Deterministic mean field games with control on the acceleration and state constraints

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Abstract

We consider deterministic mean field games in which the agents control their acceleration and are constrained to remain in a domain of $\mathbb{R}^n$. We study relaxed equilibria in the Lagrangian setting; they are described by a probability measure on trajectories. The main results of the paper concern the existence of relaxed equilibria under suitable assumptions. The fact that the optimal trajectories of the related optimal control problem solved by the agents do not form a compact set brings a difficulty in the proof of existence. The proof also requires closed graph properties of the map which associates to initial conditions the set of optimal trajectories.

1 Introduction

The theory of mean field games (MFGs for short) is more and more investigated since the pioneering works [19, 20, 21] of Lasry and Lions: it aims at studying the asymptotic behaviour of differential games (Nash equilibria) as the number of agents tends to infinity. The dynamics of the agents can be either stochastic or deterministic. Concerning the latter case, we refer to [14] for a detailed study of deterministic MFGs in which the interactions between the agents are modeled by a nonlocal regularizing operator acting on the distribution of the states of the agents. They are described by a system of PDEs coupling a continuity equation for the density of the distribution of states (forward in time) and a Hamilton-Jacobi (HJ) equation for the optimal value of a representative agent (backward in time). If the interaction cost depends locally on the density of the distribution (hence is not regularizing), then, in the deterministic case, the available theory mostly deals with so-called variational MFGs, see [15].

The major part of the literature on deterministic mean field games addresses situations when the dynamics of a given agent is strongly controllable: for example, in crowd motion models, this happens if the control of a given agent is its velocity. Under the strong controllability assumption, it is possible to study realistic models in which the agents are constrained to remain in a given region K of the space state, i.e. state constrained deterministic MFGs. An important difficulty in state constrained deterministic MFGs is

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that nothing prevents the agents from concentrating on the boundary ∂K of the state space; let us call $m(t)$ the distribution of states at time t. Even if $m(0)$ is absolutely continuous, there may exist some $t > 0$, such that $m(t)$ has a singular part supported on ∂K and the absolute continuous part of $m(t)$ with respect to Lebesgue measure blows up near ∂K. This was first observed in some applications of MFGs to macroeconomics, see [1, 2]. From the theoretical viewpoint, the main issue is that, as we have already said, the distribution of states is generally not absolutely continuous with respect to Lebesgue measure; this makes it difficult to characterize the state distribution by means of partial differential equations. These theoretical difficulties have been addressed in [10]: following ideas contained in [7, 8, 16], the authors of [10] introduce a weak or relaxed notion of equilibrium, which is defined in a Lagrangian setting rather than with PDEs. Because there may be several optimal trajectories starting from a given point in the state space, the solutions of the relaxed MFG are probability measures defined on a set of admissible trajectories. Once the existence of a relaxed equilibrium is ensured, it is then possible to investigate the regularity of solutions and give a meaning to the system of PDEs and the related boundary conditions: this was done in [11].

On the other hand, if the agents control their acceleration rather than their velocity, the strong controllability property is lost. In [3], we have studied deterministic mean field games in the whole space $\mathbb{R}^n$ with finite time horizon in which the dynamics of a generic agent is controlled by the acceleration, see also [12]. The state variable is the pair $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ where x and v respectively stand for the position and the velocity. The system of PDEs describing the MFG is then

(1.1)
$$\begin{cases} (i) & -\partial_t u - v \cdot D_x u + H(x, v, D_v u) - F[m(t)](x, v) = 0 & \text{in } \mathbb{R}^{2n} \times (0, T) \\ (ii) & \partial_t m + v \cdot D_x m - \operatorname{div}_v(D_{p_v} H(x, v, D_v u) m) = 0 & \text{in } \mathbb{R}^{2n} \times (0, T) \\ (iii) & m(x, v, 0) = m_0(x, v), u(x, v, T) = G[m(T)](x, v)\,, & \text{on } \mathbb{R}^{2n} \end{cases}$$

where $u = u(x, v, t)$, $m = m(x, v, t)$, $(x, v) \in \mathbb{R}^{2n}$, $t \in (0, T)$ and the Hamiltonian $(x, v, p_x, p_v) \mapsto -v \cdot p_x + H(x, v, p_v)$ is neither strictly convex nor coercive with respect to $p = (p_x, p_v)$. Hence the available results on the regularity of the value function u of the associated optimal control problem ([13, 14]) and on the existence of a solution of the MFG system ([14]) cannot be applied. In [3], the existence of a weak solution of the MFG system is proved via a vanishing viscosity method; the distribution of states is characterized as the image of the initial distribution by the flow associated with the optimal control.

In traffic theory and also in economics, the models may require that the position of the agents belong to a given compact subset $\overline{\Omega}$ of $\mathbb{R}^n$, and state constrained mean field games with control on the acceleration must be considered. In the present paper, we wish to investigate some examples of such mean field games and address the first step of the program followed by the authors of [10] in the strongly controllable case: we wish to prove the existence of a relaxed mean field equilibrium in the Lagrangian setting under suitable assumptions.

1.1 Our program

Most of the paper is devoted to the case when the running cost depends separately on the acceleration and on the other variables, and is quadratic with respect to the acceleration. We will assume that the acceleration can be chosen in the whole space. Thus, admissible trajectories are pairs of functions (ξ, η), $\xi \in C^1([0, T]; \overline{\Omega})$, $\eta \in W^{1,2}([0, T]; \mathbb{R}^n)$ and $\xi' = \eta$.

An example of state constrained mean field games in which the acceleration takes its values in a compact of $\mathbb{R}^n$, (the optimal value may therefore take the value $+\infty$ in the interior of the x-domain), will be studied in a forthcoming work.

In view of the applications to traffic models, we will deal with the cases when

1. Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary
2. $n = 1$ and Ω is a bounded straight line segment
3. Ω is a bounded polygonal domain of $\mathbb{R}^2$ (that we suppose convex for simplicity).

In the one-dimensional case, the simplicity of the geometry will allow us to obtain accurate information on the optimal trajectories, and in turn to get a more general existence result for the mean field game, yet under an additional assumption on the running cost. On the contrary, when dealing with the polygonal domain, we will make special assumptions in order to obtain an existence result.

Recall that the admissible states are the pairs $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$, where Ω is a bounded domain of $\mathbb{R}^n$. At first glance, we see that some restrictions will have to be imposed on the initial distribution of states: indeed, for $x \in \partial\Omega$ and v pointing outward Ω at x, there is no admissible trajectory taking the value (x, v) at $t = 0$; hence the optimal value $u(x, v, 0)$ takes the value $+\infty$; the definition of the mean field equilibrium would then be unclear if the probability that the initial state takes such values (x, v) was not zero.

As in [10], the aim is to prove the existence of relaxed MFG equilibria which are described by probability measures defined on a set of admissible trajectories. The proof involves Kakutani's fixed point theorem, see [17], applied to a multivalued map defined on a suitable convex and compact set of probability measures on a suitable set of admissible trajectories (itself endowed with the $C^1([0, T]; \mathbb{R}^n) \times C^0([0, T]; \mathbb{R}^n)$-topology). Difficulties in applying Kakutani's fixed point theorem will arise from the fact that all the optimal trajectories do not form a compact subset of $C^1([0, T]; \mathbb{R}^n) \times C^0([0, T]; \mathbb{R}^n)$ (due to the lack of strong controllability). This explains why we shall need additional assumptions, either on the support of the initial distribution of states, or, in some cases, on the running cost.

Assumptions on the support of the initial distribution of states Note that if a set of trajectories is a compact metric space, then probability measures on this set form a compact set, as required by Kakutani's theorem. Therefore, a natural strategy is to identify a compact set of trajectories which contains the optimal trajectories whose initial value belongs to the support of the initial distribution of states. In such a strategy, we therefore need to identify a modulus of continuity common to all the velocity laws of the optimal trajectories; since the running cost is quadratic in the acceleration, the more natural idea is to look for a uniform bound on the $W^{1,2}$ norms of the velocity laws of the optimal trajectories. But, due the lack of strong controllability, if x and v respectively belong to $\partial\Omega$ and to the boundary of the tangent cone to $\overline{\Omega}$ at x (the optimal value $u(x, v, 0)$ is finite), there exist sequences $(x_i, v_i)_{i\in\mathbb{N}}$ tending to (x, v) such that the optimal value $u(x_i, v_i, 0)$ blows up when $i \to \infty$; in other words, the cost of preventing the trajectories with initial value (x_i, v_i) from exiting the domain tends to $+\infty$ as $i \to \infty$. Hence, to get uniform bounds on the $W^{1,2}$ norms of the velocity law, the support of the initial distribution of states must not contain such sequences (x_i, v_i). Sufficient conditions on the support of the initial distribution will be given.

Furthermore, Kakutani's fixed point theorem requires a closed graph property for the multivalued map which maps a given point (x, v) to the set of optimal trajectories starting from (x, v). An important part of our work is therefore devoted to proving a closed graph property for the latter map. Note that this issue has its own interest in optimal control theory, independently from mean field games.

Assumptions on the running cost We will see that if $n = 1$, we will able to get rid of the above-mentioned restrictions on the support of the initial distribution of states, if an additional assumption is made on the running cost, namely that it does not favor the trajectories that exit the domain. The existence of equilibria is then proved by approximating the initial distribution m_0 by a sequence $m_{0,k}$ for which Kakutani's theorem can be applied, and by passing to the limit. To pass to the limit, accurate information on the optimal trajectories are needed. We managed to obtain them for $n = 1$ only.

1.2 Organization of the paper

The paper is organized as follows: Section 2 is devoted to state constrained optimal control problems in a bounded domain of $\mathbb{R}^n$ with a smooth boundary, and in particular to the closed graph properties of the above mentioned multivalued map. Although this issue seems to be important in several applications, we were not able to find any relevant result in the available literature. Then, Section 3 deals with an existence result for a related mean field equilibrium in the Lagrangian setting, under sufficient conditions on the support of the initial distribution of states. A variant with a non quadratic cost will be investigated as well. In Section 4, we address the case when the dynamics take place in a bounded straight line segment, $(n = 1)$: under a natural additional assumption on the running cost, we are able to prove the existence of mean field equilibria without any restriction on the initial distribution of states; the proof requires a quite careful study of the optimal trajectories. Finally, in Section 5, we discuss the case of bounded and convex polygonal domain of $\mathbb{R}^2$ and put the stress on the closed graph result which requires a special care near the corners.

2 State constrained optimal control problems in a domain of $\mathbb{R}^n$

2.1 Setting and notation

Let Ω be a bounded domain of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class C^2. For $x \in \partial\Omega$, let $n(x)$ be the unitary vector normal to $\partial\Omega$ pointing outward Ω. We will use the signed distance to $\partial\Omega$, $d : \mathbb{R}^n \to \mathbb{R}$,

$$d(x) = \begin{cases} \min_{y\in\partial\Omega} |x - y|, & \text{if} \quad x \notin \Omega, \\ -\min_{y\in\partial\Omega} |x - y|, & \text{if} \quad x \in \Omega. \end{cases}$$

Since $\partial\Omega$ is C^2, the function d is C^2 near $\partial\Omega$. In particular, for all $x \in \partial\Omega$, $\nabla d(x) = n(x)$.

Given a time horizon T and a pair $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$, we are interested in optimal

control problems for which the dynamics is of the form:

$$\left\{\begin{array}{rcll} \xi'(s) & = & \eta(s), & s\in(0,T),\\ \eta'(s) & = & \alpha(s), & s\in(0,T),\\ \xi(0) & = & x, & \\ \eta(0) & = & v. & \end{array}\right. \tag{2.1}$$

The state variable is the pair (ξ,η) and the state space is $\Xi=\overline{\Omega}\times\mathbb{R}^n$. The optimal control problem consists of minimizing the cost

$$J(\xi,\eta,\alpha)=\int_0^T\left(\ell(\xi(s),\eta(s),s)+\frac{1}{2}|\alpha|^2(s)\right)ds+g(\xi(T),\eta(T)), \tag{2.2}$$

on the dynamics given by (2.1) and staying in Ξ.

Assumption 2.1. *Here, $\ell:\Xi\times[0,T]\to\mathbb{R}$ is a continuous function, bounded from below. The terminal cost $g:\Xi\to\mathbb{R}$ is also assumed to be continuous and bounded from below. Set*

$$M=\|g_-\|_{L^\infty(\Xi)}+\|\ell_-\|_{L^\infty(\Xi\times[0,T])}. \tag{2.3}$$

It is convenient to define the set of admissible trajectories as follows:

$$\Gamma=\left\{\ (\xi,\eta)\in C^1([0,T];\mathbb{R}^n)\times AC([0,T];\mathbb{R}^n)\ :\ \left|\begin{array}{ll}\xi'(s)=\eta(s), & \forall s\in[0,T]\\ (\xi(s),\eta(s))\in\Xi, & \forall s\in[0,T]\end{array}\right.\right\}. \tag{2.4}$$

For any $(x,v)\in\Xi$, set

$$\Gamma[x,v]=\{(\xi,\eta)\in\Gamma:\ \xi(0)=x,\ \eta(0)=v\}. \tag{2.5}$$

Then, $\Gamma^{\rm opt}[x,v]$ is the set of all $(\xi,\eta)\in\Gamma[x,v]$ such that $\eta\in W^{1,2}(0,T,\mathbb{R}^n)$ and (ξ,η,η') achieves the minimum of J in $\Gamma[x,v]$.
Note that $\Gamma[x,v]=\varnothing$ if $x\in\partial\Omega$ and v points outward Ω. This is the reason why we introduce $\Xi^{\rm ad}$ as follows:

$$\Xi^{\rm ad}=\{(x,v):x\in\overline{\Omega},\ v\cdot n(x)\leqslant 0 \text{ if } x\in\partial\Omega\}\subset\Xi. \tag{2.6}$$

Lemma 2.1. *For all $(x,v)\in\Xi^{\rm ad}$, the optimal value*

$$u(x,v)=\inf_{(\xi,\eta)\in\Gamma[x,v]}J(\xi,\eta,\eta') \tag{2.7}$$

is finite. The function u is lower semi-continuous on $\Xi^{\rm ad}$.

Proof. Let us consider $(x,v)\in\Xi^{\rm ad}$. We make out two cases:

1. $x\in\Omega$ or $x\in\partial\Omega$ and $v\cdot n(x)<0$: in this case, for $\overline{t}$ small enough, the trajectory (ξ,η) defined by

$$\left\{\begin{array}{rcllll} \eta(s) & = & (1-\frac{s}{\overline{t}})v & \text{and} & \xi(s)= & x+(s-\frac{s^2}{2\overline{t}})v, \quad \text{if} \quad 0\leqslant s\leqslant\overline{t},\\ \eta(s) & = & 0 & \text{and} & \xi(s)= & x+\frac{\overline{t}}{2}v, \quad \text{if} \quad \overline{t}\leqslant s\leqslant T, \end{array}\right.$$

is admissible and $J(\xi,\eta,\eta')$ is finite.

2. $x \in \partial\Omega$ and $v \cdot n(x) = 0$. We make a simple observation that will also be used in the proof of Lemma 2.2 below: for all $x \in \partial\Omega$, there exists an open neighborhood V_x of x in $\mathbb{R}^n$, a positive number R_x and a C^2-diffeomorphism Φ_x from V_x onto $B(0, R_x)$ such that for all $y \in V_x$, the n^{th} coordinate of $\Phi_x(y)$ is $d(y)$, i.e. $\Phi_{x,n}(y) = d(y)$. Hence, $\Phi_x|_{V_x \cap \Omega}$ is a C^2-diffeomorphism from $V_x \cap \Omega$ onto $B_-(0, R_x) = B(0, R_x) \cap \{x_n < 0\}$, and $\Phi_x|_{V_x \cap \partial\Omega}$ is a C^2-diffeomorphism from $V_x \cap \partial\Omega$ onto $B(0, R_x) \cap \{x_n = 0\}$. Let us also call Ψ_x the inverse of Φ_x, which is a C^2-diffeomorphism from $B(0, R_x)$ onto V_x. Note that

$$\nabla d(y) = D\Phi_x^T(y) e_n, \quad \text{for all } y \in V_x, \tag{2.8}$$

where e_n is the n^{th} vector of the canonical basis. In particular, $n(x) = D\Phi_x^T(x) e_n$.

In the present case, let us set $\hat{x} = \Phi_x(x)$ and $\hat{v} = D\Phi_x(x)v$. It is easy to see that $\hat{x}_n = 0$ and $\hat{v}_n = 0$. Then, for $\bar{t}$ small enough, the trajectory (ξ, η) defined by $\xi(s) = \Psi_x(\hat{\xi}(s))$, $\eta(s) = \frac{d\xi}{ds}(s) = D\Psi_x(\hat{\xi}(s))\hat{\eta}(s)$ for all $s \in [0, T]$, with

$$\begin{cases} \hat{\eta}(s) = (1 - \frac{s}{\bar{t}})\hat{v} & \text{and} \quad \hat{\xi}(s) = \hat{x} + (s - \frac{s^2}{2\bar{t}})\hat{v}, & \text{if} \quad 0 \leqslant s \leqslant \bar{t}, \\ \hat{\eta}(s) = 0 & \text{and} \quad \hat{\xi}(s) = \hat{x} + \frac{\bar{t}}{2}\hat{v}, & \text{if} \quad \bar{t} \leqslant s \leqslant T, \end{cases}$$

is admissible and $J(\xi, \eta, \eta')$ is finite.

The lower semi-continuity of u on Ξ^{ad} stems from standard arguments in the calculus of variations. □

2.2 Closed graph properties

An important feature of the optimal control problem described above is the closed graph property:

Proposition 2.1. *Consider a closed subset Θ of Ξ^{ad}. Assume that for all sequences $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Theta$ and $\lim_{i\to+\infty}(x^i, v^i) = (x, v) \in \Theta$, the following holds:*
if $x \in \partial\Omega$, then

$$\left((v^i \cdot \nabla d(x^i))_+\right)^3 = o\left(\left|d(x^i)\right|\right); \tag{2.9}$$

then the graph of the multivalued map

$$\begin{aligned} \Gamma^{\text{opt}} : \Theta &\to \Gamma, \\ (x, v) &\mapsto \Gamma^{\text{opt}}[x, v] \end{aligned}$$

is closed, which means: for any sequence $(y^i, w^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(y^i, w^i) \in \Theta$ with $(y^i, w^i) \to (y, w)$ as $i \to \infty$, consider a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\text{opt}}[y^i, w^i]$; if (ξ^i, η^i) tends to (ξ, η) uniformly, then $(\xi, \eta) \in \Gamma^{\text{opt}}[y, w]$.

Remark 2.1. *In Proposition 2.1, the condition (2.9) is restrictive only for sequences $(x^i, v^i) \in \Theta$ which tend to $(x, v) \in \Theta$ such that $x \in \partial\Omega$ and v is tangent to $\partial\Omega$ at x. We will see that this assumption makes it possible to control the cost associated to the optimal trajectories starting from (x^i, v^i).*

Remark 2.2. *In Section 4.1 below, we will see that in dimension one (Ω is then a bounded straight line), and under stronger assumptions on the running cost, the closed graph properties hold for $\Theta = \Xi^{\mathrm{ad}}$.*

Remark 2.3. *In the context of mean field games, see Section 3 below, the assumptions in Proposition 2.1 will yield sufficient conditions on the support of the initial distribution for the existence of relaxed mean field equilibria.*

The proof of Proposition 2.1 relies on Lemmas 2.2 and 2.3 below.

Lemma 2.2. *Consider $(x,v) \in \Xi^{\mathrm{ad}}$, $(\xi,\eta) \in \Gamma[x,v]$ such that $\eta \in W^{1,2}(0,T;\mathbb{R}^n)$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i,v^i) \in \Xi^{\mathrm{ad}}$ and $\lim_{i\to\infty}(x^i,v^i) = (x,v)$.*
Assume that one among the following conditions is true:

1. *$x \in \Omega$*
2. *$x \in \partial\Omega$ and $v \cdot n(x) < 0$ (hence $v^i \cdot \nabla d(x^i) < 0$ for i large enough)*
3. *$x \in \partial\Omega$, $v \cdot n(x) = 0$ and one among the following properties is true:*
   (a) *for i large enough, $v^i \cdot \nabla d(x^i) \leqslant 0$*
   (b) *for i large enough, $v^i \cdot \nabla d(x^i) > 0$ (hence $d(x^i) < 0$) and*

$$\lim_{i\to\infty} \frac{(v^i \cdot \nabla d(x^i))^3}{|d(x^i)|} = 0. \tag{2.10}$$

Then there exists a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that $(\xi^i,\eta^i) \in \Gamma[x^i,v^i]$, $\eta^i \in W^{1,2}(0,T;\mathbb{R}^n)$, and (ξ^i,η^i) tends to (ξ,η) in $W^{2,2}(0,T;\mathbb{R}^n) \times W^{1,2}(0,T;\mathbb{R}^n)$, hence uniformly in $[0,T]$.

Before proving Lemma 2.2, let us define a family of third order polynomials with values in $\mathbb{R}^n$:

Definition 2.1. *Given $t > 0$ and $x,v,y,w \in \mathbb{R}^n$, let $Q_{t,x,v,y,w}$ be the unique third order polynomial with value in $\mathbb{R}^n$ such that*

$$Q_{t,x,v,y,w}(0) = x, \quad Q'_{t,x,v,y,w}(0) = v, \quad Q_{t,x,v,y,w}(t) = y, \quad Q'_{t,x,v,y,w}(t) = w. \tag{2.11}$$

It is given by

$$Q_{t,x,v,y,w}(s) = x + vs + \left(3\frac{y-x-vt}{t^2} - \frac{w-v}{t}\right)s^2 + \left(-2\frac{y-x-vt}{t^3} + \frac{w-v}{t^2}\right)s^3. \tag{2.12}$$

The first and second derivatives of $Q_{t,x,v,y,w}$ are deduced immediately from (2.12).

Proof of Lemma 2.2. We are going to see that each of the three conditions mentioned in the statement makes it possible to explicitly construct families of admissible trajectories fulfilling all the desired properties (in particular with a finite energy or cost). The more tricky situations will arise when $x \in \partial\Omega$ and $v^i \cdot \nabla d(x^i) > 0$ for i large enough, in which case the restrictive condition (2.10) will be needed. Since the construction is different in each of the three cases mentioned in Lemma 2.2, we discuss each case separately.

1. If $x \in \Omega$, then there exists $\bar{t} \in (0,T]$ and $c > 0$ such that $d(\xi(s)) < -c$ for all $s \in [0,\bar{t}]$. We construct the sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ as follows:

$$\xi^i(s) = \begin{cases} \xi(s) + Q_{\bar{t},\delta x^i,\delta v^i,0,0}(s), & \text{if} \quad 0 \leqslant s \leqslant \bar{t}, \\ \xi(s), & \text{if} \quad \bar{t} \leqslant s \leqslant T, \end{cases}$$

where $\delta x^i = x^i - x$ and $\delta v^i = v^i - v$, see Definition 2.1 for the third order polynomial $Q_{\bar{t},\delta x^i,\delta v^i,0,0}$. It is clear that for i large enough, $\xi^i(s) \in \overline{\Omega}$ for all $s \in [0,T]$, hence $(\xi^i,\eta^i) \in \Gamma[x^i,v^i]$ and $\eta^i \in W^{1,2}(0,T;\mathbb{R}^n)$. On the other hand, it can be easily checked that

$$\lim_{i\to+\infty} \int_0^T \left| \frac{d\eta^i}{dt}(s) - \frac{d\eta}{dt}(s) \right|^2 ds = 0. \tag{2.13}$$

This achieves the proof in the first case.

2. $x \in \partial\Omega$ and $v \cdot n(x) < 0$, hence for i large enough, $v^i \cdot \nabla d(x^i) < 0$. We can always assume that the latter property holds for all i.

Notation We use the same geometric arguments as in the proof of Lemma 2.1: for the neighborhood V_x mentioned there, there exists $\hat{T}$, $0 < \hat{T} \leqslant T$ such that $\xi(s) \in V_x \cap \overline{\Omega}$ for all $s \in [0,\hat{T}]$. Consider the local chart Φ_x introduced in the proof of Lemma 2.1 and call Ψ_x the inverse map, from $B(0,R_x)$ onto V_x. For $t \in [0,\hat{T}]$, let us set $\hat{\xi}(t) = \Phi_x(\xi(t))$, $\hat{\eta}(t) = \frac{d\hat{\xi}}{dt}(t) = D\Phi_x(\xi(t))\frac{d\xi}{dt}(t)$, $\hat{x} = \Phi_x(x)$ and $\hat{v} = \hat{\eta}(0) = D\Phi_x(x)v$. It is clear that $\hat{x}_n = 0$ and that $\hat{v}_n < 0$. We also set $\hat{x}^i = \Phi_x(x^i)$ and $\hat{v}^i = D\Phi_x(x^i)v^i$.

Since $\eta \in W^{1,2}(0,T)$, there exists $\bar{t} \in (0,\hat{T})$ such that for all $s \in [0,\bar{t}]$,

$$\frac{3}{2}\hat{v}_n \leqslant \hat{\eta}(s)\cdot e_n \leqslant \frac{1}{2}\hat{v}_n, \tag{2.14}$$

$$\frac{3s}{2}\hat{v}_n \leqslant (\hat{\xi}(s) - \hat{x})\cdot e_n \leqslant \frac{s}{2}\hat{v}_n. \tag{2.15}$$

For $t_i \in [0,\bar{t}]$, we set

$$\hat{\xi}^i(s) = \begin{cases} Q_{t_i,\hat{x}^i,\hat{v}^i,\hat{\xi}(t_i),\hat{\eta}(t_i)}(s), & \text{if} \quad s \in [0,t_i], \\ \hat{\xi}(s), & \text{if} \quad s \in [t_i,\hat{T}], \end{cases} \tag{2.16}$$

and $\hat{\eta}^i(s) = \frac{d\hat{\xi}^i}{dt}(s)$ for $s \in [0,\hat{T}]$. Then, we define ξ^i as follows:

$$\xi^i(s) = \begin{cases} \Psi_x\left(\hat{\xi}^i(s)\right), & \text{if} \quad s \in [0,\hat{T}], \\ \xi(s), & \text{if} \quad s \in [\hat{T},T], \end{cases} \tag{2.17}$$

and $\eta^i = \frac{d\xi^i}{dt}$. Let us first see why $(\xi^i,\eta^i) \in \Gamma[x_i,v_i]$ for t_i small enough and i large enough. A straightforward calculation shows that for $s \leqslant t_i$,

$$\begin{aligned} \hat{\xi}^i(s) - \hat{x} =& (\hat{x}^i - \hat{x})\left(1 - 3\frac{s^2}{t_i^2} + 2\frac{s^3}{t_i^3}\right) + s\hat{v}^i\left(1 - 2\frac{s}{t_i} + \frac{s^2}{t_i^2}\right) \\ &+ (\hat{\xi}(t_i) - \hat{x})\left(3\frac{s^2}{t_i^2} - 2\frac{s^3}{t_i^3}\right) + s\hat{\eta}(t_i)\left(-\frac{s}{t_i} + \frac{s^2}{t_i^2}\right). \end{aligned} \tag{2.18}$$

Let us focus on $(\hat{\xi}^i(s)-\hat{x})\cdot e_n=\hat{\xi}^i(s)\cdot e_n$: from the formula above, we see that $\hat{\xi}^i(s)\cdot e_n$ is the sum of four terms, the first three of them being nonpositive and the last one nonnegative for all $s\in[0,t_i]$. Let us consider the sum of the last three terms, namely:

$$A(s)=s\hat{v}^i\cdot e_n\left(1-2\frac{s}{t_i}+\frac{s^2}{t_i^2}\right)+\hat{\xi}(t_i)\cdot e_n\left(3\frac{s^2}{t_i^2}-2\frac{s^3}{t_i^3}\right)+s\hat{\eta}(t_i)\cdot e_n\left(-\frac{s}{t_i}+\frac{s^2}{t_i^2}\right);$$

from (2.14) and (2.15), we see that

$$\begin{aligned}\hat{\xi}(t_i)\cdot e_n\left(3\frac{s^2}{t_i^2}-2\frac{s^3}{t_i^3}\right)+s\hat{\eta}(t_i)\cdot e_n\left(-\frac{s}{t_i}+\frac{s^2}{t_i^2}\right)\leqslant& s\hat{v}\cdot e_n\left(\frac{1}{2}\left(3\frac{s^2}{t_i^2}-2\frac{s^3}{t_i^3}\right)+\frac{3}{2}\left(-\frac{s}{t_i}+\frac{s^2}{t_i^2}\right)\right)\\ =&s\hat{v}\cdot e_n\left(-\frac{3}{2}\frac{s}{t_i}+3\frac{s^2}{t_i^2}-\frac{s^3}{t_i^3}\right)\end{aligned}$$

for all $s\in[0,t_i]$. On the other hand, since $\lim_{i\to\infty}\hat{v}^i=\hat{v}$, we see that for i large enough, $\hat{v}^i\cdot e_n\leqslant 3\hat{v}\cdot e_n/4$. Hence,

$$A(s)\leqslant s\hat{v}\cdot e_n\left(\frac{3}{4}\left(1-\frac{s}{t_i}\right)^2-\frac{3}{2}\frac{s}{t_i}+3\frac{s^2}{t_i^2}-\frac{s^3}{t_i^3}\right)=s\hat{v}\cdot e_n\left(\frac{3}{4}-3\frac{s}{t_i}+\frac{15}{4}\frac{s^2}{t_i^2}-\frac{s^3}{t_i^3}\right).$$

It is easy to check that the function $\theta\mapsto\frac{3}{4}-3\theta+\frac{15}{4}\theta^2-\theta^3$ is positive for $\theta\in[0,1]$, which implies that $A(s)$ is negative for $s\in[0,t_i]$.
Hence, for i large enough, $\hat{\xi}^i(s)\cdot e_n\leqslant 0$ for all $0\leqslant s\leqslant t_i\leqslant\bar{t}$.
On the other hand, since $\lim_{i\to+\infty}\left(|\hat{x}^i-\hat{x}|+|\hat{v}^i-v|\right)=0$, $\hat{\xi}$ and $\hat{\eta}$ are continuous, (2.18) implies that there exists $I>0$ and $\tilde{t}\in(0,\bar{t}\,]$ such that, if $i\geqslant I$ and $t_i\in(0,\tilde{t})$, then $\hat{\xi}^i(s)\in\overline{B_-(0,R_x)}$ for all $s\in[0,t_i]$. Hence, for $i\geqslant I$ and $t_i\in(0,\tilde{t})$, $(\xi^i,\eta^i)\in\Gamma[x^i,v^i]$.

Let us now turn to $\left\|\frac{d\eta^i}{dt}\right\|_{L^2(0,t_i)}$: straightforward calculus shows that

$$\frac{d\eta^i}{dt}(t)=D\Psi_x(\hat{\xi}^i(t))\frac{d\hat{\eta}^i}{dt}(t)+\left(D^2\Psi_x(\hat{\xi}^i(t))\hat{\eta}^i(t)\right)\hat{\eta}^i(t).$$

This implies that

$$\left\|\frac{d\eta^i}{dt}\right\|^2_{L^2(0,t_i)}\leqslant C\left(\left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_i)}+\left\|\hat{\eta}^i\right\|^4_{L^4(0,t_i)}\right),\tag{2.19}$$

for a constant C independent of $x\in\partial\Omega$. Hereafter, C may vary from line to line.
First, we focus on $\left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_i)}$:

$$\begin{aligned}&\left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_i)}\\ =&\int_0^{t_i}\left|2\left(3\frac{\hat{\xi}(t_i)-\hat{v}^it_i-\hat{x}^i}{t_i^2}-\frac{\hat{\eta}(t_i)-\hat{v}^i}{t_i}\right)+\frac{6s}{t_i}\left(-2\frac{\hat{\xi}(t_i)-\hat{v}^it_i-\hat{x}^i}{t_i^2}+\frac{\hat{\eta}(t_i)-\hat{v}^i}{t_i}\right)\right|^2ds\\ \leqslant&2I_1+2I_2,\end{aligned}$$

where

$$I_1=\int_0^{t_i}\left|2\left(3\frac{\hat{\xi}(t_i)-\hat{v}t_i-\hat{x}}{t_i^2}-\frac{\hat{\eta}(t_i)-\hat{v}}{t_i}\right)+\frac{6s}{t_i}\left(-2\frac{\hat{\xi}(t_i)-\hat{v}t_i-\hat{x}}{t_i^2}+\frac{\hat{\eta}(t_i)-\hat{v}}{t_i}\right)\right|^2ds$$

and

$$I_2 = \int_0^{t_i} \left| 2\left(3\frac{\hat{x}-\hat{x}^i}{t_i^2} + 2\frac{\hat{v}-\hat{v}^i}{t_i} \right) + \frac{6s}{t_i}\left(-2\frac{\hat{x}-\hat{x}^i}{t_i^2} - \frac{\hat{v}-\hat{v}^i}{t_i} \right) \right|^2 ds.$$

Standard arguments yield that

$$I_1 \leqslant C \left\| \frac{d\hat{\eta}}{dt} \right\|^2_{L^2(0,t_i)},$$

for an absolute constant $C > 0$. Therefore, given $\epsilon > 0$, there exists $\hat{t} :\ 0 < \hat{t} < \bar{t}$ such that $2I_1 < \epsilon/2$ for all $t_i < \hat{t}$.
On the other hand,

$$I_2 \leqslant C_1 \left(\frac{|\hat{x}-\hat{x}^i|^2}{t_i^3} + \frac{|\hat{v}-\hat{v}^i|^2}{t_i} \right) \leqslant C \left(\frac{|x-x^i|^2}{t_i^3} + \frac{|v-v^i|^2}{t_i} \right).$$

It is possible to choose the sequence t_i such that

- $\lim_{i\to\infty} t_i = 0$,
- $\lim_{i\to\infty} \frac{|x-x^i|^2}{t_i^3} + \frac{|v-v^i|^2}{t_i} = 0$,
- $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$ for i large enough.

Such a choice of t_i yields that $\lim_{i\to\infty} \left\| \frac{d\hat{\eta}^i}{dt} \right\|^2_{L^2(0,t_i)} = 0$. On the other hand, the choice made on t_i also implies that $\lim_{i\to\infty} \frac{|x-x^i|}{t_i} = 0$, and in turn that $\|\hat{\eta}^i\|_{L^\infty(0,t_i)}$ is uniformly bounded with respect to i; therefore, the quantity $\|\hat{\eta}^i\|^4_{L^4(0,t_i)}$ tends to 0 as i tends to ∞; using (2.19), we have proved that $\lim_{i\to\infty} \left\| \frac{d\eta^i}{dt} \right\|_{L^2(0,t_i)} = 0$.
Therefore, it is possible to choose a sequence $t_i > 0$ such that the trajectories (ξ^i, η^i) are admissible for i large enough and $\lim\limits_{i\to\infty} \left\| \frac{d\eta^i}{dt} - \frac{d\eta}{dt} \right\|_{L^2(0,T)} = 0$. This achieves the proof in case 2.

3. (a) $x \in \partial\Omega$, $v \cdot n(x) = 0$ and $v^i \cdot \nabla d(x^i) \leqslant 0$ at least for i large enough. We may assume that $v^i \cdot \nabla d(x^i) \leqslant 0$ for all i. Using the same notation as in case 2, we see from (2.8) that $\hat{v}^i \cdot e_n = v^i \cdot D\Phi_x^T(x^i)e_n = v^i \cdot \nabla d(x^i) \leqslant 0$.
In the present case, the approximate trajectories will have three successive phases, see Remarks 2.4 and 2.5 for explanations on these different phases.
Given $t_{i,1} \in (0, \hat{T})$, we define $(\hat{y}^i, \hat{w}^i)$ as follows:

$$\begin{array}{rcl} \hat{y}^i & = & \hat{x}_n e_n + \pi_{e_n^\perp}\left(\hat{x}^i + \hat{v}^i t_{i,1} \right), \\ \hat{w}^i & = & \hat{v}_n e_n + \pi_{e_n^\perp}\left(\hat{v}^i \right) = \pi_{e_n^\perp}\left(\hat{v}^i \right), \end{array} \tag{2.20}$$

where $\pi_{e_n^\perp}$ stands for the orthogonal projector on $e_n^\perp = \mathbb{R}^{n-1} \times \{0\}$, and set

$$\hat{\xi}^i(s) = Q_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}(s) \quad \text{and} \quad \hat{\eta}^i(s) = \frac{d\hat{\xi}^i}{ds}(s) \qquad \text{for} \quad 0 \leqslant s \leqslant t_{i,1}.$$

Remark 2.4. *In this first phase of the approximate trajectory, i.e. for $s \in [0, t_{i,1}]$, $\pi_{e_n^\perp}\left(Q''_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}(s) \right) = 0$. The effort only lies in driving the n^{th}-components of $\hat{\xi}^i(s)$ and $\hat{\eta}^i(s)$ so that they match those of $\hat{x}$ and $\hat{v}$ at $s = t_{i,1}$.*

As above, we first check that for $t_{i,1}$ small enough and i large enough, $\hat{\xi}^i(s) \in \overline{B_-(0,R_x)}$ for all $s \in [0,t_{i,1}]$: from the definition of $Q_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}$, we see that

$$\hat{\xi}^i(s)\cdot e_n = \left(\hat{x}^i\cdot e_n\left(1+2\frac{s}{t_{i,1}}\right)+s\hat{v}^i\cdot e_n\right)\left(1-\frac{s}{t_{i,1}}\right)^2 \tag{2.21}$$

is nonpositive for $s \in [0,t_{i,1}]$. On the other hand, we see that there exist $I>0$ and $0<\tilde{t}\leqslant \hat{T}$ such that if $i\geqslant I$ and $0<t_{i,1}<\tilde{t}$, then for all $s\in[0,t_{i,1}]$, $\hat{\xi}^i(s)\in\overline{B_-(0,R_x)}$. As in case 2., we need to focus on $\left\|\frac{d\hat{\eta}^i}{dt}\right\|_{L^2(0,t_{i,1})}$: straightforward calculus shows that

$$\left\|\frac{d\hat{\eta}^i}{dt}\right\|^2_{L^2(0,t_{i,1})} \leqslant C\left(\frac{|\hat{x}^i\cdot e_n|^2}{t_{i,1}^3}+\frac{|(\hat{v}-\hat{v}^i)\cdot e_n|^2}{t_{i,1}}\right) = C\left(\frac{d^2(x^i)}{t_{i,1}^3}+\frac{|v^i\cdot\nabla d(x^i)|^2}{t_{i,1}}\right),$$

and we see as above that there exists a sequence $t_{i,1}$ such that

- $\lim_{i\to\infty} t_{i,1}=0$,
- $\lim_{i\to\infty}\frac{d^2(x^i)}{t_{i,1}^3}+\frac{|v^i\cdot\nabla d(x^i)|^2}{t_{i,1}}=0$,
- $\hat{\xi}^i(s)\in\overline{B_-(0,R_x)}$ for all $0\leqslant s\leqslant t_{i,1}$.

Taking the derivative of $\hat{\xi}^i$ and arguing as in case 2., we also see that $\lim_{i\to\infty}\|\hat{\eta}^i\|_{L^4(0,t_{i,1})}=0$, because $\lim_{i\to\infty}\frac{d(x^i)}{t_{i,1}}=0$.

Next, for $t_{i,1}<t_{i,2}<\hat{T}$, we set

$$\hat{\xi}^i(s)=\begin{cases} Q_{t_{i,1},\hat{x}^i,\hat{v}^i,\hat{y}^i,\hat{w}^i}(s), & \text{if} \quad s\leqslant t_{i,1},\\ Q_{t_{i,2}-t_{i,1},\hat{y}^i-\hat{x},\hat{w}^i-\hat{v},0,0}(s-t_{i,1})+\hat{\xi}(s-t_{i,1}), & \text{if} \quad t_{i,1}\leqslant s\leqslant t_{i,2},\end{cases} \tag{2.22}$$

and

$$\xi^i(s)=\begin{cases}\Psi_x\left(\hat{\xi}^i(s)\right), & \text{if} \quad s\leqslant t_{i,2},\\ \xi(s-t_{i,1}), & \text{if} \quad t_{i,2}\leqslant s\leqslant T.\end{cases} \tag{2.23}$$

As above $\hat{\eta}^i(s)=\frac{d\hat{\xi}^i}{dt}(s)$ for $0\leqslant s\leqslant t_{i,2}$ and $\eta^i(s)=\frac{d\xi^i}{dt}(s)$ for $0\leqslant s\leqslant T$.

Remark 2.5. *In the second phase of the approximate trajectory, i.e. for $s\in[t_{i,1},t_{i,2}]$, the components of $\hat{\xi}^i(s)$ and $\hat{\xi}(s-t_{i,1})$ parallel to e_n coincide, i.e. $Q_{t_{i,2}-t_{i,1},\hat{y}^i-\hat{x},\hat{w}^i-\hat{v},0,0}(s-t_{i,1})\cdot e_n=0$. The effort only consists of driving the projections of $\hat{\xi}^i(s)$ and $\hat{\eta}^i(s)$ on $e_n^\perp$ such that they match those of $\hat{\xi}(s-t_{i,1})$ and $\hat{\eta}(s-t_{i,1})$ at $s=t_{i,2}$. We will see that is not necessary to have $t_{i,2}$ tend to zero, because from the choice of $t_{i,1}$, the distance between $(\hat{\xi}^i(t_{i,1}),\hat{\eta}^i(t_{i,1}))$ and $(\hat{x},\hat{v})$ tends to 0 as $i\to+\infty$.*

It is possible to choose the sequence $t_{i,2}$ bounded from below by a positive constant which depends on (x,v) but not on i such that $\hat{\xi}^i(s)$ stays in $\overline{B_-(0,R_x)}$ for $s\in[t_{i,1},t_{i,2}]$. Hence, $(\xi^i,\eta^i)\in\Gamma[x^i,v^i]$.
Moreover, since $t_{i,2}$ is bounded away from 0 and $\lim_{i\to\infty}\left(|\hat{y}^i-\hat{x}|+|\hat{w}^i-\hat{v}|\right)=0$, it is not difficult to check that $\lim\limits_{i\to\infty}\left\|\frac{d\eta^i}{dt}-\frac{d\eta}{dt}\right\|_{L^2(0,T)}=0$; this achieves the proof in subcase 3.(a).

(b) $x \in \partial\Omega$, $v \cdot n(x) = 0$, $v^i \cdot \nabla d(x^i) > 0$ for all i (or for i large enough), and (2.10) holds. The trajectory ξ^i is constructed as in (2.22)-(2.23), but a further restriction on $t_{i,1}$ is needed in order to guarantee that the trajectory is admissible. Using (2.21), we see that the trajectory is admissible if

$$\frac{\hat{v}^i \cdot e_n}{|\hat{x}^i \cdot e_n|} \leqslant \frac{1}{s} + \frac{2}{t_{i,1}}, \quad \text{for all } 0 \leqslant s \leqslant t_{i,1}.$$

This happens if and only if

$$t_{i,1} \leqslant \frac{3|\hat{x}^i \cdot e^n|}{\hat{v}^i \cdot e_n} = \frac{3|d(x^i)|}{v^i \cdot \nabla d(x_i)},$$

which should be supplemented with the other two conditions as in 3.(a):

$$\lim_{i\to\infty} t_{i,1} = 0, \tag{2.24}$$

$$\lim_{i\to\infty} \frac{|d(x^i)|^2}{t_{i,1}^3} + \frac{|v^i \cdot \nabla d(x^i)|^2}{t_{i,1}} = 0. \tag{2.25}$$

If (2.10) holds, then it is possible to choose such a sequence $t_{i,1}$. Then, as in 3.(a), it is possible to choose the sequence $t_{i,2}$ bounded from below by a positive constant independent of i such that $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$; the last part of the proof is identical as in subcase 3.(a).

□

Lemma 2.3. *Consider $(x, v) \in \Xi^{\mathrm{ad}}$ and a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Xi^{\mathrm{ad}}$ and $(x^i, v^i) \to (x, v)$ as $i \to \infty$. Suppose that Assumption 2.1 and one among the three conditions in Lemma 2.2 are satisfied. Let a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ be such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\mathrm{opt}}[x^i, v^i]$. If (ξ^i, η^i) tends to (ξ, η) uniformly in $[0, T]$, then $\eta \in W^{1,2}(0, T; \mathbb{R}^n)$ and $(\xi, \eta) \in \Gamma^{\mathrm{opt}}[x, v]$.*

Proof. We need to prove that for any $(\widetilde{\xi}, \widetilde{\eta}) \in \Gamma[x, v]$ such that $\widetilde{\eta} \in W^{1,2}(0, T; \mathbb{R}^n)$,

$$J(\xi, \eta, \eta') \leqslant J(\widetilde{\xi}, \widetilde{\eta}, \widetilde{\eta}'). \tag{2.26}$$

From Lemma 2.2 applied to $(\widetilde{\xi}, \widetilde{\eta})$, there exists a sequence $(\widetilde{\xi}^i, \widetilde{\eta}^i)_{i\in\mathbb{N}}$, with $(\widetilde{\xi}^i, \widetilde{\eta}^i) \in \Gamma[x^i, v^i]$ such that $(\widetilde{\xi}^i, \widetilde{\eta}^i) \to (\widetilde{\xi}, \widetilde{\eta})$ uniformly on $[0, T]$ as $i \to \infty$, and

$$\lim_{i\to\infty} \int_0^T \left| \frac{d\widetilde{\eta}^i}{dt}(s) \right|^2 ds = \int_0^T \left| \frac{d\widetilde{\eta}}{dt}(s) \right|^2 ds.$$

On the other hand, the optimality of (ξ^i, η^i) yields that

$$J\left(\xi^i, \eta^i, \frac{d\eta^i}{dt}\right) \leqslant J\left(\widetilde{\xi}^i, \widetilde{\eta}^i, \frac{d\widetilde{\eta}^i}{dt}\right). \tag{2.27}$$

From the properties of $(\widetilde{\xi}^i, \widetilde{\eta}^i)$, the right hand side of (2.27) converges to $J\left(\widetilde{\xi}, \widetilde{\eta}, \frac{d\widetilde{\eta}}{dt}\right)$. The left side (2.27) is thus bounded. Combining this fact with the uniform convergence of (ξ^i, η^i) to (ξ, η) in $[0, T]$, we obtain that the sequence $\int_0^T \left| \frac{d\eta^i}{dt}(s) \right|^2 ds$ is bounded.

This implies that $\frac{d\eta^i}{dt} \rightharpoonup \frac{d\eta}{dt}$ in $L^2(0,T;\mathbb{R}^n)$ weakly and that $\liminf_{i\to\infty} \int_0^T \left|\frac{d\eta^i}{dt}(s)\right|^2 ds \geqslant \int_0^T \left|\frac{d\eta}{dt}(s)\right|^2 ds$. We deduce that

$$J\left(\xi,\eta,\frac{d\eta}{dt}\right) \leqslant \liminf_{i\to\infty} J\left(\xi^i,\eta^i,\frac{d\eta^i}{dt}\right).$$

Combining the information obtained above, we obtain (2.26), which achieves the proof. □

Proof of Proposition 2.1. Consider $(y,w) \in \Theta$ and a sequence $(y^i,w^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(y^i,w^i) \in \Theta$ and $(y^i,w^i) \to (y,w)$ as $i \to \infty$. Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i,\eta^i) \in \Gamma^{\rm opt}[y^i,w^i]$ and that (ξ^i,η^i) tends to (ξ,η) uniformly. Thanks to the assumption made in the statement of Proposition 2.1, possibly after the extraction of a subsequence, we may suppose that one among the three conditions in Lemma 2.2 holds. Then the conclusion follows from Lemma 2.3. □

2.3 Bounds related to optimal trajectories

Definition 2.2. *For a positive number C, let us set*

$$K_C = \{(x,v) \in \Xi : \ |v| \leqslant C\}, \tag{2.28}$$

$$\Gamma_C = \left\{(\xi,\eta) \in \Gamma : \left| \begin{array}{l} (\xi(t),\eta(t)) \in K_C, \quad \forall t \in [0,T], \\ \left\|\frac{d\eta}{dt}\right\|_{L^2(0,T;\mathbb{R}^n)} \leqslant C. \end{array} \right.\right\}. \tag{2.29}$$

Proposition 2.2. *Given $r > 0$, let us define*

$$\Theta_r = \Theta \cap K_r, \tag{2.30}$$

where K_r is defined by (2.28) and Θ is a closed subset of $\Xi^{\rm ad}$ which satisfies the assumption in Proposition 2.1.
Under Assumption 2.1, the value function u defined in (2.7) is continuous on Θ_r.
There exists a positive number $C = C(r,M)$ such that if $(x,v) \in \Theta_r$, then $\Gamma^{\rm opt}[x,v] \subset \Gamma_C$.

Remark 2.6. *The set of trajectories Γ_C is a compact subset of Γ. In the context of mean field games, see Section 3, the existence of relaxed equilibria will be obtained by applying Kakutani's fixed point theorem to a multivalued map defined on a closed set of probability measures on Γ_C.*

Proof. Take $(x,v) \in \Theta_r$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$, $(x^i,v^i) \in \Theta_r$ such that $\lim_{i\to\infty}(x^i,v^i) = (x,v)$.
From Lemma 2.1 we know that $u(x,v)$ is finite and from Assumption 2.1, that the infimum in (2.7) is achieved by a trajectory $(\xi,\eta) \in \Gamma^{\rm opt}[x,v]$.
Possibly after the extraction of a subsequence, we may assume that (x^i,v^i) satisfies one among the three points in Lemma 2.2. Then, there exists a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i,\eta^i) \in \Gamma[x^i,v^i]$, $\eta^i \in W^{1,2}(0,T;\mathbb{R}^n)$, and (ξ^i,η^i) tends to (ξ,η) in $W^{2,2}(0,T;\mathbb{R}^n) \times W^{1,2}(0,T;\mathbb{R}^n)$, hence uniformly in $[0,T]$. Hence,

$$\lim_{i\to\infty} J\left(\xi^i,\eta^i,\frac{d\eta^i}{dt}\right) = u(x,v).$$

On the other hand,

$$J\left(\xi^i, \eta^i, \frac{d\eta^i}{dt}\right) \geqslant u(x^i, v^i).$$

The latter two observations yield that

$$\limsup_{i\to\infty} u(x^i, v^i) \leqslant u(x, v).$$

This proves that u is upper-semi-continuous on Θ_r. From Lemma 2.1, u is continuous on Θ_r. Since Θ_r is a compact subset of $\mathbb{R}^n \times \mathbb{R}^n$, u is bounded on Θ_r.
Then from the definition of J and u and the boundedness of u on Θ_r, it is clear that there exists a constant $C = C(r, M)$ such that $\Gamma^{\rm opt}[x, v] \subset \Gamma_C$ for any $(x, v) \in \Theta_r$. □

3 A mean field game with control on the acceleration and state constraints

3.1 Setting and notation

The bounded domain Ω of $\mathbb{R}^n$ and the sets Ξ and $\Xi^{\rm ad}$ have been introduced in Section 2.1. Let $\mathcal{P}(\Xi)$ be the set of probability measures on Ξ.

Let $C_b^0(\Xi; \mathbb{R})$ denote the space of bounded and continuous real valued functions defined on Ξ and let $F, G : \mathcal{P}(\Xi) \to C_b^0(\Xi; \mathbb{R})$ be bounded and continuous maps (the continuity is with respect to the narrow convergence in $\mathcal{P}(\Xi)$). Let ℓ be a real valued, continuous and bounded from below function defined on $\Xi \times [0, T]$.

Let $F[m]$ and $G[m]$ denote the images by F and G of $m \in \mathcal{P}(\Xi)$. Set

$$M = \max\left(\sup_{(x,v,s)\in\Xi\times[0,T]} \ell_-(x, v, s) + \sup_{m\in\mathcal{P}(\Xi)} \|F[m]\|_{L^\infty(\Xi)}, \sup_{m\in\mathcal{P}(\Xi)} \|G[m]\|_{L^\infty(\Xi)}\right). \tag{3.1}$$

Let Γ be the set of admissible trajectories given by (2.4). It is a metric space with the distance $d((\xi, \eta), (\tilde\xi, \tilde\eta)) = \|\xi - \tilde\xi\|_{C^1([0,T];\mathbb{R}^n)}$. Let $\mathcal{P}(\Gamma)$ be the set of probability measures on Γ.
For $t \in [0, T]$, the evaluation map $e_t : \Gamma \to \Xi$ is defined by $e_t(\xi, \eta) = (\xi(t), \eta(t))$ for all $(\xi, \eta) \in \Gamma$.
For any $\mu \in \mathcal{P}(\Gamma)$, let the Borel probability measure $m^\mu(t)$ on Ξ be defined by $m^\mu(t) = e_t\sharp\mu$. It is possible to prove that if $\mu \in \mathcal{P}(\Gamma)$, then $t \mapsto m^\mu(t)$ is continuous from $[0, T]$ to $\mathcal{P}(\Xi)$, for the narrow convergence in $\mathcal{P}(\Xi)$. Hence, for all $(\xi, \eta) \in \Gamma$, $t \mapsto F[m^\mu(t)](\xi(t), \eta(t))$ is continuous and bounded by the constant M in (3.1).
With $\mu \in \mathcal{P}(\Gamma)$, we associate the cost

$$J^\mu(\xi, \eta) = \left(\begin{array}{l}\displaystyle\int_0^T \left(F[m^\mu(s)](\xi(s), \eta(s)) + \ell(\xi(s), \eta(s), s) + \frac{1}{2}\left|\frac{d\eta}{dt}(s)\right|^2\right) ds \\ \quad + G[m^\mu(T)](\xi(T), \eta(T))\end{array}\right). \tag{3.2}$$

Remark 3.1. *It is clear from (3.1) that given $\mu \in \mathcal{P}(\Gamma)$, the running cost $(y, w, s) \mapsto F[m^\mu(s)](y, w) + \ell(y, w, s)$ and the final cost $(y, w) \mapsto G[m^\mu(T)](y, w)$ satisfy Assumption 2.1, and that the constant arising in (2.3) can be chosen uniformly with respect to $\mu \in$*

$\mathcal{P}(\Gamma)$. *Hence, for all* $\mu \in \mathcal{P}(\Gamma)$, *Propositions 2.1 and 2.2 hold for the state constrained control problem related to* J^μ *and the constants arising in these propositions can be chosen uniformly with respect to* $\mu \in \mathcal{P}(\Gamma)$.

Assumption 3.1. *There exists a positive number* r *such that the initial distribution of states is a probability measure* m_0 *on* Ξ *supported in* Θ_r, *where* Θ_r *is a closed subset of* $\Xi^{\rm ad}$ *as in (2.30).*

Let $C = C(r, M)$ be the constant appearing in Proposition 2.2 (uniform w.r.t. μ), and Γ_C be the compact subset of Γ defined by (2.29); clearly, Γ_C is a Radon metric space. From Prokhorov theorem, see [5, Theorem 5.1.3], the set $\mathcal{P}(\Gamma_C)$ is compact for the narrow convergence of measures.

Let $\mathcal{P}_{m_0}(\Gamma)$, (resp. $\mathcal{P}_{m_0}(\Gamma_C)$) denote the set of probability measures μ on Γ (resp. Γ_C) such that $e_0\sharp\mu = m_0$.

Hereafter, we identify $\mathcal{P}(\Gamma_C)$ with a subset of $\mathcal{P}(\Gamma)$ by extending $\mu \in \mathcal{P}(\Gamma_C)$ by 0 outside Γ_C. Similarly, we may consider $\mathcal{P}_{m_0}(\Gamma_C)$ as a subset of $\mathcal{P}_{m_0}(\Gamma)$.

Note that for all $\mu \in \mathcal{P}(\Gamma_C)$ and for all $t \in [0, T]$, $m^\mu(t)$ is supported in K_C, where K_C is defined in (2.28).

Remark 3.2. *Note that* Γ_C *(endowed with the metric of the* $C^1 \times C^0$*-convergence of* (ξ, η)*) is a Polish space (because it is compact). The multivalued map* $\widetilde{\Gamma}^{\rm opt}$ *related for instance to* $F \equiv 0$ *and* $G \equiv 0$ *maps* Θ_r *to non empty and closed subsets of* Γ_C *(the closedness can be checked by usual arguments of the calculus of variations). Since the graph of* $\widetilde{\Gamma}^{\rm opt}$ *is closed,* $\widetilde{\Gamma}^{\rm opt}$ *is measurable. Therefore, there exists a measurable selection* $j : \Theta_r \to \Gamma_C$ *from Kuratowski and Ryll-Nardzewski theorem, [18]. Then* $j\sharp m_0$ *belongs to* $\mathcal{P}_{m_0}(\Gamma_C)$. *The set* $\mathcal{P}_{m_0}(\Gamma_C)$ *is not empty.*

3.2 Existence of a mean field game equilibrium

Lemma 3.1. *Let a sequence of probability measures* $(\mu_i)_{i\in\mathbb{N}}$, $\mu_i \in \mathcal{P}(\Gamma)$, *be narrowly convergent to* $\mu \in \mathcal{P}(\Gamma)$. *For all* $t \in [0, T]$, $(m^{\mu_i}(t))_{i\in\mathbb{N}}$ *is narrowly convergent to* $m^\mu(t)$.

Proof. For all $f \in C_b^0(\Xi; \mathbb{R})$,

$$\int_\Xi f(x,v)dm^{\mu_i}(t)(x,v) = \int_\Gamma f(\xi(t),\eta(t))d\mu_i(\xi,\eta) \quad \to \quad \int_\Gamma f(\xi(t),\eta(t))d\mu(\xi,\eta)$$
$$= \quad \int_\Xi f(x,v)dm^{\mu}(t)(x,v).$$

□

An easy consequence of Lemma 3.1 is that for $C = C(r, M)$ as in Proposition 2.2, $\mathcal{P}_{m_0}(\Gamma_C)$ is a closed subset of $\mathcal{P}(\Gamma_C)$, and is therefore compact.

Lemma 3.2. *If* $\mu \in \mathcal{P}(\Gamma_C)$, *the map* $t \mapsto m^\mu(t)$ *is* $1/2$*-Hölder continuous from* $[0, T]$ *to* $\mathcal{P}(K_C)$, *(*K_C *is defined in (2.28) and* $\mathcal{P}(K_C)$ *is endowed with the Kantorovitch-Rubinstein distance).*

Proof. Let ϕ be any Lipschitz function defined on K_C with a Lipschitz constant not larger than 1.

$$\begin{aligned}\int_{K_C} \phi(x,v)\,(dm^\mu(t_2)(x,v) - dm^\mu(t_1)(x,v)) &= \int_{\Gamma_C} (\phi(e_{t_2}(\xi,\eta)) - \phi(e_{t_1}(\xi,\eta)))\,d\mu(\xi,\eta)\\ &= \int_{\Gamma_C} (\phi(\xi(t_2),\eta(t_2)) - \phi(\xi(t_1),\eta(t_1)))\,d\mu(\xi,\eta)\\ &\leqslant \int_{\Gamma_C} (|\xi(t_2)-\xi(t_1)| + |\eta(t_2)-\eta(t_1)|)\,d\mu(\xi,\eta)\\ &\leqslant C\int_{\Gamma_C} \left(|t_2-t_1| + |t_2-t_1|^{\frac{1}{2}}\right) d\mu(\xi,\eta)\\ &\leqslant \tilde{C}\,|t_2-t_1|^{\frac{1}{2}},\end{aligned}$$

for a constant $\tilde{C}$ which depends only on C and T. □

It is useful to recall the disintegration theorem:

Theorem 3.1. *Let X and Y be Radon metric spaces, $\pi : X \to Y$ be a Borel map, μ be a probability measure on X. Set $\nu = \pi\sharp\mu$. There exists a ν-almost everywhere uniquely defined Borel measurable family of probability measures $(\mu_y)_{y\in Y}$ on X such that*

$$\mu_y(X\backslash\pi^{-1}(y)) = 0, \quad \text{for } \nu\text{-almost all } y \in Y, \tag{3.3}$$

and for every Borel function $f : X \to [0,+\infty]$,

$$\int_X f(x)d\mu(x) = \int_Y \left(\int_X f(x)d\mu_y(x)\right) d\nu(y) = \int_Y \left(\int_{\pi^{-1}(y)} f(x)d\mu_y(x)\right) d\nu(y). \tag{3.4}$$

Recall that $(\mu_y)_{y\in Y}$ is a Borel family of probability measures if for any Borel subset B of X, $Y \ni y \mapsto \mu_y(B)$ is a Borel function from Y to $[0,1]$.

It is possible to apply Theorem 3.1 with $X = \Gamma_C$, $Y = \Theta_r$, $\pi = e_0$ and $\nu = m_0$ (identifying m_0 and its restriction to Θ_r): for any $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$, there exists a m_0-almost everywhere uniquely defined Borel measurable family of probability measures $(\mu_{(x,v)})_{(x,v)\in\Theta_r}$ on Γ_C such that

$$\mu_{(x,v)}(\Gamma_C\backslash e_0^{-1}(x,v)) = 0, \quad \text{for } m_0\text{-almost all } (x,v) \in \Theta_r, \tag{3.5}$$

and for every Borel function $f : \Gamma_C \to [0,+\infty]$,

$$\begin{aligned}\int_{\Gamma_C} f(\xi,\eta)d\mu(\xi,\eta) &= \int_{\Theta_r} \left(\int_{\Gamma_C} f(\xi,\eta)d\mu_{(x,v)}(\xi,\eta)\right) dm_0(x,v)\\ &= \int_{\Theta_r} \left(\int_{e_0^{-1}(x,v)} f(\xi,\eta)d\mu_{(x,v)}(\xi,\eta)\right) dm_0(x,v).\end{aligned} \tag{3.6}$$

For $(x,v) \in \Theta_r$, m_0 supported in Θ_r and $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ (where $C = C(r,M)$ is the constant appearing in Proposition 2.2), let us set

$$\Gamma^{\mu,\mathrm{opt}}[x,v] = \left\{(\xi,\eta) \in \Gamma[x,v] \ :\ J^\mu(\xi,\eta) = \min_{(\widetilde{\xi},\widetilde{\eta})\in\Gamma[x,v]} J^\mu(\widetilde{\xi},\widetilde{\eta})\right\}. \tag{3.7}$$

Standard arguments from the calculus of variations yield that for each $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ and $(x,v) \in \Xi^{\rm ad}$, $\Gamma^{\mu,{\rm opt}}[x,v]$ is not empty. Moreover, from Proposition 2.2, $\Gamma^{\mu,{\rm opt}}[x,v] \subset \Gamma_C$ for all $(x,v) \in \Theta_r$.

Proposition 3.1. *Under the assumptions made on ℓ, F and G in paragraph 3.1, and Assumption 3.1, let $C = C(r,M)$ be chosen as in Proposition 2.2.*
Let a sequence of probability measures $(\mu_i)_{i\in\mathbb{N}}$, $\mu_i \in \mathcal{P}_{m_0}(\Gamma_C)$, be narrowly convergent to $\mu \in \mathcal{P}(\Gamma_C)$. Let $(x^i,v^i)_{i\in\mathbb{N}}$, $(x^i,v^i) \in \Theta_r$, converge to (x,v). Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i,\eta^i) \in \Gamma^{\mu_i,{\rm opt}}[x^i,v^i]$. If $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ tends to (ξ,η) uniformly, then $(\xi,\eta) \in \Gamma^{\mu,{\rm opt}}[x,v]$. In other words, the multivalued map $(x,v,\mu) \mapsto \Gamma^{\mu,{\rm opt}}[x,v]$ has closed graph.

Proof. First, from Lemma 3.1, $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ and for all $t \in [0,T]$, $(m^{\mu_i}(t))_{i\in\mathbb{N}}$ is narrowly convergent to $m^\mu(t)$. From the continuity assumptions made on F and G and the dominated convergence theorem, we deduce that

$$\begin{aligned} \int_0^T F[m^{\mu_i}(t)](\xi^i(t),\eta^i(t))dt \quad &\to \quad \int_0^T F[m^{\mu}(t)](\xi(t),\eta(t))dt, \\ G[m^{\mu_i}(T)](\xi^i(T),\eta^i(T)) &\to \quad G[m^{\mu}(T)](\xi(T),\eta(T)). \end{aligned}$$

The last part of the proof is completely similar to the proof of Proposition 2.1. It makes use of Assumption 2.1 and Lemma 3.2. □

Definition 3.1. *The probability measure $\mu \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained mean field game equilibrium associated with the initial distribution m_0 if*

$$\mathrm{supp}(\mu) \subset \bigcup_{(x,v)\in \mathrm{supp}(m_0)} \Gamma^{\mu,{\rm opt}}[x,v]. \tag{3.8}$$

Theorem 3.2. *Under the assumptions made on F and G at the beginning of paragraph 3.1 and Assumption 3.1, let $C = C(r,M)$ be chosen as in Proposition 2.2. There exists a constrained mean field game equilibrium $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$, see Definition 3.1. Moreover, $t \mapsto e_t\sharp\mu \in C^{1/2}([0,T];\mathcal{P}(K_C))$, ($K_C$ is defined in (2.28) and $\mathcal{P}(K_C)$ is endowed with the Kantorovitch-Rubinstein distance).*

Proof. The proof follows that of Cannarsa and Capuani in [10]. Define the multivalued map E from $\mathcal{P}_{m_0}(\Gamma_C)$ to $\mathcal{P}_{m_0}(\Gamma_C)$ as follows: for any $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$,

$$E(\mu) = \left\{\hat\mu \in \mathcal{P}_{m_0}(\Gamma_C) \ : \ \mathrm{supp}(\hat\mu_{(x,v)}) \subset \Gamma^{\mu,{\rm opt}}[x,v] \text{ for } m_0\text{-almost all } (x,v) \in \Xi\right\}, \tag{3.9}$$

where $(\hat\mu_{(x,v)})_{(x,v)\in\Xi}$ is the m_0-almost everywhere uniquely defined Borel measurable family of probability measures which disintegrates $\hat\mu$, see the lines after Theorem 3.1.
Then the measure $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ is a constrained mean field game equilibrium if and only if $\mu \in E(\mu)$. This leads us to apply Kakutani fixed point theorem to the multivalued map E, see [4, 17]. Several steps are needed in order to check that the assumptions of Kakutani theorem are satisfied. First of all, we recall that $\mathcal{P}_{m_0}(\Gamma_C)$ is compact.

Step 1: For any $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$, $E(\mu)$ is a non empty convex set.
First, we have already seen that $\Gamma^{\mu,{\rm opt}}[x,v] \neq \varnothing$ and that the map $(x,v) \mapsto \Gamma^{\mu,{\rm opt}}[x,v]$

has closed graph. Therefore, from [6], $(x,v) \mapsto \Gamma^{\mu,\mathrm{opt}}[x,v]$ has a Borel measurable selection $(x,v) \mapsto (\xi^{\mu}_{(x,v)}, \eta^{\mu}_{(x,v)})$. The measure $\hat{\mu}$ defined by

$$\hat{\mu}(B) = \int_{\Theta_r} \delta_{(\xi^{\mu}_{(x,v)}, \eta^{\mu}_{(x,v)})}(B) dm_0(x,v), \quad \text{for all Borel subset } B \text{ of } \Gamma_C,$$

belongs to $E(\mu)$; indeed, the total mass of $\hat{\mu}$ is one because m_0 is supported in Θ_r and $C = C(r,M)$ as in Proposition 2.2 so $E(\mu)$ is non empty.
Second, take μ^1, μ^2 in $E(\mu)$ and $\lambda \in [0,1]$. We wish to prove that $\lambda\mu_1 + (1-\lambda)\mu_2 \in E(\mu)$. It is clear that $\lambda\mu^1 + (1-\lambda)\mu^2$ belongs to $\mathcal{P}_{m_0}(\Gamma_C)$. On the other hand, since μ^1 belongs to $E(\mu)$, there exist a m_0-almost everywhere uniquely defined Borel measurable family $(\hat{\mu}^1_{(x,v)})_{(x,v)\in\Theta_r}$ of probability measures which disintegrates μ^1 and a subset A^1 of Θ_r such that $m_0(A^1) = 0$ and $\mathrm{supp}(\mu^1_{(x,v)}) \subset \Gamma^{\mu,\mathrm{opt}}[x,v]$ for all $(x,v) \in \Theta_r \backslash A^1$. Similarly, μ^2 can be disintegrated into a m_0-almost everywhere uniquely defined Borel measurable family $(\hat{\mu}^2_{(x,v)})_{(x,v)\in\Theta_r}$ of probability measures, and there exists a subset A^2 of Θ_r such that $m_0(A^2) = 0$ and $\mathrm{supp}(\mu^2_{(x,v)}) \subset \Gamma^{\mu,\mathrm{opt}}[x,v]$ for all $(x,v) \in \Theta_r \backslash A^2$. Therefore, $\lambda\mu^1 + (1-\lambda)\mu^2$ can be disintegrated as follows: for each Borel function f defined on Γ_C,

$$\begin{aligned}
&\int_{\Gamma_C} f(\xi,\eta) d\left(\lambda\mu^1 + (1-\lambda)\,\mu^2\right)(\xi,\eta)] \\
&= \int_{\Theta_r} \left(\int_{\Gamma_C} f(\xi,\eta) d\left(\lambda\mu^1_{(x,v)} + (1-\lambda)\mu^2_{(x,v)}\right)(\xi,\eta) \right) dm_0(x,v), \\
&\quad \mathrm{supp}\left(\lambda\mu^1_{(x,v)} + (1-\lambda)\mu^2_{(x,v)}\right) \subset \Gamma^{\mu,\mathrm{opt}}[x,v], \qquad \forall (x,v) \in \Theta_r \backslash (A^1 \cup A^2),
\end{aligned}$$

and $m_0(A^1 \cup A^2) = 0$. Hence, $\lambda\mu^1 + (1-\lambda)\mu^2 \in E(\mu)$, so $E(\mu)$ is convex.

Step 2: The multivalued map E has closed graph.
Consider a sequence $(\mu^i)_{i\in\mathbb{N}}$, $\mu^i \in \mathcal{P}_{m_0}(\Gamma_C)$ narrowly convergent to $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$. Let a sequence $(\hat{\mu}^i)_{i\in\mathbb{N}}$, $\hat{\mu}^i \in E(\mu^i)$ be narrowly convergent to $\hat{\mu} \in \mathcal{P}_{m_0}(\Gamma_C)$. We claim that $\hat{\mu} \in E(\mu)$.
First, there exists a m_0-almost everywhere uniquely defined Borel measurable family of probability measures $(\hat{\mu}_{(x,v)})_{(x,v)}$ on Γ_C such that (3.5) and (3.6) hold for $\hat{\mu}$ and $\hat{\mu}_{(x,v)}$. In particular, there exists a subset A of Θ_r with $m_0(A) = 0$ such that for $(x,v) \in \Theta_r \backslash A$, $\hat{\mu}_{(x,v)}(\Gamma_C \backslash e_0^{-1}(x,v)) = 0$.
Take $(x,v) \in \Theta_r \backslash A$ and $(\hat{\xi},\hat{\eta}) \in \mathrm{supp}(\hat{\mu}_{(x,v)})$.
The Kuratowski convergence theorem applied to $(\hat{\mu}^i)_i$, $\hat{\mu}$, see [9], implies that there exists a sequence $(\hat{\xi}^i,\hat{\eta}^i)_{i\in\mathbb{N}}$, $(\hat{\xi}^i,\hat{\eta}^i) \in \mathrm{supp}(\hat{\mu}^i)$, which converges to $(\hat{\xi},\hat{\eta})$ uniformly in $[0,T]$. Set $(x^i,v^i) = (\hat{\xi}^i(0),\hat{\eta}^i(0)) \in \Theta_r$. Since $\hat{\mu}^i \in E(\mu^i)$, there holds that $(\hat{\xi}^i,\hat{\eta}^i) \in \Gamma^{\mu^i,\mathrm{opt}}[x^i,v^i]$. From Proposition 3.1, we see that $(\hat{\xi},\hat{\eta}) \in \Gamma^{\mu,\mathrm{opt}}[x,v]$. Since (x,v) is any point in $\Theta_r \backslash A$, this implies that $\hat{\mu} \in E(\mu)$.

All the assumptions of Kakutani theorem are satisfied: hence, there exists $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ such that $\mu \in E(\mu)$. This achieves the proof. □

Definition 3.2. *A pair (u,m), where u is a measurable function defined on $\Xi \times [0,T]$ and $m \in C^0([0,T];\mathcal{P}(\Xi))$, is called a mild solution of the mean field game, if there exists a constrained mean fied game equilibrium μ for m_0 (see Definition 3.1) such that:*

i) $m(t) = e_t \sharp \mu$;

ii) $\forall (x,v) \in \Xi^{\rm ad}$, $u(x,v,t)$ *is given by*

$$u(x,v,t) = \inf_{(\xi,\eta,\alpha)\in\Gamma[x,v,t]} \left(\begin{array}{l} \displaystyle\int_t^T \left(F[m(s)](\xi(s),\eta(s)) + \ell(\xi(s),\eta(s),s) + \frac{1}{2}|\alpha(s)|^2 \right) ds \\ + G[m(T)](\xi(T),\eta(T)) \end{array} \right).$$

where $\Gamma[x,v,t]$ *is the set of admissible trajectories starting from* (x,v) *at* $s=t$.

A corollary of Theorem 3.2 is:

Corollary 3.1. *Under the assumptions of Theorem 3.2, there exists a mild solution* (u,m). *Moreover,* $m \in C^{\frac{1}{2}}([0,T];\mathcal{P}(K_C))$.

Remark 3.3. *Under classical monotonicity assumptions for* F *and* G, *see e.g. [10], the mild solution is unique.*

3.3 Non quadratic running costs

It is possible to generalize the results of Sections 2 and 3 to costs of the form

$$J(\xi,\eta,\alpha) = \int_0^T \left(\ell(\xi(s),\eta(s),s) + \frac{1}{p}|\alpha|^p(s) \right) ds + g(\xi(T),\eta(T)), \tag{3.10}$$

where $1 < p$, for dynamics given by (2.1) and staying in Ξ.
For brevity, we restrict ourselves to the closed graph result, whose proof is completely similar to that of Proposition 2.1. The generalization of Theorem 3.2 is then possible.

Proposition 3.2. *Consider a closed subset* Θ *of* $\Xi^{\rm ad}$. *Assume that all sequence* $(x^i,v^i)_{i\in\mathbb{N}}$ *such that for all* $i \in \mathbb{N}$, $(x^i,v^i) \in \Theta$ *and* $\lim_{i\to+\infty}(x^i,v^i) = (x,v) \in \Theta$, *the following holds: if* $x \in \partial\Omega$, *then*

$$((v^i \cdot \nabla d(x^i))_+)^{2p-1} = o\left(\left| d(x^i) \right|^{p-1} \right); \tag{3.11}$$

then the graph of the multivalued map $\Gamma^{\rm opt} : \Theta \to \Gamma$, $(x,v) \mapsto \Gamma^{\rm opt}[x,v]$ *is closed in the sense given in Proposition 2.1.*

4 One dimensional problems: more accurate results

In dimension one and for a running cost quadratic in α, it is possible to obtain more accurate results under a slightly stronger assumption on the running cost, namely that it does not favor the trajectories which exit the domain. In particular, the closed graph property can be proved to hold on the whole set $\Xi^{\rm ad}$, and concerning mean field games, no assumptions are needed on the support of m_0 by contrast with Theorem 3.2.

4.1 Optimal control problem in an interval: a closed graph property

In this paragraph, we set $\Omega = (-1,0)$ and $\Xi = [-1,0] \times \mathbb{R}$. The optimal control problem consists of minimizing $J(\xi,\eta,\eta')$ given by (2.2) on the dynamics given by (2.1) and staying in Ξ.
The definition of $\Xi^{\rm ad}$ is then modified as follows:

$$\Xi^{\rm ad} = \Xi \backslash \Big(\{0\} \times (0,+\infty) \cup \{-1\} \times (-\infty,0) \Big). \tag{4.1}$$

We make the following assumptions:

Assumption 4.1. *The running cost $\ell : \Xi \times [0,T] \to \mathbb{R}$ is a continuous function, bounded from below. The terminal cost $g : \Xi \to \mathbb{R}$ is also assumed continuous and bounded from below. Set $M = \|g_-\|_{L^\infty(\Xi)} + \|\ell_-\|_{L^\infty(\Xi\times[0,T])}$.*

Assumption 4.2. *For all $t \in [0,T]$ and $v > 0$,*

$$\ell(0,v,t) \geqslant \ell(0,0,t), \qquad \text{and} \qquad \ell(-1,-v,t) \geqslant \ell(-1,0,t). \tag{4.2}$$

An interpretation of Assumption 4.2 is that the running cost ℓ penalizes (or at least does not favor) the trajectories that exit Ξ^{ad}. In that respect, Assumption 4.2 is rather natural.

For $(x,v) \in \Xi$, let Γ, $\Gamma[x,v]$ and $\Gamma^{\mathrm{opt}}[x,v]$ be defined as follows:

$$\Gamma = \left\{ (\xi,\eta) \in C^1([0,T];\mathbb{R}) \times AC([0,T];\mathbb{R}) \ : \ \left| \begin{array}{ll} \xi'(s) = \eta(s), & \forall s \in [0,T], \\ (\xi(s),\eta(s)) \in \Xi, & \forall s \in [0,T] \end{array} \right. \right\},$$

$$\Gamma[x,v] = \{(\xi,\eta) \in \Gamma : \xi(0) = x,\ \eta(0) = v\},$$

$$\Gamma^{\mathrm{opt}}[x,v] = \mathrm{argmin}_{(\xi,\eta)\in\Gamma[x,v]} J(\xi,\eta,\eta').$$

Theorem 4.1. *Under Assumptions 4.1 and 4.2, the graph of the multivalued map $\Gamma^{\mathrm{opt}} : \Xi^{\mathrm{ad}} \to \Gamma$, $(x,v) \mapsto \Gamma^{\mathrm{opt}}[x,v]$, is closed, in the following sense: consider a sequence $(x^i,v^i)_{i\in\mathbb{N}}$, $(x^i,v^i) \in \Xi^{\mathrm{ad}}$, such that $\lim_{i\to\infty}(x^i,v^i) = (x,v) \in \Xi^{\mathrm{ad}}$. Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(\xi^i,\eta^i) \in \Gamma^{\mathrm{opt}}[x^i,v^i]$.*
If (ξ^i,η^i) tends to (ξ,η) uniformly, then $(\xi,\eta) \in \Gamma^{\mathrm{opt}}[x,v]$.

Remark 4.1. *Note that, by contrast with Proposition 2.1, Theorem 4.1 holds for Γ^{opt} and not only its restriction to a subset Θ of Ξ^{ad} satisfying suitable conditions. Hence, Theorem 4.1 is more accurate. On the other hand, it requires an additional assumption, namely Assumption 4.2.*
Note also that the result stated in Theorem 4.1, namely the closed graph property of the multivalued map Γ^{opt}, is obtained despite the fact that the value function of the optimal control problem is not continuous and not locally bounded on Ξ^{ad}. This may seem surprising at first glance. Besides, the fact that the value function is singular at some points of Ξ^{ad} will be an important difficulty in the proofs.

The proof of Theorem 4.1 relies on several lemmas.

Lemma 4.1. *Consider $(x,v) \in \Xi^{\mathrm{ad}}$, $(\xi,\eta) \in \Gamma[x,v]$ such that $\eta \in W^{1,2}(0,T;\mathbb{R})$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i,v^i) \in \Xi^{\mathrm{ad}}$ and $(x^i,v^i) \to (x,v)$ as $i \to \infty$.*
If one among the following assumptions is satisfied,

1. *$x \in \Omega$*
2. *$x = 0$, $v \leqslant 0$ and for all integer i, $v^i \leqslant 0$*
3. *$(x,v) = (0,0)$, $v^i > 0$ for all integer i and $\lim_{i\to 0} \dfrac{(v^i)^3}{|x^i|} = 0$*
4. *$x = -1$, $v \geqslant 0$ and for all integer i, $v^i \geqslant 0$*
5. *$(x,v) = (-1,0)$, $v^i < 0$ for all integer i and $\lim_{i\to 0} \dfrac{|v^i|^3}{|x^i+1|} = 0$*

then there exists a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ *such that for all* $i\in\mathbb{N}$, $(\xi^i,\eta^i)\in\Gamma[x^i,v^i]$, $\eta^i\in W^{1,2}(0,T;\mathbb{R})$, *and* (ξ^i,η^i) *tends to* (ξ,η) *in* $W^{2,2}(0,T;\mathbb{R})\times W^{1,2}(0,T;\mathbb{R})$, *hence uniformly in* $[0,T]$.

Proof. Lemma 4.1 is the counterpart of Lemma 2.2 . The proof is quite similar, so we skip it for brevity. □

Corollary 4.1. *Consider* $(x,v)\in\Xi^{\rm ad}$ *and a sequence* $(x^i,v^i)_{i\in\mathbb{N}}$ *such that for all* $i\in\mathbb{N}$, $(x^i,v^i)\in\Xi^{\rm ad}$ *and* $(x^i,v^i)\to(x,v)$ *as* $i\to\infty$. *Suppose that Assumption 4.1 and one among the five conditions in Lemma 4.1 are satisfied. Let a sequence* $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ *be such that for all* $i\in\mathbb{N}$, $(\xi^i,\eta^i)\in\Gamma^{\rm opt}[x^i,v^i]$. *If* (ξ^i,η^i) *tends to* (ξ,η) *uniformly in* $[0,T]$, *then* $\eta\in W^{1,2}(0,T;\mathbb{R})$ *and* $(\xi,\eta)\in\Gamma^{\rm opt}[x,v]$.

Proof. Corollary 4.1 is the counterpart of Lemma 2.3. The proof is identical. □

Consider $(x,v)\in\Xi^{\rm ad}$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i\in\mathbb{N}$, $(x^i,v^i)\in\Xi^{\rm ad}$ and $(x^i,v^i)\to(x,v)$ as $i\to\infty$. Because it is always possible to extract subsequences, we can say that the only cases that have not yet been addressed in Lemma 4.1 are the following:

$$\left\{\begin{array}{l}(x,v)=(0,0),\quad v^i>0,\\ \text{ and there exists a constant } C>0 \text{ s. t. for all } i\in\mathbb{N},\ \frac{(v^i)^3}{|x^i|}\geqslant C,\end{array}\right. \tag{4.3}$$

or

$$\left\{\begin{array}{l}(x,v)=(-1,0),\quad v^i<0,\\ \text{ and there exists a constant } C>0 \text{ s. t. for all } i\in\mathbb{N},\ \frac{|v^i|^3}{|x^i+1|}\geqslant C.\end{array}\right. \tag{4.4}$$

Since the two cases are symmetrical, we may concentrate on (4.3).
It is clear that (4.3) implies that $|x^i|/v^i\to 0$ as $i\to+\infty$, because $v^i\to 0$. In the case when (4.3) is satisfied, we need two technical lemmas which provide a lower bound for the cost $\int_0^T\left|\frac{d\eta^i}{dt}(s)\right|^2 ds$ of the admissible trajectories starting at (x^i,v^i):

Lemma 4.2. *Consider* $(x,v)\in\Xi^{\rm ad}$ *such that* $x<0$, $v>0$, $3|x|/v<T$, *and* $\theta\in(0,T)$. *Given a real number* $w\in[0,|x|/T]$, *set*

$$K_{\theta,w}=\left\{\eta\in W^{1,2}(0,\theta;\mathbb{R}):\left|\begin{array}{l}\eta(0)=v,\quad \eta(\theta)=w,\\ \eta(s)\geqslant w,\ \forall s\in[0,\theta],\\ x+\displaystyle\int_0^\theta\eta(s)ds\leqslant 0\end{array}\right.\right\}. \tag{4.5}$$

The quantity

$$I(\theta,w)=\inf_{\eta\in K_{\theta,w}}\frac{1}{2}\int_0^\theta\left|\frac{d\eta}{dt}(s)\right|^2 ds \tag{4.6}$$

is achieved by a function $\eta=\eta_{\theta,w}$ *and is given by*

$$I(\theta,w)=\left\{\begin{array}{ll}\dfrac{1}{2}\dfrac{(w-v)^2}{\theta}, & \text{if}\quad \theta\in\left[0,\frac{2|x|}{v+w}\right],\\ 6\dfrac{x^2}{\theta^3}+6\dfrac{x(v+w)}{\theta^2}+2\dfrac{v^2+vw+w^2}{\theta} & \text{if}\quad \theta\in\left[\frac{2|x|}{v+w},\frac{3|x|}{v+2w}\right],\\ \dfrac{2}{9}\dfrac{(v-w)^3}{|x|-w\theta}, & \text{if}\quad \theta\in\left[\frac{3|x|}{v+2w},T\right).\end{array}\right. \tag{4.7}$$

Remark 4.2. *The partition of the interval* $[0,T]$ *in (4.7) is justified by the assumptions of Lemma 4.2. Indeed*

- $3|x|/v < T$ *and* $w \geqslant 0$ *imply that* $3|x|/(v+2w) < T$
- $2|x|/(v+w) < 3|x|/(v+2w)$ *because* $0 \leqslant w \leqslant |x|/T < v/3$.

Note also that if $|x|/v \to 0$, *then* $3|x|/(v+2w) \sim 3|x|/v \ll T$.

Proof. Problem (4.6) is the minimization of a strictly convex and continuous functional under linear and continuous constraints, and the set $K_{\theta,w}$ is non empty, as we shall see below, convex and closed. Hence there exists a unique minimizer, named η again. The Euler-Lagrange necessary conditions read as follows: there exists a real number $\mu \geqslant 0$ such that η is a weak solution of the linear complementarity problem (variational inequality)

$$
\left\{
\begin{array}{rcll}
-\eta'' & \geqslant & -\mu, & \text{in } (0,\theta), \\
\eta & \geqslant & w, & \text{in } (0,\theta), \\
(-\eta''+\mu)(\eta-w) & = & 0 & \text{in } (0,\theta), \\
x+\displaystyle\int_0^\theta \eta(s)ds & \leqslant & 0, & \\
\mu & \geqslant & 0, & \\
\mu\left(x+\displaystyle\int_0^\theta \eta(s)ds\right) & = & 0, & \\
\eta(0) & = & v, & \\
\eta(\theta) & = & w. &
\end{array}
\right. \tag{4.8}
$$

The solution of (4.8) can be written explicitly. Skipping the details, it has the following form:

1. If $\theta \geqslant 3|x|/(v+2w)$, then

$$
\left\{
\begin{array}{rcll}
\eta(t) & = & v-\mu\tau t+\dfrac{\mu}{2}t^2, & 0 \leqslant t \leqslant \tau, \\
\eta(t) & = & w, & \tau < t \leqslant \theta,
\end{array}
\right. \tag{4.9}
$$

with

$$
\tau = -3\frac{x+w\theta}{v-w} \quad \text{and} \quad \mu = \frac{2(v-w)^3}{9(x+w\theta)^2}. \tag{4.10}
$$

Note that $-3\frac{x+w\theta}{v-w} \leqslant \theta$ because $\theta \geqslant \frac{3|x|}{v+2w}$. Note also that $x+\displaystyle\int_0^\theta \eta(s)ds = 0$. We see that $I(\theta,w) = \dfrac{\mu^2}{2}\displaystyle\int_0^\tau (-\tau+t)^2dt = \dfrac{\mu^2\tau^3}{6} = \dfrac{2}{9}\dfrac{(v-w)^3}{|x|-w\theta}$; we have obtained the third line in (4.7).

2. If $2|x|/(v+w) \leqslant \theta \leqslant 3|x|/(v+2w)$, then for all $t \in [0,\theta]$,

$$
\eta(t) = v+kt+\frac{\mu}{2}t^2, \tag{4.11}
$$

with

$$
k = -\frac{6x+(4v+2w)\theta}{\theta^2}, \quad \text{and} \quad \mu = 6\frac{2x+(v+w)\theta}{\theta^3}. \tag{4.12}
$$

Note that $x+\displaystyle\int_0^\theta \eta(s)ds = 0$. Easy algebra leads to $I(\theta,w) = 6\frac{x^2}{\theta^3}+6\frac{x(v+w)}{\theta^2}+2\frac{v^2+vw+w^2}{\theta}$; we have obtained the second line in (4.7).

3. If $\theta \leqslant 2|x|/(v+w)$, then for all $t \in [0,\theta]$,

$$\eta(t) = v - (v-w)\frac{t}{\theta}. \tag{4.13}$$

Then, $I(\theta, w) = \frac{1}{2}\frac{(w-v)^2}{\theta}$; we have obtained the first line in (4.7). Note that if $\theta < \frac{2|x|}{v+w}$, then $x + \displaystyle\int_0^\theta \eta(s)ds < 0$.

□

Lemma 4.3. *Consider a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that $x^i < 0$, $v^i > 0$ for all $i \in \mathbb{N}$, and $v^i \to 0$, $|x^i|/v^i \to 0$ as $i \to +\infty$. Call $I^i(\theta, w)$ the quantity given by (4.6) for $v = v^i$, $x = x^i$ and $w \in \big[0, x^i/T\big]$. Then*

$$\inf\big\{I^i(\theta, w),\ \theta \in (0,T)\big\} = \frac{2}{9}\frac{(v^i)^3}{|x^i|} + o(1), \tag{4.14}$$

where $o(1)$ is a quantity that tends to 0 as i tends to infinity (which is in fact of the order of $(v^i)^2$ or smaller).

Proof. Recall that $I^i(\theta, w)$ is given by (4.7). It is easy to see that $\theta \mapsto I^i(\theta, w)$ is decreasing on $\big(0, 2|x^i|/(v^i+w)\big]$ and increasing on $\big[3|x^i|/(v^i+2w), T\big]$.
In $\big[2|x^i|/(v^i+w), 3|x^i|/(v^i+2w)\big]$, $I^i(\theta, w) = P(1/\theta)$, where P is the third order polynomial:

$$P(z) = 6(x^i)^2 z^3 + 6x^i(v^i+w)z^2 + 2((v^i)^2 + v^i w + w^2)z.$$

The roots of the second order polynomial $P'(z) = 18(x^i)^2z^2 + 12x^i(v^i+w)z + 2((v^i)^2 + v^i w + w^2)$ are $\dfrac{v^i + w \pm \sqrt{v^i w}}{3|x^i|}$. Hence, $\theta \mapsto I^i(\theta, w)$ is decreasing in $\left[\dfrac{2|x^i|}{v^i+w}, \dfrac{3|x^i|}{v^i+w+\sqrt{v^i w}}\right]$ and increasing in $\left[\dfrac{3|x^i|}{v^i+w+\sqrt{v^i w}}, \dfrac{3|x^i|}{v^i+2w}\right]$.
Therefore, the minimizer of $\theta \mapsto I^i(\theta, w)$ on $[0, T)$ is $\theta = \frac{3|x^i|}{v^i+w+\sqrt{v^i w}}$ and the minimal value is

$$\begin{aligned}
& P\left(\tfrac{v^i+w+\sqrt{v^i w}}{3|x^i|}\right) \\
= \ & \frac{2(v^i)^3}{9|x^i|}\left(1 + \sqrt{\frac{w}{v^i}} + \frac{w}{v^i}\right)^3 - \frac{2(v^i)^3}{3|x^i|}\left(1 + \sqrt{\frac{w}{v^i}} + \frac{w}{v^i}\right)^2 + \frac{2(v^i)^3}{3|x^i|}\left(1 + \sqrt{\frac{w}{v^i}} + \frac{w}{v^i}\right) \\
& + O\left((v^i)^2\tfrac{w}{|x^i|}\right) \\
= \ & \frac{2(v^i)^3}{9|x^i|} + O\left((v^i)^2\frac{w}{|x^i|}\right).
\end{aligned}$$

□

The next lemma is the counterpart of Lemma 4.1 when (4.3) holds. By contrast with the situations considered so far, Assumption 4.2 is used.

Lemma 4.4. *Under Assumptions 4.1 and 4.2, consider a sequence $(x^i, v^i)_{i\in\mathbb{N}}$ which tends to $(x, v) = (0, 0)$ as $i \to \infty$, and which satisfies (4.3). Let a sequence $(\xi^i, \eta^i)_{i\in\mathbb{N}}$ be such that for all $i \in \mathbb{N}$, $(\xi^i, \eta^i) \in \Gamma^{\rm opt}[x^i, v^i]$. If (ξ^i, η^i) tends to (ξ, η) uniformly in $[0, T]$, then $\eta \in W^{1,2}(0, T; \mathbb{R})$ and $(\xi, \eta) \in \Gamma^{\rm opt}[x, v]$.*

Proof. The proof is more difficult than that of Lemma 4.1 because we will see that in general, the sequence $u(x^i, v^i)$ does not converge to $u(0,0)$ as $i \to \infty$, and that $\displaystyle\int_0^T \left(\frac{d\eta^i}{dt}(s)\right)^2 ds$ may tend to $+\infty$.

Step 1 : We start by building a particular competitor for the optimal control problem at (x^i, v^i). It will be used in Steps 2 and 3 below. Let us set $\widetilde{t}_i = 3|x^i|/v^i$ (observe that $\lim_{i\to\infty} \widetilde{t}_i = 0$ since $v^i \to 0$ and $(v^i)^3/|x_i| \geqslant C > 0$). As in the proof of Lemma 4.2 with $w = 0$, we construct a pair of continuous functions $(\widetilde{\xi}^i, \widetilde{\eta}^i)$ defined on $[0, \widetilde{t}_i]$ such that $-1 \leqslant \widetilde{\xi}^i \leqslant 0$ and $\frac{d\widetilde{\xi}^i}{dt} = \widetilde{\eta}^i$, and

1. $(\widetilde{\xi}^i(0), \widetilde{\eta}^i(0)) = (x^i, v^i)$
2. $\widetilde{\eta}^i(\widetilde{t}_i) = 0$
3. $\displaystyle\frac{1}{2}\int_0^{\widetilde{t}_i} \left(\frac{d\widetilde{\eta}^i}{dt}(s)\right)^2 ds \sim \frac{2}{9}\frac{(v^i)^3}{|x^i|}$,

(we have also used Lemma 4.3 with $w = 0$ and Remark 4.2). Observe that $x^i \leqslant \widetilde{\xi}^i(\widetilde{t}_i) \leqslant 0$, hence $\lim_{i\to+\infty} \widetilde{\xi}^i(\widetilde{t}_i) = 0$. Then, using the same arguments as in Lemma 2.2, it is possible to extend continuously $(\widetilde{\xi}^i, \widetilde{\eta}^i)$ to $[0,T]$ in such a way that

1. $(\widetilde{\xi}^i, \widetilde{\eta}^i) \in \Gamma[x^i, v^i]$
2. $\displaystyle\lim_{i\to\infty} \int_{\widetilde{t}_i}^T \left|\frac{d\widetilde{\eta}^i}{dt}(s) - \alpha(s - \widetilde{t}_i)\right|^2 ds = 0$, where α is an optimal control law for trajectories with initial values $(0,0)$.

Combining all the information above, we obtain that

$$J\left(\widetilde{\xi}^i, \widetilde{\eta}^i, \frac{d\widetilde{\eta}^i}{dt}\right) = \frac{2}{9}\frac{(v^i)^3}{|x^i|} + u(0,0) + o(1). \tag{4.15}$$

Step 2 : Since $(\xi^i, \eta^i) \in \Gamma^{\rm opt}[x^i, v^i]$, we know that for all $t \in [0,T]$, $\xi^i(t) = x^i + \displaystyle\int_0^t \eta^i(s)ds \leqslant 0$. We claim that there exists $t_i \in (0,T]$ such that $\eta^i(t_i) \leqslant -x^i/T$. Indeed, if it was not the case, then $\xi^i(T)$ would be larger than $x^i - T(x^i/T) = 0$, which is not true. Since η^i is continuous, we may define θ_i as the minimal time t such that $\eta^i(t) \leqslant -x^i/T$, and we see that $\eta^i(\theta_i) = -x^i/T$.

Step 2 consists of proving that

$$\lim_{i\to\infty} \theta_i = 0. \tag{4.16}$$

Suppose by contradiction that there exists $\delta > 0$ such that $\theta_i \geqslant \delta$. We may apply Lemma 4.2 with $w = |x^i|/T$. Since $v^i \to 0$ and $(v^i)^3/|x^i| \geqslant C > 0$, we see that $|x^i|/v^i \to 0$, then that $\lim_{i\to\infty} \frac{3|x^i|}{v^i + 2|x^i|/T} = 0$. Hence, for i large enough, $\theta_i \geqslant \delta > \frac{3|x^i|}{v^i+2|x^i|/T}$, and the third line of (4.7) yields

$$\frac{1}{2}\int_0^T \left(\frac{d\eta^i}{dt}(s)\right)^2 ds \geqslant \frac{2T}{9(T-\delta)}\frac{(\frac{x^i}{T} + v^i)^3}{|x^i|} = \frac{2T}{9(T-\delta)}\frac{(v^i)^3}{|x^i|} + o(1), \tag{4.17}$$

where $o(1)$ is a quantity that tends to zero as $i \to \infty$, (in fact like $(v^i)^2$).
Note that $\eta^i \geqslant -x^i/T \geqslant 0$ in $[0, \theta_i]$ yields that $\xi^i \geqslant x^i$ in $[0, \theta_i]$. Therefore

$$\lim_{i\to\infty} \|\xi^i\|_{L^\infty(0,\theta_i)} = 0. \tag{4.18}$$

Let us construct an admissible trajectory $(\widehat{\xi}^i, \widehat{\eta}^i)$ starting from $(x, v) = (0, 0)$ as follows:

1. For $s \in [0, \theta_i]$, $\widehat{\xi}^i(s) = Q_{\theta_i,0,0,\xi^i(\theta_i),\eta^i(\theta_i)}(s)$ and $\widehat{\eta}^i(s) = Q'_{\theta_i,0,0,\xi^i(\theta_i),\eta^i(\theta_i)}(s)$, see Definition 2.1
2. $(\widehat{\xi}^i(s), \widehat{\eta}^i(s)) = (\xi^i(s), \eta^i(s))$ for $s \in [\theta_i, T]$.

It is easy to check that, if $s \leqslant \theta_i$, then

$$\widehat{\xi}^i(s) = \left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\frac{s^3}{\theta_i^3} - \left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right)\frac{s^2}{\theta_i^2}, \tag{4.19}$$

$$\widehat{\eta}^i(s) = 3\left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\frac{s^2}{\theta_i^3} - 2\left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right)\frac{s}{\theta_i^2}, \tag{4.20}$$

$$\frac{d\widehat{\eta}^i}{dt}(s) = 6\left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\frac{s}{\theta_i^3} - 2\left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right)\frac{1}{\theta_i^2}. \tag{4.21}$$

Since $\eta^i(\theta_i) = -x^i/T > 0$ and $\xi^i(\theta_i) \leqslant 0$, we see that $\left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right) \geqslant 0$ and that $\left(\theta_i\eta^i(\theta_i) - 3\xi^i(\theta_i)\right) \geqslant 0$. Hence for $s \in [0, \theta_i]$, $\widehat{\xi}^i(s) = \left(\theta_i\eta^i(\theta_i) - 2\xi^i(\theta_i)\right)\left(\frac{s^3}{\theta_i^3} - \frac{s^2}{\theta_i^2}\right) + \xi^i(\theta_i)\frac{s^2}{\theta_i^2} \leqslant 0$ as the sum of two nonpositive terms. Therefore $(\widehat{\xi}^i, \widehat{\eta}^i) \in \Gamma[0, 0]$. On the other hand, using (4.18) and the fact that $\theta_i\eta^i(\theta_i) = \theta_i|x^i|/T$, then (4.19) and (4.20), we see that

$$\lim_{i\to+\infty}\left(\|\widehat{\xi}^i\|_{L^\infty(0,\theta_i)} + \|\widehat{\eta}^i)\|_{L^\infty(0,\theta_i)}\right) = 0. \tag{4.22}$$

Moreover, since $\theta_i \geqslant \delta > 0$, it is easy to check that

$$\lim_{i\to+\infty}\int_0^{\theta_i}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2 ds = 0. \tag{4.23}$$

Since $(\widehat{\xi}^i, \widehat{\eta}^i) \in \Gamma[0, 0]$,

$$\begin{aligned}
u(0,0) \leqslant\ & J(\widehat{\xi}^i, \widehat{\eta}^i, 0)\\
=\ & \int_0^T \left(\ell(\widehat{\xi}^i(s), \widehat{\eta}^i(s), s) + \frac{1}{2}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2\right) ds + g(\widehat{\xi}^i(T), \widehat{\eta}^i(T))\\
=\ & \int_0^{\theta_i} \left(\ell(\widehat{\xi}^i(s), \widehat{\eta}^i(s), s) + \frac{1}{2}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2\right) ds\\
& + \int_{\theta_i}^T \left(\ell(\xi^i(s), \eta^i(s), s) + \frac{1}{2}\left(\frac{d\eta^i}{dt}(s)\right)^2\right) ds + g(\xi^i(T), \eta^i(T))\\
=\ & u(x^i, v^i) + \int_0^{\theta_i} \left(\ell(\widehat{\xi}^i(s), \widehat{\eta}^i(s), s) + \frac{1}{2}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2\right) ds\\
& - \int_0^{\theta_i} \left(\ell(\xi^i(s), \eta^i(s), s) + \frac{1}{2}\left(\frac{d\eta^i}{dt}(s)\right)^2\right) ds.
\end{aligned}$$

Therefore

(4.24)
$$\begin{aligned} u(x^i,v^i) \geqslant \quad & u(0,0)+\frac{1}{2}\int_0^{\theta_i}\left(\frac{d\eta^i}{dt}(s)\right)^2 ds \\ & +\int_0^{\theta_i}\Big(\ell(\xi^i(s),\eta^i(s),s)-\ell(\widehat{\xi}^i(s),\widehat{\eta}^i(s),s)\Big)\,ds-\frac{1}{2}\int_0^{\theta_i}\left(\frac{d\widehat{\eta}^i}{dt}(s)\right)^2 ds. \end{aligned}$$

Let us address the terms in the right hand side of (4.24) separately.
Thanks to the continuity of ℓ, (4.18), (4.22) and Assumption 4.2, we see that

(4.25)
$$\begin{aligned} & \liminf_{i\to\infty}\int_0^{\theta_i}\Big(\ell(\xi^i(s),\eta^i(s),s)-\ell(\widehat{\xi}^i(s),\widehat{\eta}^i(s),s)\Big)\,ds \\ = \quad & \liminf_{i\to\infty}\int_0^{\theta_i}\big(\ell(0,\eta^i(s),s)-\ell(0,0,s)\big)\,ds \geqslant 0. \end{aligned}$$

Combining (4.25), (4.23) and (4.17), we obtain that

$$u(x^i,v^i) \geqslant \frac{2T}{9(T-\delta)}\frac{(v^i)^3}{|x^i|}+u(0,0)+o(1), \tag{4.26}$$

where $o(1)$ is quantity that tends to 0 as $i\to\infty$.
But for $(\widetilde{\xi}^i,\widetilde{\eta}^i)$ constructed in Step 1, $J\left(\widetilde{\xi}^i,\widetilde{\eta}^i,\frac{d\widetilde{\eta}^i}{dt}\right)\geqslant u(x^i,v^i)$. This fact and (4.15) lead to a contradiction with (4.26). We have proved (4.16).

Step 3 : Since $\lim_{i\to\infty}\theta_i=0$ and (ξ^i,η^i) converges uniformly to (ξ,η), we see that

$$\int_0^{\theta_i}\ell(\xi^i(s),\eta^i(s),s)ds=o(1).$$

Hence

(4.27)
$$\begin{aligned} u(x^i,v^i) =& J\left(\xi^i,\eta^i,\frac{d\eta^i}{dt}\right) \\ =& \frac{1}{2}\int_0^{\theta_i}\left(\frac{d\eta^i}{dt}(s)\right)^2 ds \\ & +\int_{\theta_i}^{T}\ell(\xi^i(s),\eta^i(s),s)ds+\frac{1}{2}\int_{\theta_i}^{T}\left(\frac{d\eta^i}{dt}(s)\right)^2 ds+g(\xi^i(T),\eta^i(T))+o(1). \end{aligned}$$

On the other hand, we have seen above that (4.15) implies that

$$u(x^i,v^i)\leqslant\frac{2}{9}\frac{(v^i)^3}{|x^i|}+u(0,0)+o(1). \tag{4.28}$$

From Lemma 4.3, we know that

$$\frac{1}{2}\int_0^{\theta_i}\left(\frac{d\eta^i}{dt}(s)\right)^2 ds\geqslant\frac{2}{9}\frac{(v^i)^3}{|x^i|}-o(1). \tag{4.29}$$

Combining (4.27), (4.28) and (4.29) yields that

$$\int_{\theta_i}^{T}\ell(\xi^i(s),\eta^i(s),s)ds+\frac{1}{2}\int_{\theta_i}^{T}\left(\frac{d\eta^i}{dt}(s)\right)^2 ds+g(\xi^i(T),\eta^i(T))\leqslant u(0,0)+o(1). \tag{4.30}$$

Since (ξ^i,η^i) converges uniformly to (ξ,η), (4.30) implies that $\left(\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt}\right)_{i\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T)$. Hence there exists $\phi\in L^2(0,T)$ such that, after the extraction of subsequence, $\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt}\rightharpoonup\phi$ in $L^2(0,T)$ weak. By testing with compactly supported functions in $(0,T)$, it is clear that $\phi=\frac{d\eta}{dt}$. Hence, the whole sequence $\left(\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt}\right)_{i\in\mathbb{N}}$ converges in $L^2(0,T)$ weak to $\frac{d\eta}{dt}\in L^2(0,T)$. Moreover, the weak convergence in $L^2(0,T)$ implies that

$$\int_0^T\left(\frac{d\eta}{dt}(s)\right)^2ds\leqslant\liminf_{i\to+\infty}\int_0^T\left(\mathbb{1}_{(\theta_i,T)}\frac{d\eta^i}{dt}(s)\right)^2ds.$$

This and (4.30) imply that

$$\int_0^T\ell(\xi(s),\eta(s),s)ds+\frac{1}{2}\int_0^T\left(\frac{d\eta}{dt}(s)\right)^2ds+g(\xi(T),\eta(T))\leqslant u(0,0). \tag{4.31}$$

Hence, $(\xi,\eta)\in\Gamma^{\rm opt}[0,0]$ and the above inequality is in fact an identity. The proof is achieved. □

Proof of Theorem 4.1. Consider $(x,v)\in\Xi^{\rm ad}$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i\in\mathbb{N}$, $(x^i,v^i)\in\Xi^{\rm ad}$ and $(x^i,v^i)\to(x,v)$ as $i\to\infty$. Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that for all $i\in\mathbb{N}$, $(\xi^i,\eta^i)\in\Gamma^{\rm opt}[x^i,v^i]$ and that (ξ^i,η^i) tends to (ξ,η) uniformly. Possibly after the extraction of a subsequence, we can always assume that either one among the five conditions in Lemma 4.1 or one among the two symmetrical conditions (4.3)-(4.4) holds. Then the conclusion follows from Corollary 4.1 in the former case or from Lemma 4.4 in the latter case. □

Remark 4.3. *For costs of the form*

$$J(\xi,\eta,\alpha)=\int_0^T\left(\ell(\xi(s),\eta(s),s)+\frac{1}{p}|\alpha|^p(s)\right)ds+g(\xi(T),\eta(T)), \tag{4.32}$$

with $1<p\neq 2$, it is not possible to reproduce the explicit calculations of Lemmas 4.2 and 4.3, which are crucial steps for Lemma 4.4 and finally for Theorem 4.1.

4.2 Bounds related to optimal trajectories

Proposition 4.1. *For positive numbers r and C, let us set*

$$\Theta_r = \left\{(x,v)\in\Xi:-r(x+1)\leqslant v^3\leqslant r|x|\right\}, \tag{4.33}$$

$$K_C = \{(x,v)\in\Xi:\ |v|\leqslant C\}, \tag{4.34}$$

$$\Gamma_C = \left\{(\xi,\eta)\in\Gamma:\left|\begin{array}{ll}(\xi(t),\eta(t))\in K_C, & \forall t\in[0,T],\\ \left\|\frac{d\eta}{dt}\right\|_{L^2(0,T;\mathbb{R})}\leqslant C.\end{array}\right.\right\}. \tag{4.35}$$

Under Assumption 4.1, for all $r>0$, there exists a positive number $C=C(r,M)$ (M is defined in Assumption 4.1) such that if $(x,v)\in\Theta_r$, then $\Gamma^{\rm opt}[x,v]\subset\Gamma_C$. Moreover, as $r\to+\infty$, $C(r,M)=O(\sqrt{r})$.

Proof. A possible proof consists of building a suitable map j from Θ_r to Γ. We make out different cases:

Case 1: $0 \leqslant v \leqslant -3x/T$**:** let $j(x,v) = (\widetilde{\xi}, \widetilde{\eta}) \in \Gamma[x,v]$ be defined by

$$\begin{cases} \widetilde{\eta}(t) = v\left(1 - \frac{3t}{2T}\right) & \text{and} \quad \widetilde{\xi}(t) = x + v\left(t - \frac{3t^2}{4T}\right), & \text{if} \quad 0 \leqslant t \leqslant \frac{2T}{3}, \\ \widetilde{\eta}(t) = 0 & \text{and} \quad \widetilde{\xi}(t) = x + \frac{vT}{3}, & \text{if} \quad \frac{2T}{3} \leqslant t \leqslant T. \end{cases}$$

It is easy to check that there exists a constant $\widetilde{C} = \widetilde{C}(r,M)$ such that

$$\|\widetilde{\eta}\|_{L^\infty(0,T;\mathbb{R})} \leqslant \widetilde{C}; \qquad \left\|\frac{d\widetilde{\eta}}{dt}\right\|_{L^2(0,T;\mathbb{R})} \leqslant \widetilde{C}. \tag{4.36}$$

Case 2: $-3x/T < v \leqslant r|x|^{\frac{1}{3}}$**:** in this case, we choose $j(x,v) = (\widetilde{\xi}, \widetilde{\eta}) \in \Gamma[x,v]$ where $\widetilde{\xi}' = \widetilde{\eta}$ and $\widetilde{\eta}$ is the solution of the linear complementarity problem (4.8) with $\theta = T$. Here again $(\widetilde{\xi}, \widetilde{\eta})$ satisfies (4.36) for some constant $\widetilde{C} = \widetilde{C}(r,M)$. From Lemma 4.2, we see that as $r \to +\infty$, $\widetilde{C} = O(\sqrt{r})$.

Case 3: $-3(1+x)/T \leqslant v \leqslant 0$**:** the situation is symmetric to Case 1, and $j(x,v)$ is given by the same formula.

Case 4: $-r(x+1)^{\frac{1}{3}} \leqslant v < -3(1+x)/T$**:** the situation is symmetric to Case 2, and $j(x,v)$ is constructed in the symmetric way as in Case 2.

Then, using $j(x,v)$ as a competitor for the optimal control problem leads to the desired result with a constant C that depends only on r and M and that can always be taken larger than $\widetilde{C}$.
Note that j is piecewise continuous from Θ_r to Γ. Note also that the construction of j is independent of ℓ and g. □

Remark 4.4. *Note that the sets Θ_r form an increasing family of compact subsets of Ξ^{ad} and that*

$$\bigcup_{r \geqslant 0} \Theta_r = \Xi^{\mathrm{ad}}. \tag{4.37}$$

4.3 Mean field games with state constraints

In the example considered here, we take $\Xi = [-1,0] \times \mathbb{R}$. Let $\mathcal{P}(\Xi)$ be the set of probability measures on Ξ.
Let $F, G : \mathcal{P}(\Xi) \to C_b^0(\Xi;\mathbb{R})$ be bounded and continuous maps (the continuity is with respect to the narrow convergence in $\mathcal{P}(\Xi)$) and ℓ be a continuous and bounded from below function defined on $\Xi \times [0,T]$. Set

$$M = \max\left(\sup_{(x,v,t) \in \Xi \times [0,T]} \ell_-(x,v,t) + \sup_{m \in \mathcal{P}(\Xi)} \|F[m]\|_{L^\infty(\Xi)}, \sup_{m \in \mathcal{P}(\Xi)} \|G[m]\|_{L^\infty(\Xi)}\right). \tag{4.38}$$

Assumption 4.3. *We assume that for all $t \in [0,T]$, $m \in \mathcal{P}(\Xi)$ and $v \geqslant 0$, $\ell(0,v,t) + F[m](0,v) \geqslant \ell(0,0,t) + F[m](0,0)$ and $\ell(-1,-v,t) + F[m](-1,-v) \geqslant \ell(-1,0,t) + F[m](-1,0)$.*

Using similar notations as in paragraph 3.1, we consider the cost given by (3.2). With M in (3.1), note that Proposition 4.1 can be applied to J^μ defined in (3.2) with constants $C(r,M)$ uniform in μ.

Lemma 4.5. *Let r be a positive number. Under the assumptions made above on ℓ, F and G (including Assumption 4.3), let $C = C(r, M)$ be the constant appearing in Proposition 4.1. For any probability measure m_0 on Ξ supported in Θ_r defined in (4.33), there exists a constrained mean field game equilibrium associated with the initial distribution m_0, i.e. a probability measure $\mu \in \mathcal{P}_{m_0}(\Gamma_C)$ such that (3.8) holds.*

Proof. The proof is similar to that of Theorem 3.2. We skip it. □

Remark 4.5. *Compared to Theorem 3.2, the restrictions made on the support of m_0 are weaker in Lemma 4.5, but the latter requires the additional Assumption 4.3.*
In Theorem 4.2 below, we get rid of the assumptions on the support of m_0.

Theorem 4.2. *Let m_0 be a probability measure on Ξ such that*

$$m_0(\Xi \backslash \Xi^{\mathrm{ad}}) = 0. \tag{4.39}$$

Under the assumptions made above on ℓ, F and G (including Assumption 4.3), there exists a constrained mean field game equilibrium associated with the initial distribution m_0, i.e. a probability measure $\mu \in \mathcal{P}_{m_0}(\Gamma)$ such that (3.8) holds.

Proof. From (4.37) and (4.39), there exists $n_0 > 0$ such that $m_0(\Theta_n) > 0$ for $n > n_0$. For $n > n_0$, we set $m_{0,n} = \frac{1}{m_0(\Theta_n)} m_0|_{\Theta_n}$. With a slight abuse of notation, let $m_{0,n}$ also denotes the probability on Ξ obtained by extending $m_{0,n}$ by 0 outside Θ_n, i.e. $m_{0,n}(B) = \frac{1}{m_0(\Theta_n)} m_0(B \cap \Theta_n)$, for any measurable subset B of Ξ. Let $\mu_n \in \mathcal{P}_{m_{0,n}}\left(\Gamma_{C(n,M)}\right)$ be a constrained mean field game equilibrium associated with the initial distribution $m_{0,n}$, the existence of which comes from Lemma 4.5. With a similar abuse of notations as above, let μ_n also denote the probability on Γ obtained by extending μ by 0 outside $\Gamma_{C(n,M)}$.

We claim that $\{\mu_n, n > n_0\}$ is tight in $\mathcal{P}(\Gamma)$, i.e. that for each $\epsilon > 0$, there exists a compact $K_\epsilon \subset \Gamma$ such that

$$\mu_n(\Gamma \backslash K_\epsilon) < \epsilon, \qquad \text{for each } n > n_0. \tag{4.40}$$

From the increasing character of the sequence Θ_n, (4.37) and (4.39), we observe that for each $\epsilon > 0$, there exists $n_1 > 0$ such that $m_0(\Theta_{n_1}) > 1 - \epsilon$. Let us prove (4.40) with $K_\epsilon = \Gamma_{C(n_1,M)}$.
Since for all $n > n_0$, $\mu_n \in \mathcal{P}_{m_{0,n}}(\Gamma)$ is a MFG equilibrium, we see that for all measurable $B \subset \Xi$,

$$m_{0,n}(B) = \mu_n \left\{(\xi, \eta) \in \mathrm{supp}(\mu_n) \ : \ (\xi(0), \eta(0)) \in B\right\} \leqslant \mu_n \left(\bigcup_{(x,v)\in B} \Gamma^{\mathrm{opt},\mu_n}[x, v] \right).$$

Taking $B = \Theta_{n_1}$ and using Proposition 4.1, we see that

$$m_{0,n}\left(\Theta_{n_1}\right) \leqslant \mu_n \left(\bigcup_{(x,v)\in \Theta_{n_1}} \Gamma^{\mathrm{opt},\mu_n}[x, v] \right) \leqslant \mu_n \left(\Gamma_{C(n_1,M)}\right),$$

(note that the constant $C(n_1, M)$ does not depend on μ_n).
On the other hand,

$$\begin{array}{lcll} m_{0,n}\left(\Theta_{n_1}\right) & \geqslant & m_0(\Theta_{n_1}) > 1 - \epsilon, & \text{if } n > n_1, \\ m_{0,n}\left(\Theta_{n_1}\right) & = & 1, & \text{if } n_0 < n \leqslant n_1. \end{array}$$

In both cases, $\mu_n\left(\Gamma_{C(n_1,M)}\right) \geqslant 1-\epsilon$ and therefore $\mu_n\left(\Gamma\backslash\Gamma_{C(n_1,M)}\right) \geqslant 1-\epsilon$, and the claim is proved.

Thanks to Prokhorov theorem, possibly after the extraction of subsequence that we still name μ_n, we deduce that there exists $\mu \in \mathcal{P}(\Gamma)$ such that μ_n converges narrowly to μ.

We claim that μ is a MFG equilibrium related to m_0. We already know that $\mu \in \mathcal{P}(\Gamma)$. There remains to prove that

- $\mu \in \mathcal{P}_{m_0}(\Gamma)$, i.e. that $e_0\sharp\mu = m_0$
- μ satisfies (3.8).

The fact that $e_0\sharp\mu = m_0$ stems from Lemma 3.1 and from the fact that $m_{0,n}$ narrowly converges to m_0.

In order to prove (3.8), we recall that from Kuratowski's theorem, see [5],

$$\mathrm{supp}(\mu) \subset \liminf_{n\to\infty} \mathrm{supp}(\mu_n),$$

which means that for all $(\xi,\eta) \in \mathrm{supp}(\mu)$, there exists a sequence $(\xi_n,\eta_n) \in \mathrm{supp}(\mu_n)$ such that $(\xi_n,\eta_n) \to (\xi,\eta)$ uniformly. As a consequence, setting $(x_n,v_n) = (\xi_n(0),\eta_n(0))$ and $(x,v) = (\xi(0),\eta(0))$, $\lim_{n\to\infty}(x_n,v_n) = (x,v)$ and $(\xi_n,\eta_n) \in \Gamma^{\mathrm{opt},\mu_n}[x_n,v_n]$. Applying Proposition 4.2 below, which is a generalization of Theorem 4.1, we may pass to the limit and conclude that $(\xi,\eta) \in \Gamma^{\mathrm{opt},\mu}[x,v]$, which achieves the proof. □

Proposition 4.2. *Under the assumptions made above on ℓ, F and G (including Assumption 4.3), consider a sequence $(\mu^i)_{i\in\mathbb{N}}$, $\mu^i \in \mathcal{P}(\Gamma)$, such that μ^i converges narrowly to $\mu \in \mathcal{P}(\Gamma)$. Consider a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$, $(\xi^i,\eta^i) \in \Gamma$, such that*

1. *$(\xi^i,\eta^i) \in \Gamma^{\mathrm{opt},\mu^i}[x^i,v^i]$, where $(x^i,v^i) = (\xi^i(0),\eta^i(0))$*
2. *(ξ^i,η^i) tends to $(\xi,\eta) \in \Gamma[x,v]$ uniformly, where $(x,v) = \lim_{i\to\infty}(x^i,v^i)$.*

Then $(\xi,\eta) \in \Gamma^{\mathrm{opt},\mu}[x,v]$.

Proof. We skip the proof, because it follows the same lines as that of Theorem 4.1, (see paragraph 4.1). In particular, it includes an adaptation of Lemma 4.4. The necessary modifications are obvious. □

5 State constrained optimal control problems and mean field games in a convex polygonal domain of $\mathbb{R}^2$

Let Ω be a bounded and convex domain of $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. For $x \in \overline{\Omega}$, the tangent cone to Ω at x is defined by

$$T_\Omega(x) = \left\{v \in \mathbb{R}^2 :\ x + tv \in \overline{\Omega},\ \text{for } t > 0 \text{ small enough}\right\}.$$

Note that $T_\Omega(x) = \mathbb{R}^2$ if $x \in \Omega$. A vector $v \in \mathbb{R}^2$ points outward Ω at $x \in \partial\Omega$ if $v \notin T_\Omega(x)$. Let $(\nu_i)_{0\leqslant i<N}$ be the vertices of $\partial\Omega$, labeled in such a way that $\partial\Omega = \bigcup_{i=0}^{N-1}\gamma_i$, where $\gamma_i = [\nu_i,\nu_{i+1}]$ and $\nu_N = \nu_0$. We may assume that three successive vertices are not aligned. We are going to use the notation (ν_i,ν_{i+1}) for the open straight line segment between

ν_i and ν_{i+1}. For $i \in \{0, \dots, N-1\}$, let n_i be the unitary normal vector to γ_i pointing outward Ω. It is easy to see that $T_\Omega(\nu_i) = \{x \in \mathbb{R}^2 \ : \ n_i \cdot x \leqslant 0 \text{ and } n_{i-1} \cdot x \leqslant 0\}$, setting $n_{-1} = n_{N-1}$. Since Ω is convex, $\overline{\Omega}$ coincides locally near ν_i with $\nu_i + T_\Omega(\nu_i)$.

The optimal control problem is set exactly as in Section 2: it consists of minimizing $J(\xi, \eta, \eta')$ given by (2.2) on the dynamics given by (2.1) and staying in $\Xi = \overline{\Omega} \times \mathbb{R}^2$. The set $\Xi^{\rm ad}$ is defined by (2.6).

5.1 Closed graph properties

The closed graph result given in Proposition 5.1 below is similar to that contained in Proposition 2.1, but special conditions are needed near the vertices of $\partial\Omega$:

Proposition 5.1. *Consider a closed subset Θ of $\Xi^{\rm ad}$. Assume that all sequence $(x^i, v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i, v^i) \in \Theta$ and $\lim_{i\to+\infty}(x^i, v^i) = (x, v) \in \Theta$, the following holds:*

1. *If $x \in (\nu_j, \nu_{j+1})$ for some $j \in \{0, \dots, N-1\}$ (recall that $\nu_N = \nu_0$), then*

$$(v^i \cdot n_j)_+^3 = o\Big((x - x^i) \cdot n_j\Big); \tag{5.1}$$

2. *if $x = \nu_j$ for some $j \in \{0, \dots, N-1\}$ and $v \neq 0$, then*

$$(v^i \cdot n_k)_+^3 = o\Big((x - x^i) \cdot n_k\Big), \quad \text{for } k = j-1, j, \tag{5.2}$$

   recalling that $n_{-1} = n_{N-1}$;

3. *If $x = \nu_j$ for some $j \in \{0, \dots, N-1\}$ and $v = 0$, then*

$$(v^i \cdot n_k)_+ \left(|x - x^i|^{\frac{2}{3}} + |v^i|^2\right) = o\Big((x - x^i) \cdot n_k\Big), \quad \text{for } k = j-1, j; \tag{5.3}$$

then the graph of the multivalued map $\Gamma^{\rm opt} : \Theta \to \Gamma$, $(x, v) \mapsto \Gamma^{\rm opt}[x, v]$, is closed in the sense given in Proposition 2.1.

Remark 5.1. *It is easy to find sets Θ which fulfill the assumption of Proposition 5.1, for example:*

1. *a compact subset Θ of $\mathbb{R}^2 \times \mathbb{R}^2$ contained in $\Omega \times \mathbb{R}^2$;*
2. *for $j = 0, \cdots, N-1$, let D_j be the straight line containing ν_j and ν_{j+1}; for $\rho > 1$, set*

$$S_j = \left\{(x, v) \in \overline{\Omega} \times \mathbb{R}^2 : v \cdot n_j \leqslant dist(x, D_j)^{\frac{\rho}{3}}\right\}.$$

   If Θ is a closed subset $\mathbb{R}^2 \times \mathbb{R}^2$ contained in $\left(\Xi^{\rm ad} \cap \bigcap_{j=0}^{N-1} S_j\right) \setminus \left(\bigcup_{j=0}^{N-1} \{\nu_j\} \times \mathbb{R}^2\right)$, then Θ fulfills the assumption of Proposition 5.1.
3. *For brevity, we do not supply examples of sets Θ which intersect $\bigcup_{j=0}^{N-1} \{\nu_j\} \times \mathbb{R}^2$, although it is possible to do so.*

The proof of Proposition 5.1 relies on the counterpart of Lemma 2.3 whose proof is exactly the same, and on Lemma 5.1 below which is the counterpart of Lemma 2.2:

Lemma 5.1. *Consider $(x,v) \in \Xi^{\rm ad}$, $(\xi,\eta) \in \Gamma[x,v]$ such that $\eta \in W^{1,2}(0,T;\mathbb{R}^2)$ and a sequence $(x^i,v^i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $(x^i,v^i) \in \Xi^{\rm ad}$ and $\lim_{i\to\infty}(x^i,v^i) = (x,v)$.*
Assume that one among the following seven conditions is true:

1. *$x \in \Omega$*
2. *$x \in (\nu_j,\nu_{j+1})$, $v\cdot n_j < 0$ (hence $v^i \cdot n_j < 0$ for i large enough)*
3. *$x \in (\nu_j,\nu_{j+1})$, $v\cdot n_j = 0$ and one among the following properties is true:*
   (a) *$v^i \cdot n_j \leqslant 0$ for i large enough*
   (b) *for i large enough, $v^i\cdot n_j > 0$ (hence $(x-x^i)\cdot n_j > 0$) and*
   $$\lim_{i\to\infty} \frac{(v^i\cdot n_j)^3}{(x-x^i)\cdot n_j} = 0 \tag{5.4}$$
4. *$x = \nu_j$, $v\cdot n_j < 0$ and $v\cdot n_{j-1} < 0$ (setting $n_{-1} = n_{N-1}$)*
5. *$x = \nu_j$, $v\cdot n_j = 0$ and $v\cdot n_{j-1} < 0$ and one among the two following properties holds*
   (a) *$v^i\cdot n_j \leqslant 0$ for i large enough*
   (b) *$v^i\cdot n_j > 0$ and (5.4) holds for i large enough.*
6. *$x = \nu_j$, $v\cdot n_{j-1} = 0$ and $v\cdot n_j < 0$ and one among the two following properties holds*
   (a) *$v^i\cdot n_{j-1} \leqslant 0$ for i large enough*
   (b) *$v^i\cdot n_{j-1} > 0$ and (5.4) holds (replacing j by $j-1$) for i large enough*
7. *$x = \nu_j$, $v = 0$ and*
   $$\lim_{i\to\infty} \max_{k\in\{j-1,j\}} \frac{(v^i\cdot n_k)_+}{|(x^i-x)\cdot n_k|}\left(|x-x^i|^{\frac{2}{3}} + |v^i|^2\right) = 0, \tag{5.5}$$
   with the convention that $\frac{(v^i\cdot n_k)_+}{|(x^i-x)\cdot n_k|} = 0$ if $(v^i\cdot n_k)_+ = 0$.

Then there exists a sequence $(\xi^i,\eta^i)_{i\in\mathbb{N}}$ such that $(\xi^i,\eta^i) \in \Gamma[x^i,v^i]$, $\eta^i \in W^{1,2}(0,T;\mathbb{R}^2)$, and (ξ^i,η^i) tends to (ξ,η) in $W^{2,2}(0,T;\mathbb{R}^2)\times W^{1,2}(0,T;\mathbb{R}^2)$, hence uniformly in $[0,T]$.

Proof of Lemma 5.1. Since many arguments are almost identical to those contained in the proof of Lemma 2.2, we just sketch the proof and put the stress on cases 4-7 which have no counterparts in Lemma 2.2. As for Lemma 2.2, each of the seven conditions mentioned in the statement makes it possible to explicitly construct families of admissible trajectories fulfilling all the desired properties. Since the construction is different in each of the seven cases, we discuss each case separately:

1. If $x \in \Omega$, then the construction is exactly the same as in the first case in the proof of Lemma 2.2. We do not repeat the argument.
2. $x \in (\nu_j,\nu_{j+1})$ and $v\cdot n_j < 0$, hence for i large enough, $v^i\cdot n_j < 0$. We can always assume that the latter property holds for all i. Since $\eta \in W^{1,2}(0,T)$, there exists $\bar t \in (0,T)$ such

that for all $s \in [0,\overline{t}]$, $\frac{3}{2}v\cdot n_j \leqslant \eta(s)\cdot n_j \leqslant \frac{1}{2}v\cdot n_j$ and $\frac{3s}{2}v\cdot n_j \leqslant (\xi(s)-x)\cdot n_j \leqslant \frac{s}{2}v\cdot n_j$. For $t_i \in [0,\overline{t}]$, we set

$$\xi^i(s) = \begin{cases} Q_{t_i,x^i,v^i,\xi(t_i),\eta(t_i)}(s), & \text{if} \quad s\in[0,t_i], \\ \xi(s), & \text{if} \quad s\in[t_i,T], \end{cases} \tag{5.6}$$

and the remaining arguments are very close to those in case 2. in the proof of Lemma 2.2. We skip the details.

3. (a) $x \in (\nu_j,\nu_{j+1})$, $v\cdot n_j = 0$ and $v^i\cdot n_j \leqslant 0$ at least for i large enough. We may assume that $v^i\cdot n_j \leqslant 0$ for all i. Given $t_{i,1}\in(0,T)$, we define (y^i,w^i) as follows:

$$\begin{array}{rcl} y^i &=& (x\cdot n_j)n_j + ((x^i+v^it_{i,1})\cdot n_j^\perp)n_j^\perp, \\ w^i &=& (v\cdot n_j)n_j + (v^i\cdot n_j^\perp)n_j^\perp = (v^i\cdot n_j^\perp)n_j^\perp, \end{array} \tag{5.7}$$

and, for $t_{i,1}\leqslant t_{i,2} < T$, set

$$\xi^i(s) = \begin{cases} Q_{t_{i,1},x^i,v^i,y^i,w^i}(s), & \text{if} \quad s\leqslant t_{i,1}, \\ Q_{t_{i,2}-t_{i,1},y^i-x,w^i-v,0,0}(s-t_{i,1}) + \xi(s-t_{i,1}), & \text{if} \quad t_{i,1}\leqslant s\leqslant t_{i,2}, \\ \xi(s-t_{i,1}), & \text{if} \quad t_{i,2}\leqslant s\leqslant T. \end{cases} \tag{5.8}$$

We argue as in case 3.(a) in the proof of Lemma 2.2. An important observation is that

$$(\xi^i(s)-x)\cdot n_j = \left((x^i-x)\cdot n_j\left(1+2\frac{s}{t_{i,1}}\right)+sv^i\cdot n_j\right)\left(1-\frac{s}{t_{i,1}}\right)^2 \tag{5.9}$$

is non positive for $s\in[0,t_{i,1}]$. We skip the other details.

(b) $x\in(\nu_j,\nu_{j+1})$, $v\cdot n_j = 0$, $v^i\cdot n_j > 0$ for all i (or for i large enough), and (5.4) holds. This case is the counterpart of case 3.(b) in the proof of Lemma 2.2. The trajectory ξ^i is constructed as in (5.8), but a further restriction on $t_{i,1}$ is needed in order to guarantee that the trajectory is admissible, namely that

$$t_{i,1} \leqslant \frac{3|(x^i-x)\cdot n_j|}{v^i\cdot n_j}.$$

This condition should be supplemented with the other two conditions as in 3.(a), namely that

$$\lim_{i\to\infty} t_{i,1} = 0, \tag{5.10}$$

$$\lim_{i\to\infty} \frac{|(x-x^i)\cdot n_j|^2}{t_{i,1}^3} + \frac{|(v-v^i)\cdot n_j|^2}{t_{i,1}} = 0. \tag{5.11}$$

If (5.4) holds, then it is possible to choose such a sequence $t_{i,1}$. The remaining part of the proof is as in case 3.(a).

4. $x=\nu_j$, $v\cdot n_j < 0$ and $v\cdot n_{j-1} < 0$. Since $\eta\in W^{1,2}(0,T)$, there exists $\overline{t}\in(0,T)$ such that for all $s\in[0,\overline{t}]$, $\frac{3}{2}v\cdot n_k \leqslant \eta(s)\cdot n_k \leqslant \frac{1}{2}v\cdot n_k$ and $\frac{3s}{2}v\cdot n_k \leqslant (\xi(s)-x)\cdot n_k \leqslant \frac{s}{2}v\cdot n_k$, for $k=j-1,j$. For $t_i\in[0,\overline{t}]$, we choose ξ^i as in (5.6) and the desired result is obtained as in case 2.

5. $x = \nu_j$, $v \cdot n_j = 0$ and $v \cdot n_{j-1} < 0$. We make out two subcases:

   (a) $v^i \cdot n_j \leqslant 0$ at least for i large enough: the trajectory is constructed as in (5.8), with three different stages corresponding respectively to $s \in [0, t_{i,1}]$, $s \in [t_{i,1}, t_{i,2}]$ and $s \geqslant t_{i,2}$. As in point 3, it is always possible to choose the sequence $t_{i,1}$ such that (5.10) and (5.11) hold in order to ensure the L^2 convergence of the accelerations.
   We need to prove that the trajectory is admissible for well chosen $t_{i,1}$ and $t_{i,2}$.
   Let us first check that $\xi^i(s)$ remains in $\overline{\Omega}$ for $s \in [0, t_{i,1}]$. Since (5.9) holds, we see that $(\xi^i(s) - x) \cdot n_j \leqslant 0$. On the other hand, after some algebra, we get that

$$\begin{aligned}\eta^i(s) \cdot n_{j-1} = \frac{d\xi^i}{ds}(s) \cdot n_{j-1} =& v^i \cdot n_{j-1} - \frac{s}{t_{i,1}} \left(4 - \frac{3s}{t_{i,1}}\right) \ v^i \cdot n_j \ \ n_j \cdot n_{j-1} \\ & - 6\frac{s}{t_{i,1}^2} \left(1 - \frac{s}{t_{i,1}}\right) \ (x^i - x) \cdot n_j \ \ n_j \cdot n_{j-1}.\end{aligned}$$

   Since $\lim_{i\to\infty} v^i \cdot n_j = 0$ and $\lim_{i\to\infty} v^i \cdot n_{j-1} = v \cdot n_{j-1} < 0$, we see that for i large enough,

$$\lim_{i\to\infty} \left(v^i \cdot n_{j-1} - \frac{s}{t_{i,1}} \left(4 - \frac{3s}{t_{i,1}}\right) \ v^i \cdot n_j \ \ n_j \cdot n_{j-1} \right) = v \cdot n_{j-1} < 0,$$

   uniformly with respect to $s \in [0, t_{i,1}]$. On the other hand, the conditions (5.10) and (5.11) imply that $\lim_{i\to\infty} \frac{(x^i - x) \cdot n_j}{t_i} = 0$. Combining the latter two observations yields that

$$\lim_{i\to\infty} \max_{s\in[0,t_{i,1}]} \left|(\eta^i(s) - v) \cdot n_{j-1}\right| = 0.$$

   Hence, for i large enough, $\eta^i(s) \cdot n_{j-1} < 0$ for all $s \in t_{i,1}$. This implies that $(\xi^i(s) - x) \cdot n_{j-1} \leqslant 0$ for all $s \in [0, t_{i,1}]$.
   Combining the information above and arguing essentially as in case 3.(a), we see that it is possible to choose $t_{i,1}$ satisfying (5.10) and (5.11), $t_{i,2}$ bounded away from 0 uniformly w.r.t. i, such that $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$ for i large enough and $\lim_{i\to\infty} \left\| \frac{d\eta^i}{ds} - \frac{d\eta}{ds} \right\|_{L^2(0,T)} = 0$.

   (b) $v^i \cdot n_j > 0$ at least for i large enough and (5.4) holds: again, the trajectory is constructed as in (5.8) with $t_{i,1}$ satisfying (5.10)-(5.11). As in 3.(b), a further restriction is needed on $t_{i,1}$ such that the trajectory is admissible.
   For $0 \leqslant s \leqslant t_{i,1}$, $(\xi^i(s) - x) \cdot n_j$ is given by (5.9) and is non positive if

$$t_{i,1} \leqslant \frac{3|(x^i - x) \cdot n_j|}{v^i \cdot n_j}. \tag{5.12}$$

   On the other hand, the proof that $(\xi^i(s) - x) \cdot n_{j-1} \leqslant 0$ for i large enough and all $s \in [0, t_{i,1}]$ is the same as in subcase 5.(a). Hence if the sequence $(t_{i,1})$ satisfies (5.10)-(5.11) and (5.12), then for i large enough, $\xi^i(s) \in \overline{\Omega}$ for for all $s \in [0, t_{i,1}]$. Constructing such a sequence $(t_{i,1})$ is possible thanks to (5.4).
   Then, using the fact that $v \cdot n_{j-1} < 0$ and arguing as in case 3., it is possible to choose the sequence $t_{i,2}$ bounded from below by a positive constant independent of i such that $(\xi^i, \eta^i) \in \Gamma[x^i, v^i]$ and $\lim_{i\to\infty} \left\| \frac{d\eta^i}{dt} - \frac{d\eta}{dt} \right\|_{L^2(0,T)} = 0$.

6. Same arguments as for case 5., exchanging the roles of j and $j-1$.

7. The trajectory ξ^i is constructed as follows:

$$\xi^i(s) = \begin{cases} Q_{t_i,x^i,v^i,x,0}(s), & \text{if} \quad s \leqslant t_i, \\ \xi(s-t_i), & \text{if} \quad t_i \leqslant t \leqslant T, \end{cases} \tag{5.13}$$

We see that, for $k = j-1, j$ and $s \in [0, t_i]$,

$$(\xi^i(s) - x) \cdot n_k = \left(1 - \frac{s}{t_i}\right)^2 \left(\left(1 + 2\frac{s}{t_i}\right)(x^i - x) \cdot n_k + s v^i \cdot n_k\right).$$

Hence, a sufficient condition for $\xi^i(s)$ to stay in $\overline{\Omega}$ for all $s \in [0, t_i]$ is that $t_i \leqslant 3 \min_{k \in \{j-1,j\}} \frac{(x-x^i)\cdot n_k}{(v^i \cdot n_k)_+}$, with the convention that $\frac{(x-x^i)\cdot n_k}{(v^i \cdot n_k)_+} = +\infty$ if $(v^i \cdot n_k)_+ = 0$. Then, we also need that $\lim_{i\to\infty} t_i = 0$ and that $\lim_{i\to\infty} \frac{|x-x^i|^2}{t_i^3} + \frac{|v^i|^2}{t_i} = 0$ in order to obtain that $\lim_{i\to\infty} \left\| \frac{d\eta^i}{dt} \right\|_{L^2(0,t_i)} = 0$. From (5.5), it is possible to construct a sequence $(t_i)_i$ fulfilling all the desired properties.

□

Proposition 5.2 below is the counterpart of Proposition 2.2:

Proposition 5.2. *Given $r > 0$, let us define Θ_r by (2.30) where K_r is defined by (2.28) and Θ is a closed subset of $\Xi^{\rm ad}$ which satisfies the assumption in Proposition 5.1. Under Assumption 2.1, the value function u given by (2.7) is continuous on Θ_r. There exists a positive number $C = C(r, M)$ such that if $(x, v) \in \Theta_r$, then $\Gamma^{\rm opt}[x, v] \subset \Gamma_C$, where Γ_C is defined in (2.29).*

5.2 Mean field games with state constraints

All the results obtained in Section 3 can be generalized to the case when Ω is a bounded and convex polygonal domain of $\mathbb{R}^2$, provided the initial distribution of states m_0 is supported in Θ_r defined as in Proposition 5.2.

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