



HAL
open science

Strichartz inequalities with white noise potential on compact surfaces

Antoine Mouzard, Immanuel Zachhuber

► **To cite this version:**

Antoine Mouzard, Immanuel Zachhuber. Strichartz inequalities with white noise potential on compact surfaces. 2021. hal-03198650v1

HAL Id: hal-03198650

<https://hal.science/hal-03198650v1>

Preprint submitted on 15 Apr 2021 (v1), last revised 20 Jul 2022 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Strichartz inequalities with white noise potential on compact surfaces

Antoine MOUZARD and Immanuel ZACHHUBER

Abstract

We prove Strichartz inequalities for the Schrödinger equation and the wave equation with multiplicative noise on a two-dimensional manifold. This relies on the Anderson Hamiltonian H described using high order paracontrolled calculus. As an application, it gives a low regularity solution theory for the associated nonlinear equations.

MSC 2020 – 35J10; 60H25; 58J05

Keywords – Anderson Hamiltonian; Paracontrolled calculus; White noise; Schrödinger operator; Strichartz inequalities.

Contents

1	Preliminaries	4
1.1	Strichartz inequalities on manifolds	4
1.2	Basics on paracontrolled calculus	6
1.3	Construction of the Anderson Hamiltonian	9
2	Strichartz inequalities for the stochastic Schrödinger equation	14
2.1	Strichartz inequalities for the Schrödinger group	15
2.2	Local well-posedness for multiplicative stochastic cubic NLS	18
3	Strichartz inequalities for the stochastic wave equation	21
3.1	Strichartz inequalities for the wave propagator	21
3.2	Strichartz inequalities for wave equations with rough potentials	22
3.3	The special case of the two-dimensional torus	25
3.4	Local well-posedness for the multiplicative cubic stochastic wave equation	26

Introduction

Enormous progress has been made in the last decade after Hairer introduced his theory of regularity structures [19] and the theory of paracontrolled distributions due to Gubinelli, Imkeller, and Perkowski [17] in the study of singular stochastic PDEs. A particular approach developed recently is the construction of a random stochastic operator to investigate associated PDEs. The first paper on this dealt with the continuum Anderson Hamiltonian, hereafter simply called Anderson Hamiltonian, and was by Allez and Chouk [1]. They used the latter theory to make sense of the operator on the two-dimensional torus, formally

$$H = -\Delta + \xi,$$

where ξ is spatial white noise whose regularity is just below -1 . Afterwards this approach was extended to the three dimensional torus and somewhat reformulated by Gubinelli, Ugurcan and Zachhuber in [18] and by Labbé [21] who used regularity structures and deal with both periodic and Dirichlet boundary conditions.

Naturally, substantial progress was also made in the field of singular dispersive SPDEs following the paper [13] due to Debussche and Weber on the cubic multiplicative stochastic Schrödinger equation and the paper [15] by Gubinelli, Koch and Oh on the cubic additive stochastic wave equation. Since the powerful tools from singular SPDEs are only directly applicable to parabolic

and elliptic SPDEs, these initial papers were in a not so singular regime, the former using an exponential transform to remove the most singular term and the latter using a “DaPrato-Debusche trick” to do the same. In [18], Gubinelli, Ugurcan and Zachhuber proved some sharpened results on the multiplicative Schrödinger equation and its wave analogue by reframing it in relation to the Anderson Hamiltonian as well as extending the results to dimension 3. Building on this, Strichartz estimates were shown to hold in [29] by Zachhuber for the Anderson Hamiltonian which, in a nutshell, leverage dispersion in order to allow to trade integrability in time for integrability in space, see Section 2 for a more detailed introduction. Moreover, Tzvetkov and Visciglia extended in [28] the results of [13] to a larger range of power nonlinearities. For the nonlinear wave equation with additive noise, let us mention here the follow-up paper by Gubinelli, Koch and Oh [16] in three dimensions with quadratic nonlinearity and the paper [25] by Oh, Robert and Tzvetkov which extends the results of [15] to the case of two-dimensional surfaces and is thus salient for the current paper.

In this paper, we prove Strichartz inequalities for the Schrödinger and wave equation with white noise potential on compact surfaces. Moreover, we show how this provides local well-posedness for the associated nonlinear equations in a low-regularity regimes. As for the deterministic case, the Strichartz estimates obtained depend whether the manifold has a boundary or not and are improved in the flat case of the torus. By Strichartz inequalities, we generally refer to space-time bounds on the propagators of Schrödinger and wave equations where the results on integrability are strictly better than what one gets from the Sobolev embedding so – for definiteness we consider the Schrödinger case – a bound like

$$\|e^{itH}u\|_{L^p(I,L^q)} \lesssim \|u\|_{\mathcal{H}^\alpha},$$

with $p \in [1, \infty], q > \frac{2d}{d-2\alpha}$ where d denotes the dimension and $I \subset \mathbb{R}$ is an interval. The overall approach to the Schrödinger group associated to H we follow is similar to the one in [29], where such Strichartz estimates were shown for the Anderson Hamiltonian on the two and three-dimensional torus. However, one gets sharper results in the particular case of flat geometry due to the fact that one has stronger classical Strichartz inequalities available. In the more general setting of a Riemannian compact manifold, we work with a result due to Burq, Gerard and Tzvetkov [10] which has been extended to the case with boundary by Blair, Smith and Sogge in [8]. These results can be thought of as quantifying the statement “finite frequencies travel at finite speeds – in (frequency dependent) short time the evolution is morally on flat space”. Let us also mention at this point the recent work by Huang and Sogge [20] which deals with a similar setting, however their notion of singular potential refers to low integrability while in our case singular refers rather to potentials with low regularity.

For the case of Strichartz estimates for the wave equation related to H , we follow the approach introduced by Burq, Lebeau and Planchon [11] on domains with boundary. The main idea, which is why this approach is applicable, is that all that is required is that the operator driving the wave equation satisfies some growth condition on the L^q bounds on the its eigenfunctions and one knows about the asymptotics of the eigenvalues, in their case the Laplace with boundary conditions. Since a Weyl law for H was obtained by Mouzard in [24] and our result for the Schrödinger equation gives us a suitable L^q bound on the eigenfunctions of H , their approach turns out to be enough to prove Strichartz estimates that beat the Sobolev embedding. One also gets improved results in the flat case of the two-dimensional torus since one has sharper L^q bounds coming from the sharper Strichartz estimates proved by Zachhuber in [29]. Overall this approach seems somewhat crude and we assume there to be sharper bounds possible whereas in the Schrödinger case, our result is the same as the one without noise obtained in [10] worsened only by an arbitrarily small regularity. The state of the art of Strichartz estimates for wave equations on manifolds with boundary is the paper [7], the case of manifolds without boundary being comparable to the Strichartz estimates on Euclidean space because of the finite speed of propagation. In particular, the bounds obtained on the spectral projectors of the Anderson Hamiltonian are new and of interest in themselves.

The second objective of this paper is to use the Strichartz inequalities obtained to prove local well-posedness for the associated defocussing nonlinear equations, also known as cubic multiplicative stochastic Schrödinger and wave equations. This will be done using fairly straightforward contraction arguments for which the Strichartz estimates will be crucial, this is analogous to Section 5 of [29].

We conclude the introduction by a brief outline of the construction of the Anderson Hamiltonian, see [24] for the details. It is formally given by

$$H = -\Delta + \xi$$

where ξ is the space white noise and belongs to $\mathcal{C}^{\alpha-2}$ for any $\alpha < 1$ (here \mathcal{C}^β denotes the Hölder-Besov spaces recalled in Section 1.1). The noise is only a distribution, rough almost surely everywhere as opposed to potential with a localised singularity, hence Hu is well-defined for $u \in C^\infty$ but does not belong to L^2 . The nature of the noise makes the naive candidate for the domain of H , that is the closure of

$$\{u \in C^\infty; Hu \in L^2\}$$

with respect to the domain norm unviable. This is precisely where the paracontrolled calculus comes into play, one can construct a random space $\mathcal{D}_\Xi \subset L^2$ such that almost surely

$$u \in \mathcal{D}_\Xi \implies Hu \in L^2.$$

The domain \mathcal{D}_Ξ consists of functions $u \in L^2$ paracontrolled by noise-dependent functions X_1, X_2 of the form

$$u = \tilde{\mathbb{P}}_u X_1 + \tilde{\mathbb{P}}_u X_2 + u^\sharp$$

with a remainder $u^\sharp \in \mathcal{H}^2$ and the $\tilde{\mathbb{P}}_u X_i$ are terms which are dominated by X_i in terms of regularity. In particular, smooth functions do not belong to the domain in this peculiar setting. The singularity of the product is dealt with through a renormalisation procedure which corresponds to the construction of the singular term $X \cdot \xi$ where $\Delta X = \xi$. Given a regularisation ξ_ε of the noise, the product $X_\varepsilon \cdot \xi_\varepsilon$ diverges but one gets a well-defined function after the subtraction of the diverging quantity

$$c_\varepsilon := \mathbb{E}[X_\varepsilon \cdot \xi_\varepsilon].$$

The analysis of the operator is then done with

$$L^{-1}\xi \cdot \xi := \lim_{\varepsilon \rightarrow 0} (L^{-1}\xi_\varepsilon \cdot \xi_\varepsilon - c_\varepsilon),$$

this is the Wick product, and the Anderson Hamiltonian corresponds to the limit of the family of operators

$$H_\varepsilon = -\Delta + \xi_\varepsilon - c_\varepsilon$$

as ε goes to 0. In the case of the torus, c_ε is a constant due to the invariance by translation of the noise and diverges as $\log(\varepsilon)$, for details, see Section 2.1 of [24].

While the Anderson Hamiltonian can be interpreted as the electric Laplacian $-\Delta + V$ with electric field $V = \xi$, one can consider as an analogy the magnetic Laplacian with magnetic field $B = \xi$ space white noise. This is the content of [23] where Morin and Mouzard construct

$$H = (i\partial_1 + A_1)^2 + (i\partial_2 + A_2)^2$$

on the two-dimensional torus with magnetic potential $A = (A_1, A_2)$ the Lorentz gauge associated to the white noise magnetic field. Its study is also motivated by superconductivity where H plays a specific role in the third critical field of Ginzburg-Landau theory. In particular, the first results such as the self-adjointness, discrete spectrum and Weyl law holds as for the Anderson Hamiltonian, while differences are expected to appear when one looks at finer properties. Our proof for the Strichartz inequalities for the Schrödinger group associated to the Anderson Hamiltonian also holds for the random magnetic Laplacian with white noise magnetic field. For Strichartz estimates for the magnetic Schrödinger equation, see for example [12] and references therein.

Organisation of the paper. *In the first section*, we give the context for the Strichartz inequalities on manifolds in the case of the Schrödinger equations and the heat semigroup paracontrolled calculus on manifolds, see respectively Burq, Gérard, Tzvetkov [10] and Mouzard [24]. We conclude by recalling the construction of the Anderson Hamiltonian and provide new results needed in the following. *In the second section*, we provide Strichartz inequalities for the Schrödinger group associated to the Anderson Hamiltonian and show how this gives local well-posedness for the stochastic cubic nonlinear Schrödinger equation with multiplicative white noise. *In the last section*, we use the result on the Schrödinger group to get new bounds on the eigenvalues of the Anderson Hamiltonian and use it to prove Strichartz inequalities for the wave propagator together with the Weyl-type law. Finally, we also show how this gives local well-posedness for the stochastic cubic nonlinear wave equation with multiplicative white noise and give details for the particular case of the torus where one gets improved bounds.

1 – Preliminaries

1.1 – Strichartz inequalities on manifolds

On the torus, regularity of distributions can be measured using the Littlewood-Paley decomposition. On a manifold, one has an analogue decomposition using the eigenfunctions of the Laplace-Beltrami operator Δ as a generalisation of Fourier theory, see for example Section 2 in [26] by Oh, Robert, Tzvetkov and Wang and references therein. Let (M, g) be a two-dimensional compact Riemannian manifold without boundary or with boundary and Dirichlet boundary conditions. In this framework, the Laplace-Beltrami operator $-\Delta$ is a self-adjoint positive operator with discrete spectrum

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

with the associated normalized eigenfunctions $(\varphi_n)_{n \geq 1}$ belonging to $C^\infty(M)$ with $\varphi_1 = \text{Vol}(M)^{-\frac{1}{2}}$ constant. Furthermore, the Weyl law gives the asymptotics

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \frac{\text{Vol}(M)}{4\pi}.$$

The basis $(\varphi_n)_{n \geq 1}$ of L^2 gives the decomposition

$$u = \sum_{n \geq 1} \langle u, \varphi_n \rangle \varphi_n$$

for any distribution $u \in \mathcal{D}'(M)$. On the torus, this gives the Littlewood-Paley decomposition of u where the regularity is measured by the asymptotics behavior of $\sum_{\lambda_k \sim 2^n} \langle u, \varphi_k \rangle$. On a manifold M , this is done with

$$\Delta_n := \psi(-2^{-2(n+1)}\Delta) - \psi(-2^{-2n}\Delta)$$

for $n \geq 0$ and

$$\Delta_{-1} := \psi(-\Delta)$$

with $\psi \in C_0^\infty(\mathbb{R})$ a non-negative function with $\text{supp}(\psi) \subset [-1, 1]$ and $\psi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Recall that for any function $\psi \in L^\infty(\mathbb{R})$, the operator $\psi(\Delta)$ is defined as

$$\psi(\Delta)u = \sum_{n \geq 1} \psi(\lambda_n) \langle u, \varphi_n \rangle \varphi_n$$

and this yields a bounded operator from $L^2(M)$ to itself. In this setting, Besov spaces are defined for $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$ as

$$\mathcal{B}_{p,q}^\alpha := \{u \in \mathcal{D}'(M) ; \|u\|_{\mathcal{B}_{p,q}^\alpha} < \infty\}$$

where

$$\|u\|_{\mathcal{B}_{p,q}^\alpha} := \left(\|\Delta_{-1}u\|_{L^p(M)}^q + \sum_{n \geq 0} 2^{\alpha q} \|\Delta_n u\|_{L^p(M)}^q \right)^{\frac{1}{q}}.$$

In the particular case $p = q = \infty$ these spaces are called *Hölder-Besov spaces* and we write

$$B_{\infty,\infty}^\alpha = \mathcal{C}^\alpha.$$

The case $p = q = 2$ corresponds to Sobolev spaces and we have

$$\|u\|_{\mathcal{H}^\alpha}^2 = \|\Delta_{-1}u\|_{L^2(M)}^2 + \sum_{n \geq 0} 2^{2n\alpha} \|\varphi(2^{-2n}\Delta)u\|_{L^2(M)}^2$$

where $\varphi(x) := \psi(-x^2) - \psi(-x)$. Burq, Gérard and Tzvetkov proved in [10] the bound

$$\|f\|_{L^q(M)} \lesssim \|\psi(-\Delta)f\|_{L^q(M)} + \left(\sum_{n \geq 0} \|\varphi(2^{-2n}\Delta)f\|_{L^q(M)}^2 \right)^{\frac{1}{2}}$$

using that for $\lambda \in \mathbb{R}$, we have

$$\psi(-\lambda) + \sum_{n \geq 0} \varphi(2^{-2n}\lambda) = 1.$$

Applying this to the Schrödinger group, they obtain

$$\|e^{it\Delta}v\|_{L^p([0,1],L^q)} \lesssim \|\psi(-\Delta)v\|_{L^q(M)} + \left\| \left(\sum_{k \geq 0} \|e^{it\Delta}\varphi(2^{-2k}\Delta)v\|_{L^q(M)}^2 \right)^{\frac{1}{2}} \right\|_{L^p([0,1])}$$

hence one only needs a bound for spectrally localised data. This is proved using semi-classical analysis with the use of the WKB expansion, see Proposition 2.9 from [10] and references therein which gives

$$\left(\int_J \|e^{it\Delta} \varphi(h^2\Delta)v\|_{L^q(M)}^p dt \right)^{\frac{1}{p}} \lesssim \|v\|_{L^2(M)} \quad (1)$$

for J an interval of small enough length proportional to $h \in (0, 1)$. Moreover, a well-known trick is to slice up the time interval into small pieces, this will be useful later. The previous bounds with the Minkowski inequality lead to

$$\|e^{it\Delta}v\|_{L^p([0,1],L^q)} \lesssim \|v\|_{L^2(M)} + \left(\sum_{k \geq 0} 2^{2k/p} \|\varphi(2^{-2k}\Delta)v\|_{L^2(M)}^2 \right)^{\frac{1}{2}} \lesssim \|v\|_{\mathcal{H}^{\frac{1}{p}}}.$$

This yields the following theorem.

Theorem 1.1. *Let $p \geq 2$ and $q < \infty$ such that*

$$\frac{2}{p} + \frac{2}{q} = 1.$$

Then

$$\|e^{-it\Delta}u\|_{L^p([0,1],L^q)} \lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}}}.$$

While this result is optimal on general surfaces, this can be improved in the flat case of the torus. For the Anderson Hamiltonian, analogue result was obtained by Zachhuber in [29] and we also expect the bound to be weaker on surfaces. This is indeed the case and we obtained an the same result with an arbitrary loss of regularity, this is the content of Section 2. In the case of a surface with boundary, the following result was obtained by Blair, Smith and Sogge [8].

Theorem 1.2. *Let $p \in (3, \infty]$ and $q \in [2, \infty)$ such that*

$$\frac{3}{p} + \frac{2}{q} = 1$$

Then

$$\|e^{-it\Delta}u\|_{L^p([0,1],L^q)} \lesssim \|u\|_{\mathcal{H}^{\frac{2}{p}}}$$

and

$$\left(\int_J \|e^{it\Delta} \varphi(h^2\Delta)v\|_{L^q(M)}^p dt \right)^{\frac{1}{p}} \lesssim h^{-\frac{1}{p}} \|\varphi(h^2\Delta)v\|_{L^2(M)} \quad (2)$$

for J an interval of small enough length proportional to $h \in (0, 1)$.

We end this section with two classical results that will be needed in this paper. First, one still has Bernstein Lemma with the Littlewood-Paley decomposition associated to the Laplace-Beltrami operator.

Lemma 1.3. *Let $g : M \rightarrow \mathbb{R}$ be a function which has spectral support in an interval $[a, b]$ with $0 < a < b < \infty$. Then for any $\alpha, \beta \in \mathbb{R}$ we have the following bounds which are the analogue of Bernstein's inequality on Euclidean space*

$$\|g\|_{\mathcal{H}^\alpha} \lesssim \max(b^{\alpha-\beta}, a^{\alpha-\beta}) \|g\|_{\mathcal{H}^\beta}$$

and

$$\|g\|_{\mathcal{H}^\alpha} \gtrsim \min(b^{\alpha-\beta}, a^{\alpha-\beta}) \|g\|_{\mathcal{H}^\beta}.$$

The former estimate still holds in the case where $a = 0$ and $\alpha > \beta$. We will chiefly apply these bounds to Littlewood-Paley projectors where $b = 2a = 2^j$ for $j \in \mathbb{N}$.

Proof: The condition on g means that

$$g = \sum_{\lambda_k \in [a, b]} (g, \phi_k) \phi_k$$

and we have

$$\|g\|_{\mathcal{H}^\alpha}^2 = \sum_{\lambda_k \in [a, b]} (g, \phi_k)^2 \lambda_k^{2\alpha}.$$

The upper bounds follow directly with

$$\lambda_k^{2\alpha} = \lambda_k^{2\beta} \lambda_k^{2(\alpha-\beta)} \leq \lambda^{2\beta} \max(b^{2(\alpha-\beta)}, a^{2(\alpha-\beta)})$$

and analogously for the lower bounds. □

The space \mathcal{H}^σ is an algebra only for σ large enough depending on the dimension, this can be seen with the following Proposition and the Sobolev embedding. This type of estimates are important for the dispersive equations with cubic nonlinearity considered here.

Lemma 1.4. *Let $\sigma \geq 0$. The space $\mathcal{H}^\sigma \cap L^\infty$ is an algebra and one has the bound*

$$\|f \cdot g\|_{\mathcal{H}^\sigma} \lesssim \|f\|_{\mathcal{H}^\sigma} \|g\|_{L^\infty} + \|g\|_{\mathcal{H}^\sigma} \|f\|_{L^\infty}.$$

1.2 – Basics on paracontrolled calculus

On the torus, the Littlewood-Paley decomposition can also be used to study ill-defined products. Recall that for $u \in \mathcal{D}'(\mathbb{T}^2)$, it is given by

$$u = \sum_{n \geq 0} \Delta_n f$$

where each $\Delta_n f$ is smooth and localised in frequency in an annulus of radius 2^n for $n \geq 1$ while the Fourier transform of $\Delta_0 f$ is contained in a ball around the origin. Given two distributions $u, v \in \mathcal{D}'(\mathbb{T}^2)$, the product is formally given by

$$\begin{aligned} u \cdot v &= \sum_{n, m \geq 0} \Delta_n u \cdot \Delta_m v \\ &= \sum_{n \lesssim m} \Delta_n u \cdot \Delta_m v + \sum_{n \sim m} \Delta_n u \cdot \Delta_m v + \sum_{m \lesssim n} \Delta_n u \cdot \Delta_m v \\ &=: P_f g + \Pi(f, g) + P_g f. \end{aligned}$$

The term $P_f g$ is called the paraproduct of g by f and is always well-defined while the potential singularity is encoded in the resonant term $\Pi(f, g)$. Using this decomposition, Gubinelli, Imkeller and Perkowski introduced the notion of paracontrolled calculus to develop a solution theory for singular stochastic PDEs in their seminal work [17]; this correspond to Bony's paraproduct from [9] in this flat case. On a manifold, an alternative paracontrolled calculus was developed by Bailleul and Bernicot in [4] based on the heat semigroup. Instead of Littlewood-Paley, which is discrete decomposition, they used the Calderón formula

$$u = \lim_{t \rightarrow 0} P_t u = \int_0^1 Q_t u \frac{dt}{t} + P_1 u$$

for $u \in \mathcal{D}'(M)$ with P_t the heat semigroup and $Q_t = -t\partial_t P_t$. Using Gaussian upper bounds for the heat kernel and its derivatives, this defines a continuous analogue of the Littlewood-Paley decomposition where $\sqrt{t} \simeq 2^{-n}$ and yields descriptions of Besov-Hölder and Sobolev spaces for scalar fields on manifolds. This can be used to construct a paraproduct P and a resonant product Π such that

$$u \cdot v = P_u v + \Pi(u, v) + P_v u$$

that verify the same important properties of their Fourier analogue P and Π . It was later extended to a higher order paracontrolled calculus by Bailleul, Bernicot and Frey in [5] to deal with rougher noise than the initial work of Gubinelli, Imkeller and Perkowski, again in a general geometric framework. While these different works dealt with parabolic PDEs, the paracontrolled calculus can be used to study singular random operators. It was first used by Allez and Chouk in [1] to study the Anderson Hamiltonian

$$H = -\Delta + \xi$$

on the two-dimensional torus. The same operator was constructed on the torus by Gubinelli, Ugurcan and Zachhuber in [18] on \mathbb{T}^d with $d \in \{2, 3\}$ to solve associated evolution PDEs. Labbé also constructed in [21] the operator in two and three dimensions with different boundary conditions

using regularity structures. Finally, Mouzard used the heat semigroup paracontrolled calculus in [24] to construct the operator on a two-dimensional manifold and obtained an almost sure Weyl-type law. Note that the work [24] is self-contained and a gentle introduction to the paracontrolled calculus on manifold in the spatial framework. For another example of singular random operators, see [23] where Morin and Mouzard construct the magnetic Laplacian with white noise magnetic field on \mathbb{T}^2 . With this work, we show that this approach is also well-suited for the study of dispersive PDEs.

The heat semigroup paracontrolled calculus is a theory to study PDEs with singular products on manifolds. Given a suitable family $(V_i)_{1 \leq i \leq d}$ of first order differential operators, one can construct a paraproduct P and a resonant term Π based on the heat semigroup associated to

$$L := - \sum_{i=1}^d V_i^2.$$

We briefly outline this construction here, see [5, 6] in the parabolic space-time setting and [24] in the space setting for the details. For any distribution $u \in \mathcal{D}'(M)$, the heat semigroup

$$P_t u = e^{-tL} u$$

provides a smooth approximation as t goes to 0. Introducing its derivative

$$Q_t := -t \partial_t P_t,$$

one get an analogue of the Littlewood-Paley decomposition as explained before. While the Δ_n 's enjoy proper orthogonal relation in the sense that $\Delta_n \Delta_m$ is equal to zero for $|n - m| > 1$, we only have in this continuous framework

$$Q_t Q_s = \frac{ts}{(t+s)^2} ((t+s)L)^2 e^{-tL}$$

which is indeed small if $s \ll t$ or $t \ll s$. For a given integer $b \in \mathbb{N}^*$, let

$$Q_t^{(b)} := (tL)^b e^{-tL}.$$

Then

$$Q_t^{(b)} Q_s^{(b)} = \left(\frac{ts}{(t+s)^2} \right)^b Q_{t+s}^{(b)}$$

hence the parameter b encodes a cancellation properties between different scales t and s . Furthermore, we have

$$\int_0^1 Q_t^{(b)} u \frac{dt}{t} = \lim_{t \rightarrow 0} P_t^{(b)} u = u$$

where $P_0^{(b)} = \text{Id}$ and

$$-t \partial_t P_t^{(b)} = Q_t^{(b)}.$$

In particular, we have $P_t^{(b)} = p_b(tL) e^{-tL}$ with p_b a polynomial of degree $b - 1$ such that $p_b(0) = 1$. Denote by StGC^a the family of operators $(Q_t)_{t \in [0,1]}$ of the form

$$Q_t = (t^{\frac{|I|}{2}} V_I) (tL)^j e^{-tL}$$

with $a = |I| + 2j$ and GC^a the operator with kernel satisfying Gaussian upper bounds with cancellation of order a , see Section 1.2 [24] for the definitions. We have

$$\begin{aligned} u \cdot v &= \lim_{t \rightarrow 0} P_t^{(b)} (P_t^{(b)} u \cdot P_t^{(b)} v) \\ &= \int_0^1 Q_t^{(b)} (P_t^{(b)} u \cdot P_t^{(b)} v) \frac{dt}{t} + \int_0^1 P_t^{(b)} (Q_t^{(b)} u \cdot P_t^{(b)} v) \frac{dt}{t} + \int_0^1 P_t^{(b)} (P_t^{(b)} u \cdot Q_t^{(b)} v) \frac{dt}{t} \\ &\quad + P_1^{(b)} (P_1^{(b)} u \cdot P_1^{(b)} v). \end{aligned}$$

After a number of integrations by parts, we get

$$u \cdot v = P_u v + \Pi(u, v) + P_v u$$

where $P_u v$ is a linear combination of terms of the form

$$\int_0^1 Q_t^{1\bullet} (P_t u \cdot Q_t^2 v) \frac{dt}{t}$$

and $\Pi(u, v)$ of

$$\int_0^1 P_t^\bullet (Q_t^1 u \cdot Q_t^2 v) \frac{dt}{t}$$

where $Q^1, Q^2 \in \text{StGC}^{\frac{b}{2}}$ and $P \in \text{StGC}^{[0, b]}$. In general, the operator V_i 's do not commute hence the need for the notation

$$Q_t^\bullet = \left((t^{\frac{|L|}{2}} V_I)(tL)^j e^{-tL} \right)^\bullet := (tL)^j e^{-tL} (t^{\frac{|L|}{2}} V_I)$$

which comes from the integration by parts. For simplicity we state most of the results of this Section in Besov-Hölder spaces. The following proposition gives the continuity estimates of the paraproduct and the resonant term between Sobolev and Hölder-Besov functions but they hold in the same way by replacing all the Sobolev spaces by Besov-Hölder spaces.

Proposition 1.5. *Let $\alpha, \beta \in (-2b, 2b)$ be regularity exponents.*

- *If $\alpha \geq 0$, then $(f, g) \mapsto P_f g$ is continuous from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to \mathcal{H}^β .*
- *If $\alpha < 0$, then $(f, g) \mapsto P_f g$ is continuous from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*
- *If $\alpha + \beta > 0$, then $(f, g) \mapsto \Pi(f, g)$ is continuous from $\mathcal{H}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{H}^{\alpha+\beta}$.*

While P and Π are tools to describe products, the intertwined paraproduct \tilde{P} naturally appears when formulating solutions to PDEs. The intertwining relation is

$$L \circ \tilde{P} = P \circ L$$

hence $\tilde{P}_u v$ is given as a linear combination of

$$\begin{aligned} \int_0^1 L^{-1} Q_t^{1\bullet} (P_t u \cdot Q_t^2 L v) \frac{dt}{t} &\sim \int_0^1 (tL)^{-1} Q_t^{1\bullet} (P_t u \cdot Q_t^2 (tL) v) \frac{dt}{t} \\ &\sim \int_0^1 \tilde{Q}_t^{1\bullet} (P_t u \cdot \tilde{Q}_t^2 v) \frac{dt}{t} \end{aligned}$$

with $\tilde{Q}^1 \in \text{StGC}^{\frac{b}{2}-2}$ and $\tilde{Q}^2 \in \text{StGC}^{\frac{b}{2}+2}$. The operator L is not invertible and everything here is done up to a smooth error term, see [24]. In particular, \tilde{P} has the same structure as P for large b and satisfies the same continuity estimates as P . Intuitively, the intertwined operator \tilde{P} describes solutions to elliptic PDEs of the form

$$Lu = u\xi = P_u \xi + P_\xi u + \Pi(u, \xi)$$

which rewrites

$$u = \tilde{P}_u(L^{-1}\xi) + u^\sharp$$

hence this is the operator used to describe the domain \mathcal{D}_Ξ of the Anderson Hamiltonian. The final ingredient of paracontrolled calculus is a toolbox of correctors and commutators made to express the singular product between a paracontrolled functions u and the noise ξ in a form involving only ill-defined expressions of the noise independent of u . The first one introduced by [17] is in this framework the corrector

$$\mathbf{C}(u, X, \xi) := \Pi(\tilde{P}_u X, \xi) - u\Pi(X, \xi)$$

which translates the rough paths philosophy: the multiplication of a function that locally looks like X with ξ is possible if one is given the multiplication of X itself with ξ . This is the content of the following proposition.

Proposition 1.6. *Let $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$. If*

$$\alpha + \beta < 0 \quad \text{and} \quad \alpha + \beta + \gamma > 0,$$

then \mathbf{C} extends in a unique continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}^{\alpha+\beta+\gamma}$.

While we do not give the proof, one has the following heuristic. For any $x \in M$, we have

$$\begin{aligned} C(f, g, h)(x) &= \Pi(\tilde{P}_f g, h)(x) - f(x) \cdot \Pi(g, f)(x) \\ &= \Pi(\tilde{P}_f g - f(x) \cdot g, h)(x) \\ &\simeq \Pi(\tilde{P}_{f-f(x)} g, h)(x) \end{aligned}$$

where \simeq is equal up to a smooth term since $g \simeq \tilde{P}_1 g$. Since $f \in \mathcal{C}^\alpha$ with $\alpha \in (0, 1)$, the term $f - f(x)$ allows to gain regularity in the paraproduct using that $\alpha + \beta < 0$ ending up with a term of better regularity $\alpha + \beta + \gamma > 0$. Continuity results on a number of correctors and commutators and their iterated version are also available, we refer to [24] and references therein for further details. For example, one needs the swap operator

$$S(f, g, h) = P_h \tilde{P}_f g - P_f P_h g$$

for the study of the Anderson Hamiltonian which is continuous from $\mathcal{H}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{H}^{\alpha+\beta+\gamma}$ for $\alpha, \beta \in \mathbb{R}$ and $\gamma < 0$.

1.3 – Construction of the Anderson Hamiltonian

In this Section, we recall the ideas behind the construction of the Anderson Hamiltonian with the heat semigroup paracontrolled calculus as done in [24] and state the important results we shall use without proofs. We also provide new straightforward results from the construction needed for our proof of Strichartz inequalities. The Anderson Hamiltonian on a two-dimensional manifold M is formally given by

$$H := L + \xi$$

where $-L$ is the Laplace-Beltrami operator and ξ is a spatial white noise. The noise belongs almost surely to $\mathcal{C}^{\alpha-2}$ for any $\alpha < 1$ hence the product of ξ with a generic L^2 -function is not defined almost surely. As explained, it was first constructed by Allez and Chouk in [1] on \mathbb{T}^2 . We work here with the construction on a two-dimensional manifolds from [24] using the high order paracontrolled calculus since this is the setting in which we want to prove Strichartz inequalities. Following the recent development in singular stochastic PDEs, the idea is to construct a random almost surely dense subspace \mathcal{D}_Ξ of L^2 such that the operator makes sense for $u \in \mathcal{D}_\Xi \subset L^2$ with Ξ an enhancement of the noise that depend only measurably on the noise ξ . One can then prove that H is self-adjoint with discrete spectrum

$$\lambda_1(\Xi) \leq \lambda_2(\Xi) \leq \dots \leq \lambda_n(\Xi) \leq \dots$$

where $\lambda_n(0) = \lambda_n$ corresponds to the eigenvalues of the Laplace-Beltrami operator. While the construction of the domain \mathcal{D}_Ξ relied on the notion of strongly paracontrolled functions in [1, 18], the high order paracontrolled calculus gives a finer description of the domain. In particular, it yields sharp bounds on the eigenvalues of the form

$$\lambda_n - m_\delta^1(\Xi) \leq \lambda_n(\Xi) \leq (1 + \delta)\lambda_n + m_\delta^2(\Xi)$$

for any $\delta \in (0, 1)$ which implies the almost sure Weyl-type law

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} |\{n \geq 0; \lambda_n(\Xi) \leq \lambda\}| = \frac{\text{Vol}(M)}{4\pi}.$$

We briefly present the construction of H and refer to [24] for the details.

Coming from Lyons' rough paths [22] and Gubinelli's controlled paths [14] which were developed as a pathwise approach to stochastic integration, the method used over the last decade to solve singular stochastic PDEs is to work in random subspaces of classical function spaces built from the noise tailor-made for the problem under consideration. In the context of singular random operators, this corresponds to the construction of a random dense domain $\mathcal{D}_\Xi \subset L^2$ on which the operator almost surely makes sense. In the framework of paracontrolled calculus, one considers functions u paracontrolled by noise-dependent reference functions of the form

$$u = \tilde{P}_{u'} X + u^\sharp$$

where the new unknown is (u', u^\sharp) . The function u' has to be thought as the “derivative” of u with respect to X while the error u^\sharp is a smoother remainder. The goal is to find a paracontrolled

expression for $u \in L^2$ such that $Hu \in L^2$. Let us first assume that u is smooth, then we formally get

$$\begin{aligned} Lu &= Hu - u\xi \\ &= -P_u\xi + Hu - P_\xi u - \Pi(u, \xi) \in \mathcal{H}^{\alpha-2} \end{aligned}$$

hence elliptic regularity theory gives $u \in \mathcal{H}^\alpha$ and suggests for u the paracontrolled form

$$u = \tilde{P}_u X + u^\sharp$$

with $X = -L^{-1}\xi$ and $u^\sharp \in \mathcal{H}^{2\alpha}$. Given such a function, the resonance between u and ξ can be described by

$$\begin{aligned} \Pi(u, \xi) &= \Pi(\tilde{P}_u X, \xi) + \Pi(u^\sharp, \xi) \\ &= u\Pi(X, \xi) + C(u, X, \xi) + \Pi(u^\sharp, \xi) \end{aligned}$$

using the corrector C since $\Pi(\tilde{P}_u X, \xi)$ is not defined due to lack of regularity, see Propositions 1.5 and 1.6. Since $3\alpha + 2 > 0$, the only term on the right hand side which is potentially undefined is $\Pi(X, \xi)$ and its definition is independent of the study of H , see [24] for more details including the renormalisation. Given the enhanced data

$$\Xi := (\xi, \Pi(X, \xi)) \in \mathcal{C}^{\alpha-2} \times \mathcal{C}^{2\alpha-2} =: \mathcal{X}^\alpha$$

one can define the Anderson Hamiltonian H on

$$\mathcal{D} := \{u \in L^2; u - \tilde{P}_u X \in \mathcal{H}^{2\alpha}\} \subset \mathcal{H}^\alpha$$

with

$$Hu := Lu + P_\xi u + u\Pi(X, \xi) + C(u, X, \xi) + \Pi(u^\sharp, \xi).$$

However, this gives only an unbounded operator (H, \mathcal{D}) from $\mathcal{H}^\alpha \subset L^2$ to $\mathcal{H}^{2\alpha-2}$ which is not a subspace of L^2 and thus H will not takes value in L^2 a priori. A finer description of the domain with a second order paracontrolled expansion allows to construct a dense subspace $\mathcal{D}_\Xi \subset L^2$ such that (H, \mathcal{D}_Ξ) is an unbounded operator on L^2 . Using the classical theory for unbounded operator, it is possible to prove that H is self-adjoint with pure point spectrum. In the expression for H , the roughest term is

$$P_\xi u + P_u \Pi(X, \xi) \in \mathcal{H}^{2\alpha-2}.$$

To cancel it with a paracontrolled expansion, we use the commutator S to get

$$\begin{aligned} P_\xi u &= P_\xi \tilde{P}_u X + P_\xi u^\sharp \\ &= P_u P_\xi X + S(u, X, \xi) + P_\xi u^\sharp \end{aligned}$$

hence the roughest term is

$$P_u P_\xi X + P_u \Pi(X, \xi) \in \mathcal{H}^{2\alpha-2}.$$

In the end, it is cancelled with the paracontrolled expansion

$$u = \tilde{P}_u X_1 + \tilde{P}_u X_2 + u^\sharp$$

where

$$X_1 := -L^{-1}\xi \quad \text{and} \quad X_2 := -L^{-1}(P_\xi X_1 + \Pi(X_1, \xi)).$$

Definition 1.7. We define the space \mathcal{D}_Ξ of functions paracontrolled by Ξ as

$$\mathcal{D}_\Xi := \{u \in L^2; u^\sharp := u - \tilde{P}_u X_1 - \tilde{P}_u X_2 \in \mathcal{H}^2\}.$$

A powerful tool to investigate the domain \mathcal{D}_Ξ and H is the Γ map defined as follows. The domain is given as

$$\mathcal{D}_\Xi = \Phi^{-1}(\mathcal{H}^2)$$

with

$$\Phi(u) := u - \tilde{P}_u(X_1 + X_2).$$

The map Φ is not necessarily invertible so we introduce a parameter $s > 0$ and consider the map

$$\Phi^s(u) := u - \tilde{P}_u^s(X_1 + X_2)$$

where $\tilde{\mathbb{P}}^s$ is a truncated paraproduct. In particular, $\tilde{\mathbb{P}}^s$ goes to 0 as s goes to 0 and the difference $\tilde{\mathbb{P}} - \tilde{\mathbb{P}}^s$ is smooth for any $s > 0$. This has to be thought as a frequency cut-off where one gets rid of a number of low frequencies in order to make a term small. Thus Φ^s is a perturbation of the identity of \mathcal{H}^β for any $\beta \in [0, \alpha)$ and thus invertible for $s = s(\Xi)$ small enough. We define Γ to be its inverse which is implicitly defined by

$$\Gamma u^\sharp = \tilde{\mathbb{P}}_{\Gamma u^\sharp}(X_1 + X_2) + u^\sharp$$

for any $u^\sharp \in \mathcal{H}^\beta$. It will be a crucial tool to describe the operator H since

$$\mathcal{D}_\Xi = \Phi^{-1}(\mathcal{H}^2) = (\Phi^s)^{-1}(\mathcal{H}^2) = \Gamma(\mathcal{H}^2)$$

where the equality holds because the difference $\tilde{\mathbb{P}} - \tilde{\mathbb{P}}^s$ is smooth. The maps Φ^s and Γ satisfy a number of continuity estimates that we shall use throughout this work, this is the content of the following proposition. Let

$$s_\beta(\Xi) := \left(\frac{\alpha - \beta}{m \|\Xi\|_{\mathcal{X}^\alpha} (1 + \|\Xi\|_{\mathcal{X}^\alpha})} \right)^{\frac{4}{\alpha - \beta}}$$

for any $0 \leq \beta < \alpha$.

Proposition 1.8. *Let $\beta \in [0, \alpha)$ and $s \in (0, 1)$. We have*

$$\|\Phi^s(u) - u\|_{\mathcal{H}^\beta} \leq \frac{m}{\alpha - \beta} s^{\frac{\alpha - \beta}{4}} \|\Xi\|_{\mathcal{X}^\alpha} (1 + \|\Xi\|_{\mathcal{X}^\alpha}) \|u\|_{L^2}.$$

If moreover $s < s_\beta(\Xi)$, this implies

$$\|\Gamma u^\sharp\|_{\mathcal{H}^\beta} \leq \frac{1}{1 - \frac{m}{\alpha - \beta} s^{\frac{\alpha - \beta}{4}} \|\Xi\|_{\mathcal{X}^\alpha} (1 + \|\Xi\|_{\mathcal{X}^\alpha})} \|u^\sharp\|_{\mathcal{H}^\beta}.$$

In particular, the map Φ is continuous from L^p to itself for $p \in [1, \infty]$ and \mathcal{H}^σ to itself for $\sigma \in [0, 1)$ while the same holds for Γ provided s is small enough.

Let us insist that the norm \mathcal{H}^β of $u_s^\sharp := \Phi^s(u)$ is always controlled by $\|u\|_{L^2}$ while s needs to be small depending on the noise for $\|u\|_{\mathcal{H}^\beta}$ to be controlled by $\|u_s^\sharp\|_{\mathcal{H}^\beta}$. We also define the map Γ_ε associated to the regularised noise Ξ_ε as

$$\Gamma_\varepsilon u^\sharp = \tilde{\mathbb{P}}_{\Gamma_\varepsilon u^\sharp}^s X_1^{(\varepsilon)} + \tilde{\mathbb{P}}_{\Gamma_\varepsilon u^\sharp}^s X_2^{(\varepsilon)} + u^\sharp$$

with

$$-LX_1^{(\varepsilon)} := \xi_\varepsilon \quad \text{and} \quad -LX_2^{(\varepsilon)} := \Pi(X_1^{(\varepsilon)}, \xi_\varepsilon) - c_\varepsilon + P_{\xi_\varepsilon} X_1^{(\varepsilon)}.$$

It satisfies the same bounds as Γ with constants which depend in an increasing way on $\|\Xi_\varepsilon\|_{\mathcal{X}^\alpha} \lesssim 1 + \|\Xi\|_{\mathcal{X}^\alpha}$ and the following approximation lemma holds.

Lemma 1.9. *For any $0 \leq \beta < \alpha$ and $0 < s < s_\beta(\Xi)$, we have*

$$\|\text{Id} - \Gamma \Gamma_\varepsilon^{-1}\|_{L^2 \rightarrow \mathcal{H}^\beta} \lesssim_{\Xi, s, \beta} \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

In particular, this implies the norm convergence of Γ_ε to Γ with the bound

$$\|\Gamma - \Gamma_\varepsilon\|_{\mathcal{H}^\beta \rightarrow \mathcal{H}^\beta} \lesssim_{\Xi, s, \beta} \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

In particular, this allows to prove density of the domain.

Corollary. *The domain \mathcal{D}_Ξ is dense in \mathcal{H}^β for any $\beta \in [0, \alpha)$.*

For any $u \in \mathcal{D}_\Xi$, the operator H is given by

$$Hu = Lu^\sharp + P_\xi u^\sharp + \Pi(u^\sharp, \xi) + R(u)$$

with $u^\sharp = \Phi(u) \in \mathcal{H}^2$ and R an explicit operator depending on Ξ which is continuous from \mathcal{H}^α to $\mathcal{H}^{3\alpha - 2}$. For each $s > 0$, we have a different representation of H , namely

$$Hu = H\Gamma u_s^\sharp = Lu_s^\sharp + P_\xi u_s^\sharp + \Pi(u_s^\sharp, \xi) + R(\Gamma u_s^\sharp) + \Psi^s(\Gamma u_s^\sharp)$$

with $u_s^\sharp = \Phi^s(u) \in \mathcal{H}^2$ and Ψ^s an explicit operator depending on Ξ and s continuous from L^2 to C^∞ which we henceforth include in the operator R . The operator $H\Gamma$ is thus a perturbation of L , the following proposition shows that it is a continuous operator from \mathcal{H}^2 to L^2 . In Section 2, we show that it is even a lower order perturbation of the Laplace-Beltrami operator; this will be crucial to obtain Strichartz inequalities.

Proposition 1.10. For any $\gamma \in (-\alpha, 3\alpha - 2)$ and s as above, we have

$$\|H\Gamma u_s^\sharp\|_{\mathcal{H}^\gamma} \lesssim \|\Gamma u_s^\sharp\|_{\mathcal{H}^{\gamma+2}}.$$

In particular, the result holds for $\gamma \in (-1, 1)$ since the noise belongs to $\mathcal{C}^{\alpha-2}$ for any $\alpha < 1$.

Proof: We have

$$H\Gamma u_s^\sharp = Lu_s^\sharp + \mathbb{P}_\xi u_s^\sharp + \Pi(u_s^\sharp, \xi) + R(u)$$

with $u = \Gamma u_s^\sharp$. Assume first that $0 < \gamma < 3\alpha - 2$ hence

$$\begin{aligned} \|H\Gamma u_s^\sharp\|_{\mathcal{H}^\gamma} &\lesssim \|Lu_s^\sharp\|_{\mathcal{H}^\gamma} + \|\mathbb{P}_\xi u_s^\sharp + \Pi(u_s^\sharp, \xi)\|_{\mathcal{H}^\gamma} + \|R(u)\|_{\mathcal{H}^\gamma} \\ &\lesssim \|u_s^\sharp\|_{\mathcal{H}^{\gamma+2}} + \|\xi\|_{\mathcal{C}^{\alpha-2}} \|u_s^\sharp\|_{\mathcal{H}^{\gamma+2-\alpha}} + \|R(u)\|_{\mathcal{H}^{3\alpha-2}} \end{aligned}$$

where the condition $\gamma > 0$ is needed for the resonant term and $\gamma < 3\alpha - 2$ for $R(u)$. The result follows for this case since

$$\|R(u)\|_{\mathcal{H}^{3\alpha-2}} \lesssim \|u\|_{\mathcal{H}^\alpha} \lesssim \|u_s^\sharp\|_{\mathcal{H}^\alpha} \lesssim \|u_s^\sharp\|_{\mathcal{H}^{\gamma+2}}.$$

Assume now that $-\alpha < \gamma \leq 0$. For any $\delta > 0$, we have

$$\begin{aligned} \|H\Gamma u_s^\sharp\|_{\mathcal{H}^\gamma} &\lesssim \|Lu_s^\sharp\|_{\mathcal{H}^\gamma} + \|\mathbb{P}_\xi u_s^\sharp + \Pi(u_s^\sharp, \xi)\|_{\mathcal{H}^\gamma} + \|R(u)\|_{\mathcal{H}^\gamma} \\ &\lesssim \|Lu_s^\sharp\|_{\mathcal{H}^\gamma} + \|\mathbb{P}_\xi u_s^\sharp + \Pi(u_s^\sharp, \xi)\|_{\mathcal{H}^\delta} + \|R(u)\|_{\mathcal{H}^\gamma} \\ &\lesssim \|u_s^\sharp\|_{\mathcal{H}^{\gamma+2}} + \|\xi\|_{\mathcal{C}^{\alpha-2}} \|u_s^\sharp\|_{\mathcal{H}^{\delta+2-\alpha}} + \|R(u)\|_{\mathcal{H}^{3\alpha-2}} \end{aligned}$$

using that $\gamma \leq 0 < \delta$. The proof is complete since $\gamma > -\alpha$ and δ small enough implies $\gamma + 2 > \delta + 2 - \alpha$. □

As the parameter $s > 0$ yields different representation of H , the domain \mathcal{D}_Ξ is naturally equipped with the norms

$$\|u\|_{\mathcal{D}_\Xi}^2 := \|u\|_{L^2}^2 + \|u_s^\sharp\|_{\mathcal{H}^2}^2.$$

which are equivalent to the graph norm

$$\|u\|_H^2 := \|u\|_{L^2}^2 + \|Hu\|_{L^2}^2.$$

In particular, this shows that the operator H is closed on its domain \mathcal{D}_Ξ .

Proposition 1.11. Let $u \in \mathcal{D}_\Xi$ and $s > 0$. For any $\delta > 0$, we have

$$(1 - \delta) \|u_s^\sharp\|_{\mathcal{H}^2} \leq \|Hu\|_{L^2} + m_\delta^2(\Xi, s) \|u\|_{L^2}$$

and

$$\|Hu\|_{L^2} \leq (1 + \delta) \|u_s^\sharp\|_{\mathcal{H}^2} + m_\delta^2(\Xi, s) \|u\|_{L^2}$$

with $u_s^\sharp = \Phi^s(u)$ and $m_\delta^2(\Xi, s) > 0$ an explicit constant.

In addition to this comparison between H and L in norm, one has a similar statement in the quadratic form setting.

Proposition 1.12. Let $u \in \mathcal{D}_\Xi$ and $s > 0$. For any $\delta > 0$, we have

$$(1 - \delta) \langle \nabla u_s^\sharp, \nabla u_s^\sharp \rangle \leq \langle u, Hu \rangle + m_\delta^1(\Xi, s) \|u\|_{L^2}^2$$

and

$$\langle u, Hu \rangle \leq (1 + \delta) \langle \nabla u_s^\sharp, \nabla u_s^\sharp \rangle + m_\delta^1(\Xi, s) \|u\|_{L^2}^2$$

where $u_s^\sharp = \Phi^s(u)$ and $m_\delta^1(\Xi, s) > 0$ an explicit constant.

One can show that $H\Gamma$ is the limit in norm of $H_\varepsilon \Gamma_\varepsilon$ as operators from \mathcal{H}^2 to L^2 where

$$H_\varepsilon := L + \xi_\varepsilon - c_\varepsilon$$

with c_ε a diverging function as ε goes to 0, again see Section 2.1 of [24]. In particular, one can take shift c_ε by a large enough constant to ensure that H is positive. Thus the previous proposition

implies that $\|\sqrt{H}u\|_{L^2}$ and $\|u_s^\sharp\|_{\mathcal{H}^1}$ are equivalent. The diverging quantity is needed to take care of the singularity as explained in the introduction, this is the renormalisation procedure with

$$\Pi(X_1, \xi) := \lim_{\varepsilon \rightarrow 0} \Pi(X_1^{(\varepsilon)}, \xi_\varepsilon) - c_\varepsilon$$

in $\mathcal{C}^{2\alpha-2}$. In the case of the torus, the noise is invariant by translation and the function c_ε is actually a constant that diverges as $\ln(\varepsilon)$, see [1]. This allows to prove that H is a symmetric operator as the weak limit of the symmetric operators H_ε . Being closed and symmetric, it is enough to prove that

$$(H + k)u = v$$

admits a solution for some $k \in \mathbb{R}$ to get self-adjointness for H , see Theorem X.1 in [27]. This is done using the Babuška-Lax-Milgram Theorem, see [3] and Proposition 1.12 which implies that H is almost surely bounded below. This implies self-adjointness and since the resolvent is a compact operator from L^2 to itself since $\mathcal{D}_\Xi \subset \mathcal{H}^\beta$ for any $\beta \in [0, \alpha)$.

Corollary 1.13. *The operator H is self-adjoint with discrete spectrum $(\lambda_n(\Xi))_{n \geq 1}$ which is a non-decreasing diverging sequence without accumulation points. Moreover, we have*

$$L^2 = \bigoplus_{n \geq 1} \text{Ker}(H - \lambda_n(\Xi))$$

with each kernel being of finite dimension. We finally have the min-max principle

$$\lambda_n(\Xi) = \inf_D \sup_{u \in D; \|u\|_{L^2}=1} \langle Hu, u \rangle$$

where D is any n -dimensional subspace of \mathcal{D}_Ξ ; this can also be written as

$$\lambda_n(\Xi) = \sup_{v_1, \dots, v_{n-1} \in L^2} \inf_{\substack{u \in \text{Vect}(v_1, \dots, v_{n-1})^\perp \\ \|u\|_{L^2}=1}} \langle Hu, u \rangle.$$

While the regularity of a function can be measured by its coefficients in the basis of eigenfunction of the Laplacian, the same is true for the Anderson Hamiltonian and the spaces agree if the regularity one considers is below the form domain.

Proposition 1.14. *For $\beta \in (-\alpha, \alpha)$, there exists two constants $c_\Xi, C_\Xi > 0$ such that*

$$c_\Xi \|H^{\frac{\beta}{2}} u\|_{L^2} \leq \|u\|_{\mathcal{H}^\beta} \leq C_\Xi \|H^{\frac{\beta}{2}} u\|_{L^2}.$$

Proof: Observe first that the statement is clear for $\beta = 0$, we consider only the case $\beta \in (0, \alpha)$ since the case of negative β follows by duality. Again we take $(\varphi_n)_{n \geq 1}$ and $(e_n)_{n \geq 1}$ to denote the basis of eigenfunctions of $-\Delta$ and H respectively. We have for any $v \in \mathcal{D}_\Xi$

$$\begin{aligned} \|H^{\frac{\beta}{2}} v\|_{L^2} &= \left(\sum_{n \geq 1} \lambda_n^\beta \langle v, e_n \rangle^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{n \geq 1} \lambda_n^\beta \langle v, e_n \rangle^{2\beta} \langle v, e_n \rangle^{2-2\beta} \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{n \geq 1} \lambda_n \langle v, e_n \rangle^2 \right)^{\frac{\beta}{2}} \left(\sum_{n \geq 1} \langle v, e_n \rangle^2 \right)^{\frac{1-\beta}{2}} \\ &\lesssim \|H^{\frac{1}{2}} v\|_{L^2}^\beta \|v\|_{L^2}^{1-\beta} \end{aligned}$$

using Hölder's inequality. Thus the equivalence of $\|H^{\frac{1}{2}} v\|_{L^2}$ and $\|v_s^\sharp\|_{\mathcal{H}^1}$ from Proposition 1.12, together with the continuity of Φ^s from L^2 to itself yields

$$\|H^{\frac{\beta}{2}} v\|_{L^2} \lesssim \|v_s^\sharp\|_{\mathcal{H}^1}^\beta \|v_s^\sharp\|_{L^2}^{1-\beta}.$$

Applying this with $v = \Gamma(\langle u_s^\sharp, \varphi_n \rangle \varphi_n)$ gives

$$\begin{aligned} \|H^{\frac{\beta}{2}} \Gamma(\langle u_s^\sharp, \varphi_n \rangle \varphi_n)\|_{L^2} &\lesssim \|\langle u_s^\sharp, \varphi_n \rangle \varphi_n\|_{\mathcal{H}^1}^\beta \|\langle u_s^\sharp, \varphi_n \rangle \varphi_n\|_{L^2}^{1-\beta} \\ &\lesssim |\langle u_s^\sharp, \varphi_n \rangle| \|\varphi_n\|_{\mathcal{H}^\beta} \end{aligned}$$

Thus

$$\begin{aligned} \|H^{\frac{\beta}{2}}u\|_{L^2}^2 &= \|H^{\frac{\beta}{2}}\Gamma(u_s^\sharp)\|_{L^2}^2 \leq \sum_{n \geq 1} \|H^{\frac{\beta}{2}}\Gamma(\langle u_s^\sharp, \varphi_n \rangle \varphi_n)\|_{L^2}^2 \\ &\lesssim \sum_{n \geq 1} |\langle u_s^\sharp, \varphi_n \rangle|^2 \|\varphi_n\|_{\mathcal{H}^\beta}^2 \\ &\lesssim \|u_s^\sharp\|_{\mathcal{H}^\beta}^2. \end{aligned}$$

Since $\beta \in [0, \alpha)$, we get

$$\|H^{\frac{\beta}{2}}u\|_{L^2} \lesssim \|u\|_{\mathcal{H}^\beta}.$$

from the boundedness of Γ , see Proposition 1.8. The other inequality follows from the same reasoning with

$$\|v\|_{\mathcal{H}^\beta} \lesssim \|v_s^\sharp\|_{\mathcal{H}^\beta} \lesssim \|v_s^\sharp\|_{\mathcal{H}^1}^\beta \|v_s^\sharp\|_{L^2}^{1-\beta} \lesssim \|H^{\frac{1}{2}}v\|_{\mathcal{H}^1}^\beta \|u\|_{L^2}^{1-\beta}$$

and applying this bound to $u = \sum_{n \geq 1} \langle u, e_n \rangle e_n$ and proceeding as above we get the other direction. \square

The operator H and its spectrum do not depend on $s > 0$ but the different representation of H as

$$Hu = L\Phi^s(u) + P_\xi \Phi^s(u) + \Pi(\Phi^s(u), \xi) + R(u) + \Psi^s(u)$$

yields different bounds on the eigenvalues. We state the simpler form for the bounds, see [24] for the general result. It is sharp enough to obtain an almost sure Weyl-type law from the one for the Laplace-Beltrami operator.

Proposition 1.15. *Let $\delta \in (0, 1)$. Then there exists two constants $m_\delta^1(\Xi), m_\delta^2(\Xi)$ such that*

$$\lambda_n - m_\delta^1(\Xi) \leq \lambda_n(\Xi) \leq (1 + \delta)\lambda_n + m_\delta^2(\Xi)$$

for any $n \in \mathbb{N}$. This implies the almost sure Weyl-type law

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} |\{n \geq 0; \lambda_n(\Xi) \leq \lambda\}| = \frac{\text{Vol}(M)}{4\pi}.$$

2 – Strichartz inequalities for the stochastic Schrödinger equation

For the rest of the work, we fix a parameter $s > 0$ small enough in order to have all the needed continuity estimates. Every constant may implicitly depend on s and on the norm of the enhanced noise, we do not explicate the dependence since it is not relevant at this stage. From now on, we will also use that α can be taken arbitrary close to 1 since it is given by the regularity of the spatial white noise. We consider the Schrödinger operator

$$H^\sharp := \Gamma^{-1}H\Gamma$$

which appears naturally when transforming the Schrödinger equation and the wave equation with multiplicative noise. In fact, if u solves

$$\begin{cases} i\partial_t u &= Hu, \\ u(0) &= u_0 \end{cases}$$

then $u^\sharp := \Gamma^{-1}u$ solves the transformed equation

$$\begin{cases} i\partial_t u^\sharp &= H^\sharp u^\sharp, \\ u^\sharp(0) &= \Gamma^{-1}u_0 \end{cases}$$

In this Section, we show Strichartz inequalities for the associated Schrödinger equation with an arbitrary small loss of regularity with respect to the deterministic case. Afterwards, in Section 2.2, we detail how these can be used to get a low-regularity solution theory for the nonlinear Schrödinger equation with multiplicative noise.

2.1 – Strichartz inequalities for the Schrödinger group

As was hinted at in Proposition 1.11, the transformed operator H^\sharp is a lower-order perturbation of the Laplace-Beltrami operator. We obtain the following result which is analogous to Proposition 3.2 in [29].

Proposition 2.1. *Let $0 \leq \beta < 1$. For any $\kappa > 0$, we have*

$$\|(H^\sharp - L)v\|_{\mathcal{H}^\beta} \lesssim \|v\|_{\mathcal{H}^{1+\beta+\kappa}}.$$

Proof: For $u = \Gamma u^\sharp \in \mathcal{D}_\Xi$, recall that

$$Hu = Lu^\sharp + P_\xi u^\sharp + \Pi(u^\sharp, \xi) + R(u)$$

where

$$\begin{aligned} R(u) := & \Pi(u, \Pi(X_1, \xi)) + P_{\Pi(X_1, \xi)} u + C(u, X_1, \xi) + P_u \Pi(X_2, \xi) + D(u, X_2, \xi) \\ & + S(u, X_2, \xi) + P_\xi \tilde{P}_u X_2 - e^{-L} (P_u X_1 + P_u X_2). \end{aligned}$$

Thus $H^\sharp v$ is given by

$$H^\sharp v = Lv + P_\xi v + \Pi(v, \xi) + R(\Gamma v) - \tilde{P}_{H\Gamma v}(X_1 + X_2)$$

and for any $\kappa > 0$ and $\beta \in [0, \alpha]$, we have

$$\begin{aligned} \|(H^\sharp - L)v\|_{\mathcal{H}^\beta} & \lesssim \|P_\xi v + \Pi(v, \xi)\|_{\mathcal{H}^\beta} + \|R(\Gamma v)\|_{\mathcal{H}^\beta} + \|\tilde{P}_{H\Gamma v}(X_1 + X_2)\|_{\mathcal{H}^\beta} \\ & \lesssim \|\xi\|_{\mathcal{C}^{-1-\kappa}} \|v\|_{\mathcal{C}^{\beta+1+\kappa}} + \|\Gamma v\|_{\mathcal{H}^\alpha} + \|H\Gamma v\|_{\mathcal{H}^{-1+\kappa+\beta}} \|X_1 + X_2\|_{\mathcal{H}^{1-\kappa}} \\ & \lesssim \|v\|_{\mathcal{H}^{1+\beta+\kappa}} + \|v\|_{\mathcal{H}^\alpha} + \|v\|_{\mathcal{H}^{1+\kappa+\beta}} \end{aligned}$$

using Proposition 1.10 and the proof is complete since $\alpha < 1$. □

Since the unitary group associated to H is bounded on L^2 and on the domain \mathcal{D}_Ξ of H , this implies a similar result for the “sharpened” group associated with H^\sharp in terms of classical Sobolev spaces.

Proposition 2.2. *For any $0 \leq \beta \leq 2$ and $t \in \mathbb{R}$, we have*

$$\|e^{itH^\sharp} v\|_{\mathcal{H}^\beta} \lesssim \|v\|_{\mathcal{H}^\beta}.$$

Moreover, e^{itH^\sharp} is a non-unitary strongly continuous group of L^2 bounded operators, namely

$$e^{i(t+s)H^\sharp} v = e^{itH^\sharp} e^{isH^\sharp} v$$

for all $s, t \in \mathbb{R}$ and $v \in L^2$.

Proof: For $\beta = 0$, this follows from the continuity of Γ and Γ^{-1} from L^2 to itself. For $\beta = 2$, we have

$$\begin{aligned} \|e^{itH^\sharp} v\|_{\mathcal{H}^2} & = \|\Gamma^{-1} e^{itH} \Gamma v\|_{\mathcal{H}^2} \\ & \lesssim \|H e^{itH} \Gamma v\|_{L^2} \\ & \lesssim \|e^{itH} H \Gamma v\|_{L^2} \\ & \lesssim \|H \Gamma v\|_{L^2} \\ & \lesssim \|v\|_{\mathcal{H}^2}, \end{aligned}$$

having used Proposition 1.11. The results for any $\beta \in (0, 2)$ is obtained by interpolation and the group property follows simply from the group property of e^{itH} by observing

$$e^{i(t+s)H^\sharp} v = \Gamma^{-1} e^{i(t+s)H} \Gamma v = \Gamma^{-1} e^{itH} \Gamma \Gamma^{-1} e^{isH} \Gamma v = e^{itH^\sharp} e^{isH^\sharp} v$$

□

Strichartz inequalities are refinement of the estimates from the previous proposition. The following statement is such a result, which has an arbitrary small loss of derivative coming from the irregularity of the noise in the addition to the $\frac{1}{p}$ loss from the manifold setting (without boundary) which one sees in [10]. We refer to a pair (p, q) satisfying

$$\frac{2}{p} + \frac{2}{q} = 1$$

as a Strichartz pair from here.

Theorem 2.3. *Let M be a two-dimensional compact manifold without boundary and let (p, q) be a Strichartz pair. Then for any $\varepsilon > 0$*

$$\|e^{itH^\sharp} v\|_{L^p([0,1], L^q)} \lesssim \|v\|_{\mathcal{H}^{\frac{1}{p}+\varepsilon}}.$$

First, we need to prove the following lemma. It gives the difference between the Schrödinger groups associated to H^\sharp and L from the difference between H^\sharp and L itself. Moreover it quantifies that their difference is small in a short time interval if one gives up some regularity.

Lemma 2.4. *Given $v \in \mathcal{H}^2$, we have*

$$\left(e^{i(t-t_0)H^\sharp} - e^{i(t-t_0)L} \right) v = i \int_{t_0}^t e^{i(t-s)L} (H^\sharp - L) e^{i(s-t_0)H^\sharp} v ds$$

for any $t, t_0 \in \mathbb{R}$.

Proof: The “sharpened” group yields the solution of the Schrödinger equation

$$(i\partial_t + H^\sharp)(e^{itH^\sharp} v) = 0$$

thus

$$(i\partial_t + L)(e^{itH^\sharp} v) = (L - H^\sharp)(e^{itH^\sharp} v).$$

Using the unitary group representation of the solution to the Schrödinger equation associated to L , we deduce that

$$(i\partial_t + L)(e^{itL} v - e^{itH^\sharp} v) = (H^\sharp - L)(e^{itH^\sharp} v).$$

Since the solution is equal to 0 at time 0, the mild formulation of this last equation yields

$$\left(e^{itH^\sharp} - e^{itL} \right) v = i \int_0^t e^{i(t-s)L} (H^\sharp - L) e^{isH^\sharp} v ds.$$

The result for any $t_0 \in \mathbb{R}$ follows from the same proof. □

Proof of Theorem 2.3: For $N \in \mathbb{N}^*$ to be chosen, we have

$$\|e^{itH^\sharp} v\|_{L^p([0,1], L^q)}^p = \sum_{\ell=0}^N \|e^{itH^\sharp} v\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p$$

where $t_\ell := \frac{\ell}{N}$. For $t \in [t_\ell, t_{\ell+1})$, the previous Lemma gives

$$e^{itH^\sharp} v = e^{i(t-t_\ell)H^\sharp} e^{it_\ell H^\sharp} v = e^{i(t-t_\ell)L} e^{it_\ell H^\sharp} v + i \int_{t_\ell}^t e^{i(t-s)L} (H^\sharp - L) e^{isH^\sharp} v ds.$$

Applying this with $v = \Delta_k u$ gives

$$\begin{aligned} \|\Delta_j e^{itH^\sharp} \Delta_k u\|_{L^p([0,1], L^q)}^p &\leq \sum_{\ell=0}^N \|\Delta_j e^{i(t-t_\ell)L} e^{it_\ell H^\sharp} \Delta_k u\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p \\ &\quad + \sum_{\ell=0}^N \left\| \Delta_j \int_{t_\ell}^t e^{i(t-s)L} (H^\sharp - L) e^{isH^\sharp} \Delta_k u ds \right\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p. \end{aligned}$$

Assume $N \geq 2^j$ such that $|t_{\ell+1} - t_\ell| \leq 2^{-j}$. For the first term, we have

$$\begin{aligned} \|\Delta_j e^{i(t-t_\ell)L} e^{it_\ell H^\sharp} \Delta_k u\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p &= \|e^{i(t-t_\ell)L} \Delta_j e^{it_\ell H^\sharp} \Delta_k u\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p \\ &\lesssim \|\Delta_j e^{it_\ell H^\sharp} \Delta_k u\|_{L^2}^p \\ &\lesssim 2^{-j\delta p} \|\Delta_j e^{it_\ell H^\sharp} \Delta_k u\|_{\mathcal{H}^\delta}^p \\ &\lesssim 2^{-j\delta p} 2^{-k\delta' p} \|\Delta_k u\|_{\mathcal{H}^{\delta+\delta'}}^p \end{aligned}$$

for any $\delta, \delta' \in \mathbb{R}$ using Proposition 2.2, Strichartz inequality for spectrally localised data from Section 1.1 and Bernstein's Lemma, see Lemma 1.3. For the second term, we have

$$\begin{aligned} \left\| \Delta_j \int_{t_\ell}^t e^{i(t-s)L} (H^\sharp - L) e^{isH^\sharp} \Delta_k u ds \right\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p &= \left\| \int_{t_\ell}^t e^{i(t-s)L} \Delta_j (H^\sharp - L) e^{isH^\sharp} \Delta_k u ds \right\|_{L^p([t_\ell, t_{\ell+1}], L^q)}^p \\ &\lesssim \left(\int_{t_\ell}^{t_{\ell+1}} \|e^{i(t-s)L} \Delta_j (H^\sharp - L) e^{isH^\sharp} \Delta_k u\|_{L^p([t_\ell, t_{\ell+1}], L^q)} ds \right)^p \\ &\lesssim \left(\int_{t_\ell}^{t_{\ell+1}} \|\Delta_j (H^\sharp - L) e^{isH^\sharp} \Delta_k u\|_{L^2} ds \right)^p \\ &\lesssim 2^{-j\sigma p} \left(\int_{t_\ell}^{t_{\ell+1}} \|\Delta_j (H^\sharp - L) e^{isH^\sharp} \Delta_k u\|_{\mathcal{H}^\sigma} ds \right)^p \\ &\lesssim 2^{-j\sigma p} \left(\int_{t_\ell}^{t_{\ell+1}} \|(H^\sharp - L) e^{isH^\sharp} \Delta_k u\|_{\mathcal{H}^\sigma} ds \right)^p \\ &\lesssim 2^{-j\sigma p} \left(\int_{t_\ell}^{t_{\ell+1}} \|e^{isH^\sharp} \Delta_k u\|_{\mathcal{H}^{\sigma+1+\kappa}} ds \right)^p \\ &\lesssim N^{-p} 2^{-j\sigma p} \|\Delta_k u\|_{\mathcal{H}^{1+\sigma+\kappa}}^p \\ &\lesssim N^{-p} 2^{-j\sigma p} 2^{-k\sigma' p} \|\Delta_k u\|_{\mathcal{H}^{1+\sigma+\sigma'+\kappa}}^p \end{aligned}$$

for any $\sigma \in (0, 1)$, $\sigma' \in \mathbb{R}$ and $0 < \kappa < 1 - \alpha$ where again the dyadic factors come from Bernstein's Lemma and we have used the bounds from Propositions 2.1 and 2.2 with Strichartz inequality for spectrally localised data. Summing over the sub-intervals gives

$$\|\Delta_j e^{itH^\sharp} \Delta_k u\|_{L^p([0,1], L^q)} \lesssim N^{\frac{1}{p}} 2^{-j\delta} 2^{-k\delta'} \|\Delta_k u\|_{\mathcal{H}^{\delta+\delta'}} + N^{\frac{1-p}{p}} 2^{-j\sigma} 2^{-k\sigma'} \|\Delta_k u\|_{\mathcal{H}^{1+\sigma+\sigma'+\kappa}}.$$

Let $\eta > 0$ small and take

$$N = 2^j, \quad \delta = \eta + \frac{1}{p}, \quad \delta' = \sigma = \sigma' = \eta$$

which satisfies in particular $N \geq 2^j$ and $\sigma \in [0, \alpha)$. Now it just remains to sum over j and k ; we compute

$$\begin{aligned} \left\| \sum_{j,k \geq 0} \Delta_j e^{itH^\sharp} \Delta_k u \right\|_{L^p([0,1], L^q)} &\lesssim \sum_j \sum_{k \leq j} \|\Delta_j e^{itH^\sharp} \Delta_k u\|_{L^p([0,1], L^q)} \\ &\lesssim \sum_{j \geq 0} \sum_{k \leq j} 2^{\frac{j}{p}} 2^{-j(\frac{1}{p} + \eta)} 2^{-k\eta} \|\Delta_k u\|_{\mathcal{H}^{\frac{1}{p} + 2\eta}} + 2^j \frac{1-p}{p} 2^{-j\eta} 2^{-k\eta} \|\Delta_k u\|_{\mathcal{H}^{1+2\eta+\kappa}} \\ &\lesssim \sum_{j \geq 0} 2^{-j\eta} \|\Delta_{\leq j} u\|_{\mathcal{H}^{\frac{1}{p} + 2\eta}} + 2^j \frac{1-p}{p} 2^{-j\eta} \|\Delta_{\leq j} u\|_{\mathcal{H}^{1+2\eta+\kappa}} \\ &\lesssim \|u\|_{\mathcal{H}^{\frac{1}{p} + 2\eta}} + \sum_{j \geq 0} 2^{-j\eta} 2^j \frac{1-p}{p} 2^{-j\frac{1-p}{p}} \|\Delta_{\leq j} u\|_{\mathcal{H}^{1-1+\frac{1}{p}+2\eta+\kappa}} \\ &\lesssim \|u\|_{\mathcal{H}^{\frac{1}{p} + 2\eta}} + \|u\|_{\mathcal{H}^{\frac{1}{p} + 2\eta + \kappa}}, \end{aligned}$$

having used Bernstein's inequality, Lemma 1.3, for the projector $\Delta_{\leq j}$. This completes the proof since η and κ can be taken arbitrary small. \square

Remark : We proved that Strichartz inequalities are stable under suitable perturbation, that is lower-order perturbation in the sense of previous Proposition 2.1. One can show that the magnetic Laplacian with white noise magnetic field constructed in [23] is also a lower order perturbation of

the Laplacian on the two-dimensional torus in this sense. Thus Theorem 2.3 also gives Strichartz inequalities for the associated Schrödinger group.

As Corollary, we state the inhomogeneous inequalities needed to solve the nonlinear equation. This is straightforward and we omit the proof, see [29] and references therein.

Corollary 2.5. *In the setting of Theorem 2.3, we have in addition the bound*

$$\left\| \int_0^t e^{-i(t-s)H^\sharp} f(s) ds \right\|_{L^p([0,1], L^q)} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\frac{1}{p}+\epsilon}} ds$$

for all $f \in L^1([0,1], \mathcal{H}^{\frac{1}{p}+\kappa})$.

The only ingredient in the proof of the theorem where the boundary appears is when we apply the result for the Laplacian. By using Theorem 1.2 and in particular (2) instead of (1), we immediately get the following analogous result which is of course weaker.

Theorem 2.6. *Let $p \in (3, \infty]$ and $q \in [2, \infty)$ such that*

$$\frac{3}{p} + \frac{2}{q} = 1.$$

Then for any $\kappa > 0$

$$\|e^{-itH^\sharp} u\|_{L^p([0,1], L^q)} \lesssim \|u\|_{\mathcal{H}^{\frac{2}{p}+\kappa}}.$$

and

$$\left\| \int_0^t e^{-i(t-s)H^\sharp} f(s) ds \right\|_{L^p([0,1], L^q)} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\frac{2}{p}+\kappa}} ds$$

for all $f \in L^1([0,1], \mathcal{H}^{\frac{2}{p}+\kappa})$.

2.2 – Local well-posedness for multiplicative stochastic cubic NLS

We now apply our results to the local in time well-posedness of the cubic multiplicative stochastic NLS

$$\begin{cases} i\partial_t u - Hu &= -|u|^2 u \\ u(0) &= u_0 \end{cases}$$

with $u_0 \in \mathcal{H}^\sigma$ where $\sigma \in (\frac{1}{2}, 1)$ and in the energy space, that is $u_0 \in \mathcal{D}(\sqrt{H}) = \Gamma\mathcal{H}^1$. Indeed, as for the Laplacian, the domain of \sqrt{H} can be used to make sense of the equation in a weak sense. Thus we refer to $\mathcal{D}(\sqrt{H}) = \Gamma\mathcal{H}^1$ as the energy space for the Anderson Hamiltonian, see Proposition 1.12. As explained in Section 3.2.2 of [18], their result for the equation with white noise potential is weaker than the one for the deterministic equation since Strichartz inequalities were not known in this singular case. This was a motivation for the work [29] while our result allows to solve the equation in a regime arbitrary closed to the one of the deterministic equation on a surface. This section follows the line of this last paper, we recall the core ideas nonetheless. Finally, we only consider a surface without boundary, the case with boundary is analogous using Theorem 2.6 instead of Theorem 2.3. The mild formulation is

$$u(t) = e^{-itH} u_0 + i \int_0^t e^{-i(t-s)H} (|u|^2 u)(s) ds$$

and applying the Γ^{-1} map introduced in Section 1.3 yields the mild formulation for the transformed quantity $u^\sharp = \Gamma^{-1}u$. We get

$$u^\sharp(t) = e^{-itH^\sharp} u_0^\sharp + i \int_0^t e^{-i(t-s)H^\sharp} \Gamma^{-1} (|\Gamma u^\sharp|^2 \Gamma u^\sharp)(s) ds$$

where $u_0^\sharp := \Gamma^{-1}u_0$, this is where the transformed operator $H^\sharp = \Gamma^{-1}H\Gamma$ appears naturally. Despite the seemingly complicated nonlinear expression, this new mild formulation is easier to deal with since H^\sharp is a perturbation of the Laplacian and has domain \mathcal{H}^2 , hence it is not as outlandish as H

and its domain which contains no non-zero smooth functions. Now, we have to find a bound for the map

$$\Psi(v)(t) := e^{-itH^\sharp} v_0 + i \int_0^t e^{-i(t-s)H^\sharp} \Gamma^{-1}(|\Gamma v|^2 \Gamma v)(s) ds$$

in a suitable space which allows us to get a unique fixed point. One then recovers a solution to the original equation with $u := \Gamma v$ and choosing $v_0 := \Gamma^{-1} u_0$. Since both e^{-itH^\sharp} and Γ are bicontinuous from L^p to itself for $p \in [2, \infty]$ and from \mathcal{H}^σ to itself for $\sigma \in [0, 1)$, it is natural to consider initial datum v_0 , and thus also u_0 , in \mathcal{H}^σ for $0 < \sigma < 1$. Therefore we bound $\Psi(v)$ in \mathcal{H}^σ with

$$\begin{aligned} \|\Psi(v)(t)\|_{\mathcal{H}^\sigma} &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|\Gamma v(s)^3\|_{\mathcal{H}^\sigma} ds \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|\Gamma v(s)\|_{\mathcal{H}^\sigma} \|\Gamma v(s)\|_{L^\infty}^2 ds \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|v(s)\|_{\mathcal{H}^\sigma} \|v(s)\|_{L^\infty}^2 ds \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|v\|_{L^2([0,t], L^\infty)}^2 \end{aligned}$$

where in the first and third lines we have used the continuity of e^{-itH^\sharp} and Γ and Lemma 1.4 in the second line. For $\sigma < 1$, the space \mathcal{H}^σ is not an algebra and one can not simply use its norm to bound the nonlinearity. However, one may bound it using the L^∞ -norm in space by observing that one needs less integrability in time and this is precisely the point where the Strichartz estimates turn out to be useful. As for the deterministic equation, we work with the function spaces

$$\mathcal{W}^{\beta, q}(M) = \{u \in \mathcal{D}'(M); (1 - \Delta)^{\frac{\beta}{2}} u \in L^q\}$$

with associated norm

$$\|u\|_{\mathcal{W}^{\beta, q}} := \|(1 - \Delta)^{\frac{\beta}{2}} u\|_{L^q}.$$

For $\beta \in [0, 1)$ and $q = 2$, one recovers the Sobolev spaces and the norm is equivalent to

$$\|u\|_{\mathcal{H}^\beta} = \|(1 + H)^{\frac{\beta}{2}} u\|_{L^2}$$

by Proposition 1.14. Within this framework, Strichartz inequalities from Theorem 2.3 gives us the bound

$$\|e^{-itH^\sharp} w\|_{L^p([0,1], \mathcal{W}^{\beta, q})} \lesssim \|w\|_{\mathcal{H}^{\frac{1}{p} + \beta + \kappa}},$$

for any Strichartz pair (p, q) and $\kappa > 0$. Furthermore, the space $\mathcal{W}^{\beta, q}$ is continuously embedded in L^∞ for $\beta > \frac{2}{q}$. Let $\sigma \in \mathbb{R}$ such that

$$\frac{1}{p} + \frac{2}{q} + 2\kappa \leq \sigma.$$

Thus for $0 < t \leq 1$, we get the bound

$$\begin{aligned} \|\Psi(v)\|_{L^p([0,t], \mathcal{W}^{\frac{2}{q} + \kappa, q})} &\lesssim \|v_0\|_{\mathcal{H}^{\frac{1}{p} + \frac{2}{q} + 2\kappa}} + \int_0^t \|\Gamma^{-1}(|\Gamma v|^2 \Gamma v)(s)\|_{\mathcal{H}^{\frac{1}{p} + \frac{2}{q} + 2\kappa}} ds \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^t \|\Gamma v(s)^3\|_{\mathcal{H}^\sigma} ds \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|v\|_{L^2([0,t], L^\infty)}^2 \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|v\|_{L^2([0,t], \mathcal{W}^{\frac{2}{q} + \kappa})}^2 \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + t^{\frac{p-2}{p}} \|v\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|v\|_{L^p([0,t], \mathcal{W}^{\frac{2}{q} + \kappa})}^2 \end{aligned}$$

using Corollary 2.5 in the first line, Hölder inequality in the last line and bicontinuity of Γ from \mathcal{H}^σ to itself. For $0 < t' \leq t$, we also have

$$\begin{aligned} \|\Psi(v)(t')\|_{\mathcal{H}^\sigma} &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \int_0^{t'} \|v(s)\|_{\mathcal{H}^\sigma} \|v(s)\|_{L^\infty}^2 ds \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + \|v\|_{L^\infty([0,t'], \mathcal{H}^\sigma)} \|v\|_{L^2([0,t'], L^\infty)}^2, \\ &\lesssim \|v_0\|_{\mathcal{H}^\sigma} + t'^{\frac{p-2}{p}} \|v\|_{L^\infty([0,t'], \mathcal{H}^\sigma)} \|v\|_{L^p([0,t'], \mathcal{W}^{\frac{2}{q} + \kappa})}^2. \end{aligned}$$

This gives us the combined bound

$$\|\Psi(v)\|_{L^p([0,t], \mathcal{W}^{\frac{2}{q}+\kappa, q})} + \|\Psi(v)\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \lesssim \|v_0\|_{\mathcal{H}^\sigma} + t^{\frac{p-2}{p}} \|v\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|v\|_{L^p([0,t], \mathcal{W}^{\frac{2}{q}+\kappa})}^2$$

that will be the main tool for the fixed point. Note that the restrictions

$$\frac{1}{p} + \frac{2}{q} + 2\kappa \leq \sigma \quad \text{and} \quad \frac{2}{p} + \frac{2}{q} = 1$$

gives

$$1 - \frac{1}{p} + 2\kappa \leq \sigma.$$

Since $p \geq 2$ and $\kappa > 0$ can be taken arbitrary small, this gives

$$\sigma > \frac{1}{2}$$

and leads to the following local-in-time well-posedness result. Without Strichartz estimates, even in the classical case, one could not go beyond the threshold $\sigma \geq 1$.

Theorem 2.7. *Let M be a compact surface without boundary, $\sigma > \frac{1}{2}$ and initial data $v_0 \in \mathcal{H}^\sigma$. Let $\kappa > 0$ and (p, q) a Strichartz pair such that*

$$\frac{1}{p} + \frac{2}{q} + 2\kappa \leq \sigma,$$

there exists a time $T > 0$ until which there exists a unique solution

$$v \in C([0, T], \mathcal{H}^\sigma) \cap L^p([0, T], \mathcal{W}^{\frac{2}{q}+\kappa})$$

to the mild formulation of the transformed PDE

$$\begin{cases} i\partial_t v - H^\sharp v &= -\Gamma^{-1}(|\Gamma v|^2 \Gamma v) \\ v(0) &= v_0 \end{cases}.$$

Moreover, the solution depends continuously on the initial data $v_0 \in \mathcal{H}^\sigma$.

Proof: This is a straightforward contraction argument where the main ingredient is the bound proved in the preceding arguments. By choosing the radius R of the ball and the final time appropriately, we can prove that

$$\Psi : B(0, R)_{C([0, T], \mathcal{H}^\sigma) \cap L^p([0, T], \mathcal{W}^{\frac{2}{q}+\kappa})} \rightarrow B(0, R)_{C([0, T], \mathcal{H}^\sigma) \cap L^p([0, T], \mathcal{W}^{\frac{2}{q}+\kappa})}$$

is in fact a contraction, we refer to [29] for the details. □

Finally we give the analogous result for surfaces with boundary which is of course weaker, however we still get a better result than one gets simply from using the algebra property of Sobolev spaces.

Theorem 2.8. *Let M be a compact surface with boundary, $\sigma > \frac{2}{3}$ and p, q, κ s.t.*

$$\frac{3}{p} + \frac{2}{q} = 1 \quad \text{and} \quad \frac{2}{p} + \frac{2}{q} + 2\kappa \leq \sigma.$$

For any initial datum $v_0 \in \mathcal{H}^\sigma$ there exists a unique solution

$$v \in C([0, T], \mathcal{H}^\sigma) \cap L^p([0, T], \mathcal{W}^{\frac{2}{q}+\kappa})$$

to the mild formulation of the transformed PDE up to a time $T > 0$ depending on the data which depends continuously on the initial condition.

3 – Strichartz inequalities for the stochastic wave equation

Again, we consider the “sharpened” operator

$$H^\sharp := \Gamma^{-1} H \Gamma$$

which appears naturally when transforming the wave equation with multiplicative noise. If u solves

$$\begin{cases} \partial_t^2 u + H u & = & 0 \\ (u, \partial_t u)|_{t=0} & = & (u_0, u_1) \end{cases}$$

then $u^\sharp := \Gamma^{-1} u$ solves the transformed equation

$$\begin{cases} \partial_t^2 u^\sharp + H^\sharp u^\sharp & = & 0 \\ (u^\sharp, \partial_t u^\sharp)|_{t=0} & = & (\Gamma^{-1} u_0, \Gamma^{-1} u_1) \end{cases}$$

In this Section, we show Strichartz inequalities for the associated wave equation. In the particular case of the torus, the improved Strichartz estimates for the Schrödinger equations yields better bounds on the eigenfunctions of the Anderson Hamiltonian thus improved Strichartz estimates for the wave equation. Afterwards, we detail how these can be used to get a low-regularity solution theory for the nonlinear wave equation with multiplicative noise.

3.1 – Strichartz inequalities for the wave propagator

The propagator associated to the wave equation is

$$(u_0, u_1) \mapsto \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1$$

with initial conditons $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. As for the Schrödinger equation, the following Strichartz inequalities hold on a two-dimensional compact Riemannian manifold without boundary, see [7] and the references therein. We state the result in the homogeneous case for simplicity however one directly obtains inhomogenous bounds as in Corollary 2.5. We cite the following Strichartz estimates which hold on compact surfaces respectively without and with boundary, see [7].

Theorem 3.1. *Let (M, g) be a compact two-dimensional Riemannian manifold without boundary. Let $p, q \in [2, \infty]$ such that*

$$\frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}$$

and consider

$$\frac{1}{p} + \frac{2}{q} := 1 - \sigma.$$

Then the solution to

$$\begin{cases} (\partial_t^2 - \Delta_g)u & = & 0 \\ (u, \partial_t u)|_{t=0} & = & (u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1} \end{cases}$$

satisfies the bound

$$\|u\|_{L^p([0, T], L^q)} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}}.$$

In the case where the surface M has a boundary, there is this slightly weaker result.

Theorem 3.2. *Let (M, g) be a compact two-dimensional Riemannian manifold with boundary. Let $p \in (2, \infty]$ and $q \in [2, \infty)$ such that*

$$\frac{3}{p} + \frac{1}{q} \leq \frac{1}{2}$$

and consider σ given by

$$\frac{1}{p} + \frac{2}{q} = 1 - \sigma.$$

Then the solution to

$$\begin{cases} (\partial_t^2 - \Delta_g)u & = & 0 \\ (u, \partial_t u)|_{t=0} & = & (u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1} \end{cases}$$

satisfies the bound

$$\|u\|_{L^p([0, T], L^q)} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}}.$$

3.2 – Strichartz inequalities for wave equations with rough potentials

While our proof of the Strichartz inequalities for the Schrödinger equation with white noise potential strongly relies on the deterministic result, this is not the case for the wave equation. In this case, we follow the approach from [11] for which one has two main ingredients, firstly a Weyl law for the Laplace-Beltrami operator and secondly L^q bounds on its eigenfunctions. In particular, we treat at the same time the case with and without boundary here, the only difference being that one has weaker L^q bounds on the eigenfunctions.

An analogous Weyl law for the Anderson Hamiltonian was obtained in [24], see Proposition 1.15 in Section 1.3, and the analogue of the second part follows from the Strichartz inequalities for the Schrödinger group obtained in Section 2. Let $(e_n)_{n \geq 1}$ be an orthonormal family of eigenfunctions of H associated to $(\lambda_n(\Xi))_{n \geq 1}$. Since the eigenfunctions belong to the domain \mathcal{D}_Ξ , they belong in particular to L^∞ and we have the following bounds on its L^q -norm for $q \in (2, \infty)$.

Proposition 3.3. *Let $q \in (2, \infty)$ and M a compact surface without boundary. We have*

$$\|e_n\|_{L^q} \lesssim \sqrt{\lambda_n(\Xi)}^{\frac{1}{2} - \frac{1}{q} + \kappa}$$

for any $\kappa > 0$. In particular, this implies

$$\|e_n\|_{L^q} \lesssim (1 + \sqrt{\lambda_n(0)})^{\frac{1}{2} - \frac{1}{q} + \kappa} \lesssim (1 + \sqrt{n})^{\frac{1}{2} - \frac{1}{q} + \kappa}.$$

Proof: We have

$$\|e_n\|_{L^q} = \|e^{it\lambda_n} e_n\|_{L^p([0,1], L^q)} = \|e^{itH} e_n\|_{L^p([0,1], L^q)}$$

with (p, q) a Strichartz pair. For any $\kappa > 0$, this gives

$$\begin{aligned} \|e_n\|_{L^q} &\lesssim \|e_n\|_{\mathcal{H}^{\frac{1}{p} + \kappa}} \\ &\lesssim \|\sqrt{H}^{\frac{1}{p} + \kappa} e_n\|_{L^2} \\ &\lesssim \sqrt{\lambda_n(\Xi)}^{\frac{1}{p} + \kappa} \end{aligned}$$

using Proposition 1.14 and

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{q}.$$

Finally, Proposition 1.15 gives the bound

$$\lambda_n(\Xi) \lesssim 1 + \lambda_n(0)$$

and completes the proof. □

Another important operator is the projection onto the eigenspaces of H . Let

$$\Pi_\lambda u := \sum_{\lambda_n(\Xi) \in [\lambda, \lambda+1]} \langle u, e_n \rangle e_n$$

for any $\lambda \geq 0$. The Weyl law for H together with the bounds on e_n gives the following bound on Π_λ .

Proposition 3.4. *Let $\lambda \geq 0$ and $q \in (2, \infty)$. We have*

$$\|\Pi_\lambda u\|_{L^q} \lesssim \sqrt{\lambda + 1}^{\frac{1}{2} - \frac{1}{q} + \kappa} \|u\|_{L^2}$$

for any $\kappa > 0$.

Proof: The result follows from the direct computation

$$\begin{aligned} \|\Pi_\lambda u\|_{L^q} &= \sum_{\lambda_n(\Xi) \in [\lambda, \lambda+1]} \langle u, e_n \rangle \|e_n\|_{L^q} \\ &\lesssim \sum_{\lambda_n(\Xi) \in [\lambda, \lambda+1]} \|u\|_{L^2} \|e_n\|_{L^2} \|e_n\|_{L^q} \\ &\lesssim \|u\|_{L^2} |\{\lambda_n(\Xi) \in [\lambda, \lambda+1]\}| \sqrt{\lambda + 1}^{\frac{1}{2} - \frac{1}{q} + \kappa} \\ &\lesssim \|u\|_{L^2} \sqrt{\lambda + 1}^{\frac{1}{2} - \frac{1}{q} + \kappa} \end{aligned}$$

for any $\kappa > 0$ since the almost sure Weyl-type law implies

$$\sup_{\lambda > 0} |\{\lambda_n(\Xi) \in [\lambda, \lambda + 1)\}| < \infty.$$

□

As mentioned before, this is the point where there are slightly weaker results in the case of a surface with boundary. We use Theorem 2.6 instead.

Proposition 3.5. *Let $q \in (2, \infty)$ and M a compact surface with boundary. We have*

$$\|e_n\|_{L^q} \lesssim \sqrt{\lambda_n(\Xi)}^{\frac{2}{3} - \frac{4}{3q} + \kappa}$$

for any $\kappa > 0$. In particular, this implies

$$\|e_n\|_{L^q} \lesssim (1 + \sqrt{\lambda_n(0)})^{\frac{2}{3} - \frac{4}{3q} + \kappa} \lesssim (1 + \sqrt{n})^{\frac{2}{3} - \frac{4}{3q} + \kappa}.$$

Moreover, for the operator Π_λ we have

$$\|\Pi_\lambda u\|_{L^q} \lesssim \sqrt{\lambda + 1}^{\frac{2}{3} - \frac{4}{3q} + \kappa} \|u\|_{L^2}$$

for any $\kappa > 0$.

Let B be the operator defined by

$$Be_n := \lfloor \sqrt{\lambda_n(\Xi)} \rfloor e_n$$

for any $n \geq 1$. The following proposition gives continuity estimates for the unitary groups associated to \sqrt{H} and B and bound the difference between the two operators.

Proposition 3.6. *For any $\beta \in [0, 1)$ and $t \in \mathbb{R}$, we have*

$$\|e^{it\sqrt{H}}u\|_{\mathcal{H}^\beta} \lesssim \|u\|_{\mathcal{H}^\beta}$$

and

$$\|e^{itB}u\|_{\mathcal{H}^\beta} \lesssim \|u\|_{\mathcal{H}^\beta}.$$

Moreover, the difference $B - \sqrt{H}$ is bounded on \mathcal{H}^β for any $\beta \in [0, 1)$ and the difference between the groups is given by

$$e^{itB}u - e^{it\sqrt{H}}u = -i \int_0^t e^{i(t-s)B}(\sqrt{H} - B)e^{is\sqrt{H}}u \, ds.$$

Proof: We have

$$\|e^{it\sqrt{H}}v\|_{L^2} \lesssim \|v\|_{L^2}.$$

thus

$$\|H^{\frac{\beta}{2}}e^{it\sqrt{H}}v\|_{L^2} = \|e^{it\sqrt{H}}H^{\frac{\beta}{2}}v\|_{L^2} \lesssim \|H^{\frac{\beta}{2}}v\|_{L^2}$$

for any $\beta \in (0, \alpha)$. Using Proposition 1.14, this gives

$$\|e^{it\sqrt{H}}v\|_{\mathcal{H}^\beta} \lesssim \|v\|_{\mathcal{H}^\beta}$$

and the result for e^{itB} follows from this. For the difference, $\|B - \sqrt{H}\|_{L^2 \rightarrow L^2}$ is bounded by 1 and we have

$$\begin{aligned} \|H^{\frac{\beta}{2}}(B - \sqrt{H})u\|_{L^2} &= \|(B - \sqrt{H})H^{\frac{\beta}{2}}u\|_{L^2} \\ &\leq \|H^{\frac{\beta}{2}}u\|_{L^2} \end{aligned}$$

hence the boundedness of $B - \sqrt{H}$ on \mathcal{H}^β . The result on the difference of the groups

$$e^{itB}u - e^{it\sqrt{H}}u = -i \int_0^t e^{i(t-s)B}(\sqrt{H} - B)e^{is\sqrt{H}}u \, ds$$

follows with the same reasoning as in Lemma 2.4.

□

We now have all the ingredients to prove of the Strichartz inequalities for the wave propagator associated to the Anderson Hamiltonian.

Theorem 3.7. *Let M be a compact surface without boundary $(p, q) \in [2, \infty)^2$ and $0 < \sigma < \alpha$ such that $p \leq q$ and*

$$\sigma = \frac{3}{2} - \frac{2}{p} + \frac{1}{q}.$$

Then for any $\kappa > 0$, we have the bound

$$\left\| \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 \right\|_{L^p([0,1], L^q)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{\sigma+\kappa} \times \mathcal{H}^{\sigma-1+\kappa}}.$$

Proof: We start by proving the bound for e^{itB} using the spectral decomposition

$$e^{itB}u = \sum_{n \geq 1} e^{itn} \Pi_n u$$

and then bound the difference of the two groups. First, the condition $p \leq q$ implies

$$\|e^{itB}u\|_{L^p([0,1], L^q(M))} \leq \|e^{itB}u\|_{L^q(M, L^p([0,1]))}$$

hence it is enough to bound the right hand side. Using the Sobolev embedding in the time variable and the L^q bound on the eigenvalues from Proposition 3.4, we have

$$\begin{aligned} \|e^{itB}u\|_{L^q(M, L^p([0,1]))}^2 &= \left\| \|e^{itB}u\|_{L^p([0,1])}^2 \right\|_{L^{\frac{q}{2}}(M)} \\ &\lesssim \left\| \|e^{itB}u\|_{\mathcal{H}^{\frac{1}{2}-\frac{1}{p}}([0,1])}^2 \right\|_{L^{\frac{q}{2}}(M)} \\ &\lesssim \sum_{n \geq 0} \left\| \|e^{itn} \Pi_n u\|_{\mathcal{H}^{\frac{1}{2}-\frac{1}{p}}([0,1])}^2 \right\|_{L^{\frac{q}{2}}(M)} \\ &\lesssim \sum_{n \geq 0} \|e^{itn}\|_{\mathcal{H}^{\frac{1}{2}-\frac{1}{p}}([0,1])}^2 \|\Pi_n u\|_{L^q(M)}^2 \\ &\lesssim \sum_{n \geq 0} (n+1)^{1-\frac{2}{p}} (\sqrt{n+1})^{1-\frac{2}{q}+2\kappa} \|\Pi_n u\|_{L^2}^2 \\ &\lesssim \|\sqrt{H}^{\frac{3}{2}-\frac{2}{p}-\frac{1}{q}+\kappa} u\|_{L^2}^2 \\ &\lesssim \|u\|_{\mathcal{H}^{\frac{3}{2}-\frac{2}{p}-\frac{1}{q}+\kappa}}^2 \end{aligned}$$

which gives the result for B . To obtain the proof for \sqrt{H} , we use

$$e^{itB}u - e^{it\sqrt{H}}u = -i \int_0^t e^{i(t-s)B}(\sqrt{H} - B)e^{is\sqrt{H}}u \, ds.$$

Indeed, this gives

$$\begin{aligned} \|e^{it\sqrt{H}}u\|_{L^p([0,1], L^q)} &\lesssim \|e^{itB}u\|_{L^p([0,1], L^q)} + \int_0^1 \|e^{i(t-s)B}(\sqrt{H} - B)e^{is\sqrt{H}}u\|_{L^p([0,1], L^q)} \, ds \\ &\lesssim \|u\|_{\mathcal{H}^{\sigma+\kappa}} + \int_0^1 \|(\sqrt{H} - B)e^{i(t-s)B}u\|_{\mathcal{H}^{\sigma+\kappa}} \, ds \\ &\lesssim \|u\|_{\mathcal{H}^{\sigma+\kappa}} \end{aligned}$$

for any $\kappa > 0$. The proof is directly completed from

$$\cos(t\sqrt{H}) = \frac{e^{it\sqrt{H}} + e^{-it\sqrt{H}}}{2}$$

and

$$\frac{\sin(\sqrt{H})}{\sqrt{H}} = \frac{e^{it\sqrt{H}} - e^{-it\sqrt{H}}}{2i\sqrt{H}}.$$

□

Again, the inhomogeneous inequalities follow directly and we omit the proof.

Corollary 3.8. *Let p, q, σ be as in Theorem 3.7. Then we have the following bound*

$$\left\| \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} f(s) \right\|_{L^p([0,1], L^q)} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\sigma-1+\kappa}} ds$$

for $f \in L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})$.

Moreover, we have the analogous result for surfaces with boundary which is proved analogously by using Proposition 3.5 instead of Proposition 3.4.

Theorem 3.9. *Let M be a compact surface with boundary and $p, q \in [2, \infty)$ s.t. $p \leq q$ and*

$$\sigma = \frac{5}{3} - \frac{2}{p} - \frac{4}{3q} \in (0, \alpha).$$

Then for any $\kappa > 0$, we have the bound

$$\left\| \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}v \right\|_{L^p([0,1], L^q)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{\sigma+\kappa} \times \mathcal{H}^{\sigma-1+\kappa}} + \|v\|_{L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})}$$

for initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ and inhomogeneity $v \in L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})$.

3.3 – The special case of the two-dimensional torus

We discuss here how the procedure from the previous Section can be applied to get improved results on the torus. The only difference is that instead of Theorem 2.3, we have

$$\|e^{-itH}v\|_{L^4([0,1], L^4(\mathbb{T}^2))} \lesssim \|v\|_{\mathcal{H}^\kappa}$$

for any $\kappa > 0$, which was the central result of [29]. This implies improved L^q bounds on the eigenfunctions, we omit the details since the proof is identical to before.

Proposition 3.10. *Consider the Anderson Hamiltonian H on the two-dimensional torus and let $q \in [4, \infty)$. Then we have the bound*

$$\|e_n\|_{L^q(\mathbb{T}^2)} \lesssim \sqrt{\lambda_n(\Xi)}^{1-\frac{4}{q}+\kappa} \lesssim (1+\sqrt{n})^{1-\frac{4}{q}+\kappa}.$$

Moreover, we get the bound for the spectral projector

$$\|\Pi_\lambda u\|_{L^q(\mathbb{T}^2)} \lesssim \sqrt{\lambda+1}^{1-\frac{4}{q}+\kappa} \|u\|_{L^2}.$$

Since the Weyl law holds in the same way in the particular case of the torus, we get the following improved result by the identical proof as before using Proposition 3.10.

Theorem 3.11. *Let $2 \leq p \leq q < \infty$ such that $q \geq 4$ and let*

$$\sigma = \frac{3}{2} - \frac{1}{p} - \frac{4}{q}.$$

Then for any $\kappa > 0$, we have the bound

$$\left\| \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}v \right\|_{L^p([0,1], L^q)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^{\sigma+\kappa} \times \mathcal{H}^{\sigma-1+\kappa}} + \|v\|_{L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})}.$$

for initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ and inhomogeneity $v \in L^1([0,1], \mathcal{H}^{\sigma-1+\kappa})$.

Remark: *Due to the nature of the bound from [29], there is no gain by allowing q to be smaller than 4. Indeed, the loss of derivative in that case is arbitrarily small and one does not gain anything by interpolating with the trivial L^2 bound.*

3.4 – Local well-posedness for the multiplicative cubic stochastic wave equation

Now we use the results from the previous Section to prove local well-posedness of stochastic multiplicative wave equations of the form

$$\begin{cases} \partial_t^2 u + Hu &= -u|u|^2 \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1) \end{cases}$$

in a low-regularity regime both on general two-dimensional surfaces with boundary and in the special case of the two-dimensional torus. While we have the classical Sobolev embedding

$$\mathcal{H}^\nu \hookrightarrow L^{\frac{2}{1-\nu}}$$

for $\nu \in [0, 1)$, we also make use of the following dual Sobolev bound

$$\forall \sigma \in (0, 1], \quad L^{\frac{2}{2-\sigma}} \hookrightarrow \mathcal{H}^{\sigma-1}$$

which is true on general manifolds, see for example the book by Aubin [2]. Using this, we make a preliminary computation meant to show how far we get by using *only the Sobolev embedding result*. Then we will see how the bounds in Theorems 3.7 and 3.11 give better results on general manifolds and on the torus respectively. We first rewrite the equation under the mild formulation

$$u(t) = \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}u(s)^3 ds.$$

Then apply the dual Sobolev bound for $\sigma \in (0, 1]$ and $p = \frac{2}{2-\sigma} \in (1, 2]$ to get

$$\begin{aligned} \|u(t)\|_{\mathcal{H}^\sigma} &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u^3\|_{L^1([0,t], \mathcal{H}^{\sigma-1})} \\ &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u^3\|_{L^1([0,t], L^p)} \\ &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u\|_{L^\infty([0,t], L^{\frac{2}{1-\sigma}})} \|u\|_{L^2([0,t], L^4)}^2, \end{aligned}$$

having applied Hölder with $\frac{1}{2} + \frac{1-\sigma}{2} = \frac{2-\sigma}{2}$. Finally, the Sobolev embedding gives

$$\|u(t)\|_{\mathcal{H}^\sigma} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|u\|_{L^2([0,t], \mathcal{H}^{\frac{1}{2}})}^2.$$

This can then lead to a solution by fixed point by choosing $\sigma \geq \frac{1}{2}$. Clearly this can be improved by using more subtle bounds than the Sobolev embedding. The Strichartz inequalities from the previous section allow us to get local well-posedness below, this is the content of the following theorems; As before we separately state the cases of surfaces without boundary, with boundary, and the special case of the torus which are proved in precisely the same way, just using Theorems 3.7, 3.9 and 3.11 respectively.

Theorem 3.12. *Let M be a compact surface without boundary and $\sigma \in (\frac{1}{4}, \frac{1}{2})$ and $\delta > 0$ sufficiently small. Then for any initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ there exists a time $T > 0$ depending on the data such that there exists a unique solution*

$$u \in C([0, T], \mathcal{H}^\sigma) \cap L^{\frac{2}{1-\delta}}([0, T], L^4)$$

to the mild formulation of the multiplicative cubic stochastic wave equation. Moreover, the solution depends continuously on the initial data (u_0, u_1) .

Proof: As usual, this is proved in a standard way using the Banach fixed point theorem. Define the map

$$\Psi(u)(t) := \cos(t\sqrt{H})u_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}u(s)^3 ds.$$

For $t > 0$, we have as above

$$\begin{aligned} \|u(t)\|_{\mathcal{H}^\sigma} &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|u\|_{L^2([0,t], L^4)}^2 \\ &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + t^\delta \|u\|_{L^\infty([0,t], \mathcal{H}^\sigma)} \|u\|_{L^{\frac{2}{1-\delta}}([0,t], L^4)}^2 \end{aligned}$$

using Hölder inequality in the last line for $\delta \in (0, 1)$. We then apply Theorem 3.7 with $p = \frac{2}{1-\delta}$ and $q = 4$ and obtain

$$\begin{aligned} \|\Psi(u)\|_{L^{\frac{2}{1-\delta}}([0,T],L^4)} &\lesssim \|u_0\|_{\mathcal{H}^{\frac{3}{2}-(1-\delta)-\frac{1}{4}+\kappa}} + \|u_1\|_{\mathcal{H}^{\frac{1}{2}-(1-\delta)-\frac{1}{4}+\kappa}} + \|u^3\|_{L^1([0,T],\mathcal{H}^{\frac{1}{2}-(1-\delta)-\frac{1}{4}+\kappa})} \\ &\lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + \|u^3\|_{L^1([0,T],\mathcal{H}^{\sigma-1})} \end{aligned}$$

using that $\sigma > \frac{1}{4}$ and $\delta < \sigma - \frac{1}{4}$ gives $\frac{3}{2} - (1-\delta) - \frac{1}{4} + \kappa \leq \sigma$ for $\kappa > 0$ small enough. Finally, we get

$$\|\Psi(u)\|_{L^{\frac{2}{1-\delta}}([0,T],L^4)} \lesssim \|u_0\|_{\mathcal{H}^\sigma} + \|u_1\|_{\mathcal{H}^{\sigma-1}} + T^\delta \|u\|_{L^\infty([0,T],\mathcal{H}^\sigma)} \|u\|_{L^{\frac{2}{1-\delta}}([0,T],L^4)}^2$$

as above. Thus we can get a fixed point in

$$C([0, T], \mathcal{H}^\sigma) \cap L^{\frac{2}{1-\delta}}([0, T], L^4)$$

in the usual way for $T > 0$ small enough. □

In a completely analogous way we get the following result for the case of surfaces with boundary using the Strichartz estimates from Theorem 3.9.

Theorem 3.13. *Let M be a compact surface without boundary and $\sigma \in (\frac{1}{3}, \frac{1}{2})$ and $\delta > 0$ sufficiently small. Then for any initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ there exists a time $T > 0$ depending on the data such that there exists a unique solution*

$$u \in C([0, T], \mathcal{H}^\sigma) \cap L^{\frac{2}{1-\delta}}([0, T], L^4)$$

to the mild formulation of the multiplicative cubic stochastic wave equation. Moreover, the solution depends continuously on the initial data (u_0, u_1) .

In the special case of the two-dimensional torus, the range of regularities in which we obtain well-posedness is greater using the improved bound from Theorem 3.11 as one might expect. This is the following Theorem, we again omit the proof since it is similar to the previous one. Note that this result gives the maximal range of regularities for which the regularity measured in terms of H agrees with the usual regularity, see Proposition 1.14.

Theorem 3.14. *Let $M = \mathbb{T}^2$, $\sigma \in (0, \frac{1}{2})$ and $\delta > 0$ sufficiently small. Then for any initial data $(u_0, u_1) \in \mathcal{H}^\sigma \times \mathcal{H}^{\sigma-1}$ there exists a time $T > 0$ such that there exists a unique solution*

$$u \in C([0, T], \mathcal{H}^\sigma(\mathbb{T}^2)) \cap L^{\frac{2}{1-\delta}}([0, T], L^4(\mathbb{T}^2))$$

to the mild formulation of the multiplicative cubic stochastic wave equation. Moreover, the solution depends continuously on the initial data (u_0, u_1) .

References

- [1] R. Allez and K. Chouk, *The continuous Anderson Hamiltonian in dimension two*. arXiv:1511.02718, (2015).
- [2] T. Aubin, *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics, Berlin: Springer, (1998).
- [3] I. Babuška, *Error-bounds for finite element method*. Numerische Mathematik, **16**(4):322–333, (1971).
- [4] I. Bailleul and F. Bernicot, *Heat semigroup and singular PDEs*. J. Funct. Anal., **270**:3344–3452, (2016).
- [5] I. Bailleul and F. Bernicot and D. Frey, *Spacetime paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations*. Ann. Scient. Éc. Norm. Sup., **51**:1399–1457, (2018).

- [6] I. Bailleul and F. Bernicot, *High order paracontrolled calculus*. Forum Math. Sigma, **7**(e44):1–94, (2019).
- [7] M. D. Blair, H. F. Smith, C. D. Sogge, *Strichartz estimates for the wave equation on manifolds with boundary* Annales de l’Institut Henri Poincaré C, Analyse non linéaire, Volume 26, Issue 5, 2009, Pages 1817-1829.
- [8] M. D. Blair, H. F. Smith, C. D. Sogge. *Strichartz estimates and the nonlinear Schrödinger equation on manifolds with boundary* Mathematische Annalen 354.4 (2012): 1397-1430.
- [9] J.M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*. Ann. Scient. Éc. Norm. Sup., **14**(4):209–246, (1981).
- [10] N. Burq, P. Gérard and N. Tzevtkov, *Strichartz Inequalities and the Nonlinear Schrödinger Equation on Compact Manifolds*. American Journal of Mathematics, **126**(3):569–605, (2004).
- [11] N. Burq, G. Lebeau, and F. Planchon. *Global existence for energy critical waves in 3-D domains* Journal of the American Mathematical Society 21.3 (2008): 831-845.
- [12] P. D’Ancona, L. Fanelli, L. Vega and N. Visciglia, *Endpoint Strichartz estimates for the magnetic Schrödinger equation*. Journal of Functional Analysis, **258**(10): 3227-3240, (2010).
- [13] A. Debussche and H. Weber, *The Schrödinger equation with spatial white noise potential*. Electron. J. Probab., **23**(28):16 pp, (2018).
- [14] M. Gubinelli *Controlling rough paths* Journal of Functional Analysis 216.1 (2004): 86-140.
- [15] M. Gubinelli, H. Koch and T. Oh. *Renormalization of the two-dimensional stochastic nonlinear wave equations*. Transactions of the American Mathematical Society 370.10 (2018): 7335-7359.
- [16] M. Gubinelli, H. Koch and T. Oh. *Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity*. arXiv preprint arXiv:1811.07808 (2018).
- [17] M. Gubinelli, P. Imkeller and N. Perkowski, *Paracontrolled distributions and singular PDEs*. Forum Math. Pi, **3**(e6):1–75, (2015).
- [18] M. Gubinelli and B. Ugurcan and I. Zachhuber, *Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions*. Stoch. PDE: Anal. Comp., **8**:82–149 (2020).
- [19] M. Hairer, *A theory of regularity structures*. Invent. Math., **198**(2):269–504, (2014).
- [20] X. Huang and C. D. Sogge, *Quasimode and Strichartz estimates for time-dependent Schrödinger equations with singular potentials* arXiv preprint arXiv:2011.04007 (2020).
- [21] C. Labbé, *The continuous Anderson Hamiltonian in $d \leq 3$* . J. Funct. Anal., **277**(9):3187-3235, (2019).
- [22] Terry J. Lyons *Differential equations driven by rough signals*. Revista Matemática Iberoamericana 14.2 (1998): 215-310.
- [23] L. Morin and A. Mouzard, *2D random magnetic Laplacian with white noise magnetic field*. arXiv:2101.05020, (2021).
- [24] A. Mouzard, *Weyl law for the Anderson Hamiltonian on a two-dimensional manifold*. arXiv:2009.03549, (2020).
- [25] T. Oh, T. Robert and N. Tzvetkov *Stochastic nonlinear wave dynamics on compact surfaces* arXiv:1904.05277 (2019)
- [26] T. Oh, T. Robert, N. Tzvetkov and Y. Wang, *Stochastic quantization of Liouville conformal field theory*. arXiv:2004.04194, (2020).
- [27] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier, Analysis, self-adjointness*. Academic Press Harcourt Brace Jovanovich, Publishers, New York-London, (1975).
- [28] N. Tzvetkov and N. Visciglia, *Two dimensional nonlinear Schrödinger equation with spatial white noise potential and fourth order nonlinearity*. arXiv:2006.07957, (2020).
- [29] I. Zachhuber, *Strichartz estimates and low-regularity solutions to multiplicative stochastic NLS*. arXiv:1911.01982, (2019).

- A. Mouzard – Univ. Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
E-mail: antoine.mouzard@ens-rennes.fr
- I. Zachhuber – , FU Berlin, D-14195 Berlin, Germany
E-mail: immanuel.zachhuber@fu-berlin.de