# Diophantine Gaussian excursions and random walks

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November 24, 2021

**Abstract** We establish general asymptotic upper and lower bounds for the volume variance of Euclidean Gaussian nodal excursions in terms of the random walk associated to the spectral measure. These bound are sharp in several situations, and under mild assumptions, the variance is at least linear.

To obtain sublinear variances, we focus on the case where the spectral measure is purely atomic, and show that the associated irrational random walk on the multi-dimensional torus comes back more often close to 0 when the atoms are well approximable by rational tuples. Hence the excursion behaviour strongly depends on the diophantine properties of the atoms, it has fluctuations which power can be arbitrarily close from the maximum $2d$ (quadratic fluctuations), whereas if the atoms are badly approximable the excursion is strongly hyperuniform, meaning the variance asymptotic power is minimal, $(d-1)$, corresponding to the window boundary measure. Also, given any reasonable variance asymptotic behaviour, there are uncountably many sets of spectral atoms that realise it.

The versatility of the variance formula is illustrated by other examples where the spectral measure support can have higher dimension, in particular it is able to capture the variance cancellation phenomenon of Gaussian random waves, and it also yields that there are no hyperuniform isotropic Gaussian excursions.

**Keywords:** Gaussian fields, nodal excursion, random walk, diophantine approximation, hyperuniformity, Gaussian random waves, variance cancellation.

**AMS 2010** 60G15, 60G50, 11J13, 34L20

## 1 Introduction

The primary motivation of this article is to study the variance of the excursion volume for Euclidean stationary Gaussian fields, and exhibit a class of models that realise a presribed asymptotic variance behavior. It turns out that this can only be achieved by spectral measures with a low dimensional support, hence we consider measures with a finite support. This investigation requires to study a random walk which behaviour depends on the diophantine properties of the spectral atoms. To conduct this program, we establish two unrelated results which are of independent interest, corresponding to Sections 2 and 3.

The first result, Theorem 2.1, deals with the volume of general Euclidean Gaussian fields excursions. The main finding is that the variance magnitude is strongly related to the probability of the associated random walk to return around 0. The second result, Theorem 3.1, contains bounds for random walks with irrational increments. The combination of those two results yields variance asymptotics for diophantine Gaussian excursions, as detailed in Section 4, culminating with Theorem 4.1, which can be seen as the main result of this paper.

The results about diophantine random walks are of independent interest and can be projected onto the torus and actually add some uniform estimates to the existing literature (see Section 1.3).

The fact that Theorem 2.1 has a more general scope and can be applied in various situations is illustrated by the variance cancellation phenomenon for a fundamentally different model, the Gaussian random wave, see Section 1.5.

The rest of the introduction presents some aspects and corollaries of the important theoretical results of this paper (Theorems 2.1 and 3.1). Their proofs necessarily call to subsequent sections.

## 1.1 Gaussian excursions volume variance

The main actors of this article are centred stationary real Gaussian random fields $\{X(\mathbf{t}); \mathbf{t} \in \mathbb{R}^d\}$, which law is invariant under translations of $\mathbb{R}^d$. See the monograph [1] for a comprehensive expositions of main properties and fundamental results about Gaussian fields and their geometry. It is known that they are completely characterised by their reduced covariance function

$$C(\mathbf{t}) = \mathbb{E}(X(0)X(\mathbf{t})), \mathbf{t} \in \mathbb{R}^d,$$

or by their *spectral measure*, i.e. the unique finite symmetric measure $\boldsymbol{\mu}$ on $\mathbb{R}^d$ such that $C$ admits the representation

$$C(\mathbf{t}) = \int_{\mathbb{R}^d} e^{-i\mathbf{t}\cdot\mathbf{x}} \boldsymbol{\mu}(d\mathbf{x}), \mathbf{t} \in \mathbb{R}^d, \tag{1.1}$$

where $\cdot$ denotes the standard scalar product.

Excursions of Gaussian processes on the real line have often been studied through their number of crossings with the axis [19, 10, 11, 32, 20]. Elementary considerations yield that the average number of crossings on an interval is proportionnal to the length of the interval. Furthermore, if $\boldsymbol{\mu}$ contains more than one (symmetrised) atom, the variance of the number of crossings is quadratic [22, 2]. We focus here on the Lebesgue measure of the nodal excursions

$$\{X > 0\} = \{\mathbf{t} \in \mathbb{R}^d : X(\mathbf{t}) > 0\}.$$

Here again, the field centering and an application of Fubini's theorem yields that the expectation is proportionnal to the volume:

$$\mathbb{E}(\mathscr{L}^d(A \cap \{X > 0\})) = \frac{\mathscr{L}^d(A)}{2}, A \subset \mathbb{R}^d$$

where $\mathscr{L}^d$ is the $d-$dimensional Legesgue measure. We give in Section 2 general upper and lower bounds for the variance of the excursion volume

$$V_{\boldsymbol{\mu}}(T) = \mathbf{Var}(\mathscr{L}^d(\{X > 0\} \cap B_d(0, T)))$$

where $B_d(0, T)$ is the centered ball with radius $T$. These bounds imply in particular that if $X$ is isotropic, or more generally if $\boldsymbol{\mu}$'s support has dimension $\geqslant 1$ and spans the whole space, the volume has at least linear variance, i.e. larger than $T^d$, and hyperuniform behaviour is unreachable (see Section 1.4).

If on the other hand $\boldsymbol{\mu}$'s support is finite, a wide class of asymptotic behaviours are reachable.

**Corollary 1.1.** *Let $\psi : \mathbb{N}^* \to [0, 1]$ a function decaying regularly faster than $q^{-\frac{1+2d}{1+d}}$ (Definition 4.1). There are uncountably many finite sets $\Sigma \subset \mathbb{R}^d$ such that if $\boldsymbol{\mu}$ is symmetric with support $\Sigma$, there are $0 < c_- \leqslant c_+ < \infty$ such that for $T > 0$ sufficiently large*

$$c_- T^{2d} \psi^{-1}(T)^{-(1+2d)} \overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\mu}}(T) \leqslant c_+ T^{2d} \psi^{-1}(T)^{-(1+2d)}$$

*where $\psi^{-1}$ is the pseudo-inverse of $\psi$ (defined at (3.1)) and $\overset{\textbf{i.o.}}{\leqslant}$ means that the inequality is true for a sequence $T_k \to \infty$.*

The consequence is that we have a parametric model which achieves any reasonable asymptotic variance between the minimal *surface-scaling order*, in $T^{d-1}$, and the maximal *quadratic order*, in $T^{2d}$, see Corollary 4.1 for details. For sublinear variances (below $T^d$), the excursions are hence *hyperuniform*, contributing to the already large research body on the subject (see Section 1.4).

The result above is an immediate corollary of Theorem 4.1 with $m = 1$, where $\boldsymbol{\mu}$ is of the form

$$\boldsymbol{\mu} = \sum_{k=1}^{d} \sum_{i=0}^{m} \bar{\delta}_{\omega_i \mathbf{e}_k}, \text{ where } \bar{\delta}_a = \frac{1}{2}(\delta_a + \delta_{-a}), a \in \mathbb{R}^d,$$

and the atoms $\omega_i, i = 1, \ldots, m$ are $\psi$-approximable, i.e. roughly speaking such that for infinitely many $q = (q_i) \in \mathbb{Z}^m$, $\sum_i \omega_i q_i$ is $\psi(q)$-close to an integer (and this is not true for $\varphi << \psi$). Let us give formal definitions, as they will be useful throughout the introduction and the paper: say that $\omega \in \mathbb{R}^m$ is

$\psi$-BA (Badly Approximable), if for some $r > 0$ (1.2)

$$|p - \omega \cdot q| \geqslant 2\psi(q) \text{ for all } p \in \mathbb{Z}, q \in \mathbb{Z}^m \setminus B_m(0, r)$$

and $\psi$-WA (Well approximable), if for some $c > 0$ (1.3)

$$|p - \omega \cdot q| < c\psi(q) \text{ for infinitely many } p \in \mathbb{Z}, q \in \mathbb{Z}^m, q \equiv 1,$$

where $q \equiv 1$ means that $\sum_{i=1}^m q_i$ is an odd number. The proof consists in (i) expressing the variance in terms of the behaviour around 0 of the diophantine random walk which increment measure is $\boldsymbol{\mu}$ (see Theorem 2.1) and (ii) studying this random walk with the help of results from diophantine approximation theory, see Theorem 3.1. Independently, we also consider Gaussian random waves (Theorem 1.1) and short range fields (Proposition 2.1) to illustrate the wide scope of this method.

More refined results from diophantine approximation theory actually yield the quantity of tuples $(\omega_i)$ yielding a given asymptotics variance, and we build at Section 4.3 mixtures of such Gaussian models with a random support $\Sigma$ giving a prescribed asymptotic variance. The need for models that yield any prescribed variance asymptotics is explained in [7], along with another such procedure based on Fourier transforms.

## 1.2 Background and motivation

Properties of excursions and level sets of continuous random Gaussian functions have been studied under many different instances. The zero set of a one-dimensional Gaussian stationary process is the subject of an almost century long line of research, starting with the seminal works of Kac & Rice [19], or Cramer & Leadbetter [10], and followed by many other authors mainly interested by second order behaviour, see the major contributions [11, 32, 20]. In higher dimensions, zeros of Gaussian entire functions [16, 27] and nodal sets of high enery Gaussian harmonics on a compact manifold [21, 38, 23] (and their Euclidean counterpart the Random Wave Model [26]) have attracted a lot of attention from both physicists and mathematicians. Random trigonometric Gaussian polynomials, i.e. independent Gaussian coefficients multiplied by trigonometric monomials based on a fundamental frequency, have also been studied in the asymptotics of the large degree, see for instance [39] and references therein. We propose here a crucial modification of the spectral measure support: instead of taking frequencies in a proportionnal relation, we choose finitely many frequencies which are incommensurable; this specificity allows for instance to reach all possible behaviours for the variance asymptotics (see Theorem 4.1).

A different approach to our results is through the lens of hyperuniform models, defined at section 1.4. In the last decades, physicist have put in evidence states of matter intermediate between crystals and liquids, where the medium exhibits apparent disorder at the local scale, but fluctuations are suppressed at large scales. This denotes in some sense a long-range compensation of the medium behaviour, and is considered by physicists a *new state of matter*, see the works by S. Torquato and his co-authors [36, 37] that introduce the topic and expose the main tools and discoveries. Even though the focus was primarily on atomic measures, this concept has been then generalised to other random measures, in particular bi-phased random media [35, 34]. Such heterogeneous materials abound in nature and synthetic situations. Examples include composite and porous media, metamaterials, biological media (e.g., plant and animal tissue), foams, polymer blends, suspensions, granular media, cellular solids, colloids.

The Gaussian realm provides models for many types of phenomena, and the present work yields Gaussian hyperuniform random sets, i.e. which variance on a large window is asymptotically negligible with respect to the window volume (see Section 1.4). The model we present here shares some similarities with perturbed lattices, in the sense that the long range correlations are very strong, but its disorder state is also one step above as one cannot write it as the (perturbed) repetition of a given pattern. It shares with quasi-crystals the property of *almost periodicity*, defined below, and exhibits a spectrum reminiscent of quasi-crystals, see Fig 1. Any asymptotic variance can be achieved, yielding in particular hyperuniform models. According to the typology established in [36, 6.1.2], the model is type-I hyperuniform for almost all choice of parameters; but uncountably many choices of the parameters will yield type-II or actually any type of hyperuniformity. We also give randomised versions of the model not involving diophantine parameters which exhibit different types of hyperuniformity.

As it turns out, a non-isotropic model is necessary to obtain a hyperuniform behaviour (Proposition 1.1). We use the general variance formula of Theorem 2.1 to study Gaussian random waves in any dimension and prove a variance cancellation phenomenon.

## 1.3 Diophantine random walk on the torus

We derive in Section 3 results about diophantine random walks on $\mathbb{R}^d$, which ultimately lead to variance estimates for Gaussian diophantine excursions. The current section discusses the connections with the existing litterature for the diophantine Gaussian random walks on the torus, and is completely disconnected from the results about Gaussian fields discussed above.

Let $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ be a basis of $\mathbb{R}^d$, $m \geqslant 1, \boldsymbol{\mu}$ be a symmetric measure on $\mathbb{R}^d$ parametrised by its support $\boldsymbol{\omega} = (\omega_{[k]})_{1 \leqslant k \leqslant d} \in (\mathbb{R}^m)^d$ via

$$\boldsymbol{\mu} = \frac{1}{d(m+1)} \sum_{k=1}^{d} \sum_{i=0}^{m} \bar{\delta}_{\omega_{[k],i} \mathbf{e}_k}, \tag{1.4}$$

with $\omega_{[k],0} = 1$ by convention, and let $\overline{\mathbf{U}}_n$ be the corresponding random walk on the torus

$$\overline{\mathbf{U}}_n = \{\sum_{i=1}^{n} X_i\}$$

where the $X_i$ are independent and identically distributed with law $\boldsymbol{\mu}$ and $\{\mathbf{x}\} = (\{x_{[k]}\}) \in [0, 1[^d$ is the fractional part in $\mathbb{R}^d$.

It is clear that if $\boldsymbol{\omega}$'s components are well approximable by rationals, the same goes for the increments of the random walk, hence it is likely to come back closer to 0 faster. The study of random walks on a group started on finite arithmetic groups with the works of Diaconis, Saloff-Coste, Rosenthal, Porod, (see references in [33]) and results for such irrational random walks in the continuous settings were then achieved by Diaconis [12], and finally Su [33], who gave the optimal speed of convergence of the law of $\overline{\mathbf{U}}_n$ in an appropriate distance. Then Prescott and Su [30] extended the study in higher dimensional tori.

The novelty of our approach is to consider estimates as $\varepsilon \to 0$ uniformly in $n$; we show in Section 3 that for a given $\varepsilon$, irrelevant of the number of steps $n$, there is a probablity always smaller than $c_+\varepsilon^{\frac{m}{m+\eta}}$ that the walk on the torus ends up in $B_d(0,\varepsilon)$ after $n$ steps, where $\eta \geqslant 0$ is such that the $\omega_k, 1 \leqslant k \leqslant d$ are $q^{-(m+\eta)}$-(Badly approximable).

**Remark 1.1.** *This value is actually very sensititive to the probability of vanishing coordinates $\overline{\mathbf{U}}_{n,[k]}$ of $\overline{\mathbf{U}}_n$, in the sense that it decays slowly in $\varepsilon$ because of the fast recurrence to* 0 *on the axes: for $p < d$*

$$\mathbb{P}(\overline{\mathbf{U}}_{n,[1]} = \overline{\mathbf{U}}_{n,[2]} = \cdots = \overline{\mathbf{U}}_{n,[p]} = 0) \sim n^{-p/2}.$$

*A heuristic argument is that the symmetric random walk on $\mathbb{Z}$ has a probability $\sim n^{-1/2}$ to come back to* 0 *in $2n$ steps, and the components are almost independent up to the parity relation $n \equiv \sum_{k=1}^{d} \mathbf{U}_{n,[k]}$ (see Lemma 3.2).*

In the light of the remark above, only non-vanishing coordinates matter in the speed of decay as $\varepsilon \to 0$. Denote by $[\![d]\!]$ the set $\{1,2,\ldots,d\}$. Define for $K \subset [\![d]\!], K \neq \emptyset$, the projected ball $B_K(\varepsilon) := B_d(0,\varepsilon) \cap H_K$ where

$$H_K := \{\mathbf{y} = (y_{[k]})_{1 \leqslant k \leqslant d} \in \mathbb{R}^d : y_{[k]} \neq 0, k \in K \text{ and } y_{[k]} = 0, k \notin K\}.$$

Then we have according to Theorem 3.1-(i):

**Corollary 1.2.** *For some $c < \infty$, uniformly on $n, \varepsilon$,*

$$\mathbb{P}(\overline{\mathbf{U}}_n \in B_K(\varepsilon)) \leqslant c n^{-\frac{(d-|K|)m}{2}} \varepsilon^{\frac{|K|m}{m+\eta}}$$

Regarding the dependance in $\varepsilon$, the random walk hence comes back to 0 faster on subspaces with fewer coordinates equal to 0 (the dependence as $n$ increases is opposite). The most interesting part of the convegence, i.e. where the magnitude is not dominated by coordinates equal to 0, seems to happen on the domain $H_{[\![d]\!]}$ of points with non-vanishing coordinates. More precise results are derived in (3.3), in Section 3, dedicated to irrational random walks; the results are derived in particular in terms of the optimal function $\psi$ such that $\boldsymbol{\omega}$'s components are $\psi$-(Badly Approximable).

Lower bounds are more unstable, hence we consider the smoothed estimate, for $\beta > 2$,

$$\mathbf{I}_\beta(\varepsilon) = \sum_{n \geqslant n_\varepsilon} n^{-\beta/2} \mathbb{P}(0 < \|\overline{\mathbf{U}}_n\| < \varepsilon) \tag{1.5}$$

where $n_\varepsilon \geqslant 1$ grows sufficiently slowly (see Theorem 3.1-(ii)). To have matching upper and lower bounds, we assume that for some fixed $\omega \in \mathbb{R}^m$, $\omega_{[k]} = \omega$ for $1 \leqslant k \leqslant d$, and that $\omega$ is $\psi$-WA and $\psi$-BA. We have the following corollary of Theorem 3.1-(i),(ii):

**Corollary 1.3.** *There are $0 < c_- < c_+ < \infty$ such that*

$$c_- \varepsilon^{\frac{\beta-2+dm}{m+\eta}} \overset{\textbf{i.o.}}{\leqslant} \mathbf{I}_\beta(\varepsilon) \leqslant c_+ \varepsilon^{\frac{\beta-2+dm}{m+\eta}}$$

*as $\varepsilon \to 0$.*

In Section 3, similar results (but with different magnitudes in $n$) are actually derived first for the random walk $\mathbf{U}_n = \sum_{i=1}^n X_i$ itself, and passed on to $\overline{\mathbf{U}}_n$ to yield the aforementionned results. A quantity $\mathbf{J}_\beta(\varepsilon)$ related to $\mathbf{U}_n$ and analogue to $\mathbf{I}_\beta(\varepsilon)$ is estimated and used in Section 2 to determine the variance of the excursion of the Gaussian field which spectral measure is $\boldsymbol{\mu}$, as discussed previously.

## 1.4 Hyperuniform models

We have just observed, for some values of the parameter $\boldsymbol{\omega}$, the suppression of the variance at large scales, also called hyperuniformity phenomenon. A more general mathematical indicator of hyperuniformity is through the structure factor, or more generally the behaviour around zero of the Fourier transform of the associated random measure. Folowing [29], we use the integrated structure factor to characterize hyperuniformity.

**Definition 1.1.** *Let $E$ be a random subset of $\mathbb{R}^d$. The structure factor of $E$, when it exists, is the measure $\mathcal{S}$ on $\mathbb{R}^d$ defined through test functions $\varphi$ smooth with compact support via*

$$\int_{\mathbb{R}^d} \varphi(\mathbf{t}) \mathrm{Cov}(\mathbf{1}_{\{0 \in E\}}, \mathbf{1}_{\{\mathbf{t} \in E\}}) d\mathbf{t} = \int_{\mathbb{R}^d} \hat{\varphi}(\mathbf{x}) \mathcal{S}(d\mathbf{x})$$

*where $\hat{\varphi}$ is the classical Fourier transform of $\varphi$. Then say that $E$ is* hyperuniform *if $\mathcal{S}(B_d(0,\varepsilon)) \leqslant c_+ \varepsilon^{\alpha+d}, \varepsilon > 0$ for some $\alpha > 0$, and* strongly hyperuniform *if furthermore $\alpha > 1$.*

The precise definition of a random set is not precised here as many existing theories can fit in the previous definition (see for instance [25]). The hyperuniformity of $E$ corresponds under some hypotheses to the suppression of the variance at large scales, i.e.

$$\lim_{T \to \infty} \frac{\mathscr{L}^d(E \cap TW)}{T^d} \to 0$$

for sufficiently regular shapes $W \subset \mathbb{R}^d$, and the strong hyperuniformity corresponds to a variance of minimal magnitude, proportionnal to the window boundary measure, i.e.

$$\sup_T \frac{\mathscr{L}^d(E \cap TW)}{T^{d-1}} < \infty,$$

see for instance [29], or the survey in preparation [9]. Our first result is that natural models of Gaussian fields will not yield hyperuniform excursions.

**Proposition 1.1.** *Let $X$ be some centred stationary Gaussian field on $\mathbb{R}^d$ with spectral measure $\boldsymbol{\mu}$. Assume that for some odd integer $n \geqslant 1, \varepsilon > 0, x \in \mathbb{R}^d, c > 0$, $\boldsymbol{\mu}^n(B_d(x,\varepsilon)) \geqslant c\varepsilon^d$. Then for some $c_- > 0, T$ sufficiently large,*

$$V_{\boldsymbol{\mu}}(T) \geqslant c_- T^d.$$

*This is for instance the case if $X$ is isotropic, i.e. if $X$'s law is invariant under rotations, or equivalently if $\boldsymbol{\mu}$ is invariant under rotations.*

The proof requires tools and notation from Section 2 and is at Section 2.3. As illustrated by the proof, to obtain sublinear variance, the spectral measure's support must have essentially dimension smaller than 1, hence we consider finite atomic support. Let $\boldsymbol{\mu}$ be of the form (1.4) with $\omega_{[k]} \in \mathbb{R}^m$ that is $q^{-(m+\eta)}-$(Badly Approximable) for some $m \geqslant 1, \eta \geqslant 0$, for $1 \leqslant k \leqslant d$, and $X_{\boldsymbol{\omega}}$ the Gaussian field which spectral measure is $\boldsymbol{\mu}$.

**Proposition 1.2.** *Let $\alpha = \frac{1+d(1-\eta)}{m+\eta}$. Then $E = \{X_{\boldsymbol{\omega}} > 0\}$ admits a structure factor $\mathcal{S}$ satisfying*

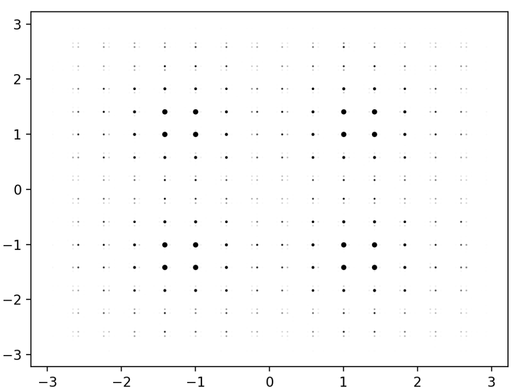
$$\mathcal{S}(B_d(0,\varepsilon)) \leqslant c_+ \varepsilon^{d+\alpha}, \varepsilon \to 0.$$

*Hence if $\eta < 1 + \frac{1}{d}$, $E$ is hyperuniform, and if $\eta \leqslant 1 - \frac{m}{d+1}$, $E$ is strongly hyperuniform.*

The proof is at Section 3.1. If the $\omega_{[k]}$ are $\psi$-(Simultaneously Well Approximable) (see Section 3), which is the case if for instance the $\omega_{[k]}$ are all equal to a $\psi$-WA tuple $\omega \in \mathbb{R}^m$, the right hand side is optimal, see (3.7). We give an approximate representation at figure 1 in a special case.

This kind of spectrum is reminiscent of Bragg peaks in quasi-crystals [35], and more generally of almost periodic fields, for which we give a definition here: a field $X : \mathbb{R}^d \to \mathbb{R}$ is *almost periodic* if for any sequence of vectors $\mathbf{t}_n \to \infty$, there is a subsequence $\mathbf{t}_{n'}$ such that $\|X - X(\mathbf{t}_{n'} + \cdot)\|_\infty \to 0$, see for instance the monograph [8]. Covariance functions and random Gaussian fields considered in this paper are $\|\cdot\|_\infty$-limits of trigonometric polynomials, and as such they are almost periodic. On the other hand, their excursions, seen as $\{0,1\}-$valued functions, are not almost periodic in this sense, mainly because of the discontinuities at the set boundary. On the other hand, they are likely almost periodic for weaker norms, and could hence be seens as *almost periodic sets*, but this question is outside the scope of the current article.

Figure 1: Structure factor for $d = 1, m = 1, \omega_{[1]} = \omega_{[2]} = \sqrt{2}$, i.e. $\boldsymbol{\mu} = \bar{\delta}_{\mathbf{e}_1} + \bar{\delta}_{\sqrt{2}\mathbf{e}_1} + \bar{\delta}_{\mathbf{e}_2} + \bar{\delta}_{\sqrt{2}\mathbf{e}_2}$




## 1.5 Variance cancellation for Gaussian random waves

Let us give another (and unrelated) application of Theorem 2.1 in the context of ergodic isotropic fields. This example is mainly derived to illustrate the sensibility of Theorem 2.1, and could likely be deduced from (more precise) results on the sphere by Rossi [31].

Let $d \geqslant 2, \mathbb{S}^{d-1} = \{\mathbf{t} \in \mathbb{R}^d : |\mathbf{t}| = 1\}$ the $d$-dimensional unit sphere and $\boldsymbol{\mu}_d$ the Haar distribution on the sphere, i.e. the unique probability measure on $\mathbb{S}^{d-1}$ invariant under rotations. Let $X_d(\mathbf{t})$ be a centred Gaussian random field with spectral measure $\boldsymbol{\mu}_d$ and reduced covariance

$$C_d(\mathbf{t}) = \int_{\mathbb{S}^{d-1}} \exp(-\imath \mathbf{t} \cdot \mathbf{x}) \boldsymbol{\mu}_d(d\mathbf{x}) = c_d \frac{\mathsf{B}_{\frac{d}{2}-1}(\|\mathbf{t}\|)}{\|\mathbf{t}\|^{\frac{d}{2}-1}}, \mathbf{t} \in \mathbb{R}^d \tag{1.6}$$

for some $c_d > 0$, see [15, (21)] and Example 2.1 for the definition of the Bessel function of the first kind $\mathsf{B}_a, a \in \mathbb{R}$.

The field $X_d$, called *Berry's random wave model*, is of central importance as it is the unique stationary isotropic field satisfying $\Delta X_d = -X_d$ a.s. [28]. It can be seen as a local approximation of random eigenfunctions of the Laplacian on compact $d-$dimensional smooth manifolds, of high interest in the physics litterature. Since nodal statistics are local quantities, it makes sense to expect analogies between the behaviours of random waves on different manifolds as $T \to \infty$. Such random Gaussian harmonics have been recently heavily studied in dimension 2 in the mathematics literature, especially on the sphere or the torus [21, 24, 14, 23], and the Euclidean version $X_d$ has also been investigated, through its percolation properties [26] or the statistical properties of its nodal lines [28, 14]. An interesting feature of Gaussian random harmonics is the variance cancellation phenomenon, i.e. the very small asymptotic fluctuations of some statistics of the excursion set at the level $u = 0$ in the high energy limit, compared to other levels $u \neq 0$. First conjectured by Berry [4] for the length of the excursion boundary on the torus (*nodal lines*), it has then been observed and deeply analysed in several other instances [21, 23].

We prove here that a variance cancellation at the level $u = 0$ also occurs for the nodal excursions of the Euclidean Gaussian random waves in any dimension. More specifically, while the excursion volume is overfluctuating for levels $u \neq 0$ (i.e. the volume of large windows is negligible with respect to the variance of the excursions of $X_d$ restricted to this window), the fluctuations are linear for $u = 0$, as would be the case for fluctuations of short range random fields such as the Bargmann-Fock field (see Example 2.2). For $T > 0$, the rescaled version $X_{d,T}(\mathbf{t}) = X_d(T\mathbf{t})$ satisfies $\Delta X_{d,T} = -T^2 X_{d,T}$ and hence can be compared to random harmonics with same wavelength on compact manifolds.

**Theorem 1.1.** *Denote by $V^u(T)$ the variance of $\mathscr{L}^d(B_d(0,1) \cap \{X_{d,T} > u\})$. For $u \neq 0$, there is $c_u > 0$ such that*

$$c_u T^{1-d} \leqslant V^u(T), T > 0$$

*and there is $0 < c_- \leqslant c_+ < \infty$ such that*

$$c_- T^{-d} \leqslant V^0(T) \leqslant c_+ T^{-d}, T > 0.$$

The proof is at Section 2.2. This result can be compared with similar results on the sphere, see the work of Marinucci and Wigman in dimension 2 [24], and then of Rossi [31] in dimension $d \geqslant 2$, who study the excursion volume (also called *defect volume* after centering) of spherical Gaussian harmonics $X$ satisfying $\Delta_{\mathbb{S}^2} X = -\ell(\ell + d - 1)X, \ell \in \mathbb{N}$, where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on the sphere. They also obtain a variance of magnitude $\ell^{1-d}$ at levels $u \neq 0$ and $\ell^{-d}$ at the level $u = 0$, echoing experimental results from Blum, Gnutzmann and Smilansky [5]. Hence the present results are consistent with those obtained on the sphere.

**Remark 1.2.** *Various cancellation phenomena have been explained by the cancellation of the second order Wiener chaos of the corresponding functional, see the seminal work [23]. The proof of the previous result hence sheds a different light on this phenomenon, in terms of the support dimension for low order powers $\boldsymbol{\mu}^2, \boldsymbol{\mu}^3$ of the spectral measure.*

# 2 General variance estimates

Let $X$ be some real centred stationary Gaussian field on $\mathbb{R}^d$, denote its spectral measure by $\boldsymbol{\mu}$ and its reduced covariance function by $C$ (see Section 1.1). We study here the statistic

$$M_u^\gamma = \int_{\mathbb{R}^d} \mathbf{1}_{\{X(\mathbf{t})>u\}} \gamma(\mathbf{t}) d\mathbf{t}$$

where $u \in \mathbb{R}$ and $\gamma$ is some measurable *window function*, bounded with compact support with non-empty interior. Define

$$\hat{\gamma}(\mathbf{x}) = \int_{\mathbb{R}^d} e^{-i\mathbf{t}\cdot\mathbf{x}} \gamma(\mathbf{t}) d\mathbf{t}, \mathbf{x} \in \mathbb{R}^d,$$

and $\gamma_T(\mathbf{t}) = \gamma(T^{-1}\mathbf{t}), T > 0$. The variance is

$$V_{\boldsymbol{\mu}}^{\gamma,u}(T) := \mathbf{Var}(\mathscr{L}^d(M_u^{\gamma_T}))$$

and we use the shortcut notation $V_{\boldsymbol{\mu}}^{\gamma} := V_{\boldsymbol{\mu}}^{\gamma,0}$. The most prominent example is the unit sphere indicator $\gamma^d = \mathbf{1}_{\{B_d(0,1)\}}(\cdot)$, and in this case $\gamma^d$ is also implicit in the notation $V_{\boldsymbol{\mu}}^{\gamma^d,u} = V_{\boldsymbol{\mu}}^u, V_{\boldsymbol{\mu}}^0 = V_{\boldsymbol{\mu}}$.

Introduce the notation

$$A = \Theta B$$

for two quantities $A, B$ to mean that $c_- A \leqslant B \leqslant c_+ B$ for some $0 < c_- \leqslant c_+ < \infty$, on their domains of definition. Without loss of generality, we use the convention

$$C(0) = \boldsymbol{\mu}(\mathbb{R}^d) = 1$$

as it allows to adopt the probability formalism and eases certain arguments, denote by $\mathbf{U}_n$ the random walk which increment has law $\boldsymbol{\mu}$ (i.e. $\mathbf{U}_n$'s law is $\boldsymbol{\mu}^n$). Define the function

$$\mathbf{K}(\varepsilon) = \sum_{n \in \mathbb{N} \text{ odd}} \frac{\binom{n}{2n}}{4^n(2n+1)} \mathbb{P}(\|\mathbf{U}_n\| \leqslant \varepsilon)$$

(this function is related to the function $\mathbf{J}_3$ from Section 3 via the relation $\mathbf{K}(\varepsilon) = \Theta \mathbf{J}_3(\varepsilon)$). Say that $\boldsymbol{\mu}$ is $\mathbb{Z}-$free if $\mathbb{P}(\mathbf{U}_{2n+1} = 0) = 0$ for $n \in \mathbb{N}$. For $r > 0$, denote by $c_r^+, c_r^-$ respectively the supremum and infimum of $\|\hat{\gamma}(\mathbf{x})\|^2$ for $\mathbf{x} \in B_d(0, 4r)$. Remark that $c_r^-, c_r^+ \to |\hat{\gamma}(0)|^2 > 0$ as $r \to 0$.

**Theorem 2.1.** *Assume $\boldsymbol{\mu}$ is $\mathbb{Z}-$free and let $r > 0$.*

*(i) For $T > 0$*

$$c_r^- T^{2d} \mathbf{K}(rT^{-1}) \leqslant V_{\boldsymbol{\mu}}^{\gamma}(T). \tag{2.1}$$

*(ii) If in addition for some $c_1 < \infty$, $|\hat{\gamma}(\mathbf{x})| \leqslant c_1 \|\mathbf{x}\|^{-\frac{d+1}{2}}$ for $\|\mathbf{x}\| > 4r$, then*

$$V_{\boldsymbol{\mu}}^{\gamma}(T) \leqslant c_r^+ T^{2d} \mathbf{K}(rT^{-1}) + c_1 T^{d-1} \int_0^{(T/r)^{d+1}} \mathbf{K}(y^{-\frac{1}{d+1}}) dy. \tag{2.2}$$

*In particular if $\gamma = \gamma^d$ and $\mathbf{K}(\varepsilon) = \Theta \varepsilon^\alpha$ as $\varepsilon \to 0$ for some $\alpha > 0$, then $\alpha \leqslant d + 1$ and*

$$V_{\boldsymbol{\mu}}(T) = \Theta T^{2d-\alpha}, T > 0.$$

*(iii) For $u \neq 0, T > 0$,*

$$V_{\boldsymbol{\mu}}^{\gamma,u}(T) \geqslant 2^{2d} c_r^- \alpha_{2,u} T^{2d} \mathbb{P}(\|\mathbf{U}_2\| < rT^{-1}).$$

*In particular if $\boldsymbol{\mu}$ has an atom at $\mathbf{x}_0$, letting $r \to 0$ yields*

$$0 < 2^{2d} |\hat{\gamma}(0)|^2 \alpha_{2,u} \boldsymbol{\mu}(\{\mathbf{x}_0\})^2 T^{2d} \leqslant V_{\boldsymbol{\mu}}^{\gamma,u}(T) \leqslant T^{2d}.$$

The proof, deferred to Section 2.1, is based on a local formula for the variance of Gaussian excursions volume, which gives the exact formula (2.4). This expression is then decomposed in two terms, one of them is proportionnal to $\mathbf{K}(rT^{-1})$, and the other term gives rise to the second term in the right hand side of (2.2). We show that these two terms have the same magnitude in three different settings: (a) when $\boldsymbol{\mu}$ has finitely many incommensurate atoms (Theorem 3.1 and Proposition 4.1), (b) when $\boldsymbol{\mu}$ is the Haar measure on the unit sphere (Theorem 1.1), and (c) when $\boldsymbol{\mu}$ is the Fourier transform of an integrable function (Example 2.2); see also the end of Section 2.1 for a general argument. The second term results from the upper bound

$$\int_0^{(T/r)^{d+1}} \mathbf{K}(\lambda(y)) dy \leqslant \int_0^{(T/r)^{d+1}} \mathbf{K}(cy^{-\frac{1}{d+1}}) dy)$$

where $\lambda$ is a pseudo-inverse of $\hat{\gamma}^2$. Since $\hat{\gamma}$ usually experiences oscillations at $\infty$ (see for instance Example 2.1), obtaining a simple asymptotic equivalent of this term requires more involved computations, but doing so would provide an accurate lower bound on the variance.

**Remark 2.1.** *The fact that if $\mathbf{K}(\varepsilon) = \Theta \varepsilon^{\alpha}$ as $\varepsilon \to 0$ for some $\alpha > 0$ then $\alpha \leqslant d+1$ (point (ii)), is a non-trivial fact about general random walks on $\mathbb{R}^d$.*

**Example 2.1.** *For the unit sphere indicator, we have the classical formula ([15, Chap. 1.5])*

$$\hat{\gamma}^d(\mathbf{x}) = \kappa_d \|\mathbf{x}\|^{-d/2} \mathsf{B}_{d/2}(\|\mathbf{x}\|)$$

*where $\kappa_d = \mathscr{L}^d(B_d(0,1))$ and $\mathsf{B}_a$ is the Bessel function of the first kind with parameter $a$*

$$\mathsf{B}_a(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+a+1)} \left(\frac{r}{2}\right)^{2m+a}, r \geqslant 0.$$

*In particular, $\hat{\gamma}^d(\mathbf{x}) \sim \kappa_d \Gamma(d/2+1)^{-1} > 0$ in $0$ and*

$$\hat{\gamma}^d(\mathbf{x}) \sim \kappa_d (2/\pi)^{1/2} \|\mathbf{x}\|^{-\frac{d+1}{2}} \cos(\|\mathbf{x}\| + \Delta_d)$$

*as $\mathbf{x} \to \infty$, for some $\Delta_d \in \mathbb{R}$. It is known [13] that the first zero of $\mathsf{B}_a, a \geqslant 1/2$ is larger than the first zero of $\mathsf{B}_{1/2}$, which is $\pi$ , hence we can take $r = \frac{1}{2}$ in Theorem 2.1.*

**Example 2.2.** *The most studied Gaussian fields are probably those with an integrable reduced covariance function*

$$\int_{\mathbb{R}^d} C(\mathbf{t})d\mathbf{t} < \infty,$$

*such as the Bargmann-Fock field, where $C(\mathbf{t}) = e^{-\mathbf{t}^2}$. Let us emphasise that the following result is far from new, but, along with theorems 4.1 and 1.1, it illustrates the variety of situations where Theorem 2.1 can be applied.*

**Proposition 2.1.** *Let $\boldsymbol{\mu}$ be a spectral measure with integrable covariance. We have*

$$V_{\boldsymbol{\mu}}(T) = \Theta T^d, T > 0.$$

*Proof.* The integrability of $C$ yields that $\boldsymbol{\mu}$ admits a continuous bounded density function with respect to $\mathscr{L}^d$, denoted by $\hat{C}$, satisfying $\|\hat{C}\|_{L^1} = 1$. Hence, denoting by $\hat{C}^{*n}$ the $n$-fold self convolution of $\hat{C}$,

$$\int \mathbf{1}_{\{B_d(0,\varepsilon)\}}(\mathbf{z})\boldsymbol{\mu}^n(d\mathbf{z}) \leqslant \mathscr{L}^d(B_d(0,\varepsilon))\|\hat{C}^{*n}\|_\infty$$

and classical properties of the convolution product yield that $\|\hat{C}^{*n}\|_\infty \leqslant \|\hat{C}\|_\infty \|\hat{C}\|_{L^1}^{n-1} = \|\hat{C}\|_\infty$, hence

$$\mathbf{K}(\varepsilon) \leqslant \|\hat{C}\|_\infty (\sum_n \alpha_n)\varepsilon^d.$$

Theorem 2.1-(ii) then gives the upper bound.

For the lower bound, let $\mathbf{x} \in \mathbb{R}^d, r > 0$ be such that $\hat{C} > 0$ on $B_d(\mathbf{x}, r)$. It is then easy to show by induction that $\hat{C}^{*n} > 0$ on $B_d(\mathbf{x}, nr)$, hence for $n \geqslant r^{-1}(|\mathbf{x}| + 1)$ and $\varepsilon \leqslant 1$, $B_d(0,\varepsilon) \subset B_d(\mathbf{x}, nr)$ and

$$\mathbb{P}(\|\mathbf{U}_n\| \leqslant \varepsilon) = \int_{B_d(0,\varepsilon)} C^{*n}(\mathbf{t})d\mathbf{t} \geqslant c'\varepsilon^d$$

for some $c' > 0$, Theorem 2.1-(i) concludes the proof. □

## 2.1 Proof of Theorem 2.1

The starting point is the following lemma, straightforward consequence of [6, Lemma 2].

**Lemma 2.1.** *We have for every $u \in \mathbb{R}$ coefficients $\alpha_{n,u} \geqslant 0, n \in \mathbb{N}$ such that for two centred standard Gaussian variables $X, Y$ with correlation $\rho$*

$$\Gamma_u(\rho) := \mathrm{Cov}(\mathbf{1}_{\{X>u\}}, \mathbf{1}_{\{Y>u\}}) = \sum_{n=1}^{\infty} \alpha_{n,u}\rho^n = \frac{1}{2\pi}\int_0^\rho \frac{1}{\sqrt{1-r^2}} \exp\left(-\frac{u^2}{1+r}\right) dr \tag{2.3}$$

*in particular,* $\Gamma_0(\rho) = \arcsin(\rho)$ *with* $\alpha_{2n,0} = 0$ *and*

$$\alpha_{2n+1} := \alpha_{2n+1,0} = \frac{\binom{n}{2n}}{4^n(2n+1)} = \Theta n^{-3/2}.$$

*We also have* $\alpha_{2,u} \neq 0$ *for* $u \neq 0$*.*

Let $\mathbf{U}_n = \sum_{i=1}^n X_i$ where the $X_i$ are independent and identically distributed with law $\boldsymbol{\mu}$. Denote by $\gamma^{\star 2}$ the auto-convolution of $\gamma$ with itself, and by $\boldsymbol{\mu}^n$ the law of $\mathbf{U}_n$. We have by Lemma 2.1

$$\begin{aligned}
V^{\gamma}_{\boldsymbol{\mu}}(T) &= \int_{(\mathbb{R}^d)^2} \Gamma_u(C(\mathbf{t}-\mathbf{s}))\gamma(\mathbf{t}/T)\gamma(\mathbf{s}/T)d\mathbf{t}d\mathbf{s} \\
&= \int_{(\mathbb{R}^d)^2} \Gamma_u(C(\mathbf{z}))\gamma\left(\frac{\mathbf{z}+\mathbf{w}}{2T}\right)\gamma\left(\frac{\mathbf{w}-\mathbf{z}}{2T}\right)d\mathbf{w}d\mathbf{z} \\
&= \int_{\mathbb{R}^d} \Gamma_u(C(\mathbf{z}))\gamma^{\star 2}_{2T}(2\mathbf{z})d\mathbf{z} \\
&= \sum_{n\in\mathbb{N}} \alpha_{n,u} \int C(\mathbf{z})^n \gamma^{\star 2}_{2T}(2\mathbf{z})d\mathbf{z} \\
&= \sum_{n\in\mathbb{N}} \alpha_{n,u} \int \boldsymbol{\mu}^n(d\mathbf{z})\hat{\gamma}_{2T}(2\mathbf{z})^2 d\mathbf{z} \text{ using (1.1)} \\
&= \sum_{n\in\mathbb{N}} \alpha_{n,u} \int \boldsymbol{\mu}^n(d\mathbf{z})(2T)^{2d}\hat{\gamma}(4T\mathbf{z})^2 d\mathbf{z} \\
&= (2T)^{2d} \sum_{n\in\mathbb{N}} \alpha_{n,u}\mathbb{E}(\hat{\gamma}(4T\mathbf{U}_n)^2) \qquad (2.4) \\
&= 2^{2d}(v_T^{(1)} + v_T^{(2)})
\end{aligned}$$

where, with $A_1 = [0, r], A_2 = ]r, \infty]$

$$v_T^{(i)} = T^{2d} \sum_{n\in\mathbb{N}} \alpha_{n,u}\mathbb{E}(\hat{\gamma}(4T\mathbf{U}_n)^2 \mathbf{1}_{\{\|T\mathbf{U}_n\|\in A_i\}}).$$

For the case $u \neq 0$, point (iii) simply comes by lower bounding by the term corresponding to $n = 2$.

Let us now focus on the case $u = 0$. Remark first that since $\boldsymbol{\mu}$ is $\mathbb{Z}$-free, $\mathbb{P}(\mathbf{U}_n = 0) = 0$ for $n$ odd, hence

$$v_T^{(1)} \geqslant T^{2d} c_r^- \mathbf{K}(rT^{-1})$$

hence (2.1) is proved. For the second point, the hypothesis on $\gamma$ yields

$$\begin{aligned}v_T^{(2)} \leqslant& T^{2d}\sum_{n \text{ odd}} \alpha_n \mathbb{E}(c_1^2\|T\mathbf{U}_n\|^{-d-1}\mathbf{1}_{\{\|T\mathbf{U}_n\|>r\}})\\ =&c_1^2T^{2d}\sum_{n \text{ odd}}\alpha_n\int_0^{r^{-d-1}}\mathbb{P}((T\|\mathbf{U}_n\|)^{-d-1}>y)dy\\ =&c_1^2T^{2d}T^{-d-1}\sum_{n \text{ odd}}\alpha_n\int_0^{(T/r)^{d+1}}\mathbb{P}(\|\mathbf{U}_n\|<y^{-\frac{1}{d+1}})dy\\ =&c_1^2T^{d-1}\int_0^{(T/r)^{d+1}}\mathbf{K}(y^{-\frac{1}{d+1}})dy.\end{aligned}$$

To conclude the proof of (ii), let us assume that $c_-\varepsilon^\alpha \leqslant \mathbf{K}(\varepsilon) \leqslant c_+\varepsilon^\alpha$ as $\varepsilon \to 0$ for some $0 < c_- \leqslant c_+ < \infty$, and let us prove that $\alpha \leqslant d+1$. If $\alpha > d+1$, then the second term on the right hand side of 2.2 is negligible with respect to the first one (recall that $\mathbf{K}$ is uniformly bounded), we have in particular

$$c_r^- c_- r^\alpha \leqslant T^{\alpha-2d}V_{\boldsymbol{\mu}}(T) \leqslant c_r^+ c_+ r^\alpha + o_{T\to\infty}(T).$$

Since this is true for all $r$, we have in particular for $0 < r_1 < r_2$

$$c_{r_1}^- c_- r_1^\alpha \leqslant c_{r_2}^+ c_+ r_2^\alpha$$

which is impossible if we let $r_2$ go to 0 faster than $r_1$.

## 2.2 Proof of Theorem 1.1

We study the field at the original scale $X_d$, it is then straightforward to deduce the results for $X_{d,T}, T > 0$. We need to estimate $\mathbb{P}(\|\mathbf{U}_n\| \leqslant \varepsilon)$ for $n \geqslant 1$, where $\mathbf{U}_n$ is the random walk which increment measure is $\boldsymbol{\mu}_d$. Equation (1.6) and the universal bound $|B_a(\mathbf{t})| \leqslant \Theta\|\mathbf{t}\|^{-1/2}, \mathbf{t} \in \mathbb{R}^d$ yield $|C_d(\mathbf{t})| \leqslant \Theta(1+\|\mathbf{t}\|)^{-\frac{d-1}{2}}$. Then

$$\begin{aligned}\mathbb{P}(\|\mathbf{U}_n\| \leqslant \varepsilon) = \int \mathbf{1}_{\{B_d(0,\varepsilon)\}}(\mathbf{z})\boldsymbol{\mu}_d^n(d\mathbf{z}) &\leqslant \varepsilon^d \int \mathbf{1}_{\{B_d(0,1)\}}(\mathbf{x})\left|\int_{\mathbb{R}^d} C(\mathbf{t})^n e^{i\varepsilon\mathbf{x}\mathbf{t}}d\mathbf{t}\right| d\mathbf{x}\\ &\leqslant \Theta\varepsilon^d\int_{\mathbb{R}^d}(1+\|\mathbf{t}\|)^{-n\frac{d-1}{2}}d\mathbf{t},\end{aligned}$$

hence $\mathbb{P}(\|\mathbf{U}_n\| \leqslant \varepsilon) \leqslant c_{5+}\varepsilon^d < \infty$ for $n \geqslant 5$ (and $n \geqslant 3$ if $d \geqslant 4$). We still have to deal with $1 \leqslant n \leqslant 4$, and independently with the lower bounds. Let us analyse the self-convoluted measures $\boldsymbol{\mu}_d^n, n \geqslant 1$. They are related by the recurrence relation, based on the isotropy of the measures $\boldsymbol{\mu}^n, n \geqslant 1$,

$$\begin{aligned}\boldsymbol{\mu}_d^{n+1}(B_d(0,r)) =& \int_{\mathbb{S}^{d-1}\times\mathbb{R}^d}\mathbf{1}_{\{s+x\in B_d(0,r)\}}\boldsymbol{\mu}_d^n(ds)\boldsymbol{\mu}_d(dx)\\ =&\boldsymbol{\mu}_d(\mathbb{S}^{d-1})\boldsymbol{\mu}_d^n(\{x : x+e_1 \in B_d(0,r)\}) = \boldsymbol{\mu}_d^n(B_d(-e_1,r)) = \boldsymbol{\mu}_d^n(B_d(e_1,r))\end{aligned} \tag{2.5}$$

where $e_1$ is some vector of $\mathbb{S}^{d-1}$, e.g. $e_1 = (1, 0, \ldots, 0)$. Let $r > 0$. Hence

$$\boldsymbol{\mu}_d^2(B_d(0,r)) = \boldsymbol{\mu}_d(B_d(e_1, r)) \tag{2.6}$$

which is equivalent to $\mathscr{L}^{d-1}(B_{d-1}(0,r))$ as $r \to 0$. Theorem 2.1-(iii) implies in the case $u \neq 0$ that

$$V_{\boldsymbol{\mu}}^{\gamma,u}(T) \geqslant 2^{2d} c_r^- \alpha_{2,u} T^{2d} \mu_d^2(B_d(0, rT^{-1})) \geqslant c_{2,-} T^{d+1}, \varepsilon > 0$$

for some $c_{2,-} > 0$ (for $r > 0$ sufficiently small, see Example 2.1).

Since $\alpha_{2,0} = \alpha_{4,0} = 0$ (Lemma 2.1), to treat the case $u = 0$ it remains to study $\boldsymbol{\mu}_d^3$ (only for $d = 2$ and $d = 3$). Using (2.5)-(2.6) easily yields $0 < c_{3,-}\varepsilon^d \leqslant \boldsymbol{\mu}_d^3(B_d(0,\varepsilon)) \leqslant c_{3,+}\varepsilon^d < \infty$ as $\varepsilon \to 0$. Hence

$$\alpha_3 c_{3,-} \varepsilon^d \leqslant \mathbf{K}(\varepsilon) = \sum_{n \geqslant 3, n \text{ odd}} \alpha_n \mathbb{P}(\|\mathbf{U}_n\| \leqslant \varepsilon) \leqslant \alpha_3 c_{3,+} \varepsilon^d + \sum_{n=5}^{\infty} \alpha_n c_{5+} \varepsilon^d$$

gives the desired upper and lower bounds for $u = 0$ (using Theorem 2.1-(i),(ii)).

## 2.3 Proof of Proposition 1.1

The statement in the case $\boldsymbol{\mu}^n(B(0,\varepsilon)) \geqslant c\varepsilon^d, n$ odd, follows immediately from (2.1). If $x \neq 0$, we have $\mu^{n+m}(B_d(0,\varepsilon)) \geqslant c'\varepsilon^d$ for $m > |x|/\varepsilon$ even, for some $c' > 0$. Let us prove that this is the case if $\boldsymbol{\mu}$ is isotropic. There is $b > 0$ such that $\boldsymbol{\mu}(A_b) > 0$ where

$$A_b = \{\mathbf{t} \in \mathbb{R}^d : \|t\| \in [b, b+1]\}.$$

Up to lower bounding $\boldsymbol{\mu}$ by $\boldsymbol{\mu}\mathbf{1}_{A_d}$, assume without loss of generality that $\boldsymbol{\mu}$'s support is contained in $A_b$. By isotropy there is a measure $\nu$ on $[b, b+1]$ such that $\boldsymbol{\mu}$ can be decomposed in $\boldsymbol{\mu} = \boldsymbol{\mu}_d \times \nu$ in polar coordinates, where $\boldsymbol{\mu}_d$ is the uniform measure on the $d$-dimensional sphere (see Section 2.2). We have

$$\begin{aligned} C(\mathbf{t}) &= \int_{\mathbb{R}^d} \exp(-i\mathbf{x} \cdot \mathbf{t}) \boldsymbol{\mu}(d\mathbf{x}) \\ &= \int_b^{b+1} B_0(r\|\mathbf{t}\|)\nu(dr) \\ &\leqslant \int_b^{b+1} \Theta(1 + r\|t\|)^{-\frac{1}{2}} \nu(dr) \\ &\leqslant \Theta(1 + b\|t\|)^{-\frac{1}{2}}, \mathbf{t} \in \mathbb{R}^d. \end{aligned}$$

It follows that $C^{2d+1} \in L^1(\mathbb{R}^d)$, hence $\boldsymbol{\mu}^{2d+1}$ has a bounded continuous density $f$, there is in particular $\mathbf{t} \in \mathbb{R}^d, r > 0, c > 0$ such that $f \geqslant c > 0$ on $B(\mathbf{x}, r)$. For $m \geqslant 1$, $\mu^{(2d+1)m}$ hence has a positive density on $B(\mathbf{x}, mr)$, and for $m$ sufficiently large, $\mu^{(2d+1)m}$ has a positive density on $B(0,1)$. Since $n = (2d+1)m$ is odd for $m$ odd, we indeed have $\mu^n(B_d(0,\varepsilon)) \geqslant c'\varepsilon^d$ for some $c' > 0$.

# 3 Irrational random walks

We consider a random walk in $\mathbb{R}^d$ which increment measure $\boldsymbol{\mu}$ is symmetric with finite support. For technical reasons, it is simpler to assume that $\boldsymbol{\mu}$ has atoms along some $d$ linearly independent unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$, with the same number of atoms in each direction: for some $m \geqslant 1$, let $\boldsymbol{\omega} = (\omega_{[k],i})_{\substack{1 \leqslant k \leqslant d \\ 1 \leqslant i \leqslant m}} \in (\mathbb{R}^m)^d$ and $\boldsymbol{\mu}$ be of the form (1.4) with $\omega_{[k],0} = 1$ by convention. We are interested in the associated random walk

$$\mathbf{U}_n := \sum_{k=1}^{n} X_k$$

where the $X_k$ are independent and identically distributed with law $\boldsymbol{\mu}$, hence centred. The study of $\mathbf{U}_n$ is related to the random walk on the torus

$$\overline{\mathbf{U}}_n = \mathbf{U}_n - [\mathbf{U}_n] \in [0,1[^d,$$

which has been intensively studied, the consequences of the current results to the random walk on the torus are discussed at Section 1.3. To avoid degenerate behaviour, we assume that $\boldsymbol{\mu}$ is $\mathbb{Z}$-free, i.e. there is no $\mathbf{q} \in \mathbb{Z}^M \setminus \{0\}$ such that $\sum_{i=1}^{M} \mathbf{q}_i \boldsymbol{\omega}_i = 0$, where $M = (m+1)d$. In general we further assume that the $\omega_{[k]}$ are $\psi$-BA for some non-vanishing function $\psi$, which automatically implies that $\boldsymbol{\mu}$ is $\mathbb{Z}$-free.

According to the Central Limit Theorem, the law of the renormalised sum $n^{-1/2}\mathbf{U}_n$ weakly converges to a Gaussian measure (see also Lemma 3.2 for precise estimates), and the law $\overline{\boldsymbol{\mu}_n}$ of $\overline{\mathbf{U}}_n$ is known to converge to Lebesgue measure on $[0,1[^d$ [33]. But if we zoom in further on this convergence around 0, it becomes very erratic. We estimate the following quantities below:

$$\begin{aligned} p_n^{\mathbf{x}}(\varepsilon) =& \mathbb{P}(0 < \|\mathbf{U}_n - \mathbf{x}\| \leqslant \varepsilon), \varepsilon > 0, \mathbf{x} \in \mathbb{Z}^d, \\ \bar{p}_n(\varepsilon) =& \sum_{\mathbf{x} \in \mathbb{Z}^d} p_n^{\mathbf{x}}(\varepsilon) = \mathbb{P}(0 < \|\overline{\mathbf{U}}_n\| \leqslant \varepsilon). \end{aligned}$$

**Remark 3.1.** *In general, if the sum $n + \sum_{k=1}^{d} \mathbf{x}_{[k]}$ is even, $\mathbb{P}(\mathbf{U}_n = \mathbf{x})$ is in $n^{-\frac{M}{2}}$ and dominates $p_n^{\mathbf{x}}(\varepsilon)$ for $\varepsilon \to 0$, which is why it is estimated separately. For odd values, since $\boldsymbol{\mu}$ is $\mathbb{Z}$-free, $\mathbb{P}(\mathbf{U}_n = \mathbf{x}) = 0$, hence $p_n^{\mathbf{x}}(\varepsilon)$ is simply $\mathbb{P}(\|\mathbf{U}_n - \mathbf{x}\| \leqslant \varepsilon)$. A fine analysis of the recurrence around 0 yields that the rate strongly depends on the number of coordinates equal to 0, expressed through*

$$p_n^{K,\mathbf{x}}(\varepsilon) = \mathbb{P}(\mathbf{U}_n - \mathbf{x} \in B_K(0,\varepsilon)), \ \bar{p}_n^{K}(\varepsilon) = \mathbb{P}(\overline{\mathbf{U}}_n \in B_K(0,\varepsilon)).$$

*We show below for instance that if along each direction $k$ of $\mathbb{R}^d$, $\boldsymbol{\mu}$'s support is made up of a vector $(\omega_{[k],i})_{1 \leqslant i \leqslant m}$ which is $q^{-(m+\eta)}$-(Badly Approximable) for some $m \geqslant 1$, $\eta \geqslant 0$ (see 1.2), then for some $c < \infty$,*

$$\bar{p}_n^{K}(\varepsilon) \leqslant c n^{-m\frac{d-|K|}{2}} \varepsilon^{\frac{m|K|}{m+\eta}}, \quad K \subset [\![d]\!], n \in \mathbb{N}, 0 < \varepsilon < \frac{1}{2},$$

*so that it is really the number of vanishing coordinates that determines the recurrence probabilities.*

To avoid the technicality mentionned in the previous remark and obtain lower bounds, we consider the smoothed estimates for $\beta > 0$, with $p_n^{\mathbf{0}} = p_n^{[\![d]\!],\mathbf{0}}$

$$\mathbf{J}_\beta(\varepsilon) := \sum_{n \geqslant n_\varepsilon, n \in \mathbb{N}, n \text{ odd}} n^{-\beta/2} p_n^{\mathbf{0}}(\varepsilon)$$

where $n_\varepsilon$ does not grow too fast as $\varepsilon \to 0$ and serves the purpose to show that it is the series tail that actually matters.

**Remark 3.2.** *Considering this statistic also allows to suppress the erratic behaviour in $n$, and we can prove that $\mathbf{J}_\beta(\varepsilon)$ and $\mathbf{I}_\beta(\varepsilon)$ (defined in the introduction at (1.5)) both behave in $\varepsilon^{\frac{m}{m+\eta}}$ and find a matching lower bound. The summation over odd $n$ in $\mathbf{J}_\beta(\varepsilon)$ is adapted to estimating the volume variance of Gaussian nodal excursions (see Remark 3.3).*

If the atoms are different in different directions, we need to generalise the concept of $\psi$-WA: say that $\boldsymbol{\omega} = (\omega_{[k]})_{1 \leqslant k \leqslant d}$ is $\psi$-SWA' (Simultaneously Well Approximable) if for some $c > 0$, for infinitely many $q^j \in \mathbb{Z}^m, j \geqslant 1$, there exist $p_{[k]}^j \in \mathbb{Z}, 1 \leqslant k \leqslant d$ such that

$$|p_{[k]}^j - \omega_{[k]} \cdot q^j| < c\psi(|q^j|), 1 \leqslant k \leqslant d.$$

Say that $\boldsymbol{\omega}$ is $\psi$-SWA if furthermore $\sum_{k=1}^d (p_{[k]}^j + \sum_{i=1}^m q_i^j)$ is odd. The need to distinguish between $\psi$-SWA and $\psi$-SWA' is discussed in Remark 3.3.

**Theorem 3.1.** *Let $\psi$ be some mapping $\mathbb{N}_* \to (0,1]$ converging to $0$, and let $\psi^{-1}$ be its pseudo-inverse defined by*

$$\psi^{-1}(\varepsilon) = \min\{q \in \mathbb{N}^* : \psi(q) \leqslant \varepsilon\}, \varepsilon > 0. \tag{3.1}$$

*Let $\beta > 0$. There is $0 < c < \infty$ depending on $d, m, \psi, \beta$ such that the following holds:*
*(i) assume each $\omega_{[k]}$ is $\psi$-BA. We have for $\mathbf{x} \in \mathbb{Z}^d, K \subset [\![d]\!], 0 < \varepsilon < \frac{1}{2}, n \in \mathbb{N}^*$*

$$p_n^{K,\mathbf{x}}(\varepsilon) \leqslant c n^{-d/2} n^{-\frac{(d-|K|)m}{2}} \psi^{-1}(\varepsilon)^{-m|K|} \exp(-cn^{-1}\|\mathbf{x}\|^2) \tag{3.2}$$

$$\bar{p}_n^K(\varepsilon) \leqslant c n^{\frac{-(d-|K|)m}{2}} \psi^{-1}(\varepsilon)^{-m|K|} \tag{3.3}$$

$$\mathbf{I}_\beta(\varepsilon) \leqslant c\psi^{-1}(\varepsilon)^{-\beta-dm+2} \tag{3.4}$$

$$\mathbf{J}_\beta(\varepsilon) \leqslant c\psi^{-1}(\varepsilon)^{-\beta-d(m+1)+2} \tag{3.5}$$

*(ii) Assume $\boldsymbol{\omega}$ is $\psi$-SWA'. Then if $n_\varepsilon \leqslant \psi^{-1}(\varepsilon)^2, \varepsilon > 0$,*

$$\psi^{-1}(\varepsilon)^{-\beta-dm+2} \overset{\textbf{i.o.}}{\leqslant} c\mathbf{I}_\beta(\varepsilon), \varepsilon \to 0. \tag{3.6}$$

*(iii) Assume $\boldsymbol{\omega}$ is $\psi$-SWA . Then if $n_\varepsilon \leqslant \psi^{-1}(\varepsilon)^2, \varepsilon > 0$,*

$$\psi^{-1}(\varepsilon)^{-\beta-d(m+1)+2} \overset{\textbf{i.o.}}{\leqslant} c\mathbf{J}_\beta(\varepsilon), \varepsilon \to 0. \tag{3.7}$$

**Remark 3.3.** *The assumption that $\boldsymbol{\omega}$ is $\psi$-SWA is stronger than $\psi$-SWA', and also less natural, which might cause confusions. The reason why $\boldsymbol{\omega}$ has to be $\psi-$SWA instead of $\psi-$SWA' at point (iii) is because summands are odd in the definition of $\mathbf{J}_\beta$. The assumption $\psi$-SWA' and function $\mathbf{I}_\beta$ are introduced only because they are more natural in the context of random walks, but they are useless for giving lower bounds for Gaussian excursions variances. As supported by Proposition 4.5, this subtlety does not influence final results about Gaussian excursions, hence one would like a general result from diophantine approximation that states that $\psi-$SWA' tuples are also $\psi-$SWA, but that is most likely not true.*

**Example 3.1.** *An example intensively used in this article is $\beta = 3, \psi(q) = cq^{-(m+\eta)}, \eta \geqslant 0$, for which*

$$\psi^{-1}(\varepsilon)^{-\beta-d(m+1)+2} = c'\varepsilon^{\frac{1+d(m+1)}{m+\eta}}, \varepsilon > 0. \tag{3.8}$$

## 3.1 Proof of Proposition 1.2

Recall from Lemma 2.1 that

$$\mathrm{Cov}(\mathbf{1}_{\{0\in E\}}, \mathbf{1}_{\{\mathbf{t}\in E\}}) = \sum_{n \text{ odd}} \alpha_n C(\mathbf{t})^n, \mathbf{t} \in \mathbb{R}^d$$

where $C$ is the reduced covariance function of $X_{\boldsymbol{\omega}}$. Hence we are looking for $\mathcal{S}$ satisfying for $\varphi$ smooth with compact support

$$\begin{aligned}\int_{\mathbb{R}^d} \hat{\varphi}(\mathbf{x})\mathcal{S}(d\mathbf{x}) &= \int_{\mathbb{R}^d} \varphi(\mathbf{t}) \sum_{n \text{ odd}} \alpha_n C(\mathbf{t})^n d\mathbf{t} \\ &= \int_{\mathbb{R}^d} \varphi(\mathbf{t}) \sum_{n \text{ odd}} \alpha_n \left( \int_{\mathbb{R}^d} e^{i\mathbf{t}\cdot\mathbf{x}} \boldsymbol{\mu}(d\mathbf{t}) \right)^n d\mathbf{x}\end{aligned}$$

hence

$$\mathcal{S} = \sum_{n \text{ odd}} \alpha_n \boldsymbol{\mu}^n.$$

Then using (3.5) in the context of Example 3.1,

$$\mathcal{S}(B_d(0,\varepsilon)) \leqslant c \sum_{n \text{ odd}} \alpha_n \boldsymbol{\mu}^n(B_d(0,\varepsilon)) \leqslant c'\varepsilon^{\frac{1+d(m+1)}{m+\eta}}.$$

## 3.2 Proof of Theorem 3.1

**Notation.** We specify here the notation $A = \Theta B$ to indicate that there are finite constants $c, c' > 0$ depending on $m, d, \psi, \beta$ and not (further) on $\boldsymbol{\omega}, \varepsilon, T, n$ such that $A \leqslant cB, B \leqslant c'A$.

Also, for a $d$-tuple of vectors of $\mathbb{R}^{m+1}$, $\bar{\mathbf{x}} = (\bar{x}_{[1]}, \dots, \bar{x}_{[d]}) \in (\mathbb{R}^{m+1})^d$ with $\bar{x}_{[k]} = (x_{[k],0}, \dots, x_{[k],m}) \in \mathbb{R}^{m+1}$, remove the bar when the 0-th component is removed from each vector:

$$x_{[k]} = (x_{[k],1}, \dots, x_{[k],m}), \qquad \mathbf{x} = (x_{[1]} \dots, x_{[d]}).$$

Euclideand norms in $\mathbb{R}^m$ are denoted by a single bar and in $(\mathbb{R}^m)^d$ by two bars:

$$|x_{[k]}|^2 = \sum_{i=1}^{m} x_{[k],i}^2, \ \|\mathbf{x}\|^2 = \sum_{k=1}^{d} |x_{[k]}|^2.$$

We also define for $q \in \mathbb{Z}^m, \omega \in \mathbb{R}^m$

$$\mathrm{d}_q(\omega) = \inf_{p \in \mathbb{Z}} |p - q \cdot \omega|.$$

**Lemma 3.1.** *Let $\omega \in \mathbb{R}^m$ that is $\psi$-BA. For $1/2 \geqslant \varepsilon > 0$, define*

$$I_\varepsilon(\omega) := \{q \in \mathbb{Z}^m \setminus \{0\} : 0 < \mathrm{d}_q(\omega) \leqslant \varepsilon\}.$$

*Let $q^{(N)}, N \in \mathbb{N}^*$, the elements of $I_\varepsilon$ ordered by increasing radius. Then*

$$|q^{(N)}| \geqslant \Theta N^{\frac{1}{m}} \psi^{-1}(\varepsilon). \tag{3.9}$$

*In particular, we prove the following estimate:*

$$\sum_{q \in I_\varepsilon} \exp(-\Theta n^{-1} |q|^2) \leqslant \sum_{N=1}^{\infty} \exp(-\Theta n^{-1} N^{\frac{2}{m}} \psi^{-1}(\varepsilon)^2) \leqslant \Theta n^{\frac{m}{2}} \psi^{-1}(\varepsilon)^{-m}. \tag{3.10}$$

*Proof.* The starting point is that for $q \in I_\varepsilon$, since $\varepsilon > \varepsilon/2 \geqslant \mathrm{d}_q(\omega)/2 \geqslant \psi(|q|)$, we have $|q| \geqslant \rho := \psi^{-1}(\varepsilon)$. And the triangular inequality yields for $q \neq q' \in I_\varepsilon$,

$$2\psi(q - q') \leqslant \mathrm{d}_{q-q'}(\omega) \leqslant 2\varepsilon,$$

hence $|q - q'| \geqslant \rho$ as well. It follows that all $q \in I_\varepsilon$ are pairwise distant by more than $\rho$, and the balls $B_m(q, \rho/2), q \in I_\varepsilon$ are disjoint. Hence for $N_0 \in \mathbb{N}^*$, the total $\mathscr{L}^m-$ measure occupied by the $B_m(q^{(N)}, \rho/2), N \leqslant N_0$ is larger than $\Theta N_0 \rho^m$. This volume is necessarily smaller than the volume of the ball with radius $|q^{(N_0)}| + \rho/2 \leqslant 2|q^{(N_0)}|$, hence

$$\Theta N_0 \rho^m \leqslant \Theta |q^{(N_0)}|^m$$

which yields (3.9). Finally (3.10) follows from

$$\sum_{N=1}^{\infty} \exp(-\Theta n^{-1} (N^{\frac{1}{m}} \psi^{-1}(\varepsilon))^2) \leqslant 2 \int_{1/2}^{\infty} \exp(-\Theta (n^{-\frac{m}{2}} \psi^{-1}(\varepsilon)^m y)^{\frac{2}{m}}) dy \leqslant \Theta n^{\frac{m}{2}} \psi^{-1}(\varepsilon)^{-m}.$$

□

*Proof of Theorem 3.1.* Let $M = d(m+1)$. The proof is based on the study of the symmetric random walk $(S_n)_n$ on $\mathbb{Z}^M$ with independent increments defined by $S_0 = 0$ and

$$\mathbb{P}(S_{n+1} = S_n \pm \mathbf{e}_j) = \frac{1}{2M}, 1 \leqslant j \leqslant M,$$

where $(\mathbf{e}_j)_j$ is some basis of $\mathbb{R}^d$. Following the notation introduced above, denote also $\bar{\omega}_{[k]} = (1, \omega_{[k]})$ and $\bar{\boldsymbol{\omega}} = (\bar{\omega}_{[k]})_k$.

For $\bar{q}_{[k]} = (q_{[k],0}, q_{[k]}) \in \mathbb{Z}^{m+1}, 1 \leqslant k \leqslant d, \bar{\mathbf{q}} = (\bar{q}_{[1]}, \dots, \bar{q}_{[d]}) \in (\mathbb{Z}^{m+1})^d \approx \mathbb{Z}^M$, denote by $q_{[k]} = (q_{[k],1}, \dots, q_{[k],m}) \in \mathbb{Z}^m, \mathbf{q} = (q_{[k]})_k \in (\mathbb{Z}^m)^d$. We define

$$\bar{\mathbf{q}} \otimes \bar{\boldsymbol{\omega}} := \mathbf{q}_0 - (\bar{q}_{[k]} \cdot \bar{\omega}_{[k]})_{k=1,\dots,d}$$

where $\mathbf{q}_0 = (q_{[k],0})_{k=1}^d$, so that we have the equality in law $\mathbf{U}_n \overset{(d)}{=} S_n \otimes \bar{\boldsymbol{\omega}}$.

We use the notation, for $\mathbf{x} = (x_{[1]}, \dots, x_{[d]}) \in \mathbb{Z}^d, K \subset [\![d]\!]$,

$$\bar{\mathbf{I}}_\varepsilon^{\mathbf{x},K}(\boldsymbol{\omega}) = \bar{\mathbf{I}}_\varepsilon^{\mathbf{x},K} = \{\bar{\mathbf{q}} \in \mathbb{Z}^M : \bar{q}_{[k]} = 0, k \notin K \text{ and } 0 < |\bar{q}_{[k]} \cdot \bar{\omega}_{[k]} - x_{[k]}| \leqslant \varepsilon, k \in K\}.$$

For $\varepsilon < 1/2$, an element $\bar{\mathbf{q}} \in \bar{\mathbf{I}}_\varepsilon^{\mathbf{x},K}$ satisfies the following for $k \in K$:

$$|q_{[k],0} - q_{[k]} \cdot \omega_{[k]} - x_{[k]}| < \varepsilon,$$

hence since $x_{[k]} \in \mathbb{Z}$ and $q_{[k]} \in I_\varepsilon(\omega_{[k]})$, $\omega_{[k]} \cdot q_{[k]}$ is $\varepsilon$-close to $\mathbb{Z}$. It follows that $q_{[k],0}$ depends explicitly on other coordinates

$$\begin{aligned} q_{[k],0} =& q_{[k],0}(x_{[k]}, q_{[k]}) := \operatorname{argmin}_{p \in \mathbb{Z}} |p - q_{[k]} \cdot \omega_{[k]} - x_{[k]}| \\ \mathbf{q}_0 =& \mathbf{q}_0(\mathbf{x}, \mathbf{q}) := (q_{[k],0})_k. \end{aligned} \tag{3.11}$$

In particular, $|x_{[k]}| \leqslant |q_{[k],0}| + |q_{[k]} \cdot \omega_{[k]}| + 1$, and

$$\|\bar{\mathbf{q}}\|^2 = \|\mathbf{q}_0\|^2 + \|\mathbf{q}\|^2 \geqslant \max(\|\mathbf{q}\|^2, \|\mathbf{q}\|^2 + \Theta(\|\mathbf{x}\|^2 - 1)) \geqslant \Theta(\|\mathbf{q}\|^2 + \|\mathbf{x}\|^2). \tag{3.12}$$

We also have the one-to-one correspondance

$$\mathbf{I}_\varepsilon^{\mathbf{x},K}(\boldsymbol{\omega}) := \{\mathbf{q} \in \prod_{k \in K} \mathbb{Z}^m : (\mathbf{q}_0(\mathbf{x}; \mathbf{q}), \mathbf{q}) \in \bar{\mathbf{I}}_\varepsilon^{\mathbf{x},K}(\boldsymbol{\omega}), k \in K\} \equiv \{\mathbf{0}\}^{d-|K|} \times \prod_{k \in K} I_\varepsilon(\omega_{[k]}). \tag{3.13}$$

**Proof of (i):** By the Gaussian approximation Lemma 3.2 (below), and

(3.12),

$$
\begin{aligned}
p_n^{\mathbf{x},K}(\varepsilon) &= \sum_{\bar{\mathbf{q}}\in\bar{\mathbf{I}}_\varepsilon^{\mathbf{x},K}} \mathbb{P}(S_n=\bar{\mathbf{q}}) = \sum_{\mathbf{q}\in\mathbf{I}_\varepsilon^{\mathbf{x},K}} \mathbb{P}(S_n=(\mathbf{q}_0(\mathbf{x},\mathbf{q}),\mathbf{q}))\\
&\leqslant \Theta \sum_{\mathbf{q}\in\mathbf{I}_\varepsilon^{K,\mathbf{x}}} n^{-\frac{M}{2}}\exp(-\Theta n^{-1}(\|\mathbf{q}\|^2+\|\mathbf{q}_0\|^2))\\
&\leqslant \Theta n^{-\frac{M}{2}} \sum_{\mathbf{q}\in\mathbf{I}_\varepsilon^{K,\mathbf{x}}} \exp(-\Theta n^{-1}\|\mathbf{q}\|^2)\exp(-\Theta n^{-1}\|\mathbf{x}\|^2))\\
&\leqslant \Theta n^{-\frac{M}{2}}\exp(-\Theta n^{-1}\|\mathbf{x}\|^2)\prod_{k\in K}\sum_{q_{[k]}\in I_\varepsilon(\omega_{[k]})}\exp(-\Theta n^{-1}|q_{[k]}|^2)) \text{ by (3.13)} \qquad (3.14)\\
&\leqslant \Theta n^{-\frac{d(m+1)}{2}}\exp(-\Theta n^{-1}\|\mathbf{x}\|^2)(n^{\frac{m}{2}}\psi^{-1}(\varepsilon)^{-m})^{|K|} \quad \text{with (3.10)},\\
&\leqslant \Theta n^{-d/2} n^{-\frac{(d-|K|)m}{2}}\psi^{-1}(\varepsilon)^{-m|K|}\exp(-\Theta n^{-1}\|\mathbf{x}\|^2)
\end{aligned}
$$

and (3.2) is proved.

The bound (3.3) immediately stems from $\bar{p}_n^K=\sum_{\mathbf{x}\in\mathbb{Z}^d}p_n^{\mathbf{x},K}$ and Lemma 3.3 (after summing over $i\in\{0,1\}$). Hence using (3.10), and (3.14) with $\mathbf{x}=0$

$$
\begin{aligned}
\mathbf{I}_\beta^{\mathbf{0}}(\varepsilon) &:= \sum_{n\geqslant n_\varepsilon} n^{-\beta/2} p_n^{\mathbf{0}}(\varepsilon)\\
&= \sum_{n\geqslant n_\varepsilon} n^{-\beta/2}\sum_{K\neq\emptyset} p_n^{\mathbf{0},K}(\varepsilon)\\
&\leqslant \Theta\mathbf{K}_\beta(\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
\text{with } \mathbf{K}_\beta(\varepsilon) &:= \sum_{n\geqslant n_\varepsilon} n^{-\beta/2-\frac{M}{2}} \sum_{K\subset[\![d]\!],K\neq\emptyset}\prod_{k\in K}\sum_{N_k=1}^{\infty}\exp(-\Theta n^{-1}(N_k^{\frac{1}{m}}\psi^{-1}(\varepsilon))^2)\\
&\leqslant \sum_{n\geqslant n_\varepsilon} n^{-\beta/2-\frac{M}{2}} \sum_{K\subset[\![d]\!],K\neq\emptyset}\sum_{N_k\geqslant 1,k\in K}^{\infty}\exp(-\Theta n^{-1}\sum_{k\in K}(N_k^{\frac{1}{m}}\psi^{-1}(\varepsilon))^2)\\
&\leqslant \Theta \sum_{K\subset[\![d]\!],K\neq\emptyset}\sum_{N_k\geqslant 1,k\in K}\sum_{n\geqslant n_\varepsilon}\int_n^{n+1/2}(z-1/2)^{-\beta/2-\frac{M}{2}}\exp(-\Theta z^{-1}\sum_{k\in K}\psi^{-1}(\varepsilon)^2N_k^{2/m})dz\\
&\leqslant \Theta \sum_{K\subset[\![d]\!],K\neq\emptyset}\sum_{N_k\geqslant 1,k\in K}(\sum_{k\in K}\psi^{-1}(\varepsilon)^2N_k^{2/m})^{1-\beta/2-\frac{M}{2}}\int_0^\infty y^{\beta/2+\frac{M}{2}-2}\exp(-\Theta y)dy\\
&\leqslant \Theta(\psi^{-1}(\varepsilon)^2)^{1-\beta/2-\frac{M}{2}}\max_{K\subset[\![d]\!],K\neq\emptyset}\int_{[1,\infty]^{|K|}}(\sum_{k\in K}x_k^{\frac{2}{m}})^{1-\beta/2-\frac{M}{2}}\prod_{k\in K}dx_k\\
&\leqslant \Theta\psi^{-1}(\varepsilon)^{2-M-\beta}\max_{1\leqslant p\leqslant d}\int_{[1,\infty]^p}(\sum_{k=1}^p y_k)^{1-\beta/2-\frac{M}{2}}\prod_{k=1}^p y_k^{\frac{m}{2}-1}dy_k\\
&\leqslant \Theta\psi^{-1}(\varepsilon)^{2-d(m+1)-\beta}\max_{1\leqslant p\leqslant d}\int_1^\infty(\Theta r)^{1-\beta/2-\frac{(m+1)d}{2}}r^{mp/2-p}r^{p-1}dr
\end{aligned}
$$

and the integral converges if $\beta/2 > 1 - d/2$. Since there are less terms in $\mathbf{J}_\beta(\varepsilon)$ than in $\mathbf{I}^{\mathbf{0}}_\beta(\varepsilon)$, the upper bound holds and (3.5) is proved.

With the same computations, using first (3.14), and then(3.10), and Lemma 3.3,

$$\begin{aligned}\mathbf{I}_\beta(\varepsilon) &= \sum_{n\geqslant n_\varepsilon} n^{-\beta/2} \sum_{\mathbf{x}\in\mathbb{Z}^d} \sum_{K\neq\emptyset} p_n^{\mathbf{x},K}(\varepsilon)\\ &\leqslant \sum_{n\geqslant n_\varepsilon} n^{-\beta/2-\frac{M}{2}} \sum_{\mathbf{x}\in\mathbb{Z}^d} \exp(-\Theta n^{-1}\mathbf{x}^2) \sum_{K\subset[\![d]\!],K\neq\emptyset} \prod_{k\in K} \sum_{N_k=1}^{\infty} \exp(-\Theta n^{-1}(N_k^{\frac{1}{m}}\psi^{-1}(\varepsilon))^2)\\ &\leqslant \Theta \mathbf{K}_{\beta-d}(\varepsilon)\end{aligned}$$

provided $\beta/2 > 1$ , which proves (3.4).

Let us conclude with the proof of **(iii)**, the proof of **(ii)** is similar and easier. There are by hypothesis infinitely many $q^j \in \mathbb{Z}^m, j \geqslant 1$ and $p^j_{[k]} \in \mathbb{Z}, 1 \leqslant k \leqslant d$, such that $\bar{\mathbf{q}}^j := ((p^j_{[k]}, q^j))_k \equiv 1$ and

$$|p^j_{[k]} - \omega_{[k]} \cdot q^j| \leqslant c_W \psi(|q^j|) =: c_W \varepsilon_j$$

(we have $\varepsilon_j \to 0$ because $\psi$ converges to 0 by hypothesis). We have in particular with Cauchy-Schwarz inequality

$$\|\bar{\mathbf{q}}^j\| \leqslant \sum_{k=1}^{d} (|p^j_{[k]}| + |q^j|) \leqslant \sum_{k=1}^{d} (|\omega_k||q^j| + 1 + |q^j|) \leqslant \Theta|q^j|$$

and clearly the other inequality as well $|q^j| \leqslant \|\bar{\mathbf{q}}^j\|$.

Then, by Lemma 3.2, with $\tilde{n}_j := c_{\inf}^{-1}|\bar{\mathbf{q}}^j| \vee n_{\varepsilon_j}$

$$\begin{aligned}\mathbf{J}_\beta(\varepsilon_j) = \sum_{n\geqslant n_{\varepsilon_j}, n \text{ odd}} n^{-\beta/2} p_n^{\mathbf{0}}(\varepsilon_j) &\geqslant \sum_{n\geqslant n_{\varepsilon_j}, n \text{ odd}} n^{-\beta/2} \mathbb{P}(S_n = \bar{\mathbf{q}}^j)\\ &\geqslant \Theta \sum_{n\geqslant \tilde{n}_j, n\equiv\bar{\mathbf{q}}^j\equiv 1} n^{-\beta/2} n^{-\frac{d(m+1)}{2}} \exp(-\Theta n^{-1}\|\bar{\mathbf{q}}^j\|^2)\\ &\geqslant \Theta \int_{[\tilde{n}_j/2]}^{\infty} y^{-\beta/2-d\frac{m+1}{2}} \exp(-\Theta y^{-1}|q^j|^2) dy\\ &\geqslant \Theta |q^j|^{2-\beta-d(m+1)} \int_0^{\Theta|q^j|^2\tilde{n}_j^{-1}} z^{\beta/2+d\frac{m+1}{2}-2} \exp(-\Theta z) dz\\ &\geqslant \Theta \psi^{-1}(\varepsilon_j)^{2-\beta-d(m+1)}\end{aligned}$$

provided $\beta > 0$, because $|q^j|^2\|\bar{\mathbf{q}}^j\|^{-1} \geqslant \Theta > 0$ and $n_{\varepsilon_j} \leqslant \psi^{-1}(\varepsilon_j)^2$ yields (recalling $\psi(|q^j|) = \varepsilon_j$)

$$|q^j|^2 n_{\varepsilon_j}^{-1} \geqslant \psi^{-1}(\varepsilon_j)^2 \psi^{-1}(\varepsilon_j)^{-2} = 1,$$

hence (3.7) is proved. The proof of (3.6) is similar without the requirement that $\bar{\mathbf{q}}^j \equiv 1$, hence the sum is over all $n \geqslant n_{\varepsilon_j}$ (even and odd).

□

## 3.3 Gaussian approximation

The following lemma quantifies how much $S_n$ is close to a Gaussian distribution.

**Lemma 3.2.** *Let $\theta_0 \in (0, \frac{1}{2}), M \in \mathbb{N}$ and $S_n$ be the symmetric random walk on $\mathbb{Z}^M$ with weights $\theta_i \in (\theta_0, 1-\theta_0), 1 \leqslant i \leqslant M$, summing to 1, i.e.*

$$\mathbb{P}(S_{n+1} = S_n \pm \mathbf{e}_i) = \frac{\theta_i}{2}, 1 \leqslant i \leqslant M, n \in \mathbb{N}.$$

*For $\mathbf{q} = (\mathbf{q}_i) \in \mathbb{Z}^M, n \in \mathbb{N}$, write $\mathbf{q} \equiv n$ if $\sum_{i=1}^M \mathbf{q}_i$ and $n$ have the same parity, and remark that $\mathbb{P}(S_n = \mathbf{q}) = 0$ if $\mathbf{q} \not\equiv n$. There is a constant $c_{\text{inf}} > 0$ such that for $\mathbf{q} \in \mathbb{Z}^M$*

$$\mathbb{P}(S_n = \mathbf{q}) \leqslant \Theta n^{-\frac{p}{2}} \exp(-\Theta n^{-1} \|\mathbf{q}\|^2) \tag{3.15}$$
$$\mathbf{1}_{\{\|\mathbf{q}\| \leqslant c_{\text{inf}} n\}} \mathbb{P}(S_n = \mathbf{q}) \geqslant \Theta n^{-\frac{p}{2}} \exp(-\Theta n^{-1} \|\mathbf{q}\|^2) \mathbf{1}_{\{|\mathbf{q}| \leqslant c_{\text{inf}} n\}} \text{ for } \mathbf{q} \equiv n.$$

**Remark 3.4.** *The constants involved in this result depend also on $\theta_0$.*

*Proof.* Let $N_i$ be the number of times direction $i$ has been chosen in the random walk, and let $B_i \leqslant N_i$ be the number of $+\mathbf{e}_i$ increments, hence $N_i - B_i$ is the number of $-\mathbf{e}_i$ increments. The $i$-th component of $S_n$ is therefore $S_{n,i} := 2B_i - N_i$. We have $N_i \sim \mathcal{B}(n, \theta_i), B_i \sim \mathcal{B}(N_i, 1/2)$, and the $B_i$ are independent conditionally on $\mathbf{N} := (N_i)_i$. Hence for $|\varepsilon| \leqslant c_{\text{Bin}}$, from Lemma 3.4

$$\mathbb{P}(B_i = [N_i(1/2 + \varepsilon)] \mid \mathbf{N}) = \Theta \exp(-\Theta N_i \varepsilon^2) N_i^{-1/2}.$$

Let $\mathbf{q} = (\mathbf{q}_i) \in \mathbb{Z}^M$ such that for $1 \leqslant i \leqslant p, |\mathbf{q}_i| \leqslant c_{\text{Bin}} N_i$, let $\varepsilon_i = N_i^{-1} \mathbf{q}_i$,

$$\begin{aligned}\mathbb{P}(S_{n,i} = \mathbf{q}_i \mid \mathbf{N}) &= \mathbb{P}(B_i = N_i/2 + \mathbf{q}_i/2 \mid \mathbf{N}) \\ &= \begin{cases} 0 \text{ if } N_i \not\equiv \mathbf{q}_i \\ \Theta N_i^{-1/2} \exp(-\Theta N_i \varepsilon_i^2) = \Theta N_i^{-1/2} \exp(-\Theta N_i^{-1} \mathbf{q}_i^2) \text{ otherwise.} \end{cases}\end{aligned}$$

Let

$$c_{\text{inf}} := c_{\text{Bin}} (\min_i \theta_i - c_{\text{Bin}}) > 0.$$

If for all $i$, $N_i > (\theta_i - c_{\text{Bin}})n$ and $|\mathbf{q}_i| < c_{\text{inf}} n$, then $|\mathbf{q}_i| < c_{\text{Bin}} N_i$ (and $N_i = \Theta n$) and we have the lower bound

$$\begin{aligned}\mathbb{P}(S_n = \mathbf{q}) =& \mathbb{E}(\mathbb{P}(S_n = \mathbf{q} \mid \mathbf{N})) \\ =& \mathbb{E}(\mathbf{1}_{\{\mathbf{q}_i \equiv N_i, \forall i\}} \mathbb{P}(S_n = \mathbf{q} \mid \mathbf{N})) \\ \geqslant& \mathbb{E}(\mathbf{1}_{\{\mathbf{q}_i \equiv N_i, N_i > (\theta_i - c_{\text{Bin}})n, \forall i\}} \mathbb{P}(S_n = \mathbf{q} \mid \mathbf{N})) \\ \geqslant& \mathbb{E}(\mathbf{1}_{\{\mathbf{q}_i \equiv N_i, N_i > (\theta_i - c_{\text{Bin}})n, \forall i\}} \Theta \prod_i N_i^{-\frac{1}{2}} \exp(-\Theta N_i^{-1} \mathbf{q}_i^2)) \\ \geqslant& \mathbb{E}(\mathbf{1}_{\{\mathbf{q}_i \equiv N_i, N_i > (\theta_i - c_{\text{Bin}})n, \forall i\}} \Theta n^{-\frac{p}{2}} \exp(-\Theta n^{-1} \mathbf{q}^2)) \\ \geqslant& \Theta n^{-\frac{p}{2}} \exp(-\Theta n^{-1} \mathbf{q}^2) \mathbb{P}(\mathbf{q}_i \equiv N_i, N_i > (\theta_i - c_{\text{Bin}})n, \forall i).\end{aligned}$$

Since $\sum_i N_i \equiv n$, if we don't have $\sum_i \mathbf{q}_i \equiv n$, we can't have $N_i \equiv \mathbf{q}_i, \forall i$. Otherwise, asymptotically a fraction $2^{-p}$ of admissible tuples $\mathbf{N} \in [\![n]\!]^M$ are such that $N_i \equiv \mathbf{q}_i, \forall i$, hence $\mathbb{P}(\mathbf{q}_i \equiv N_i, N_i > (\theta_i - c_{\mathrm{Bin}})n, \forall i) = \Theta \mathbf{1}_{\{\mathbf{q} \equiv n\}} \mathbb{P}(N_i > (\theta_i - c_{\mathrm{Bin}})n, \forall i)$ and the latter probability converges to 1 thanks to Lemma 3.4 , hence the lower bound is proved.

The upper bound is a bit delicate. Let us start by the trivial bound, if $|\mathbf{q}_i| > n$ for some $i$,

$$\mathbb{P}(S_n = \mathbf{q}) = 0 \leqslant \Theta n^{-1/2} \exp(-\Theta n^{-1} \mathbf{q}_i^2).$$

Assume henceforth that $|\mathbf{q}_i| \leqslant n$ for all $i$. Let $\Omega$ be the event that for some $i$, $N_i < \theta_i(1 - c_{\mathrm{Bin}})n$. On $\Omega^c$, $N_i = \Theta n$ for all $i$, hence by Lemma 3.2

$$\mathbb{P}(S_{n,i} = \mathbf{q}_i \mid \Omega^c) \leqslant \Theta n^{-1/2} \exp(-\Theta n^{-1} \mathbf{q}_i^2).$$

Finally, in all cases,

$$\begin{aligned}
\mathbb{P}(S_n = \mathbf{q}) \leqslant & \mathbb{E}(\mathbf{1}_{\{\Omega^c\}} \prod_i \mathbb{P}(S_{n,i} = \mathbf{q}_i \mid \mathbf{N})) + \mathbb{P}(\Omega) \\
\leqslant & \mathbb{E}(\mathbf{1}_{\{\Omega^c\}} \Theta \prod_i n^{-\frac{1}{2}} \exp(-\Theta n^{-1} \mathbf{q}_i^2)) + \mathbb{P}(\Omega) \\
\leqslant & n^{-p/2} \exp(-\Theta n^{-1} \mathbf{q}^2) + \mathbb{P}(\Omega).
\end{aligned}$$

Then Lemma 3.4 with $\varepsilon = -c_{\mathrm{Bin}}$ yields, using the decreasing of binomial probabilities around the mean,

$$\begin{aligned}
\mathbb{P}(\Omega) \leqslant & \sum_i \sum_{k < [n(\theta_i - c_{\mathrm{Bin}})]} \mathbb{P}(N_i = k) \\
\leqslant & \sum_i n \mathbb{P}(N_i = [n(\theta_i + \varepsilon)]) \\
\leqslant & \Theta n^{1/2} \exp(-\Theta n) \\
\leqslant & \Theta n^{-\frac{p}{2}} \exp(-\Theta n/2) \\
\leqslant & \Theta n^{-\frac{p}{2}} \exp(-\Theta n^{-1} \mathbf{q}^2),
\end{aligned}$$

using $\|\mathbf{q}\| \leqslant n$, which concludes the proof of (3.15). □

**Lemma 3.3.** *For $i \in \{0, 1\}$*

$$\sum_{\mathbf{x} \in \mathbb{Z}^d, \mathbf{x} \equiv i} \exp(-\Theta n^{-1} \mathbf{x}^2) = \Theta n^{d/2}.$$

*where $\mathbf{x} \equiv i$ means that $\sum_{k=1}^d \mathbf{x}_{[k]}$ has the same parity as $i$.*

*Proof.* The lower bound stems from $y^2 \geqslant \min_{\mathbf{x}\in\mathbb{Z}^d\cap B(y,2),\mathbf{x}\equiv i \text{ or } \mathbf{x}=0} \mathbf{x}^2, y \in \mathbb{R}^d$, and

$$\begin{aligned}\Theta n^{d/2} \leqslant \int_{\mathbb{R}^d} \exp(-\Theta n^{-1}y^2)dy \leqslant &\int \max_{\mathbf{x}\in\mathbb{Z}^d\cap B(y,2),\mathbf{x}\equiv i \text{ or } \mathbf{x}=0} \exp(-\Theta n^{-1}\mathbf{x}^2)dy\\ \leqslant &4^d \sum_{\mathbf{x}\in\mathbb{Z}^d,\mathbf{x}\equiv i \text{ or } \mathbf{x}=0} \exp(-\Theta n^{-1}\mathbf{x}^2)\\ \leqslant &4^d (\sum_{\mathbf{x}\in\mathbb{Z}^d,\mathbf{x}\equiv i} \exp(-\Theta n^{-1}\mathbf{x}^2)+1)\end{aligned}$$

because at most a mass $4^d$ of $y's$ are within distance 2 from some $\mathbf{x} \in \mathbb{Z}^d$. For the upper bound, for $\mathbf{x} \in \mathbb{Z}^d \setminus \{0\}$, there is at least one unit cube $C_{\mathbf{x}}$ with integer coordinates within the $2^d$ cubes that touch $\mathbf{x}$ such that for all $y \in C_{\mathbf{x}}$, $y^2 \leqslant \mathbf{x}^2$. Hence

$$\begin{aligned}\sum_{\mathbf{x}\equiv i} \exp(-\Theta n^{-1}\mathbf{x}^2) \leqslant \sum_{\mathbf{x}\equiv i,\mathbf{x}\neq 0} \exp(-\Theta n^{-1}\mathbf{x}^2)+1 \leqslant \sum_{\mathbf{x}\in\mathbb{Z}^d\setminus\{0\}} \int_{C_{\mathbf{x}}} \exp(-\Theta n^{-1}y^2)dy+1\\ \leqslant 2^d \int_{\mathbb{R}^d} \exp(-\Theta n^{-1}y^2)dy+1 \leqslant \Theta n^{d/2}.\end{aligned}$$

□

## 3.4 Binomial estimates

**Lemma 3.4.** *Let $\theta_0 < \frac{1}{2}$. There is a constant $c_{\text{Bin}} \in (0,1)$ depending on $\theta_0$ such that for $\theta \in (\theta_0, 1-\theta_0), B \sim \mathcal{B}(m,\theta)$, for $-c_{\text{Bin}} \leqslant \varepsilon_m = \varepsilon \leqslant c_{\text{Bin}}$*

$$\mathbb{P}(B = [m(\theta+\varepsilon)]) = \Theta m^{-1/2} \exp\left(-\Theta m\varepsilon^2\right)$$

*where the constants involved in $\Theta$ depend on $\theta_0$, and not on $\theta, m, \varepsilon$.*

*Proof.* Let $c_0 = \min(\frac{1}{2}, \frac{1-\theta_0}{2})$, and $\varepsilon \in (-c_0\theta, c_0\theta)$. Let then $k = [m(\theta+\varepsilon)]$. By Stirling's formula,

$$\begin{aligned}\mathbb{P}(B=k) =&\Theta \frac{\sqrt{m}}{\sqrt{k}\sqrt{m-k}} \theta^k (1-\theta)^{m-k} \frac{m^m}{k^k(m-k)^{m-k}}\\ =&\Theta m^{-1/2} \frac{\theta^k(1-\theta)^{m-k}}{\sqrt{(\theta+\varepsilon)(1-\theta-\varepsilon)}} \frac{m^m}{(\theta m)^k (m(1-\theta))^{m-k} \left(\frac{k}{\theta m}\right)^k \left(\frac{m-k}{m(1-\theta)}\right)^{m-k}}\\ =&\Theta m^{-1/2} \frac{1}{\sqrt{\theta}} \left(1+\frac{\varepsilon}{\theta}\right)^{-k} \left(1-\frac{\varepsilon}{1-\theta}\right)^{k-m}\\ =&\Theta m^{-1/2} \theta_0^{-1/2} \exp(\gamma_{\varepsilon,\theta})\end{aligned}$$

where

$$\begin{aligned}\gamma_{\varepsilon,\theta} =& - m(\theta+\varepsilon)(\frac{\varepsilon}{\theta} - \frac{\varepsilon^2}{2\theta^2} + O(\varepsilon^3)) - m((1-\theta)-\varepsilon)(-\frac{\varepsilon}{1-\theta} + \frac{\varepsilon^2}{2(1-\theta)^2} + O(\varepsilon^3))\\ =& m\frac{\varepsilon^2}{2\theta} - \frac{m\varepsilon^2}{\theta} + O(m\varepsilon^3 + m\varepsilon^4) - \frac{m\varepsilon^2}{2(1-\theta)} - \frac{m\varepsilon^2}{1-\theta} + O(m\varepsilon^3 + m\varepsilon^4)\\ =& \frac{-m\varepsilon^2}{2\theta} - \frac{3m\varepsilon^2}{2(1-\theta)} + O(m\varepsilon^3)\\ =& - \Theta m\varepsilon^2\end{aligned}$$

for $|\varepsilon|$ sufficiently small.

# 4 Variance asymptotics for diophantine Gaussian excursions

We consider symmetric spectral measures whose support contains incommensurate atoms. For $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$, denote by $X_{\boldsymbol{\omega}}$ a centered stationary Gaussian random field which spectral measure is $\boldsymbol{\mu}$, as defined in (1.4). The excursion volume variance is denoted by

$$V_{\boldsymbol{\omega}}(T) := \mathbf{Var}(\mathscr{L}^d(\{X_{\boldsymbol{\omega}} > 0\} \cap B_d(0,T))).$$

## 4.1 Regular asymptotics

We will use the assumptions that $\boldsymbol{\omega}$ is $\psi$-BA and / or $\psi$-WA (Section 1.1) with functions $\psi$ of the following form :

**Definition 4.1.** *Say that $\psi : \mathbb{N}^* \to (0,1]$ is regular (or $\tau$-regular) if it is of the form $\psi(q) = q^{-\tau}L(q), q \in \mathbb{N}^*$, where $\tau > 0$ and $L$ does not vanish and is slowly varying in the sense that $|L(q) - L(q+1)| = o(L(q)q^{-1})$ as $q \to \infty$.*

This property yields that $\ln(L(q))/\ln(q) \to 0$ as $q \to \infty$, hence $L$ is dominated by any power of $q$ and $\tau$ is uniquely defined. We introduce the pseudo-inverse $\psi^{-1} : (0,1] \mapsto \mathbb{N}$ by (3.1). We can show that for every finite $r > 0$ there are finite $c_i > 0$ such that $c_1\psi(q) \leqslant \psi(rq) \leqslant c_2\psi(q)$ and $c_3\psi^{-1}(\varepsilon) \leqslant \psi^{-1}(r\varepsilon) \leqslant c_4\psi^{-1}(\varepsilon)$ on their domains of definition. Remark that $q\psi(q)$ is strictly non-increasing for sufficiently large $q$ if $\tau > 1$.

Our most precise and general result concerns the case where the frequencies $\omega_{[k]}$ of $\boldsymbol{\mu}$ are the same in all $d$ directions, i.e. $\omega_{[k],i} = \omega_{[1],i}$, it yields stationary random sets in $\mathbb{R}^d$ with any reasonable asymptotic prescribed variance behaviour.

**Theorem 4.1.** *Let $\tau > 0, \psi$ $\tau$-regular and $\omega \in \mathbb{R}^m$ that is $\psi$-WA and $\psi$-BA, and $\boldsymbol{\omega} := (\omega, \omega, \dots, \omega) \in (\mathbb{R}^m)^d$. Then with $\tau^* = \frac{1+d(m+1)}{1+d}$, as $T \to \infty$,*

$$\begin{aligned}
\frac{c_- T^{2d}}{\psi^{-1}(T^{-1})^{1+d(m+1)}} &\overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\omega}}(T) \leqslant \frac{c_+ T^{2d}}{\psi^{-1}(T^{-1})^{1+d(m+1)}} = o(T^{2d}) \quad && \text{if } \tau > \tau^* \\
c_- T^{d-1} &\overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\omega}}(T) \leqslant c_+ T^{d-1} \ln(T) && \text{if } \psi(q) = q^{-\tau^*} \\
c_- \frac{T^{d-1}}{\ln(T)^\alpha} &\leqslant V_{\boldsymbol{\omega}}(T) \leqslant c_+ T^{d-1} && \text{if } \psi(q) \geqslant q^{-\tau^*} \ln(q)^{\frac{1}{d}}
\end{aligned}$$

*for some $0 < c_- \leqslant c_+ < \infty$ depending on $d, m, \psi$, and $\alpha = \frac{1+d(m+1)}{d\tau^*}$. If $m = 1$ and $\tau > 1$, there are uncountably many $\omega \in \mathbb{R}$ satisfying the assumption.*

To prove the upper bound we need the following computation.

**Proposition 4.1.** *Let $\psi(q) = q^{-\tau} L(q)$ be regular (Definition 4.1) and assume $\boldsymbol{\mu}$ is of the form (1.4) with $\boldsymbol{\omega}$ that is $\psi$-BA. Then as $T \to \infty$,*

$$\max\left\{ T^{2d} \mathbf{J}_3(T^{-1}), \quad T^{d-1} \int_0^{T^{d+1}} \mathbf{J}_3(y^{-\frac{1}{d+1}}) dy \right\} \leqslant \begin{cases} \Theta T^{d-1} \ \textit{if } \psi(q) \geqslant q^{-\tau^*} \ln(q)^{1/d} \\ \Theta T^{d-1} \ln(T) \ \textit{if } \psi(q) = q^{-\tau^*} \\ \Theta T^{2d} \psi^{-1}(T^{-1})^{-1-d(m+1)} \ \textit{if } \tau > \tau^* \end{cases}$$

*Proof.* According to (3.5) in Theorem 3.1-(i),

$$\mathbf{J}_3(\varepsilon) \leqslant \Theta \psi^{-1}(\varepsilon)^{-1-d(m+1)}, \varepsilon > 0,$$

which yields that $T^{2d} \mathbf{J}_3(T^{-1})$ admits an upper bound consistent with the claim.

To deal with the other term, assume without loss of generality that $\psi$ is extended to a smooth strictly non-increasing function $z^{-\tau} L(z) : [a, \infty) \to (0, 1]$ for some $a \geqslant 1$, such that $L'(z) = o(z^{-1} L(z))$ (the contribution of the integral on $(0, a)$ is uniformly bounded). Make the change of variables $z = \psi^{-1}(y^{-\frac{1}{d+1}})$, i.e. $\psi(z)^{-d-1} = y$, let $Z = \psi^{-1}(T^{-1})$.

$$\int_a^{T^{d+1}} \mathbf{J}_3(y^{-\frac{1}{d+1}}) dy \leqslant \Theta \int_a^{T^{d+1}} \psi^{-1}(y^{-\frac{1}{d+1}})^{-1-d(m+1)} dy = \Theta \int_\Theta^Z z^{-1-d(m+1)} (\psi(z)^{-d-1})' dz.$$

The hypotheses on $\psi$ yield

$$\begin{aligned}
(\psi(z)^{-d-1})' =& (d+1)(\tau z^{-\tau-1} L(z) - z^{-\tau} L'(z)) \psi(z)^{-d-2} = (d+1)(\tau z^{-1} \psi(z) - z^{-\tau} o(z^{-1} L(z))) \psi(z)^{-d-2} \\
&\sim_{z\to\infty} (d+1) \tau z^{-1} \psi(z)^{-d-1}. \qquad (4.1)
\end{aligned}$$

In the case $\tau \leqslant \tau^*$, the previous two displays yield

$$\int_a^{T^{d+1}} \mathbf{J}_3(y^{-\frac{1}{d+1}}) dy \leqslant \Theta \int_\Theta^Z z^{-2-d(m+1)} \psi(z)^{-(d+1)} dz$$

and the integral converges if $\psi(q) > q^{-\tau^*} \ln(q)^{1/d}$, and if $\psi(q) = q^{-\tau^*}$ it behaves in $\ln(Z) = \Theta \ln(T)$.

Let us turn to the case $\tau > \tau^*$. Let $\tau' \in (\tau^*, \tau)$, we have by (4.1) as $z \to \infty$

$$
\begin{aligned}
(z^{-1-d(m+1)}\psi(z)^{-d-1})' =& z^{-1-d(m+1)}(\psi(z)^{-d-1})' - (1+d(m+1))z^{-2-d(m+1)}\psi(z)^{-d-1} \\
\geqslant & z^{-1-d(m+1)}(\psi(z)^{-d-1})' - \frac{1+d(m+1)}{d+1} z^{-1-d(m+1)}(\psi(z)^{-d-1})' \\
\geqslant & z^{-1-d(m+1)}(\psi(z)^{-d-1})'(1-\frac{\tau*}{\tau'})
\end{aligned}
$$

which results in

$$
\int_a^{T^{d+1}} \mathbf{J}_3(y^{-\frac{1}{d+1}})dy \leqslant \frac{\Theta}{1-\tau^*/\tau'}[z^{-1-d(m+1)}\psi(z)^{-d-1}]_\Theta^Z = \Theta\psi^{-1}(T^{-1})^{-1-d(m+1)}T^{d+1}
$$

which allows to conclude. □

*Proof of Theorem 4.1.* Apply first Theorem 2.1 to the measure $\boldsymbol{\mu}$ to have bounds on the variance in terms of the function $\mathbf{J}_3$, recalling that $\mathbf{J}_3 = \Theta\mathbf{K}$ (with $\gamma$ as the unit ball indicator function, see Example 2.1). Proposition 4.1 yields the upper bound. Then lower bounds for $\mathbf{J}_3$ are derived in Theorem 3.1, noticing that $\boldsymbol{\omega}$ is $\psi$-BA and $\psi$-SWA thanks to Proposition 4.6. In the case $\psi(q) = q^{-\tau^*}\ln(q)^{\frac{1}{d}}$, $\psi^{-1}$ is not explicit, but we use the bound

$$
\psi^{-1}(\varepsilon) \geqslant c\varepsilon^{-\frac{1}{\tau^*}}|\ln(\varepsilon)|^{\frac{1}{d\tau^*}}, \varepsilon > 0
$$

for some $c > 0$.

Regarding the non-vacuity of Theorem 4.1, if $m = 1$ and $\tau > 1$, it is a standard fact in diophantine approximation that the set of $\omega$ that are $\psi$-BA and $\psi$-WA is uncountable when $q\psi(q)$ is non-increasing at infinity, see the seminal construction based on continued fractions by Jarnìk [18] and Proposition 4.5.

□

For $m \geqslant 2$, most studies concern power functions $\psi(q) = q^{-\tau}$ and are discussed in Corollary 4.1, but can likely be extended to more general functions $\psi$.

**Remark 4.1.** *The presence of the term $T^{d-1}$ on the right hand side, proportionnal to the surface measure of the observation window, is natural as random stationary measures applied to a large window are usually not expected to have a variance behaviour lower than the boundary measure. No rigourous general result in this direction is known by the author, Beck [3] gives a formal proof in the case of point processes. See also [36], which classifies hyperuniform behaviours in three types: type I have asymptotic variance in $T^{d-1}$, type II in $T^{d-1}\ln(T)$, and type III gathers all other sublinear behaviours, which actually correspond to the three cases above. Intermediate behaviours between $T^{d-1}$ and $T^{d-1}\ln(T)$ can likely be obtained by the same method.*

*It is likely that the upper bound in $T^{d-1}\ln(T)$ is sharp if $\psi(q) = q^{-\tau^*}$, proving it rigourously would require a lower bound for $v_T^{(2)}$ in the proof of Theorem 1.1, which raises some technical difficulties because of the cosine term.*

**Remark 4.2.** *This type of behaviour is really specific of nodal excursions. The volume variance for excursions $\{X > u\} \cap B_d(0,T)$ always behave in $T^{2d}$ if $u \neq 0$, see Theorem 2.1. The phenomenon of variance cancellation at $u = 0$ is heavily documented for Gaussian random waves (see Section 1.5).*

## 4.2 Power functions

Prominent examples are provided by power functions, for which we introduce a special notation: say that $\omega \in \mathbb{R}^m$ is $(\tau)$-BA (resp. $(\tau)$-WA) if it is $cq^{-\tau}$-BA (resp. $cq^{-\tau}$-WA) for some finite $c > 0$. These considerations are further developped and commented in the Appendix 3.4, let us simply mention that for $\eta > 0$, $\mathscr{L}^m$-a.a. $\omega \in \mathbb{R}^m$ is $(m+\eta)$-BA and $(m)$-WA. For any $\eta \geqslant 0$, there are uncountably many $\omega \in \mathbb{R}^m$ that are $(m+\eta)$-BA and $(m+\eta)-$WA. There are also uncountably many *Liouville numbers*, i.e. $\omega \in \mathbb{R}$ that are $(\tau)-$WA for any $\tau > 0$.

The following corollary examines different regimes, depending on the relation between $\tau, d$ and $m$, it is a consequence of Theorem 4.1 for $\psi(q) = q^{-\tau}$.

**Corollary 4.1.** *For $\omega \in \mathbb{R}^m, \tau > 0$, let $\boldsymbol{\omega} = (\omega, \dots, \omega) \in (\mathbb{R}^m)^d, \tau^* = \frac{1+d(m+1)}{1+d}$.*

*(i) If $d \geqslant m$, for $\tau \in (m, \tau^*)$, $\mathscr{L}^m$-a.a. $\omega \in \mathbb{R}^m$ is $(\tau)$-BA, and for some $c_+ < \infty$*

$$V_{\boldsymbol{\omega}}(T) \leqslant c_+ T^{d-1}, T > 0.$$

*(ii) If $d < m$, since for $\mathscr{L}^m$-a.a. $\omega \in \mathbb{R}^m$, $\omega$ is $(m)$-WA and $(m+\eta)-$BA for $\eta > 0$, we have*

$$c_- T^{d - \frac{1+d}{m}} \overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\omega}}(T) \leqslant c_+ T^{d - \frac{1+d}{m+\eta}}$$

*for some $0 < c_-, c_+ < \infty$.*

*(iii) Let $m = 1$. For $d - 1 \leqslant \beta < 2d$, the set of $\omega \in \mathbb{R}$ such that for some $0 < c_- \leqslant c_+ < \infty$*

$$c_- T^{\beta} \overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\omega}}(T) \leqslant c_+ T^{\beta}, T \geqslant 1$$

*is uncountable (there is $\tau \geqslant 1$ such that $\beta = 2d - \frac{1+2d}{\tau}$, and uncountably many $\omega$ are $(\tau)$-WA and $(\tau)$-BA ).*

*(iv) Let $m = 1$. For $\omega$ a* Liouville number, *for every $\varepsilon > 0$, for some $c_- > 0$,*

$$c_- T^{2d-\varepsilon} \overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\omega}}(T).$$

*(v) In all cases, $V_{\boldsymbol{\omega}}(T) = o(T^{2d})$.*

*All constants $c_-, c_+$ involved only depend on $d, m, \tau$.*

**Remark 4.3.** *An interesting observation in dimension $d = 1$ is that in (i), the variance of the excursion indicator is bounded, while its derivative in the distributional sense, i.e. the number of zeros, has maximal quadratic variance, in $T^2$ (see [22, Theorem 2-(iii)]).*

If the $\omega_{[k]}$ differ along the directions $1 \leqslant k \leqslant d$, an application of Theorem 3.1, Proposition 4.1, and Theorem 2.1 similar to the proof of Theorem 4.1 yields that the upper bound corresponds to the worst upper bound among the $\omega_{[k]}$:

**Corollary 4.2.** *Let $\psi : \mathbb{N}_* \to (0, 1]$ regular. Assume $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$ is such that each $\omega_{[k]}$ is $\psi$-BA, $1 \leqslant k \leqslant d$. Then the same upper bounds as in Theorem 4.1 hold. In particular*

*(i) if $d \geqslant m$, for $\mathscr{L}^{md}$-a.a. $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$, $V_{\boldsymbol{\omega}}(T) \leqslant c_+ T^{d-1}$ for some $c_+ < \infty$.*

*(ii) For every $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$, $V_{\boldsymbol{\omega}}(T) = o(T^{2d})$*

Theorem 3.1-(iii) and Theorem 2.1 yield:

**Corollary 4.3.** *Assume that for some function $\psi : \mathbb{N}* \to (0, 1]$ converging to 0, $\boldsymbol{\omega}$ is $\psi$-SWA. Then for some $c_- > 0$*

$$c_- T^{2d} \psi^{-1}(T^{-1})^{-1-d(m+1)} \overset{\textbf{i.o.}}{\leqslant} V_{\boldsymbol{\omega}}(T)$$

*where $\psi^{-1}$ denotes the pseudo-inverse of $\psi$ (see (3.1)).*

Thanks to Groshev's theorem (see the Appendix 3.4), for $\eta > 0$, $\mathscr{L}^{md}$-a.a. $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$ is $|q|^{-m/d}$-SWA but not $|q|^{-m/d-\eta}$-SWA.

## 4.3 A randomised model

It is easy to build randomised models that exploit the metric results of diophantine approximation to yield hyperuniform models that are more stable, i.e. not subject to subtle diophantine properties of the parameters.

**Proposition 4.2.** *Let $\Omega$ be a real random variable which law is continuous with respect to Lebesgue measure, and let $a_k^i, i \geqslant 0, k \geqslant 1$ be independent and identically distributed standard Gaussian variables. Define*

$$X(\mathbf{t}) = \frac{1}{2d} \sum_{k=1}^{d} (a_k^0 \cos(t_{[k]}) + a_k^1 \sin(t_{[k]}) + a_k^2 \cos(\Omega t_{[k]}) + a_k^3 \sin(\Omega t_k)), \mathbf{t} = (t_{[k]}) \in \mathbb{R}^d,$$

*$M_T = \mathscr{L}^d(\{X > 0\} \cap B_d(0, T))$ and $V(T) = \mathbf{Var}(M_T)$. Then for some $c_+ < \infty$,*

$$V(T) \leqslant c_+ T^{d-1}.$$

*Proof.* Since the Gaussian field is centered, for any fixed $\omega \in \mathbb{R}$,

$$\mathbb{E}(M_T \mid \Omega = \omega) = \mathscr{L}^d(B_d(0, T))/2$$

is deterministic. We also know that a.a. $\omega \in \mathbb{R}$ is (2)-BA, and if we condition by $\Omega = \omega$, $X$ is the Gausian field with reduced covariance $\frac{1}{2d}\sum_{k=1}^{d}(\cos(t_{[k]}) + \cos(\omega t_{[k]}))$. Hence the conditional variance formula and Corollary 4.1 yield

$$\begin{aligned} V(T) =& \mathbb{E}(\mathbf{Var}(M_T \mid \Omega)) + \mathbf{Var}(\mathbb{E}(M_T \mid \Omega)) \\ & \leqslant c_+ T^{d-1}, \end{aligned}$$

we emphasize that $c_+$ depends only on $d, m, \tau = 2$, and not further on $\omega$. □

The same arguments with $1 \leqslant m < d$ yield the following:

**Proposition 4.3.** *Let $(\Omega_0, \dots, \Omega_m)$ a random $(m+1)-$tuple of vectors with continuous law with respect to $\mathscr{L}^{(m+1)d}$, and*

$$X(\mathbf{t}) = \frac{1}{d(m+1)} \sum_{k=1}^{d} \sum_{i=0}^{m} (a_k^{2i} \cos(\Omega_i t_k) + a_k^{2i+1} \sin(\Omega_i t_k)).$$

*Then the variance is bounded by $c_+ T^{d-1}$ if $d > m$.*

Along similar lines, exploiting Corollary 4.1-(iii) with $m > d$ yields randomised models which variance is in $T^{\beta}$ for some $d - 1 < \beta < 2d$.

**Remark 4.4.** *Similar models in the context of random walks (Section 3) yield interesting examples of random walks in a random environment.*

□

# Appendix: Diophantine approximation

The core of the paper is provided by results from diophantine approximation, we explain here basic principles and results, as well as the more advanced ones we will need. The quality of the approximation of a tuple $\omega \in \mathbb{R}^m$ is measured by the numbers

$$\mathrm{d}_q(\omega) = \inf_{p \in \mathbb{Z}} |p - q \cdot \omega|, q \in \mathbb{Z}^m.$$

Given $\psi : \mathbb{N}^* \to [0, 1]$, the definitions of $\psi-$BA, $\psi-$WA , $\psi$-SWA', $\psi-$SWA based on this distance are given in the introduction and we complete this picture with the following definition: $\omega \in \mathbb{R}^m$ is $\psi$-WA' if for some $c_\omega < \infty$, for infinitely many $p \in \mathbb{Z}, q \in \mathbb{Z}^m$,

$$d_q(\omega) \leqslant c_\omega \psi(q).$$

Proposition 4.5, at the end of this section, yields that most quantitative statements available in the literature about $\psi$-SWA' tuples also hold for $\psi$-SWA tuples. The most basic, yet useful result is the Dirichlet principle:

**Proposition 4.4.** *Let $m \geqslant 1$. There is $c_m < \infty$ such that for $N \in \mathbb{N}^*$, one can find $q, q' \in B_N := (\mathbb{Z} \cap [-N, N])^m$ distinct such that for $\omega \in \mathbb{R}^m$,*

$$\mathrm{d}_{q-q'}(\omega) \leqslant N^{-m} \leqslant c_m \|q - q'\|^{-m},$$

*which yields that $\omega$ is $(m)$-WA' and if $\omega$ is $(m+\eta)$-BA, then necessarily $\eta \geqslant 0$.*

*Proof.* Simply remark that if one divides $[0,1]$ in $M := |B_N| - 1$ bins of size $M^{-1}$, out of the $|B_N|$ values $\mathrm{d}_q(\omega), q \in B_N$, at least two of them will end up in the same bin, yielding for some $q, q' \in B_N$ distinct

$$\mathrm{d}_{q-q'}(\omega) \leqslant |\mathrm{d}_q(\omega) - \mathrm{d}_{q'}(\omega)| \leqslant M^{-1} \leqslant N^{-m}.$$

The second inequality comes from $|q - q'| \leqslant 2\sqrt{m}N \leqslant \sqrt{m}2^{1-m}M^{1/m}$. □

Another fundamental but more technical result is the Khintchine-Groshev theorem, we do not include the proof here, see the latest improvement by Hussain and Yusupova [17].

**Theorem 4.2** (Khintchine-Groshev)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}_+$ tending to 0 such that*

$$\sum_{q \in \mathbb{Z}^m} \psi(|q|)^d < \infty.$$

*Then the set of $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$ that are $\psi$-SWA' is $(\mathscr{L}^m)^d-$negligible. If on the other hand the sum diverges then $(\mathscr{L}^m)^d-$a.a. $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$ is $\psi$-SWA', in the case $m = d = 1$ $\psi$ needs furthermore to be monotonic.*

The theorem yields that $(\mathscr{L}^m)^d-$a.a. $\boldsymbol{\omega}$ has irrationality index $\tau(\boldsymbol{\omega}) = m/d$, where the irrationality index of some $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$ is defined by

$$\tau(\boldsymbol{\omega}) := \inf\{\tau : \boldsymbol{\omega} \text{ is not } \tau\text{-SWA'}\} = \sup\{\tau : \boldsymbol{\omega} \text{ is } \tau\text{-SWA'}\}.$$

In particular, for $\mathscr{L}^m$-a.a. $\omega \in \mathbb{R}^m$, $\omega$ is $(m+\eta)-$BA for each $\eta > 0$. The following result yields that the situation is the same if SWA' is replaced by SWA. Actually, for most statements about the quantity of existing $\psi$-SWA' arrays, there are about as many $\psi$-SWA arrays. More precisely, we show that for every $\boldsymbol{\omega}$ that is $\psi$-SWA', there is a $\psi$-SWA array $\boldsymbol{\omega}'$ in the finite neighbourhood

$$\mathcal{N}(\boldsymbol{\omega}) = \{\boldsymbol{\omega}' = (\omega'_{[k]})_{k=1}^d : \ (\exists k : \omega'_{[k]} = 2\omega_{[k]} \text{ or } \exists i : \omega'_{[1],i} = \omega_{[1],i} + 1 \text{ or } \boldsymbol{\omega}' = \boldsymbol{\omega})\}.$$

**Proposition 4.5.** *Let $\boldsymbol{\omega} \in (\mathbb{R}^m)^d$ that is $\psi$-SWA' for $\psi : \mathbb{N} \to \mathbb{R}_+$ non-increasing, then there is $\boldsymbol{\omega}'$ in $\mathcal{N}(\boldsymbol{\omega})$ that is $\psi$-SWA.*

*Proof.* Either $\boldsymbol{\omega}$ is $\psi$-SWA or there are by definition $c > 0$ and infinitely many distinct $p_{[k]} = p^j_{[k]} \in \mathbb{Z}, q = q^j \in \mathbb{Z}^m, j \geqslant 1, 1 \leqslant k \leqslant d$ such that

$$\sum_k (p_{[k]} + \sum_i q_i) = \sum_k p_{[k]} + d \sum_i q_i \equiv 0$$

and

$$|p_{[k]} - \omega_{[k]} \cdot q| < c\psi(|q|), 1 \leqslant k \leqslant d.$$

Let $m_j \in \mathbb{N}$ maximal such that $2^{m_j}$ divides each $p_{[k]}, 1 \leqslant k \leqslant d$ and each $q_i, 1 \leqslant i \leqslant m$, and let $\tilde{p}_{[k]} = 2^{-m_j} p_{[k]}, \tilde{q}_i = 2^{-m_j} q_i$. Since $\psi$ is non-increasing and $|\tilde{q}| \leqslant |q|$,

$$|\tilde{p}_k - \omega_{[k]} \cdot \tilde{q}| = 2^{-m_j} |p_{[k]} - \omega_{[k]} \cdot q| < 2^{-m_j} c\psi(|q|) \leqslant c\psi(|\tilde{q}|), 1 \leqslant k \leqslant d.$$

It is important to precise that there are infinitely many pairwise distinct couples $(\tilde{p}^j, \tilde{q}^j)$ with $\tilde{p}^j = (\tilde{p}^j_{[k]})_k$, otherwise there is $j_0$ and $m'_j \to \infty$ such that for infinitely many $j$, $p^j = 2^{m'_j} \tilde{p}^{j_0}, q^j = 2^{m'_j} \tilde{q}_{j_0}$, which contradicts $|p^j_{[k]} - \omega_{[k]} \cdot q^j| \to 0$.

If there are infinitely many couples $(\tilde{p}, \tilde{q}) \equiv 1$, then $\boldsymbol{\omega}$ is $\psi$-SWA and the proof is complete. Hence let us suppose in the following that there are infinitely many couples $(\tilde{p}, \tilde{q}) \equiv 0$. The maximality of $m_j$ and the drawer principle then yield that there is either $k_0$ such that for infinitely many couples $(\tilde{p}, \tilde{q})$, $\tilde{p}_{[k_0]} \equiv 1$, or $i_0$ such that for infinitely many couples, $\tilde{q}_{i_0} \equiv 1$.

In the case where $\tilde{p}_{[k_0]} \equiv 1$, let

$$\omega'_{[k],i} = \begin{cases} 2\omega_{[k_0],i} \text{ if } k = k_0 \\ \omega_{[k],i} \text{ otherwise,} \end{cases} \quad p'_{[k_0],i} = \begin{cases} 2\tilde{p}_{[k_0],i} \text{ if } k = k_0 \\ \tilde{p}_{[k],i} \text{ otherwise,} \end{cases} \quad 1 \leqslant i \leqslant m.$$

We have $|p'_{[k]} - \omega'_{[k]} \cdot \tilde{q}| \leqslant 2\psi(|\tilde{q}|), 1 \leqslant k \leqslant d$ for infinitely many couples $(p', \tilde{q})$, and $(p', \tilde{q}) = (\tilde{p}, \tilde{q}) + (p_{[k_0]i}, 0) \equiv 1$, hence $\boldsymbol{\omega'} := (\omega'_{[k]})_{1 \leqslant k \leqslant d}$ is $\psi$-SWA.

In the case where $\tilde{q}_{i_0} \equiv 1$, let

$$\omega'_{[k],i} = \begin{cases} \omega_{[1],i} + 1 \text{ if } k = 1, i = i_0 \\ \omega_{[k],i} \text{ otherwise} \end{cases} , p'_{[k]} = \begin{cases} p_{[1]} + q_{i_0} \text{ if } k = 1 \\ p_{[k]} \text{ otherwise} \end{cases} .$$

Then

$$p'_{[k]} - \omega'_{[k]} \cdot \tilde{q} = \begin{cases} \tilde{p}_{[1]} + \tilde{q}_{i_0} - \omega_{[1]} \cdot \tilde{q} - \tilde{q}_{i_0} = \tilde{p}_1 - \omega_1 \cdot \tilde{q} \text{ if } k = 1 \\ \tilde{p}_{[k]} - \omega_{[k]} \cdot \tilde{q} \text{ otherwise} \end{cases}$$

and

$$(p', \tilde{q}) \equiv (0, \tilde{q}_{i_0}) \equiv 1$$

for infinitely many couples $(p', \tilde{q})$, hence $\boldsymbol{\omega'}$ is $\psi$-SWA. □

The next result is useful for tensorizing variance estimates.

**Proposition 4.6.** *If $\omega \in \mathbb{R}^m$ is $\psi$-WA, $\boldsymbol{\omega} = (\omega, \dots, \omega) \in (\mathbb{R}^m)^d$ is $\psi$-SWA.*

*Proof.* Since $\omega$ is assumed to be $\psi$-WA, there is a sequence $(p^j, q^j)_j$ such that $|p^j - \omega \cdot q^j| < c_W \psi(|q^j|)$ and $(p^j, q^j) \equiv 1$. Hence with $q^j_{[k]} = q^j, p^j_{[k]} = p^j$,

$$|p^j_{[k]} - q^j_{[k]} \cdot \omega| = |p^j - q^j \cdot \omega| < c_W \psi(q^j)$$

but $\sum_{k=1}^{d}(p^j_{[k]} + \sum_{i=1}^{m} q^j_{[k],i}) = d(p^j + \sum_i q^j_i)$ is odd only if $d$ is odd. If $d$ is even, choose instead $p^j_{[1]} = 2p^j, q^j_{[1]} = 2q^j$, so that $|p^j_{[1]} - q^j_{[1]} \cdot \omega| < 2c_W\psi(|q^j|)$, and $\sum_{k=1}^{d}(p^j_{[k]} + \sum q^j_{[k],i}) = (2d+1)(p^j + q^j)$ is indeed odd. This sequence demonstrates taht $\boldsymbol{\omega}$ is $\psi$-SWA.

□

# Acknowledgements

I am indebted to M. A. Klatt, with whom I had many discussions about hyperuniformity and Gaussian excursions. I also wish to thank S. Torquato for comments on the final manuscript. I am particularly grateful to Yann Bugeaud for insights about diophantine approximation.

This study was supported by the IdEx Université de Paris, ANR-18-IDEX-0001.

# References

[1] R. J. Adler and J. E. Taylor. *Random Fields and Geometry.* Springer, 2007.

[2] E. Assaf, J. Buckley, and N. Feldheim. An asymptotic formula for the variance of the number of zeroes of a stationary gaussian process. arXiv:2101.04052.

[3] J. Beck. Irregularities of distribution. i. *Acta Math.*, 159:1–49, 1987.

[4] Michael V Berry. Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. *Journal of Physics A: Mathematical and General*, 35(13):3025, 2002.

[5] G. Blum, S. Gnutzmann, and U. Smilansky. Nodal domains statistics: A criterion for quantum chaos. *Phys. Rev. Lett.*, 88(114101), 2002.

[6] A. Bulinski, E. Spodarev, and F. Timmermann. Central limit theorems for the excursion set volumes of weakly dependent random fields. *Bernoulli*, 18(1):100–118, 2012.

[7] D. Chen and S. Torquato. Designing disordered hyperuniform two-phase materials with novel physical properties. *Acta Materialia*, 142:152–161, 2018.

[8] C. Corduneanu, N. Gheorghiu, and V. Barbu. *Almost periodic functions.* Chelsea Publishing Company, 1989.

[9] S. Coste. Order, fluctuations, rigidities. https://scoste.fr/assets/survey_hyperuniformity.pdf.

[10] H. Cramér and M. R. Leadbetter. *Stationary and Related Stochastic Processes*. Wiley, 1967.

[11] J. Cuzick. A central limit theorem for the number of zeros of a stationary Gaussian process. *Ann. Prob.*, 4(4):547–556, 1976.

[12] P. Diaconis. Group representations in probability and statistics. *Inst. Math. Stat. Lect. Notes*, 11, 1988.

[13] A. Elbert. Some recent results on the zeros of bessel functions and orthogonal polynomials. *J. Comp. Appl. Math.*, 133:65–83, 2001.

[14] A. Estrade and J. R. Leon. A central limit theorem for the Euler characteristic of a Gaussian excursion set. *Ann. Prob.*, 44(6):3849–3878, 2016.

[15] I. I. Gikhman and A. V. Skorokhod. *Introduction to the theory of random processes, Vol. I.* W. B. Saunders Company, 1965.

[16] Ben Hough, Khrishnapur, Peres, and Viràg. *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. University Lecture Series. Institute of Mathematical Statistics, 2009.

[17] M. Hussain and T. Yusupova. A note on the weighted khintchine-groshev theorem. *J. de théorie des nombres de Bordeaux*, 26(2):385–397, 2014.

[18] V. Jarník. über die konvexe hülle von $n$ zufällig gewählten punkten iiber die simultanen diophantische approximationen. *Math. Z*, 33:505–543, 1931.

[19] M. Kac. On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.*, 49:314–320, 1943.

[20] M. Kratz and J. R. Leon. Hermite polynomial expansion for non-smooth functionals of stationary gaussian processes: Crossings and extremes. *Stoc. Proc. Appl.*, 66:237–252, 1997.

[21] M. Krishnapur, P I. P. Kurlberg, and Wigman. Nodal length fluctuations for arithmetic random waves. *Annals of Mathematics*, 177(2):699–737, 2013.

[22] R. Lachièze-Rey. Variance linearity for real gaussian zeros. arXiv:2006.10341.

[23] D. Marinucci, G. Peccati, M. Rossi, and I. Wigman. Non-universality of nodal length distribution for arithmetic random waves. *Geom. Funct. Anal*, 26:926–960, 2016.

[24] D. Marinucci and I. Wigman. The defect variance of random spherical harmonics. *J. Phys. A: Math. and Theor.*, 44(35):355206, 2011.

[25] I. Molchanov. *Theory of random sets*. Springer-Verlag, London, 2005.

[26] S. Muirhead, A. Rivera, and H. Vanneuville. The phase transition for planar gaussian percolation models without FKG. https://arxiv.org/abs/2010.11770, 2020.

[27] F. Nazarov and M. Sodin. Fluctuations in random complex zeroes: Asymptotic normality revisited. *Int. Math. Res. Notic.*, 24:5720–5759, 2011.

[28] I. Nourdin, G. Peccati, and M. Rossi. Nodal statistics of planar random waves. *Comm. Math. Phys.*, 369:99–151, 2019.

[29] E. C. Oğuz, J. E. S. Socolar, P. J. Steinhardt, and S. Torquato. Hyperuniformity of quasicrystals. *Physical Review B*, 95(054119), 2017.

[30] T. Prescott and F. E. Su. Random walks on the torus with several generators. *Rand. Struct. Alg.*, 25(3):336–345, 2004.

[31] M. Rossi. The defect of random hyperspherical harmonics. *J. Theor. Prob.*, 32:2135–2165, 2019.

[32] E. Slud. Multiple Wiener-Ito integral expansions for level-crossing-count functionals. *Prob. Th. Rel. Fields*, 87:349–364, 1991.

[33] F. E. Su. Convergence of random walks on the circle generated by an irrational rotation. *Trans. AMS*, 350(9):3717–3741, 1998.

[34] S. Torquato. Disordered hyperuniform heterogeneous materials. *J. Phys.: Condens. Matter*, 28(414012), 2016.

[35] S. Torquato. Hyperuniformity and its generalizations. *Phys. Rev. E*, 94(022122), 2016.

[36] S. Torquato. Hyperuniform states of matter. *Physics Reports*, 745:1–95, 2018.

[37] S. Torquato and F. H. Stillinger. Local density fluctuations, hyperuniform systems, and order metrics. *Phys. Rev. E*, 68(041113):1–25, 2003.

[38] I. Wigman. Fluctuations of the nodal length of random spherical harmonics. *Comm. Maths Phys.*, 298:787–831, 2010.

[39] I. Wigman and A. Granville. The distribution of the zeros of random trigonometric polynomials. *Amer. J. Math.*, 133(2):295–357, 2011.