{R}$ by $\theta_{\gamma}(w, g) = (2\beta)^{-1}(1 + \gamma\mathbf{L})[-\tau_{\gamma}(w) - \gamma c_{\infty} + \sigma\sqrt{\gamma}g]$. Then, using that

$$\mathbf{A} = \left\{ \theta_{\gamma}(w, G) - \alpha/(2\beta) \leq \tilde{G} \leq \theta_{\gamma}(w, G) + \alpha/(2\beta) \right\},$$

we have

$$\begin{aligned}
&\mathbb{P}(\{2\sigma\sqrt{\gamma}G \geq \tau_{\gamma}(w) + \gamma c_{\infty}\} \cap \mathbf{A}) \\
&= \int_{\mathbb{R}} \mathbb{1}_{[0, +\infty)} \left(g - \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right) \left[\Phi \left(\theta_{\gamma}(w, g) + \frac{\alpha}{2\beta} \right) - \Phi \left(\theta_{\gamma}(w, g) - \frac{\alpha}{2\beta} \right) \right] \varphi(g) dg.
\end{aligned}$$

Since $\alpha/(2\beta) \leq 1$ by Lemma 29 we have, for any $a \in \mathbb{R}$,

$$\Phi \left(a + \frac{\alpha}{2\beta} \right) - \Phi \left(a - \frac{\alpha}{2\beta} \right) \geq 1 - 2\Phi \left(-\frac{\alpha}{2\beta} \right) - \frac{a^2\alpha}{2\sqrt{2\pi}\beta} e^{-\alpha^2/(8\beta^2)},$$

which implies

$$\begin{aligned}
(61) \quad &\mathbb{P}(\{2\sigma\sqrt{\gamma}G \geq \tau_{\gamma}(w) + \gamma c_{\infty}\} \cap \mathbf{A}) \geq \Phi \left(-\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right) \left(1 - 2\Phi \left(-\frac{\alpha}{2\beta} \right) \right) \\
&\quad - \int_{\mathbb{R}} \mathbb{1}_{[0, +\infty)} \left(g - \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right) \frac{\theta_{\gamma}^2(g, w)\alpha}{2\sqrt{2\pi}\beta} e^{-\alpha^2/(8\beta^2)} \varphi(g) dg.
\end{aligned}$$

Therefore, we obtain using that $\mathbb{E}[\mathbb{1}_{[0, +\infty)}(G)G^2] = 1/2$,

$$\begin{aligned}
&\left(1 - 2\Phi \left(-\frac{\alpha}{2\beta} \right) \right) \Phi \left(-\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right) - \mathbb{P}(\{2\sigma\sqrt{\gamma}G \geq \tau_{\gamma}(w) + \gamma c_{\infty}\} \cap \mathbf{A}) \\
&\leq \int_{\mathbb{R}} \mathbb{1}_{[0, +\infty)} \left(g - \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right) \frac{\theta_{\gamma}^2(g, w)\alpha}{2\sqrt{2\pi}\beta} e^{-\alpha^2/(8\beta^2)} \varphi(g) dg \\
&\leq \frac{\alpha\gamma(1+\gamma\mathbf{L})^2\sigma^2}{\beta^3} e^{-\alpha^2/(8\beta^2)} \left[\left(\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right)^2 \Phi \left(-\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right) \right. \\
&\quad \left. - \frac{1}{\sqrt{2\pi}} \frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \exp \left(-\left(\frac{\tau_{\gamma}(w) + \gamma c_{\infty}}{2\sigma\sqrt{\gamma}} \right)^2 / 2 \right) + 1/8 \right].
\end{aligned}$$

Combining this inequality and (60) in (59) concludes the proof. \square

Under **H1** and **H2**, define $(\alpha_k)_{k \geq 1}$, $(\beta_k)_{k \geq 1}$ for any $k \geq 1$ by

$$(62) \quad \alpha_k = \gamma c_\infty \sum_{i=0}^{k-1} (1 + \gamma L)^{-i}, \quad \beta_k^2 = \gamma \sigma^2 \sum_{i=0}^{k-1} (1 + \gamma L)^{-2i}.$$

LEMMA 31. Assume **H1** and **H2** hold. For any $\gamma > 0$, $k \geq 1$, we have

$$(63) \quad \begin{aligned} k\gamma c_\infty e^{-k\gamma L} &\leq \alpha_k \leq k\gamma c_\infty, & (k\gamma)^{1/2} \sigma e^{-k\gamma L} &\leq \beta_k \leq (k\gamma)^{1/2} \sigma, \\ [(k\gamma)^{1/2} c_\infty / \sigma] e^{-k\gamma L} &\leq \alpha_k / \beta_k \leq [(k\gamma)^{1/2} c_\infty / \sigma] e^{k\gamma L} \\ [c_\infty \gamma^{1/2} / (\sigma^3 k^{1/2})] e^{-k\gamma L} &\leq \gamma \alpha_k / \beta_k^3 \leq [c_\infty \gamma^{1/2} / (\sigma^3 k^{1/2})] e^{3k\gamma L} \end{aligned}$$

$$(64) \quad \gamma \sum_{i=1}^{k-1} \{\alpha_k / \beta_k^3\} \leq [2c_\infty (k\gamma)^{1/2} / \sigma^3] e^{3k\gamma L},$$

where $(\alpha_k)_{k \geq 1}, (\beta_k)_{k \geq 1}$ are defined in (62).

PROOF. Let $k \geq 1$. Using for any $i \in \mathbb{N}$, $e^{-i\gamma L} \leq (1 + \gamma L)^{-i} \leq 1$, we have

$$(65) \quad k\gamma c_\infty e^{-k\gamma L} \leq \gamma c_\infty \sum_{i=0}^{k-1} (1 + \gamma L)^{-i} \leq k\gamma c_\infty.$$

In the same way, using for any $i \in \mathbb{N}$, $e^{-2i\gamma L} \leq (1 + \gamma L)^{-2i} \leq 1$, we obtain

$$(66) \quad k\gamma \sigma^2 e^{-2k\gamma L} \leq \gamma \sigma^2 \sum_{i=0}^{k-1} (1 + \gamma L)^{-2i} \leq k\gamma \sigma^2.$$

Combining (65) and (66) completes the proof of the first four inequalities. Then, (64) is a simple consequence of (63) and a comparison test. \square

LEMMA 32. Assume **H1** and **H2** hold. Let $\bar{\delta} \in (0, \{L^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$, with the convention $1/0 = +\infty$. For any $\gamma \in (0, \bar{\gamma}]$, $n \in \{0, \dots, n_\gamma\}$, $n_\gamma = \lfloor \bar{\delta}/\gamma \rfloor$, and $w \in [0, +\infty)$, it holds

$$(67) \quad \int_{[0, +\infty)} \mathbb{1}_{(0, +\infty)}(\tilde{w}) Q_\gamma^{n+1}(w, d\tilde{w}) \leq 1 - 2\Phi\left(-\frac{\tau_\gamma(w) + \alpha_{n+1}}{2\beta_{n+1}}\right) + \zeta \sum_{k=1}^n \frac{\gamma \alpha_k}{\beta_k^3}.$$

where $(\alpha_k)_{k \geq 1}, (\beta_k)_{k \geq 1}$ and ζ are defined in (62) and (58) respectively and Φ is the cumulative distribution of the one-dimensional Gaussian distribution with mean 0 and variance 1.

PROOF. Let $\gamma \in (0, \bar{\gamma}]$, $w \in [0, +\infty)$. Note that for any $n \in \{0, \dots, n_\gamma\}$, by (62) and Lemma 31,

$$(68) \quad \alpha_n / (2\beta_n) \leq 1.$$

Then, by Lemma 26, (67) holds for $n = 0$. Assume it holds for $n \in \{0, \dots, n_\gamma - 1\}$. Then, we get

$$\int_{[0, +\infty)} \mathbb{1}_{(0, +\infty)}(\tilde{w}) Q_\gamma^{n+1}(w, d\tilde{w}) \leq \int_{[0, +\infty)} \left[1 - 2\Phi\left(-\frac{\tau_\gamma(w) + \alpha_n}{2\beta_n}\right) \right] Q_\gamma(w, d\tilde{w}) + \zeta \sum_{k=1}^{n-1} \frac{\gamma \alpha_k}{\beta_k^3}.$$

The proof is then concluded by a straightforward induction using Lemma 30 and (68). \square

THEOREM 33. Assume **H1** and **H2** hold. Let $R \geq 0$. For any $\gamma \in (0, \bar{\gamma}]$ and $\bar{\delta} \in (0, \{\mathbf{L}^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$, $\mu_\gamma((0, R)) \leq \eta_R c_\infty$, where

$$(69) \quad \eta_R = [\bar{\delta} + \bar{\gamma}]^{1/2} \left[\frac{\zeta e^{3(\bar{\delta} + \bar{\gamma})\mathbf{L}}}{\sigma^3} + \frac{e^{(\bar{\delta} + \bar{\gamma})\mathbf{L}}}{2\sqrt{2\pi}\sigma} \right] / \Phi \left(-\frac{(1 + \bar{\gamma}\mathbf{L})R + (\bar{\delta} + \bar{\gamma})c_\infty}{2\bar{\delta}^{1/2}\sigma e^{-(\bar{\delta} + \bar{\gamma})\mathbf{L}}} \right).$$

PROOF. Let $\bar{\delta} \in (0, \{\mathbf{L}^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$ and $\gamma \in (0, \bar{\gamma}]$. Set $n_\gamma = \lfloor \bar{\delta}/\gamma \rfloor$. Note that $\bar{\delta} \leq \gamma(n_\gamma + 1) \leq \bar{\delta} + \bar{\gamma}$. By Lemma 32, Proposition 9, integrating (67) with respect to μ_γ and using that $\tau_\gamma(0) = 0$, $\Phi(-t) \leq 1/2$ for any $t \geq 0$, gives

$$(70) \quad \begin{aligned} \mu_\gamma((0, +\infty)) &\leq \int_{\mathbb{R}} \left\{ 1 - 2\Phi \left(-\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) \right\} d\mu_\gamma(w) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3} \\ &\leq 1 - 2\Phi \left(-\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3} \\ &\quad + 2 \int_{(0, +\infty)} \left\{ \Phi \left(-\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) - \Phi \left(-\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) \right\} d\mu_\gamma(w) \\ &\leq 1 - 2\Phi \left(-\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) + \mu_\gamma((0, +\infty)) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3} - 2 \int_{(0, R_1)} \Phi \left(-\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) d\mu_\gamma(w). \end{aligned}$$

Rearranging the terms yields

$$(71) \quad 2 \int_{(0, R_1)} \Phi \left(-\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) d\mu_\gamma(w) \leq 1 - 2\Phi \left(-\frac{\alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) + \zeta \sum_{k=1}^{n_\gamma} \frac{\gamma \alpha_k}{\beta_k^3}.$$

In addition by **H1** using Lemma 31 and $t \mapsto \Phi(-t)$ is decreasing on \mathbb{R} , we have

$$(72) \quad \begin{aligned} \int_{(0, R_1)} \Phi \left(-\frac{\tau_\gamma(w) + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) d\mu_\gamma(w) &\geq \Phi \left(-\frac{(1 + \gamma\mathbf{L})R_1 + \alpha_{n_\gamma+1}}{2\beta_{n_\gamma+1}} \right) \mu_\gamma((0, R_1)) \\ &\geq \Phi \left(-\frac{(1 + \bar{\gamma}\mathbf{L})R_1 + (\bar{\delta} + \bar{\gamma})c_\infty}{2\bar{\delta}^{1/2}\sigma e^{-(\bar{\delta} + \bar{\gamma})\mathbf{L}}} \right) \mu_\gamma((0, R_1)), \end{aligned}$$

Using that $t \mapsto 1 - 2\Phi(-t)$ is $\sqrt{2/\pi}$ -Lipschitz and combining (71), (72) and Lemma 31 we get that

$$(73) \quad \begin{aligned} \Phi \left(-\frac{(1 + \bar{\gamma}\mathbf{L})R_1 + (\bar{\delta} + \bar{\gamma})c_\infty}{2\bar{\delta}^{1/2}\sigma e^{-(\bar{\delta} + \bar{\gamma})\mathbf{L}}} \right) \mu_\gamma((0, R_1)) &\leq \alpha_{n_\gamma+1}/(2\sqrt{2\pi}\beta_{n_\gamma+1}) + \zeta \gamma \sum_{k=1}^{n_\gamma} \{\alpha_k/\beta_k^3\} \\ &\leq c_\infty [\bar{\delta} + \bar{\gamma}]^{1/2} \left[\frac{\zeta e^{3(\bar{\delta} + \bar{\gamma})\mathbf{L}}}{\sigma^3} + \frac{e^{(\bar{\delta} + \bar{\gamma})\mathbf{L}}}{2\sqrt{2\pi}\sigma} \right], \end{aligned}$$

which implies that $\mu_\gamma((0, R_1)) \leq \eta c_\infty$ and completes the proof. \square

PROOF OF THEOREM 11. Let $\bar{\delta} \in (0, \{\mathbf{L}^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$ and $\gamma \in (0, \bar{\gamma}]$. By Proposition 8 and using μ_γ is invariant for Q_γ , we obtain

$$\int_{\mathbb{R}} w d\mu_\gamma(w) \leq (1 - \gamma\mathbf{m}) \int_{[R_1, +\infty)} w d\mu_\gamma(w) + (1 + \gamma\mathbf{L}) \int_{(0, R_1)} w d\mu_\gamma(w) + \gamma c_\infty$$

Then, rearranging the terms in this inequality yields

$$\begin{aligned} \int_{[R_1, +\infty)} w \, d\mu_\gamma(w) &\leq R_1 \mu_\gamma((0, R_1)) \mathbb{L}/\mathfrak{m} + c_\infty/\mathfrak{m} \\ \int_{[0, +\infty)} w \, d\mu_\gamma(w) &\leq R_1 \mu_\gamma((0, R_1)) (1 + \mathbb{L}/\mathfrak{m}) + c_\infty/\mathfrak{m}, \end{aligned}$$

which, combined with Theorem 33 applied to $R \leftarrow R_1$, concludes the proof of the first inequality in (17). Finally, by (70), using that $t \mapsto 1 - 2\Phi(-t)$ is $\sqrt{2/\pi}$ -Lipschitz, we have

$$\begin{aligned} (74) \quad \mu_\gamma((0, +\infty)) &\leq (\sqrt{2\pi}\beta_{n_\gamma+1})^{-1} \int_{[0, +\infty)} \{(1 + \bar{\gamma}\mathbb{L})w + \alpha_{n_\gamma+1}\} d\mu_\gamma(w) + \zeta\gamma \sum_{k=1}^{n_\gamma} \{\alpha_k/\beta_k^3\} \\ &\leq (c_\infty c_1(1 + \bar{\gamma}\mathbb{L}) + \alpha_{n_\gamma+1})/(\sqrt{2\pi}\beta_{n_\gamma+1}) + \zeta\gamma \sum_{k=1}^{n_\gamma} \{\alpha_k/\beta_k^3\}. \end{aligned}$$

This finishes the proof using $n_\gamma = \lfloor \bar{\delta}/\gamma \rfloor$, $\bar{\delta} \leq \gamma(n_\gamma + 1) \leq \bar{\delta} + \bar{\gamma}$ and Lemma 31. \square

5.1.7. *Proof of Theorem 12.* For any $a > 0$, define $\mathcal{V}_a^* : \mathbb{R} \rightarrow [0, +\infty)$ for any $w \in \mathbb{R}$ by

$$(75) \quad \mathcal{V}_a^*(w) = \mathbb{1}_{(0, +\infty)}(w) \exp(aw).$$

PROPOSITION 34. Assume **H1** and **H2** hold. Let $a > 0$. Then, for any $w \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$,

$$(76) \quad Q_\gamma \mathcal{V}_a^*(w)/\mathcal{V}_a^*(w) \leq \lambda_a^\gamma \mathbb{1}_{[R_a, +\infty)}(w) + \exp\{\gamma a(\mathbb{L}w + 2\sigma^2 a + c_\infty)\} \mathbb{1}_{[0, R_a)}(w),$$

where

$$(77) \quad R_a = 1 \vee R_1 \vee [(4a\sigma^2 + 2c_\infty)/\mathfrak{m}], \quad \lambda_a = \exp(-a\mathfrak{m}R_a/2).$$

PROOF. Set for any $w \in \mathbb{R}$, $\tau_\gamma^\infty = \tau_\gamma(w) + \gamma c_\infty$. By definition (13) and using the change of variable $g \mapsto \sigma\gamma^{1/2}g$ twice, we have

$$\begin{aligned} Q_\gamma \mathcal{V}^*(w) &= \int_{\mathbb{R}} \exp(a\{\tau_\gamma^\infty(w) - 2\sigma\gamma^{1/2}g\}) \{1 - \bar{p}_{\sigma^2\gamma}(\tau_\gamma^\infty(w), g)\} \varphi(g) dg \\ &= \int_{-\infty}^{\tau_\gamma^\infty(w)/2} \exp(a\{\tau_\gamma^\infty(w) - 2g\}) \{\varphi_{\sigma^2\gamma}(g) - \varphi_{\sigma^2\gamma}(\tau_\gamma(w) - g)\} dg \\ &\leq (2\pi)^{-1/2} \int_{-\infty}^{\sigma\gamma^{1/2}\tau_\gamma^\infty(w)/2} \exp(a\{\tau_\gamma^\infty(w) - 2\sigma\gamma^{1/2}g\} - g^2/2) dg \\ &= \exp(a\tau_\gamma^\infty(w) + 2a^2\sigma^2\gamma) (2\pi)^{-1/2} \int_{-\infty}^{\sigma\gamma^{1/2}\tau_\gamma^\infty(w)/2} \exp(-\{g + 2a\sigma\gamma^{1/2}\}^2/2) dg \\ &\leq \exp(a\tau_\gamma^\infty(w) + 2a^2\sigma^2\gamma) \Phi(\sigma\gamma^{1/2}\tau_\gamma^\infty(w)/2 + 2a\sigma\gamma^{1/2}) \leq e^{a\tau_\gamma^\infty(w) + 2a^2\sigma^2\gamma}. \end{aligned}$$

This concludes the proof of (76) for $w \in [0, R_a)$ using **H1**. In the case $w \in [R_a, +\infty)$, we only need to use that by **H1** and definition of R_a (77), $a\tau_\gamma^\infty(w) + 2a^2\sigma^2\gamma \leq a(1 - \mathfrak{m}\gamma)w + ac_\infty + 2a^2\sigma^2\gamma \leq aw - a\gamma\mathfrak{m}R_a/2$. \square

PROOF OF THEOREM 12. Let $\bar{\delta} \in (0, \{\mathbf{L}^{-1} \wedge (\sigma e^{-1}/c_\infty)^2\}]$ and $\gamma \in (0, \bar{\gamma}]$. We show that (17) holds with

$$c_3 = (B_a/\lambda_a)^{\bar{\gamma}} a(\mathbf{L}R_a + 2\sigma^2 a + c_\infty) \eta_{R_a} \exp(aR_a) / |\log(\lambda_a)|,$$

where $B_a = \exp\{a(\mathbf{L}R_a + 2\sigma^2 a + c_\infty)\}$, λ_a, R_a are defined in (77) and η_R in (69).

By Proposition 34 and since $\mathcal{V}^*(0) = 0$ and μ_γ is invariant for Q_γ , we have

$$\int_{(0,+\infty)} \mathcal{V}_a^*(w) d\mu_\gamma(w) \leq \lambda_a^\gamma \int_{[R_a,+\infty)} \mathcal{V}_a^*(w) d\mu_\gamma(w) + B_a^\gamma \int_{(0,R_a)} \mathcal{V}_a^*(w) d\mu_\gamma(w),$$

setting $B_a = \exp\{a(\mathbf{L}R_a + 2\sigma^2 a + c_\infty)\}$. Rearranging terms yields

$$\begin{aligned} \int_{(0,+\infty)} \mathcal{V}_a^*(w) d\mu_\gamma(w) &\leq \{B_a^\gamma - \lambda_a^\gamma\} / \{1 - \lambda_a^\gamma\} \int_{(0,R_a)} \mathcal{V}^*(w) d\mu_\gamma(w) \\ &\leq \{B_a^\gamma \lambda_a^{-\gamma} - 1\} / \{\lambda_a^{-\gamma} - 1\} \exp(aR_a) c_\infty \eta_{R_a}, \end{aligned}$$

where we have used Theorem 33 applied to $R \leftarrow R_a$ in the last inequality. The proof is then completed upon using that for any $t \geq 0$, $t \leq e^t - 1 \leq te^t$. \square

5.1.8. Proof of Theorem 13.

LEMMA 35. Assume **H1** and **H2** hold. For any $w \in [0, +\infty)$ and $\alpha, \beta \in [0, +\infty)$, $\beta > 0$,

$$\int_{(0,+\infty)} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tilde{w}) + \alpha}{2\beta} \right) \right] Q_\gamma(w, d\tilde{w}) \leq 1 - 2\Phi \left(-\frac{\tau_\gamma(w) + \gamma c_\infty + \alpha/(1 + \gamma \mathbf{L})}{2\sqrt{\sigma^2 \gamma + \beta/(1 + \gamma \mathbf{L})^2}} \right),$$

where Φ are the density and the cumulative distribution function of the one-dimensional Gaussian distribution with mean 0 and variance 1.

PROOF. Let $\alpha, \beta \geq 0$, $\beta > 0$. By (13), we have

$$\begin{aligned} &\int_{(0,+\infty)} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tilde{w}) + \alpha}{2\beta} \right) \right] Q_\gamma(w, d\tilde{w}) \\ &= \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \right] (1 - \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g)) \varphi(g) dg. \end{aligned}$$

By **H1**, $t \mapsto 1 - 2\Phi(-t)$ is increasing, we have setting $\psi_\gamma(w) = \tau_\gamma(w) + \gamma c_\infty + \alpha/(1 + \gamma \mathbf{L})$,

$$1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \leq 1 - 2\Phi \left(-\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma}g}{2\beta/(1 + \gamma \mathbf{L})} \right).$$

Using [15, Lemma 20], Lemma 27 and Lemma 28, we get

$$\begin{aligned} &\int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\tau_\gamma(\tau_\gamma(w) + \gamma c_\infty - 2\sigma\sqrt{\gamma}g) + \alpha}{2\beta} \right) \right] (1 - \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g)) \varphi(g) dg \\ &\leq \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma}g}{2\beta/(1 + \gamma \mathbf{L})} \right) \right] (1 - \bar{p}_{\sigma^2 \gamma}(\psi_\gamma(w), g)) \varphi(g) dg \\ &\quad - \int_{\mathbb{R}} \left[1 - 2\Phi \left(-\frac{\psi_\gamma(w) - 2\sigma\sqrt{\gamma}g}{2\beta/(1 + \gamma \mathbf{L})} \right) \right] \bar{p}_{\sigma^2 \gamma}(\tau_\gamma(w) + \gamma c_\infty, g) \varphi(g) dg \\ &\leq 1 - 2\Phi \left(-\frac{\psi_\gamma(w)}{2\sqrt{\sigma^2 \gamma + \beta/(1 + \gamma \mathbf{L})^2}} \right), \end{aligned}$$

which completes the proof. \square

We consider in what follows that $(W_k)_{k \in \mathbb{N}}$ is the canonical process on $([0, +\infty)^\mathbb{N}, \mathcal{B}([0, +\infty))^{\otimes \mathbb{N}})$ and for any $w \in [0, +\infty)$, \mathbb{P}_w and \mathbb{E}_w correspond to the probability and expectation respectively, associated with Q_γ and the initial condition δ_w on this space.

LEMMA 36. Assume **H1** and **H2** hold. For any $k \in \mathbb{N}$ and $w \in (0, +\infty)$,

$$\mathbb{P}_w \left(\min_{i \in \{0, \dots, k+1\}} W_i > 0 \right) \leq 1 - 2\Phi \left[-\frac{\tau_\gamma(w) + \alpha_{k+1}}{2\beta_{k+1}} \right],$$

where $\alpha_{k+1}, \beta_{k+1}$ are given in (62).

PROOF. The proof is by induction on $k \in \mathbb{N}$. The proof for $k = 0$ follows from Lemma 26. Assume that the result holds for $k - 1 \in \mathbb{N}$ and for any $w \in (0, +\infty)$. Then, by the Markov property and the assumption hypothesis, for any $w \in (0, +\infty)$,

$$(78) \quad \mathbb{P}_w \left(\min_{i \in \{0, \dots, k\}} W_i > 0 \right) = \mathbb{E}_w \left[\mathbb{1}_{(0, +\infty)}(W_1) \mathbb{P}_{W_1} \left(\min_{i \in \{0, \dots, k-1\}} W_i > 0 \right) \right] \\ \leq \mathbb{E}_x \left[\mathbb{1}_{(0, +\infty)}(W_1) \left\{ 1 - 2\Phi \left(-\frac{\tau_\gamma(W_1) + \alpha_k}{2\beta_k} \right) \right\} \right].$$

The proof is then completed upon using Lemma 35. \square

LEMMA 37. Assume **H1** and **H2** hold. Then, for any $k \in \mathbb{N}$, $w, \tilde{w} \in [0, +\infty)$,

$$\|\delta_w Q_\gamma^{k+1} - \delta_{\tilde{w}} Q_\gamma^{k+1}\|_{\text{TV}} \leq 1 - 2\Phi \left[-\frac{\tau_\gamma(w \vee \tilde{w}) + \alpha_{k+1}}{2\beta_{k+1}} \right],$$

where $\alpha_{k+1}, \beta_{k+1}$ are given in (62).

PROOF. We consider again $(G_k)_{k \geq 1}$ and $(U_k)_{k \geq 1}$ two independent sequences of i.i.d. standard Gaussian and $[0, 1]$ -uniform random variables respectively. Define the Markov chains $(W_k)_{k \in \mathbb{N}}$ and $(\tilde{W}_k)_{k \in \mathbb{N}}$ starting from $w \in [0, +\infty)$ and $\tilde{w} \in [0, +\infty)$ respectively, for any $k \in \mathbb{N}$, $W_{k+1} = \mathcal{G}_\gamma(W_k, G_{k+1}, U_{k+1})$ and $\tilde{W}_{k+1} = \mathcal{G}_\gamma(\tilde{W}_k, G_{k+1}, U_{k+1})$. Note that the case $w = \tilde{w}$ is trivial so we only consider the converse and assume that $w < \tilde{w}$, $\tilde{w} > 0$. Then, we obtain by Proposition 6 that almost surely $W_k \leq \tilde{W}_k$ for any $k \in \mathbb{N}$, which implies that

$$\|\delta_w Q_\gamma^{k+1} - \delta_{\tilde{w}} Q_\gamma^{k+1}\|_{\text{TV}} \leq \mathbb{P}(W_{k+1} \neq \tilde{W}_{k+1}) \leq \mathbb{P} \left(\min_{i \in \{0, \dots, k+1\}} \tilde{W}_i > 0 \right).$$

Indeed, if $\min_{i \in \{0, \dots, k+1\}} \tilde{W}_i = 0$, then there exists $i \in \{0, \dots, k+1\}$, $\tilde{W}_i = 0$ which implies since $\tilde{W}_i \geq W_i \geq 0$ that $\tilde{W}_i = W_i$ and therefore $\tilde{W}_{k+1} = W_{k+1}$ by definition of the two processes. The proof is then completed by Lemma 36. \square

PROPOSITION 38. Assume **H1** and **H2** hold. Let $t > 0$. Then, for any $w, \tilde{w} \in [0, +\infty)$,

$$\|\delta_w Q_\gamma^{\lceil t_0/\gamma \rceil} - \delta_{\tilde{w}} Q_\gamma^{\lceil t_0/\gamma \rceil}\|_{\text{TV}} \leq 1 - 2\Phi \left[-L^{1/2} \frac{w \vee \tilde{w} + t_0 c_\infty}{2\sigma \{1 - e^{-2L(t_0 + \tilde{\gamma})}\}^{1/2}} \right].$$

PROOF. Note that by (62) for any $k \in \mathbb{N}$, $\alpha_{k+1} \leq k\gamma c_\infty$ and $\beta_{k+1}^2 = (\sigma^2(1 + \gamma L)/L) \{1 - (1 + \gamma L)^{-2k}\} \geq (\sigma^2(1 + \gamma L)/L) \{1 - e^{-2k\gamma L}\}$ using that $(1 + t) \leq e^t$ for any $t \in \mathbb{R}$. The proof is then completed using Lemma 37, **H1** and the previous bounds for $k \leftarrow \lceil t_0/\gamma \rceil$. \square

LEMMA 39. Assume **H1** and **H2** hold. Let $a > 0$. Then, for any $w \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$,

$$(79) \quad Q_\gamma \mathcal{V}_1^*(w) \leq (1 - \gamma \mathfrak{m}) \mathcal{V}_1^*(w) + \gamma \{\mathbf{L} + \mathfrak{m}\} R_1 + \gamma c_\infty ,$$

$$(80) \quad Q_\gamma \mathcal{V}_a^*(w) \leq \lambda_a^\gamma \mathcal{V}_a^*(w) + \gamma \alpha_a \mathbb{1}_{[0, R_1]}(w) ,$$

where Q_γ is defined by (13), for any $w \in \mathbb{R}$, $\mathcal{V}_1^*(w) = |w|$, \mathcal{V}_a^* by (75), λ_a, R_a are defined by (77) and

$$(81) \quad \alpha_a = a \exp\{a R_a + \bar{\gamma} a (\mathbf{L} R_a + 2\sigma^2 a + c_\infty)\} [\mathbf{L} R_a + 2\sigma^2 a + c_\infty] .$$

PROOF. (79) is a simple consequence of Proposition 8. Note that for $w \in \mathbb{R}$, $w \geq R_a$, (80) holds by Proposition 34. In the case $w < R_a$, using Proposition 34 and $e^t - 1 \leq t e^t$ for any $t \geq 0$, we obtain $Q_\gamma \mathcal{V}^*(w) \leq \lambda_a^\gamma \mathcal{V}^*(w) + \mathcal{V}^*(w) [\lambda_a^{-\gamma} \exp\{\gamma a (\mathbf{L} w + 2\sigma^2 a + c_\infty)\} - 1] \leq \lambda_a^\gamma \mathcal{V}^*(w) + a \mathcal{V}^*(w) \exp\{\gamma a (\mathbf{L} w + 2\sigma^2 a + c_\infty)\} [\mathbf{L} w + 2\sigma^2 a + c_\infty]$, which completes the proof. \square

Define $\mathcal{V}_1, \mathcal{V}_a : \mathbb{R} \rightarrow [0, +\infty)$ for any $w \in \mathbb{R}$ by

$$(82) \quad \mathcal{V}_1(w) = 1 + \Lambda_1 |w| , \quad \mathcal{V}_a(w) = 1 + \Lambda_a \mathbb{1}_{(0, +\infty)}(w) \exp(aw) .$$

PROPOSITION 40. Assume **H1** and **H2** hold. Let $a > 0$ and $t_0 > 0$. Then, for any $w \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$,

$$Q_\gamma^{[t_0/\gamma]} \mathcal{V}_1(w) \leq \lambda_1 \mathcal{V}_1^*(w) + \beta_1 , \quad Q_\gamma^{[t_0/\gamma]} \mathcal{V}_a(w) \leq \lambda_a \mathcal{V}_a^*(w) + \beta_a ,$$

where Q_γ is defined by (13), $\mathcal{V}_1, \mathcal{V}_a$ by (82), λ_a by (77), α_a by (81) and

$$\lambda_1 = e^{-\mathfrak{m}} , \quad \beta_1 = (t_0 + \bar{\gamma}) [\{\mathbf{L} + \mathfrak{m}\} R_1 + c_\infty] + 1 , \quad \beta_a = (t_0 + \bar{\gamma}) \alpha_a + 1 .$$

PROOF OF THEOREM 13. By Lemma 39 and an easy induction, and using that $1 - t \leq e^{-t}$ for any $t \geq 0$, we have for any $k \in \mathbb{N}$,

$$Q_\gamma^k \mathcal{V}_1^*(w) \leq \lambda_1^{k\gamma} \mathcal{V}_1^*(w) + k\gamma [\{\mathbf{L} + \mathfrak{m}\} R_1 c_\infty] , \quad Q_\gamma^k \mathcal{V}_a^*(w) \leq \lambda_a^{k\gamma} \mathcal{V}_a^*(w) + k\gamma \alpha_a .$$

Using that $Q_\gamma^k 1 \equiv 1$ for any $k \in \mathbb{N}$ completes the proof. \square

PROOF OF THEOREM 13. Let $t_0 > 0$. We only show the result for $\mathcal{V} = \mathcal{V}_1$. The result for $\mathcal{V} = \mathcal{V}_a$, $a > 0$ is similar upon replacing λ_1 and β_1 by λ_a and β_a given in Proposition 40 respectively.

Define $\delta_1 = 4\beta_1/(1 - \lambda_1) - 1$ and $M_1 = \sup\{w \in [0, +\infty) : \mathcal{V}_1(w) \leq \delta_1\}$ which is well defined since $\lim_{w \rightarrow +\infty} \mathcal{V}_1(w) = +\infty$. Define in addition,

$$\varepsilon_1 = 2\Phi \left[-\mathbf{L}^{1/2} \frac{M_1 + t_0 c_\infty}{2\sigma \{1 - e^{-2\mathbf{L}(t_0 + \bar{\gamma})}\}^{1/2}} \right] < 1 .$$

Then, $\{\mathcal{V}_1 \leq \delta_1\}$ is a $(1, \varepsilon)$ -Doebelin set for Q_γ and $\lambda_1 + 2\beta_1/(1 + \delta_1) < 1$. Therefore, [11, Theorem 19.4.1] implies that for any $k \in \mathbb{N}$,

$$\left\| \delta_w Q_\gamma^{[t_0/\gamma]k} - \mu_\gamma \right\|^V \leq C \rho^k \{\mathcal{V}_1(w) + \mu_\gamma(\mathcal{V}_1)\} ,$$

where a bound on $\mu_\gamma(\mathcal{V}_1)$ is provided by Theorem 11 and

$$\log(\rho) = \log(1 - \varepsilon_1) \log(\bar{\lambda}_1) / \{\log(1 - \varepsilon_1) + \log(\bar{\lambda}_1 - \log(\bar{\beta}_1))\}$$

$$\bar{\lambda}_1 = \lambda_1 + 2\beta_1/(1 + \delta_1) , \quad \bar{\beta}_1 = \lambda_1 \beta_1 + \delta_1$$

$$C = \{\lambda_1 + 1\} / [1 + \bar{\beta}_1 / \{(1 - \varepsilon_1)(1 - \bar{\lambda}_1)\}] .$$

\square

5.2. Postponed proofs of Section 3.

5.2.1. *Proof of Proposition 15.* The proof is an easy consequence of Lemma 43 below and the definition of $(\mathbf{W}_t^{(n)})_{t \geq 0}$. Before stating and proving Lemma 43, we need the following technical results.

LEMMA 41. *Assume A 1. Then, for any $q \in [1, +\infty)$, we have for any $\gamma \in (0, \bar{\gamma}]$,*

$$\mathbb{E}[|W_1 - \tau_\gamma(W_0) - \gamma c_\infty|^q] \leq (4\sigma^2\gamma)^{q/2} \left\{ \mathbf{m}_q + 2 \sup_{u \geq 0} [u^q \Phi(-u)] \right\},$$

where W_1 is defined by (12), \mathbf{m}_q is the q -th moment of the standard Gaussian distribution and Φ is its cumulative distribution function.

PROOF. Let $w_0 \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$. By definition (13) and (10), we have setting $\bar{\tau}_\gamma^\infty(w_0) = \{\tau_\gamma(w_0) + \gamma c_\infty\} / (2\sqrt{\sigma^2\gamma})$,

$$\begin{aligned} \int_{\mathbb{R}_+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) &= (4\sigma^2\gamma)^{q/2} \int_{-\infty}^{\bar{\tau}_\gamma^\infty(w_0)} |g|^q \varphi(g) dg \\ &\quad - (4\sigma^2\gamma)^{q/2} \int_{\bar{\tau}_\gamma^\infty(w_0)}^{+\infty} |g|^q \varphi(g) dg + \int_{\mathbb{R}} |\tau_\gamma(w_0) + \gamma c_\infty|^q \varphi(2\bar{\tau}_\gamma^\infty(w_0) - g) \wedge \varphi(g) dg \\ &\leq (4\sigma^2\gamma)^{q/2} \mathbf{m}_q + 2^{p+1} \sigma^p \gamma^{p/2} [\bar{\tau}_\gamma^\infty(w_0)]^q \Phi\{-\bar{\tau}_\gamma^\infty(w_0)\}, \end{aligned}$$

which completes the proof. \square

LEMMA 42. *Assume A 1. Then, there exists $C \geq 0$ such that for any $k \in \mathbb{N}$ and $\gamma \in (0, \bar{\gamma}]$, $\mathbb{E}[W_k^4] \leq e^{Ck\gamma} \{\mathbb{E}[W_0^4] + 1\}$, where $(W_k)_{k \in \mathbb{N}}$ is defined by (12).*

PROOF. By A 1, (12) and (13) we have that for any $w_0 \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$, setting $W_0 = w_0$ and $\kappa_\infty(w_0) = \kappa(w_0) + c_\infty$,

$$\begin{aligned} \int_{\mathbb{R}_+} w_1^4 Q_\gamma(w_0, dw_1) &= \mathbb{E}[W_1^4] \leq \mathbb{E}[(w_0 + \gamma \kappa_\infty(w_0) - 2\sqrt{\sigma^2\gamma} G_1)^4] \\ &= \{w_0 + \gamma \kappa_\infty(w_0)\}^4 + 6\sigma^2\gamma \{w_0 + \gamma \kappa_\infty(w_0)\}^2 + 3\sigma^4\gamma^2. \end{aligned}$$

By A 1, for any $\ell \in \{2, 4\}$, we have that for any $w_0 \in [0, +\infty)$, $\gamma \in (0, \bar{\gamma}]$,

$$\{w_0 + \gamma \kappa_\infty(w_0)\}^\ell \leq w_0^\ell + 2^{\ell-1} \ell \gamma (1 + L_\kappa)^\ell [|w_0|^\ell + c_\infty].$$

Therefore, we obtain that there exists some constant $C_1, C_2 \geq 0$, such that for any $w_0 \in [0, +\infty)$, $\gamma \in (0, \bar{\gamma}]$,

$$\int_{\mathbb{R}_+} w_1^4 Q_\gamma(w_0, dw_1) \leq w_0^4 + C_1 \gamma \{1 + w_0^2 + w_0^4\} \leq (1 + \gamma C_2) w_0^4 + \gamma C_2.$$

By an easy induction, we get then that for any $w_0 \in [0, +\infty)$, $\gamma \in (0, \bar{\gamma}]$ and $k \in \mathbb{N}$,

$$\int_{\mathbb{R}_+} w_1^4 Q_\gamma^k(w_0, dw_1) \leq (1 + C_2\gamma)^k w_0^4 + C_2\gamma \sum_{i=0}^{k-1} (1 + C_2\gamma)^i \leq e^{k\gamma C_2} [w_0^4 + C_2],$$

which completes the proof by the Markov property. \square

LEMMA 43. *Assume A 1. Then, there exists $C \geq 0$ such that for any $k \in \mathbb{N}$ and $\gamma \in (0, \bar{\gamma}]$, $\mathbb{E}[\max_{\ell \in \{0, \dots, k\}} |W_\ell - W_0|^4] \leq C(k\gamma)^2 e^{Ck\gamma} \{\mathbb{E}[W_0^4] + 1\}$, where $(W_k)_{k \in \mathbb{N}}$ is defined by (12).*

PROOF. Assume that $\mathbb{E}[W_0^4] < +\infty$, otherwise the results holds. Denote by $(\mathcal{F}_k)_{k \in \mathbb{N}}$ the filtration associated with $(W_k)_{k \in \mathbb{N}}$. We consider the following decomposition for any $\ell \in \mathbb{N}$,

$$W_\ell - W_0 = A_\ell + B_\ell, \quad A_\ell = \sum_{i=0}^{\ell-1} \Delta M_i, \quad B_\ell = \sum_{i=0}^{\ell-1} H_i,$$

where using that $\mathbb{E}[W_{i+1}|\mathcal{F}_i] = \tau_\gamma^\infty(W_i)$ by Lemma 25 and the Markov property,

$$(83) \quad \Delta M_i = W_{i+1} - \mathbb{E}[W_{i+1}|\mathcal{F}_i] = W_{i+1} - \tau_\gamma^\infty(W_i), \quad H_i = \tau_\gamma^\infty(W_i) - W_i.$$

Then, using Young's inequality, we get for any $\gamma \in (0, \bar{\gamma}]$ and $k \in \mathbb{N}$,

$$(84) \quad \max_{\ell \in \{0, \dots, k\}} [W_\ell - W_0]^4 \leq 2^3 \{ \max_{\ell \in \{0, \dots, k\}} A_\ell^4 + \max_{\ell \in \{0, \dots, k\}} B_\ell^4 \}.$$

We now bound the two last terms in the right hand side of this equation. First, by A1 and Young's inequality, we get for any $\gamma \in (0, \bar{\gamma}]$ and $k \in \mathbb{N}$,

$$(85) \quad \mathbb{E}[\max_{\ell \in \{0, \dots, k\}} B_\ell^4] \leq k^3 \sum_{i=0}^{k-1} H_i^4 \leq 2^3 (k\gamma)^4 (1 + L_\kappa)^4 \{ \max_{i \in \{0, \dots, k-1\}} \mathbb{E}[W_i^4] + c_\infty^4 \}.$$

In addition, by definition (83), $(\Delta M_i)_{i \in \mathbb{N}}$ are $(\mathcal{F}_i)_{i \in \mathbb{N}}$ -martingale increments. It follows by Burkholder inequality [5, Theorem 3.2] and Young's inequality that there exists $C_4 \geq 0$ satisfying for any $k \in \mathbb{N}$ and $\gamma \in (0, \bar{\gamma}]$,

$$\mathbb{E}[\max_{\ell \in \{0, \dots, k\}} A_\ell^4] \leq C_4 \mathbb{E}[\{\sum_{i=0}^{k-1} \Delta M_i^2\}^2] \leq C_4 k \sum_{i=0}^{k-1} \mathbb{E}[\Delta M_i^4].$$

Therefore by Lemma 41, we get that

$$\mathbb{E}[\max_{\ell \in \{0, \dots, k\}} A_\ell^4] \leq C_4 \mathbb{E}[\{\sum_{i=0}^{k-1} \Delta M_i^2\}^2] \leq C_4 (4\sigma^2 k\gamma)^2 \left\{ \mathbf{m}_4 + 2 \sup_{u \geq 0} [u^4 \Phi(-u)] \right\},$$

where \mathbf{m}_4 is the fourth moment of the standard Gaussian distribution. Combining this result with (85) and using Lemma 42 in (84) concludes the proof. \square

5.2.2. *Proof of Proposition 16.* To show this result, we use the Komolgorov criteria [21, Corollary 14.9]: for any $T \geq 0$, there exist $C_T \geq 0$ such that for any $n \in \mathbb{N}$ and $s, t \in [0, +\infty)$, $s \leq t$,

$$\mathbb{E} \left[\left| \mathbf{W}_t^{(n)} - \mathbf{W}_s^{(n)} \right|^4 \right] \leq C_T (t - s)^2.$$

Note that denoting $k_1^{(n)} = \lceil s/\gamma_n \rceil$ and $k_2^{(n)} = \lfloor t/\gamma_n \rfloor$, we have by (19)

$$\begin{aligned} & \mathbb{E} \left[\left| \mathbf{W}_t^{(n)} - \mathbf{W}_s^{(n)} \right|^4 \right] \\ & \leq \begin{cases} (t-s)\gamma_n^{-1} \{W_{k_1^{(n)}+1} - W_{k_1^{(n)}}\}^4 & \text{if } k_1^{(n)} < k_2^{(n)} \\ 3^3 [\{W_{k_2^{(n)}+1} - W_{k_1^{(n)}}\}^4 + \{W_{k_2^{(n)}} - W_{k_1^{(n)}}\}^4 + \{W_{k_1^{(n)}} - W_{k_1^{(n)}-1}\}^4] & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 43, Lemma 42 and the Markov property complete the proof.

5.2.3. *Proof of Proposition 18.* We preface the proof by the following technical lemma.

LEMMA 44. Assume A1. Then, for any $q \in [1, +\infty)$, we have

(a) for any $\gamma \in (0, \bar{\gamma}]$ and $w_0 \in [0, +\infty)$,

$$(86) \quad -2(4\sigma^2\gamma)^{q/2} \int_{\bar{\tau}_\gamma^\infty(w_0)}^{+\infty} |g|^q \varphi(g) dg \\ \leq \int_{\mathbb{R}_+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) - (4\sigma^2\gamma)^{q/2} \mathbf{m}_q \leq 0,$$

where $\bar{\tau}_\gamma^\infty(w_0) = \{\tau_\gamma(w_0) + \gamma c_\infty\} / (2\sqrt{\sigma^2\gamma})$, Q_γ is defined by (13), \mathbf{m}_q is the q -th moment of the standard Gaussian distribution. and φ is its probability density function;

(b) for any $\gamma \in (0, \bar{\gamma}]$,

$$\left| \int_{\mathbb{R}_+} |w_1 - \gamma c_\infty|^q Q_\gamma(0, dw_1) \right| \leq 3(\gamma c_\infty)^q.$$

PROOF. (a) Let $w_0 \in [0, +\infty)$ and $\gamma \in (0, \bar{\gamma}]$. By definition (13) and (10), we have setting $\bar{\tau}_\gamma^\infty(w_0) = \{\tau_\gamma(w_0) + \gamma c_\infty\} / (2\sqrt{\sigma^2\gamma})$,

$$(87) \quad \int_{\mathbb{R}_+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) = (4\sigma^2\gamma)^{q/2} \int_{-\infty}^{\bar{\tau}_\gamma^\infty(w_0)} |g|^q \varphi(g) dg \\ - (4\sigma^2\gamma)^{q/2} \int_{\bar{\tau}_\gamma^\infty(w_0)}^{+\infty} |g|^q \varphi(g) dg + \int_{\mathbb{R}} |\tau_\gamma(w_0) + \gamma c_\infty|^q \varphi(2\bar{\tau}_\gamma^\infty(w_0) - g) \wedge \varphi(g) dg.$$

Therefore, we obtain that

$$\int_{\mathbb{R}_+} |w_1 - \tau_\gamma(w_0) - \gamma c_\infty|^q Q_\gamma(w_0, dw_1) - (4\sigma^2\gamma)^{q/2} \mathbf{m}_q \\ = -2(4\sigma^2\gamma)^{q/2} \int_{\bar{\tau}_\gamma^\infty(w_0)}^{+\infty} |g|^q \varphi(g) dg + 2(4\sigma^2\gamma)^{q/2} [\{\bar{\tau}_\gamma(w_0)\}^q \Phi(-\bar{\tau}_\gamma(w_0))].$$

Using that $\int_{\bar{\tau}_\gamma^\infty(w_0)}^{+\infty} |g|^q \varphi(g) dg \geq \{\bar{\tau}_\gamma^\infty(w_0)\}^q \Phi(-\bar{\tau}_\gamma^\infty(w_0))$ completes the proof.

(b) By (87) and since $\tau_\gamma(0) = 0$ and $\bar{\tau}_\gamma^\infty(0) = \gamma c_\infty / (2\sqrt{\sigma^2\gamma})$,

$$\left| \int_{\mathbb{R}_+} |w_1 - \tau_\gamma(0) - \gamma c_\infty|^q Q_\gamma(0, dw_1) \right| \leq 2(4\sigma^2\gamma)^{q/2} \int_0^{\frac{\gamma c_\infty}{2\sqrt{\sigma^2\gamma}}} |g|^q \varphi(g) dg + (\gamma c_\infty)^q.$$

Using that $(4\sigma^2\gamma)^{q/2} \int_0^{\gamma c_\infty / (2\sqrt{\sigma^2\gamma})} |g|^q \varphi(g) dg \leq (\gamma c_\infty)^q$ completes the proof. \square

PROOF OF PROPOSITION 18. **Proof of (24).** Let $\varphi \in C^\infty(\mathbb{R}^d)$ satisfying (23), $N \in \mathbb{N}$, $(t_1, \dots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \dots \leq t_N \leq s < t$, $\psi : \mathbb{R}_+^N \rightarrow \mathbb{R}$, continuous and bounded. Note that we only need to show that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left| \mathbb{E} \left[\varphi(\mathbf{W}_t^{(n)}) - \varphi(\mathbf{W}_s^{(n)}) - \int_s^t \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) du \middle| \mathcal{G}_s^{(n)} \right] \right| \right],$$

setting for any $n \in \mathbb{N}$, $u \in [0, +\infty)$, $\mathcal{G}_u^{(n)} = \mathcal{F}_{[u/\gamma_n]}^{(n)}$, where $(\mathcal{F}_k^{(n)})_{k \in \mathbb{N}}$ is the filtration corresponding to $(W_k^{(n)})_{k \in \mathbb{N}}$.

Define $k_1^{(n)} = \lceil t/\gamma_n \rceil$ and $k_2^{(n)} = \lceil s/\gamma_n \rceil$ and consider the following decomposition

$$(88) \quad \mathbb{E} \left[\varphi(\mathbf{W}_t^{(n)}) - \varphi(\mathbf{W}_s^{(n)}) - \int_s^t \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) du \middle| \mathcal{G}_s^{(n)} \right] = \mathbb{E} \left[A_1^{(n)} + A_2^{(n)} + A_3^{(n)} \middle| \mathcal{G}_s^{(n)} \right]$$

$$A_1^{(n)} = \varphi(\mathbf{W}_t^{(n)}) - \varphi(W_{k_1^{(n)}}^{(n)}) - \{\varphi(\mathbf{W}_s^{(n)}) - \varphi(W_{k_2^{(n)}}^{(n)})\}$$

$$A_2^{(n)} = - \int_s^t \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) du + \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathcal{A}\varphi(W_k^{(n)})$$

$$A_3^{(n)} = \varphi(W_{k_1^{(n)}}^{(n)}) - \varphi(W_{k_2^{(n)}}^{(n)}) - \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathcal{A}\varphi(W_k^{(n)}).$$

We deal with these three terms separately.

First since φ satisfies (23), by the fundamental theorem of calculus, there exists $C \geq 0$ such that for any $w_0, w_1 \in \mathbb{R}$, $|\varphi(w_1) - \varphi(w_0)| \leq C \max(|w_0|, |w_1|) |w_0 - w_1|$. By (19), we get that there exists $C \geq 0$ such that for any $n \in \mathbb{N}$, almost surely,

$$|A_1^{(n)}| \leq C \gamma_n \{ \max_{i \in \{k_1^{(n)}, k_1^{(n)}+1, k_2^{(n)}, k_2^{(n)}+1\}} |W_i^{(n)}|^2 + 1 \}.$$

This implies by Lemma 41 and the Lebesgue convergence theorem that

$$(89) \quad \lim_{n \rightarrow +\infty} \mathbb{E}[|A_1^{(n)}|] = 0.$$

Regarding $A_2^{(n)}$, we consider the decomposition,

$$A_2^{(n)} = A_{2,1}^{(n)} + A_{2,2}^{(n)},$$

$$A_{2,1}^{(n)} = \int_s^{k_2^{(n)}\gamma_n} \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) du + \int_t^{k_1^{(n)}\gamma_n} \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) du$$

$$A_{2,2}^{(n)} = - \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \int_{k\gamma_n}^{(k+1)\gamma_n} \{ \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) - \mathcal{A}\varphi(W_k^{(n)}) \} du.$$

Since φ satisfies (23), by Lemma 41 and the Lebesgue dominated convergence theorem, we get that $\lim_{n \rightarrow +\infty} \mathbb{E}[|A_{2,1}^{(n)}|] = 0$. In addition, we have by definition of \mathcal{A} (22) that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$, $u \in (k\gamma_n, (k+1)\gamma_n)$,

$$\left| \mathcal{A}\varphi(\mathbf{W}_u^{(n)}) - \mathcal{A}\varphi(W_k^{(n)}) \right| \leq B_{u,k} + 2\mathbb{1}_{A_k^c} \sigma^2 \sup_{\mathbb{R}} |\varphi''|$$

$$B_{u,k}^{(n)} = (|\kappa(W_k^{(n)})| + c_\infty) |\varphi'(\mathbf{W}_u^{(n)}) - \varphi'(W_k^{(n)})|$$

$$+ |\varphi'(\mathbf{W}_u^{(n)})| |\kappa(\mathbf{W}_u^{(n)}) - \kappa(W_k^{(n)})| + 2\mathbb{1}_{A_k^c} \sigma^2 |\varphi''(\mathbf{W}_u^{(n)}) - \varphi''(W_k^{(n)})|,$$

where $A_k = \{W_k^{(n)} = 0, W_{k+1}^{(n)} \neq 0\} \cup \{W_{k+1}^{(n)} = 0, W_k^{(n)} \neq 0\}$. Note that using that φ' and φ'' are Lipchitz and $\sup_{\tilde{w} \in [0, +\infty)} |\varphi'|(w)/(1+|w|) < +\infty$ by (23), A 1 and (19), we get that there exists $C \geq 0$ such that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$, $u \in (k\gamma_n, (k+1)\gamma_n)$,

$$|B_{u,k}^{(n)}| \leq C \gamma_n \{ |W_k^{(n)}|^2 + |W_{k+1}^{(n)}|^2 + 1 \},$$

which implies by Lemma 41 that

$$\lim_{n \rightarrow +\infty} \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \int_{k\gamma_n}^{(k+1)\gamma_n} \mathbb{E}[|B_{u,k}^{(n)}|] du = 0.$$

To conclude that $\lim_{n \rightarrow +\infty} \mathbb{E}[|A_2^{(n)}|] = 0$, it remains to show that

$$(90) \quad \lim_{n \rightarrow +\infty} \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{E} \left[\mathbb{1}_{A_k^c} \right] = 0.$$

Note that using that by definition, $(W_k^{(n)})_{k \in \mathbb{N}}$ is a Markov chain with Markov kernel Q_{γ_n} (13), the Markov property implies that for any $n \in \mathbb{N}$ and $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$,

$$(91) \quad \begin{aligned} \mathbb{E} \left[\mathbb{1}_{A_k^c} \right] &\leq \mathbb{P} \left(W_k^{(n)} = 0, W_{k+1}^{(n)} \neq 0 \right) + \mathbb{P} \left(W_k^{(n)} \neq 0, W_{k+1}^{(n)} = 0 \right) \\ &= 1 - 2\Phi[-c_\infty \sqrt{\gamma_n}/(2\sigma)] + 2\mathbb{E} \left[\mathbb{1}_{\mathbb{R}_+^*}(W_k^{(n)}) \Phi[-\tau_{\gamma_n}^\infty(W_k^{(n)})/\{2(\sigma^2 \gamma_n)^{1/2}\}] \right], \end{aligned}$$

where $\tau_{\gamma_n}^\infty(w) = \tau_{\gamma_n}(w) + \gamma_n c_\infty$, for any $w \in [0, +\infty)$. Since $1 - 2\Phi(-u) \leq u$ for any $u \in [0, +\infty)$, we get that there exists $C \geq 0$ such that for any $n \in \mathbb{N}$, $\gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} 1 - 2\Phi[-c_\infty \sqrt{\gamma_n}/(2\sigma)] \leq C((t-s) + \gamma_n) \gamma_n^{1/2}$ and therefore

$$(92) \quad \lim_{n \rightarrow +\infty} \gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \{1 - 2\Phi[-c_\infty \sqrt{\gamma_n}/(2\sigma)]\} = 0.$$

Regarding the second term in (91), consider the sequence of measurable functions defined for any $n \in \mathbb{N}$, $\omega \in \Omega$, $k \in \mathbb{N}$ by

$$f_n(\omega, k) = \gamma_n \mathbb{1}_{\{k_2^{(n)}, \dots, k_1^{(n)} - 1\}}(k) \mathbb{1}_{\mathbb{R}_+^*}(W_k^{(n)}) \Phi[-\tau_{\gamma_n}^\infty(W_k^{(n)})/\{2(\sigma^2 \gamma_n)^{1/2}\}],$$

on the measure space $(\Omega \times \mathbb{N}, \mathcal{F} \otimes 2^{\mathbb{N}}, \mathbb{P} \otimes \nu_c)$, where $2^{\mathbb{N}}$ is the power set of \mathbb{N} and ν_c is the counting measure on \mathbb{N} . Note that $\mathbb{P} \otimes \nu_c$ almost everywhere, $\lim_{n \rightarrow +\infty} f_n(\omega, k) = 0$ and in addition, $\sum_{k \in \mathbb{N}} \int_{\Omega} f_n(\omega, k) d\mathbb{P}(\omega) \leq (t-s) + \gamma_n$. Therefore by the Lebesgue dominated convergence theorem, we obtain that $\lim_{n \rightarrow +\infty} \sum_{k \in \mathbb{N}} \int_{\Omega} f_n(\omega, k) d\mathbb{P}(\omega) = 0$ which implies by definition,

$$\lim_{n \rightarrow +\infty} 2\gamma_n \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{E} \left[\mathbb{1}_{\mathbb{R}_+^*}(W_k^{(n)}) \Phi[-\tau_{\gamma_n}^\infty(W_k^{(n)})/\{2(\sigma^2 \gamma_n)^{1/2}\}] \right] = 0.$$

This result combined with (92) and (91) in (90) shows that

$$(93) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left[|A_2^{(n)}| \right] = 0.$$

Finally we deal with $A_3^{(n)}$ from the decomposition

$$A_3^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \varphi(W_{k+1}^{(n)}) - \varphi(W_k^{(n)}) - \gamma_n \mathcal{A}\varphi(W_k^{(n)}).$$

Set for any $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$, $\Delta W_k^{(n)} = W_{k+1}^{(n)} - W_k^{(n)}$. Using that φ is three times continuously differentiable, we get by Taylor's theorem with Lagrange reminder, that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$, there exists $u_k \in [0, 1]$ satisfying

$$\begin{aligned} \varphi(W_{k+1}^{(n)}) - \varphi(W_k^{(n)}) - \gamma_n \mathcal{A}\varphi(W_k^{(n)}) &= \varphi'(W_k^{(n)}) \Delta W_k^{(n)} \\ &\quad + (\varphi''(W_k^{(n)})/2) \{\Delta W_k^{(n)}\}^2 + 6^{-1} \varphi^{(3)}(u_k W_{k+1}^{(n)} + (1 - u_k) W_k^{(n)}) \{\Delta W_k^{(n)}\}^3. \end{aligned}$$

It follows from the definition (12), A1 and Young's inequality, setting $\tau_{\gamma_n}^\infty(w) = \tau_{\gamma_n}(w) + c_\infty$ and $\kappa^\infty(w) = \kappa(w) + c_\infty$ that for any $n \in \mathbb{N}$, $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$,

$$|\Delta W_k^{(n)}|^3 \leq 4\{\gamma_n^3 |\kappa^\infty(W_k^{(n)})|^3 + |W_{k+1}^{(n)} - \tau_{\gamma_n}^\infty(W_k^{(n)})|^3\}.$$

It follows then using the definition of \mathcal{A} (22), (23), Lemma 43 and Lemma 41 that

$$(94) \quad \lim_{n \rightarrow +\infty} \mathbb{E}[|A_3^{(n)} - A_4^{(n)}|] = 0, \quad \text{where } A_4^{(n)} = A_{4,1}^{(n)} + A_{4,2}^{(n)},$$

$$A_{4,1}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} [\varphi'(W_k^{(n)}) \Delta W_{k+1}^{(n)} - \gamma_n \varphi'(W_k^{(n)}) \kappa^\infty(W_k^{(n)})]$$

$$A_{4,2}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} [(\varphi''(W_k^{(n)})/2) \{\Delta W_{k+1}^{(n)}\}^2 - 2\gamma_n \sigma^2 \varphi''(W_k^{(n)}) \mathbb{1}_{\mathbb{R}_+^*}(W_k^{(n)})].$$

Note that by Lemma 25 and the Markov property, we have that for any $n \in \mathbb{N}$ and $k \in \{k_2^{(n)}, \dots, k_1^{(n)} - 1\}$, $\varphi'(W_k^{(n)}) \mathbb{E}[\Delta W_{k+1}^{(n)} | \mathcal{F}_k^{(n)}] - \gamma_n \varphi'(W_k^{(n)}) \kappa^\infty(W_k^{(n)}) = 0$, which implies that

$$(95) \quad \mathbb{E}[A_{4,1}^{(n)} | \mathcal{G}_s^{(n)}] = \mathbb{E}[A_{4,1}^{(n)} | \mathcal{F}_{k_2^{(n)}}^{(n)}]$$

$$= \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{E}[\varphi'(W_k^{(n)}) \mathbb{E}[\Delta W_{k+1}^{(n)} | \mathcal{F}_k^{(n)}] - \gamma_n \varphi'(W_k^{(n)}) \kappa^\infty(W_k^{(n)}) | \mathcal{F}_{k_2^{(n)}}^{(n)}] = 0.$$

We now show that $\lim_{n \rightarrow +\infty} \mathbb{E}[|\mathbb{E}[A_{4,2}^{(n)} | \mathcal{G}_s^{(n)}]|] = 0$ using the decomposition

$$(96) \quad A_{4,2}^{(n)} = A_{4,2,1}^{(n)} + A_{4,2,2}^{(n)},$$

$$A_{4,2,1}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{1}_{\mathbb{R}_+^*}(W_k^{(n)}) (\varphi''(W_k^{(n)})/2) [\{\Delta W_{k+1}^{(n)}\}^2 - 4\sigma^2 \gamma_n]$$

$$A_{4,2,2}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \mathbb{1}_{\{0\}}(W_k^{(n)}) (\varphi''(0)/2) \{\Delta W_{k+1}^{(n)}\}^2.$$

Then, by Lemma 44-(a), we have using the Markov property that

$$(97) \quad |\mathbb{E}[A_{4,2,1}^{(n)} | \mathcal{G}_s^{(n)}]| = |\mathbb{E}[A_{4,2,1}^{(n)} | \mathcal{F}_{k_2^{(n)}}^{(n)}]| \leq 8\sigma^2 \sup_{\mathbb{R}} \{|\varphi''|\} \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} \tilde{f}_n(\omega, k),$$

$$\text{where } \tilde{f}_n(\omega, k) = \gamma_n \mathbb{1}_{\{k_2^{(n)}, \dots, k_1^{(n)}-1\}}(k) \mathbb{1}_{\mathbb{R}_+^*}(W_k^{(n)}(\omega)) \Upsilon(\bar{\tau}_\gamma^\infty(W_k^{(n)}(\omega))),$$

$\Upsilon(u) = \int_u^{+\infty} |g|^q \varphi(g) dg$, $\bar{\tau}_\gamma^\infty(W_k^{(n)}(\omega)) = \tau_{\gamma_n}^\infty(W_k^{(n)}(\omega)) / \{2(\sigma^2 \gamma_n)^{1/2}\}$. $(\tilde{f}_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions defined $(\Omega \times \mathbb{N}, \mathcal{F} \otimes 2^{\mathbb{N}}, \mathbb{P} \otimes \nu_c)$. Note that $\mathbb{P} \otimes \nu_c$ almost everywhere, $\lim_{n \rightarrow +\infty} \tilde{f}_n(\omega, k) = 0$ and in addition,

$$\sum_{k \in \mathbb{N}} \int_{\Omega} \tilde{f}_n(\omega, k) d\mathbb{P}(\omega) \leq [(t-s) + \gamma_n] \int_{\mathbb{R}} |g|^q \varphi(g) dg.$$

Therefore, we obtain that $\lim_{n \rightarrow +\infty} \sum_{k \in \mathbb{N}} \int_{\Omega} \tilde{f}_n(\omega, k) d\mathbb{P}(\omega) = 0$ by the Lebesgue dominated convergence theorem, which implies by (97) that

$$(98) \quad \lim_{n \rightarrow +\infty} \mathbb{E}[|\mathbb{E}[A_{4,2,1}^{(n)} | \mathcal{G}_s^{(n)}]|] = 0.$$

We now consider $A_{4,2,2}$. By Lemma 44-(b) and the Markov property, it follows that

$$|\mathbb{E}[A_{4,2,2}^{(n)} | \mathcal{G}_s^{(n)}]| = |\mathbb{E}[A_{4,2,2}^{(n)} | \mathcal{F}_{k_2^{(n)}}^{(n)}]| \leq (3\varphi''(0)/2)[(t-s) + \gamma_n]\gamma_n c_{\infty}^2,$$

showing that $\lim_{n \rightarrow +\infty} \mathbb{E}[|\mathbb{E}[A_{4,2,2}^{(n)} | \mathcal{G}_s^{(n)}]|] = 0$. Combining this result, (98)-(96)-(95)-(94), we get that $\lim_{n \rightarrow +\infty} \mathbb{E}[|\mathbb{E}[A_3^{(n)} | \mathcal{G}_s^{(n)}]|] = 0$. Plugging this result and (93)-(89) in (88) completes the proof.

Proof of (25). The proof follows exactly the same lines as (24) but we use that the only different and non negligible term is $A_{4,2}^{(n)}$ which becomes

$$A_{4,2}^{(n)} = \sum_{k=k_2^{(n)}}^{k_1^{(n)}-1} [(\varphi''(W_k^{(n)})/2)[\{\Delta W_{k+1}^{(n)}\}^2 - 4\gamma_n \sigma^2]].$$

Using (86) in Lemma 44, the assumption that $\varphi''(w) \geq 0$ for any $w \in \mathbb{R}$, and the Markov property, we get that for any $n \in \mathbb{N}$, $\mathbb{E}[A_{4,2}^{(n)} | \mathcal{G}_s] \leq 0$, which concludes the proof. \square

5.2.4. Proof of Theorem 17.

PROPOSITION 45. Assume **A 1**. Let μ_{∞} be a limit point of $(\mu_n)_{n \in \mathbb{N}}$. Then, μ_{∞} -almost everywhere, $\inf_{t \in [0, +\infty)} W_t \geq 0$.

PROOF. Without loss of generality, we assume that $(\mu_n)_{n \in \mathbb{N}}$ converges to μ_{∞} . Since $\omega \mapsto \inf_{t \in [0, +\infty)} \omega_t$ is continuous, $F = \{\omega \in \mathbb{W} : \inf_{t \in [0, +\infty)} \omega_t \geq 0\}$ is closed. Therefore, by the Portmanteau theorem [22, Theorem 13.16], we obtain that $\mu_{\infty}(F) \geq \limsup_{n \rightarrow +\infty} \mu_n(F) = 1$. \square

PROOF OF THEOREM 17. Recall that we denote by $(\mu_n)_{n \in \mathbb{N}}$ the sequence of distribution on \mathbb{W} associated with $\{(\mathbf{W}_t^{(n)})_{t \geq 0} : n \in \mathbb{N}\}$. Let μ_{∞} be a limit point of this sequence for the convergence in distribution. Without loss of generality, we assume that $(\mu_n)_{n \in \mathbb{N}}$ converges in distribution to μ_{∞} . Note that by Proposition 15, for any continuous function $F : \mathbb{W} \rightarrow \mathbb{R}$ such that $|F|(\omega) \leq C_T \{1 + \sup_{t \in [0, T]} |\omega_t|^{\delta_c}\}$ for $\delta_c \in [0, 4)$, $T, C_T \geq 0$, then F is uniformly integrable for $(\mu_n)_{n \in \mathbb{N}}$ and therefore (see e.g. [1, Lemma 5.1.7.])

$$(99) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{W}} F d\mu_n = \int_{\mathbb{W}} F d\mu_{\infty}.$$

We divide then the proof into two parts. **First part:** we first show that under μ_{∞} , $(M_t)_{t \geq 0}$ is a $(W_t)_{t \geq 0}$ -martingale. Considering

$$F_1 : \omega \mapsto \left(\varphi_1(\omega_t) - \varphi_1(\omega_s) - \int_s^t \mathcal{A}\varphi_1(\omega_u) du \right) \psi(\omega_{t_1}, \dots, \omega_{t_N}),$$

and applying Proposition 18-(24) to $\varphi_1(w) = w$ for any $w \in \mathbb{R}$, since $\mathcal{A}\varphi_1$ is continuous under **A 1**, for any $N \in \mathbb{N}$, $(t_1, \dots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \dots \leq t_N \leq s < t$, $\psi : \mathbb{R}_+^N \rightarrow \mathbb{R}$, continuous and bounded,

$$\mathbb{E}^{\mu_{\infty}} [(M_t - M_s) \psi(W_{t_1}, \dots, W_{t_N})] = 0,$$

where $\mathbb{E}^{\mu_\infty}[\cdot]$ is the expectation under μ_∞ on $(\mathbb{W}, \mathcal{W})$. We obtain by the monotone class theorem and [27, Theorem 2.3, Chapter 0] that the first part of the result holds, *i.e.* $(M_t)_{t \geq 0}$ defined by (21) is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale on $(\mathbb{W}, \mathcal{W}, (\mathcal{W}_t)_{t \geq 0}, \mu_\infty)$.

Second part: It remains to show that under μ_∞ , $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale. We first establish setting $\varphi_2(w) = w^2$ for $w \in \mathbb{R}$, that

$$\tilde{N}_t = \varphi_2(W_t) - \varphi_2(W_0) - \int_0^t \mathcal{A}\varphi_2(W_u) du,$$

is a $(\mathcal{W}_t)_{t \geq 0}$ -submartingale, which easily implies that $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -submartingale. Let $N \in \mathbb{N}$, $(t_1, \dots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \dots \leq t_N \leq s < t$, $\psi: \mathbb{R}_+^N \rightarrow \mathbb{R}$, continuous, nonnegative and bounded. Then, consider $F_2^+ = F_{2,1}^+ + F_{2,2}^+$ on \mathbb{W} with :

$$F_{2,1}^+ : \omega \mapsto \left\{ \varphi_2(\omega_t) - \varphi_2(\omega_s) - 2 \int_s^t \omega_u (\kappa(\omega_u) + c_\infty) du \right\} \psi(\omega_{t_1}, \dots, \omega_{t_N})$$

$$F_{2,2}^+ : \omega \mapsto -4\sigma^2 \left\{ \int_s^t \mathbb{1}_{\mathbb{R}_+^*}(\omega_u) du \right\} \psi(\omega_{t_1}, \dots, \omega_{t_N})$$

Note that it is easy to check that $F_{2,1}^+$ is continuous and $F_{2,2}^+$ is bounded lower semi-continuous on \mathbb{W} , *i.e.* for any $(\omega^n)_{n \in \mathbb{N}}$ converging to ω^∞ in \mathbb{W} endowed with the uniform convergence on compact set, $\liminf_{n \rightarrow +\infty} F_{2,2}^+(\omega^n) \geq F_{2,2}^+(\omega^\infty)$. Therefore, we obtain by the Portmanteau theorem [22, Theorem 13.16] and (99) that

$$\int_{\mathbb{W}} F_{2,1}^+ d\mu = \lim_{n \rightarrow \infty} \int_{\mathbb{W}} F_{2,1}^+ d\mu_n, \text{ and } \int_{\mathbb{W}} F_{2,2}^+ d\mu \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{W}} F_{2,2}^+ d\mu_n.$$

Therefore, Proposition 18-(24) applied with $\varphi \leftarrow \varphi_2$ implies that

$$0 = \limsup_{n \rightarrow +\infty} \int_{\mathbb{W}} F_2^+ d\mu_n \leq \int_{\mathbb{W}} F_2^+ d\mu.$$

Using the same arguments as before, we obtain that under μ_∞ , $(\tilde{N}_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -submartingale. Then, it is easy to verify that $(N_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -submartingale. We complete then the proof by showing that $(N_t)_{t \geq 0}$ is also a $(\mathcal{W}_t)_{t \geq 0}$ -supermartingale under μ_∞ . To do so, we need the following lemma.

LEMMA 46. *Assume A 1. Then, for any limit point μ_∞ of $(\mu_n)_{n \in \mathbb{N}}$, μ_∞ -almost everywhere, $t \mapsto \langle M \rangle_t - 4\sigma^4 t$ is nonincreasing, where $(\langle M \rangle_t)_{t \geq 0}$ is the quadratic variation of $(M_t)_{t \geq 0}$.*

PROOF. Let $N \in \mathbb{N}$, $(t_1, \dots, t_N, s, t) \in [0, +\infty)^{N+2}$, $0 \leq t_1 \leq \dots \leq t_N \leq s < t$, $\psi: \mathbb{R}_+^N \rightarrow \mathbb{R}$, continuous, nonnegative and bounded. Consider now the continuous map

$$F_2^- : \omega \mapsto \left\{ \varphi_2(\omega_t) - \varphi_2(\omega_0) - \int_0^t \tilde{\mathcal{A}}\varphi_2(\omega_u) du \right\} \psi(\omega_{t_1}, \dots, \omega_{t_N}).$$

Then, by (99) and Proposition 18-(25), we get $\lim_{n \rightarrow \infty} \int_{\mathbb{W}} F_2^- d\mu_n \leq 0$. Using that under μ_∞ $(M_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale, we get that $(M_t^2 - 4\sigma^2 t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -supermartingale. By the Doob-Meyer decomposition [21, Theorem 22.5], under μ_∞ , there exists a unique nondecreasing, locally integrable and predictable process $(C_t)_{t \geq 0}$, such that $(M_t^2 - 4\sigma^2 t + C_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale. In addition, under μ_∞ , by [27, Theorem 1.8, Chapter IV], the quadratic variation $(\langle M \rangle_t)_{t \geq 0}$ of $(M_t)_{t \geq 0}$ is a finite variation process satisfying $(M_t^2 - \langle M \rangle_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale therefore $(M_t^2 - 4\sigma^2 t - (\langle M \rangle_t - 4\sigma^2 t))_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale. Therefore, $(\langle M \rangle_t - 4\sigma^2 t - C_t)_{t \geq 0}$ is a $(\mathcal{W}_t)_{t \geq 0}$ -martingale and a finite variation process. By [27, Proposition 1.2, Chapter IV], μ_∞ -almost everywhere, for any $t \in [0, +\infty)$, $\langle M \rangle_t - 4\sigma^2 t + C_t = 0$, which completes the proof. \square

By Lemma 46, denoting by $(\langle M \rangle_t)_{t \geq 0}$, the quadratic variation of $(M_t)_{t \geq 0}$, see [27, Theorem 1.8, Chapter IV], μ_∞ -almost everywhere, $t \mapsto \langle M \rangle_t - 4\sigma^2 t$ is nondecreasing and therefore we get that for any $s, t \in [0, +\infty)$, $s \leq t$, μ_∞ -almost everywhere,

$$(100) \quad \int_s^t \mathbb{1}_{\mathbb{R}_+^*}(W_u) d\langle M \rangle_u \leq 4\sigma^2 \int_s^t \mathbb{1}_{\mathbb{R}_+^*}(W_u) du.$$

In addition, by the occupation times formula [27, Corollary 1.6, Chapter VI] applied twice and Proposition 45, μ_∞ -almost everywhere,

$$\langle M \rangle_t = \int_0^t d\langle M \rangle_u = \int_0^t \mathbb{1}_{\mathbb{R}_+}(W_u) d\langle W \rangle_u = \int_{\mathbb{R}_+} L_t^a da = \int_{\mathbb{R}_+^*} L_t^a da = \int_0^t \mathbb{1}_{\mathbb{R}_+^*}(W_u) d\langle M \rangle_u$$

Using this result and (100), we get that $\langle M \rangle_t - \langle M \rangle_s \leq 4\sigma^2 \int_s^t \mathbb{1}_{\mathbb{R}_+^*}(W_u) du$, for any $s, t \in [0, +\infty)$, $s \leq t$. Therefore since $(M_t^2 - \langle M \rangle_t)_{t \geq 0}$ is a $(W_t)_{t \geq 0}$ -martingale under μ_∞ , we conclude that $(N_t)_{t \geq 0}$ is a $(W_t)_{t \geq 0}$ -supermartingale which completes the proof. \square

5.3. Postponed proofs of Section 4.

PROOF OF PROPOSITION 20. From **AO 3** we know that $z_\theta(t) \in \mathbf{K}$ for all $t \in [0, T]$. In particular, using that for all $0 \leq s \leq t \leq T$, $z_\theta(t) - z_\theta(s) = \int_s^t F_\theta(z_\theta(u), u) du$, we get that

$$(101) \quad \|z_\theta(t) - z_\theta(s)\| \leq \mathbf{C}_F(t - s).$$

Let $k \in \mathbb{N}$ be such that $\tilde{z}_\theta^h(kh) \in \mathbf{K}$ (this is for instance the case of $k = 0$). Then, for $t \in [kh, (k+1)h]$, using by (32) that

$$z_\theta(t) - \tilde{z}_\theta^h(t) = z_\theta(kh) - \tilde{z}_\theta^h(kh) + \int_{kh}^t \left(F_\theta(z_\theta(s), s) - F_\theta(\tilde{z}_\theta^h(kh), kh) \right) ds,$$

and setting $f(t) = \|z_\theta(t) - \tilde{z}_\theta^h(t)\|$, we get by (101) and **AO 3** for any $h > 0$ and $t \in [kh, (k+1)h]$,

$$f(t) \leq (1 + \mathbf{L}'_F h) f(kh) + \mathbf{L}'_F (1 + \mathbf{C}_F) \frac{h^2}{2}.$$

Assuming that $h \leq \bar{h}$ where \bar{h} is sufficiently small so that

$$\frac{1}{2} \mathbf{L}'_F (1 + \mathbf{C}_F) \bar{h} T e^{\mathbf{L}'_F T} < \delta,$$

we get by a direct induction that, for all $k \in \{0, \dots, \lfloor T/h \rfloor\}$, $\tilde{z}_\theta^h(kh) \in \mathbf{K}$ and for all $t \in (kh, (k+1)h]$, $t \in [0, T]$,

$$f(t) \leq \frac{1}{2} \mathbf{L}'_F (1 + \mathbf{C}_F) h^2 (k+1) e^{\mathbf{L}'_F h(k+1)}$$

Conclusion follows with

$$\mathbf{C} = \frac{1}{2} \mathbf{L}'_F (1 + \mathbf{C}_F) N T e^{\mathbf{L}'_F T}.$$

\square

PROOF OF PROPOSITION 22. We have by **AO 2** and (37), $T_\gamma(\theta) = \theta - \gamma \nabla \nabla U(\theta)$ with

$$(102) \quad \nabla U(\theta) = \nabla U_0(\theta) + \sum_{i=1}^N \mathbf{s}_i(z_\theta(t_i))$$

where z_θ solves (30) and \mathbf{s}_i are defined in (37). For all $\theta, \tilde{\theta} \in \mathbb{R}^d$ and $t \in [0, T]$,

$$z_\theta(t) - z_{\tilde{\theta}}(t) = \int_0^t (F_\theta(z_\theta(s), s) - F_{\tilde{\theta}}(z_{\tilde{\theta}}(s), s)) ds$$

and thus, using (33), Grönwall's inequality implies that

$$\|z_\theta(t) - z_{\tilde{\theta}}(t)\| \leq \mathbf{L}_F \|\theta - \tilde{\theta}\| t e^{\mathbf{L}'_F t} \quad \forall t \geq 0.$$

In particular, by (102) and (37),

$$\|\nabla U(\theta) - \nabla U(\tilde{\theta})\| \leq \left(\mathbf{L}_U + \mathbf{L}_s \mathbf{L}_F \sum_{i=1}^N t_i e^{\mathbf{L}'_F t_i} \right) \|\theta - \tilde{\theta}\|.$$

Moreover, similarly, we get if $\|\theta - \tilde{\theta}\| \geq R_U$,

$$\langle \theta - \tilde{\theta}, \nabla U(\theta) - \nabla U(\tilde{\theta}) \rangle \geq \mathbf{m}_U \|\theta - \tilde{\theta}\|^2 - N \mathbf{C}_s \|\theta - \tilde{\theta}\|.$$

Combining the last two estimates yields **H1**. Finally, **H2** follows using **AO4**, (35) and (37) from

$$\|\nabla U(\theta) - \tilde{b}_h(\theta)\| \leq \mathbf{L}_s \sum_{i=1}^N \|z_\theta(t_i) - \Psi_i^h(\theta)\| \leq \mathbf{C}_\Psi \mathbf{L}_s h^\alpha.$$

□

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