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Elliptic Functions according to Eisenstein and Kronecker : An Update.

Newly Found Notes of Lectures by Kronecker on the work of Eisenstein

Pierre Charollois
Robert Sczech

July 18, 2016

Abstract

This article introduces a set of recently discovered lecture notes from the last course of Leopold Kronecker, delivered a few weeks before his death in December 1891. The notes, written by F. von Dalwigk, elaborate on the late recognition by Kronecker of the importance of the "Eisenstein summation process", invented by the "companion of his youth", in order to deal with conditionally convergent series known today as Eisenstein series. We take this opportunity to give a brief update of the well known book by André Weil (1976) that brought these results of Eisenstein and Kronecker back to light. We believe that Eisenstein's approach to the theory of elliptic functions was in fact a very important part of Kronecker's planned proof of his visionary "Jugendtraum".

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1 Introduction

Born in 1823, Leopold Kronecker died in Berlin on December 29, 1891[1], at the age of 68, precisely 15 days after delivering his last lecture of his university course entitled "On elliptic

[1] According to the tombstone on his grave, Kronecker died on December 23rd, 1891.

functions depending on two pairs of real variables". We gathered that historical information from recently discovered lecture notes at the library of the University of Saarbrücken. Before discussing the content and the author of these handwritten notes, we wish to recall the circumstances of their discovery. The original discovery is due to Professor Franz Lemmermeyer, who started the task of retyping the text written in old style German cursive handwriting, but did not finish the job. The manuscript then got lost during a library move. Being interested to learn more about Kronecker's work and insight into his so called "liebster Jugendtraum", we decided to enlist the help of Simone Schulze at the library in finding it. After a search effort which lasted a few weeks, the complete set of notes was finally found and the library even produced a high quality digital copy which is now available to the public [Cw]. We thank Ms. Schulze for her help and we also thank Franz-Josef Rosselli who undertook the effort to translate the cursive German handwriting (prevailing in the 19th century) into modern German typeface [Cw].

The manuscript was written by Friedrich von Dalwigk, a graduate student at that time who was just finishing his dissertation on theta functions of many variables [vD]. Later, von Dalwigk became Professor for applied mathematics at the University of Marburg (1897-1923). Although the course consisted only of six lectures (due to the premature death of Kronecker), the whole manuscript is more than 120 pages, with many appendices, partly written by von Dalwigk himself relying on published papers of Kronecker as well as on unpublished papers from Kronecker's "Nachlass". Dalwigk explicitly mentions the word "Nachlass" in the manuscript. He is likely to have had access to that specific document from his colleague Hensel in Marburg, who was in possession of all scientific papers of Kronecker at that time. As we learned from Hasse and Edwards [Ed], the personal papers of Kronecker were lost in the chaotic events surrounding WWII. Most probably, they were destroyed by a fire caused by exploding munitions in an old mine near Göttingen. That mine was used to store some of the collections of the Göttingen library in 1945. This dramatic event is only part of the long lasting spell put on the posterity of Eisenstein's ideas, as predicted by André Weil in his essay [We1].

At any rate, the care and the amount of detail included in the manuscript is extraordinary and leads us to think that it was written with the ultimate intention to publish it as a book.

We know from a letter of Kronecker to his friend Georg Cantor who at that time served as the first president of the newly founded German Mathematical Association (DMV) and who invited Kronecker to deliver the opening address at the first annual meeting of the DMV in 1891, that Kronecker intended to talk at that meeting about the "forgotten" work of Eisenstein. In that letter, Kronecker apologizes for not being able to attend the meeting due to the death of his wife Fanni. It is very likely that the lecture notes in question are an expanded version of his intended talk.

Kronecker was indeed a great mathematician with fundamental contributions to algebra and number theory. We mention here only his "Jugendtraum" which, historically, gave rise to class field theory and to Hilbert's 12th problem (the analytic generation of all abelian extensions of a given number field), one of the great outstanding problems in classical algebraic number theory. The celebrated conjectures of Stark (published in a sequence of 4 papers

during the 1970's) offer a partial solution to a problem with ultimate roots in the work of Kronecker and Eisenstein.

In passing, we wish to mention two standard references about the work of Kronecker in number theory. The first chapter of Siegel's Lectures on Advanced Number Theory [Si] is devoted to the so called Kronecker limit formulas with applications (Kronecker's solution of Pell's equation). The second reference is the book by André Weil [We2] on elliptic functions according to Eisenstein and Kronecker. The lecture notes under review can be classified roughly as an extension and elaboration of the material discussed by Weil. Besides resurrecting the ideas of Eisenstein from final oblivion, the book of Weil is also a valuable source for many anecdotes about Eisenstein and Kronecker. Our favorite one is the story that Kronecker, in his public PhD defense talk, claimed that mathematics is both science and art. His friend Eisenstein challenged Kronecker publicly by claiming mathematics is art only.

2 Kronecker and the work of Eisenstein

Except for a letter to Dedekind dated March 15, 1880, there is no comprehensive statement of the Jugendtraum in the papers of Kronecker. In that letter, Kronecker reports on his recent progress towards a proof of his conjecture (the Jugendtraum) that all abelian extensions of an imaginary quadratic field F are generated by division values of suitable elliptic functions admitting complex multiplication by elements of F together with the corresponding singular moduli, that is, the values of the j-invariant of the corresponding elliptic curves. He expresses the hope to complete the proof soon. In closing, he regrets having to postpone the problem of finding the analogue of singular moduli for arbitrary complex number fields (Hilbert's 12th problem) until the case of imaginary quadratic fields is completely resolved. 10 years later, in his lectures on elliptic functions, he does not mention his work on the Jugendtraum at all. Instead, he concentrates on reviewing and generalizing the work of Eisenstein. In the following, we wish to give a hypothetical explanation of why the approach of Eisenstein may have been an important step in Kronecker's envisioned proof of his Jugendtraum. Namely, we are going to carry out Kronecker's program in the simpler setting of abelian extensions of the field of rational numbers by modifying a basic example given by Eisenstein in his great paper [Eis2].

Let u be a complex number which is not an integer. Then the coset $\mathbb{Z}+u \subset \mathbb{C} \setminus \mathbb{Z}$ does not contain the zero element so all the terms of the series

$$\phi(u) = \sum_{m \in \mathbb{Z}+u} \frac{1}{m}$$

are well defined, but the series does not converge absolutely. It is therefore necessary to specify an order of summation. Following Eisenstein, we define

$$\phi(u) = \lim_{t \to +\infty} \sum_{\substack{m \in \mathbb{Z}+u, \\ |m|<t}} \frac{1}{m} = \frac{1}{u} + \sum_{n=1}^{\infty} \left(\frac{1}{u+n} + \frac{1}{u-n} \right) = \frac{1}{u} + \sum_{n=1}^{\infty} \frac{2u}{u^2-n^2}.$$

The last series on the right converges absolutely. The function ϕ is odd and 1-periodic, hence $\phi(\frac{1}{2}) = 0$. Next, we consider the special value $\phi(\frac{1}{4})$ and we obtain

$$\begin{aligned}\phi\left(\frac{1}{4}\right) &= \sum_{m\in\mathbb{Z}+\frac{1}{4}} \frac{1}{m} = 4\sum_{\ell\in 4\mathbb{Z}+1} \frac{1}{\ell} \\ &= 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\dots\right) = 4\int_0^1 \frac{dx}{1+x^2} = \pi.\end{aligned}$$

We are now ready to derive the fundamental property of the ϕ function, its addition formula. To this end, we let u, v, w be three complex numbers, none of which is a rational integer, such that $u + v + w = 0$. Then the equation

$$p + q + r = 0 \tag{1}$$

with $p \in \mathbb{Z} + u, q \in \mathbb{Z} + v, r \in \mathbb{Z} + w$ has infinitely many solutions which can be obtained by letting p, q (or q, r or r, p) run independently. Since $pqr \neq 0$, Equation (1) becomes

$$\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp} = 0.$$

It is the starting point of Eisenstein's method, which he learnt from high school teacher Schellbach, to average this rational identity over all solutions of Equation (1). Since the resulting double series are conditionally convergent, we need again to pay attention to the ordering of the series. To this end, we choose three non-zero fixed real parameters α, β, γ such that $\alpha + \beta + \gamma = 0$. Then

$$\alpha p - \beta q = \gamma q - \alpha r = \beta r - \gamma p,$$

which allows us to write

$$\sum_{|\alpha p-\beta q|<t} \frac{1}{pq} + \sum_{|\gamma q-\alpha r|<t} \frac{1}{qr} + \sum_{|\beta r-\gamma p|<t} \frac{1}{rp} = 0.$$

It is easy to see that each of these series converges absolutely for every $t > 0$. In order to pass to the limit $t \to \infty$, we need the following lemma :

Lemma 2.1.

$$\lim_{t\to\infty}\left(\sum_{|\alpha p-\beta q|<t} \frac{1}{pq}\right) = \phi(u)\phi(v) + \pi^2 \operatorname{sign}\alpha \operatorname{sign}\beta.$$

For a generalization and a proof of this lemma, we refer to [Scz3, Th.2]. As a corollary, we obtain

$$\phi(u)\phi(v) + \phi(v)\phi(w) + \phi(w)\phi(u) = \pi^2, \tag{2}$$

which is the addition formula for the function ϕ. To prove (2) without using Lemma 2.1, it is enough to show that the left side is independent of u, v because the specialization

$u = v = \frac{1}{4}, w = -\frac{1}{2}$ provides the value π^2. That was the approach taken by Eisenstein in his original proof of (2).

To study arithmetic applications, we eliminate the period π by introducing the function

$$c(u) = \frac{\phi(u)}{\phi(1/4)} = \phi(u)/\pi,$$

and its Cayley transform

$$\mathbf{e}(u) = \frac{c(u) + i}{c(u) - i}.$$

The addition formula (2) implies immediately

$$\mathbf{e}(u)\mathbf{e}(v) = \mathbf{e}(u + v).$$

From here it follows easily that $\mathbf{e}(u) = e^{2\pi i u}$ and $c(u) = \cot(\pi u)$. Iterating the addition formula for the cotangent function,

$$c(u + v) = \frac{c(u)c(v) - 1}{c(u) + c(v)},$$

we obtain the formula

$$c(nu) = \frac{U_n(c(u))}{V_n(c(u))}, \ n = 1, 2, 3, 4, \ldots \tag{3}$$

with polynomials $U_n, V_n \in \mathbb{Z}[t]$ given explicitly by

$$U_n(t) = \mathrm{Re}\,(t + i)^n, \ V_n(t) = \mathrm{Im}\,(t + i)^n.$$

A formula of type (3) was called a "transformation formula" in the 19th century. The elliptic analogs of (3) played a prominent role in the work of Kronecker. Taking $u \in \mathbb{Q}$ with denominator $n > 1$, we conclude that the numbers $c(u)$ are the roots of $V_n(t) = 0$, that is, $c(u)$ is a real algebraic number whenever u is a non-integral rational number. One can refine this result and prove more precisely :

Proposition 2.2. *The number $c(u)$ is an algebraic integer iff n is not the power of an odd prime. If $n = p^k$ with an odd prime p and $k > 0$, then $p\,c(u)$ is an algebraic integer. Moreover, $c(u)$ is a unit iff neither n nor $n/2$ is a power of an odd prime.*

Theorem 2.3 (Kronecker-Weber)**.** *The set of numbers $c(\mathbb{Q} \setminus \mathbb{Z})$ generates the real subfield of the maximal abelian extension of $\mathbb{Q}$.*

Kronecker was interested in generalizing these results to the case of elliptic functions with period lattice being an ideal in an imaginary quadratic field. One of the difficulties he was facing is that the addition formula for elliptic functions is in general algebraic and not rational like in the case of the cotangent function. To the best of our knowledge, the proof of the Jugendtraum as envisioned by Kronecker has never been completed[2]. Except

[2]see the comment linked to interpretation c) in [HH].

for the case of imaginary quadratic fields and the case of the rational number field (both closely related to the work of Kronecker), we do not even know whether Hilbert's 12th has a solution. This is partly related to a classical theorem of Weierstrass asserting that every meromorphic function with algebraic addition law is either elliptic, circular or rational.

Instead of the series defining $\phi(u)$, Kronecker preferred to study the more general series

$$\phi(u,\xi) = \sum_{n=-\infty}^{\infty} \frac{\mathbf{e}(-n\xi)}{u+n} = 2\pi i \, \frac{\mathbf{e}(\xi u)}{\mathbf{e}(u)-1}, \; 0 < \xi < 1, \tag{4}$$

again, with a complex number u which is not an integer. The introduction of the second variable ξ is very natural and suggested by Fourier analysis. Note that the limiting case

$$\lim_{\substack{\xi=0 \\ \xi>0}} \phi(u,\xi) = \phi(u) - i\pi$$

relates $\phi(u,\xi)$ to the Eisenstein function ϕ. The right side of Equation (4) is essentially the generating function for the Bernoulli polynomials. Expanding the left side into a power series in u, one obtains the Fourier expansion of the Bernoulli polynomials. The above proof of the addition formula for the cotangent function applies to $\phi(u,\xi)$ as well and yields addition formulas for the Bernoulli polynomials. Due to the factor $\mathbf{e}(-n\xi)$ in the numerator, the convergence of this series is slightly better than that of the cotangent series, but is still conditional. A substantial part of the lecture notes is devoted to the study of the elliptic analog of (4), written in Kronecker's notation,

$$Ser\,(\xi,\eta,u,\tau) = \lim_{N\to+\infty} \lim_{M\to+\infty} \sum_{n=-N}^{N} \sum_{m=-M}^{M} \frac{\mathbf{e}(-m\xi+n\eta)}{u+n\tau+m}, \tag{5}$$

where ξ,η is a pair of real variables and τ is a point in the upper half space. The complex variable u must be restricted to the complement of the lattice $\mathbb{Z}+\tau\mathbb{Z}$ in $\mathbb{C}$. Writing $u = \sigma\tau+\rho$ as a linear combination of τ and 1 with real coefficients σ,ρ, this series can be viewed as a function of two pairs of real variables $(\xi,\eta),(\sigma,\rho)$ and these are the two pairs of real variables referred to in the title of the lecture notes.

Various alternative ways to sum the conditionally convergent series (5) are discussed in the manuscript. It is a remarkable fact that in all cases, the value obtained for the sum is independent of the limiting process chosen. Kronecker's main result expresses these series in terms of Jacobi theta series. Let

$$\begin{aligned}\vartheta(z,\tau) = \vartheta_1(z,\tau) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \mathbf{e}\left(\frac{n^2\tau}{2} + n(z-\frac{1}{2})\right) \\ &= 2\,q^{\frac{1}{8}} \sin(\pi z) \prod_{n\geq 1} (1-q^n)(1-q^n\mathbf{e}(z))(1-q^n\mathbf{e}(-z)),\end{aligned}$$

with a complex variable z, a point τ in the upper half plane and $q = \mathbf{e}(\tau)$.

Theorem 2.4. *(Kronecker). Suppose* $0 < \Im u < \Im \tau$, $0 < \xi < 1$. *Then*

$$Ser\,(\xi, \eta, u, \tau) = \mathbf{e}(\xi u) \frac{\vartheta'(0,\tau)\vartheta(u+\eta+\xi\tau,\tau)}{\vartheta(u,\tau)\vartheta(\eta+\xi\tau,\tau)}. \tag{6}$$

This result is reminiscent of the so called limit formula of Kronecker which is not discussed in the lecture notes but deserves to be stated here : let τ, τ' be two complex numbers with $\Im\tau > 0, \Im\tau' < 0$, and let $0 \leq \xi, \eta < 1$ be two real numbers, not both equal to zero. Writing $u = \eta - \xi\tau, v = \eta - \xi\tau'$, the second limit formula is the identity

$$\frac{(\tau - \tau')}{2\pi i} \sum_{m,n} \frac{\mathbf{e}(m\xi + n\eta)}{(m\tau + n)(m\tau' + n)} = -\log \frac{\vartheta(u,\tau)\vartheta(v,-\tau')}{\eta(\tau)\eta(-\tau')} - \pi i \frac{(u-v)^2}{\tau - \tau'}, \tag{7}$$

where $\eta(\tau)$ refers to the Dedekind eta function, and the term $(m, n) = (0, 0)$ needs to be excluded from the series. Again, convergence is only conditional so a specific order of summation like in (5) needs to be observed.

Historically, the notion of complex multiplication of elliptic functions appeared for the first time in the work of Abel. Pages 64-67 of Kronecker's lecture notes are devoted to the task of expressing the elliptic functions used by Abel in terms of the Kronecker series $Ser(\xi, \eta, u, \tau)$. This is perhaps one of the highlights of the manuscript and deserves special attention. It is very likely that Kronecker's conception of the Jugendtraum was the result of a close study of the work of Abel.

3 Recent developments.

It would be wise to leave aside the complete discussion of the legacy of Eisenstein or Kronecker to serious historians. We offer instead a brief survey of several recent developments that feature Eisenstein's summation process and Kronecker's Theorem 2.4 for the series $Ser\,(\xi, \eta, u, \tau)$.

3.1 Algebraicity and p-adic interpolation of Eisenstein-Kronecker numbers.

It was already observed by Weil in 1976 that the combination of the methods of Eisenstein and Kronecker gives a direct access to the Damerell's classical result (1971) on the algebraicity of values of L-functions attached to a Hecke Grössencharakter of an imaginary quadratic field.

Weil also anticipated that their methods would extend to the investigation of the p-adic properties of these algebraic numbers. In the following 10 years, the works of Manin-Višik, Katz and Yager among others provided the expected p-adic interpolation of this family of special values. To give a taste of the results in question, we wish to introduce a recent work by Bannai-Kobayashi (2010), based on Kronecker's series, that enables a similar construction.

Our first task is to connect the series Ser to a series that includes a s-parameter pertaining to the style of Hecke. Let τ be a complex number in the upper half plane, and let $A = \frac{\Im\tau}{\pi}$

be the area of the fundamental domain of the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ divided by π. Let ψ be the character $\psi(z) := e^{\frac{z-\overline{z}}{A}}$. For $a \geq 0$ an integer, we introduce the Kronecker-Eisenstein series as

$$K_a^*(z, w, s, \tau) := \sum_{\lambda \in \Lambda}^{*} \frac{(\overline{z} + \overline{\lambda})^a}{|z + \lambda|^{2s}} \psi(z\overline{w}), \ \mathrm{Re}(s) > a/2 + 1,$$

where the $*$ means that the summation excludes $\lambda = -z$ if $z \in \Lambda$. It is a continuous function of the parameters $z \in \mathbb{C} \setminus \Lambda, w \in \mathbb{C}$, and it has meromorphic continuation to the whole s-plane, with possible poles only at $s = 0$ (if $a = 0$ and $z \in \Lambda$) and at $s = 1$ (if $a = 0$ and $w \in \Lambda$). Moreover, it satisfies a functional equation relating the value at s to the value at $a + 1 - s$. Write $w = \eta + \xi\tau$ with real variables ξ, η, and abbreviate the central value $K_1^*(z, w, 1, \tau)$ by $K(z, w)$ when there is no ambiguity on the lattice. This specific function is related to Kronecker's series by the identity

$$K(z, w) = K_1^*(z, w, 1, \tau) = Ser\,(\xi, \eta, z, \tau), \ \text{where } w = \eta + \xi\tau,$$

at least if $z, w \notin \Lambda$, using [We1, §5, p. 72]. In this new set of notations, Theorem 2.4 can be restated as

$$K(z, w) = e^{z\overline{w}/A}\,\Theta(z, w),$$

where

$$\Theta(z, w) := \frac{\vartheta'(0, \tau)\vartheta(z + w, \tau)}{\vartheta(z, \tau)\vartheta(w, \tau)} \tag{8}$$

denotes the meromorphic function appearing as the ratio in Equation (6). It will play a major role in the remainder of this text, so we name it the "Kronecker theta function", in agreement with the terminology in [BK].

Given a pair of integers $a \geq 0, b > 0$ and $z_0, w_0 \in \mathbb{C}$, the Eisenstein-Kronecker numbers are defined as

$$e_{a,b}^*(z_0, w_0, \tau) = K_{a+b}^*(z_0, w_0, b, \tau).$$

When $b > a + 2$, these numbers include in particular the values of the absolutely convergent partial Hecke L-series

$$e_{a,b}^*(0, 0, \tau) = \sum_{\lambda \in \mathbb{Z} + \tau\mathbb{Z}} \frac{\overline{\lambda}^a}{\lambda^b}.$$

They should be considered as elliptic analogs of the Bernoulli numbers. As such, the Eisenstein-Kronecker numbers can be packaged in a generating series that is the elliptic analog of the cotangent function and its relative $\phi(u, \xi)$. To obtain that nice two-variables generating series, it is enough to translate and slightly alter the Kronecker theta function in order to define

$$\Theta_{z_0, w_0}(z, w) = e^{-\frac{(z_0+z)\overline{w_0} + w\overline{z_0}}{A}}\Theta(z_0 + z, w_0 + w).$$

Its Laurent expansion around $z = w = 0$ displays exactly the collection of Eisenstein-Kronecker numbers.

Proposition 3.1. *Fix $z_0, w_0 \in \mathbb{C}$. We have the following Laurent expansion near $z = w = 0$:*

$$\Theta_{z_0,w_0}(z,w) = \psi(w_0\overline{z_0})\frac{\delta(z_0)}{z} + \frac{\delta(w_0)}{w} + \sum_{a\geq 0, b>0} (-1)^{a+b-1}\frac{e^*_{a,b}(z_0,w_0,\tau)}{a!A^a} z^{b-1}w^a, \qquad (9)$$

where $\delta(u) = 1$ if $u \in \Lambda$ and 0 otherwise.

If in addition τ is a CM point and z_0, w_0 are torsion points over the lattice Λ, then these coefficients are algebraic.

Theorem 3.2. *Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be lattice in $\mathbb{C}$. Assume that the complex torus $\mathbb{C}/\Lambda$ has complex multiplication by the ring of integers of an imaginary quadratic field k, and possess a Weierstrass model $E : y^2 = 4x^3 - g_2x - g_3$ defined over a number field F. Fix $N > 1$ an integer. For z_0, w_0 two complex numbers such that $Nz_0, Nw_0 \in \Lambda$, the Laurent expansion in (9) has coefficients in $F(E[4N^2])$. In particular, the rescaled Eisenstein-Kronecker numbers*

$$e^*_{a,b}(z_0,w_0,\tau)/A^a$$

are algebraic.

We refer to [BK], Th. 1 and Cor. 2.11 for the proofs. The above construction enables Bannai and Kobayashi to recover Damerell's result along the way.

Let p be a prime number. Bernoulli numbers and Bernoulli polynomials satisfy a whole collection of congruences modulo powers of p, known as the "Kummer congruences". These congruences are incorporated in the construction of the Kubota-Leopoldt p-adic zeta function $\zeta_p(s)$, $s \in \mathbb{Z}_p$, which interpolates the values of the Riemann zeta function at negative integers, as given by the Bernoulli numbers.

Similarly, Eisenstein-Kronecker numbers satisfy a collection of congruences that are the building blocks for the construction of two-variables p-adic L-functions. Since Bannai-Kobayashi also have a generating series at their disposal, they can interpolate the Eisenstein-Kronecker numbers p-adically, not only when p splits in k, like in the work of Katz (ordinary case), but also when p is inert in k (supersingular case).

It seems appropriate to remark in passing that in this p-adic setting, the Kronecker limit formula (7) also has a counterpart. It has been originally obtained by Katz (1976), and proved to be crucial in the study of Euler systems attached to elliptic units.

To complete this p-adic picture and get started with a recurring theme for sections to come, we would like to mention a generalization by Colmez-Schneps [CS] of the above construction of Manin-Višik and Katz. Colmez and Schneps consider the case where the Hecke character is attached to an extension F of degree n over the imaginary quadratic field k (the previous setting thus corresponds to $n = 1$). The field F might not necessary be a CM field. Building on the techniques of [Co], they can construct a p-adic L-function for F by interpolating the algebraic numbers arising from the Laurent expansion of certain linear combinations of *products of n generating series* of Kronecker's type, each of them being evaluated at torsion points over the lattice Λ. The juxtaposition of n copies of Proposition 3.1-Theorem 3.2 allows them to bootstrap the case $n = 1$ to arbitrary $n \geq 1$ using their identity [CS, Eq. (31)].

3.2 Periods of Hecke eigenforms.

Let (z_0, w_0) be a fixed pair of N-torsion points over the lattice Λ. As functions of the τ variable, the modular forms $e^*_{a,b}(z_0, w_0, \tau)$ are Eisenstein series for the principal congruence subgroup $\Gamma(N)$. In particular, the Laurent expansion (9) for the translated Kronecker theta function at $z = w = 0$ is a generating series for Eisenstein series of increasing weight and fixed level. Its decomposition under the action of the Hecke operators algebra thus possess only Eisenstein components. From this perspective, interesting new phenomenons start to appear when one considers *a product of two Kronecker theta functions*.

Such a product encodes all periods polynomials of modular forms of all weights, at least in the level one case[3]. This is the content of the main result of Zagier's paper [Za1], which we now describe.

Let M_k be the $\mathbb{C}$-vector space of modular forms of weight $k \geq 4$ on $SL_2(\mathbb{Z})$, and let $S_k \subset M_k$ be the subspace of cusp forms, equipped with the Petersson scalar product (f, g) and its basis of normalised Hecke eigenforms $\mathcal{B}_k^{\mathrm{cusp}}$. The period polynomial attached to $f \in S_k$ is the polynomial of degree $\leq k-2$ defined by

$$r_f(X) = \int_0^{i\infty} f(\tau)(\tau - X)^{k-2} d\tau.$$

The Eichler-Shimura-Manin theory implies that both maps $f \mapsto r_f^{\mathrm{ev}}$ and $f \mapsto r_f^{\mathrm{od}}$ assigning to f the even and the odd part of r_f are both injective. Moreover, if f is a normalized Hecke eigenform, then the two-variables polynomial

$$R_f(X, Y) := \frac{r_f^{\mathrm{ev}}(X) r_f^{\mathrm{od}}(Y) + r_f^{\mathrm{od}}(X) r_f^{\mathrm{ev}}(Y)}{(2i)^{k-3}(f, f)} \in \mathbb{C}[X, Y]$$

transforms under $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ as $R_{\sigma(f)} = \sigma(R_f)$, so R_f has coefficients in the number field generated by the Fourier coefficients of f. As a consequence, for each integer $k > 0$, the finite sum

$$C_k^{\mathrm{cusp}}(X, Y, \tau) := \frac{1}{(k-2)!} \sum_{f \in \mathcal{B}_k^{\mathrm{cusp}}} R_f(X, Y) f(\tau),$$

belongs to $\mathbb{Q}[X, Y][[q]]$. Zagier starts to complete this cuspidal term by a contribution that arises from the Eisenstein series in M_k, using the following convenient recipe. For any even $k > 0$, let B_k be the usual k-th Bernoulli number, and let

$$E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{m \geq 1} \Big(\sum_{d|m} d^{k-1}\Big) q^m$$

be the normalized weight k Eisenstein series for $SL_2(\mathbb{Z})$. Its pair of period polynomials in $X^{-1}\mathbb{Q}[X]$ is defined by

$$r_{E_k}^{\mathrm{od}}(X) = X^{k-2} - 1, \; r_{E_k}^{\mathrm{ev}}(X) = \sum_{h=0}^{k} \frac{B_h}{h!} \frac{B_{k-h}}{(k-h)!} X^{h-1},$$

[3] A very recent preprint [CPZ] indicates that a similar result, slightly weaker, also holds for arbitrary squarefree level.

and they make up the contribution of Eisenstein series by the rule

$$C_k^{\mathrm{Eis}}(X,Y,\tau) := -\left(r_{E_k}^{\mathrm{od}}(X) r_{E_k}^{\mathrm{ev}}(Y) + r_{E_k}^{\mathrm{ev}}(X) r_{E_k}^{\mathrm{od}}(Y)\right) E_k(\tau).$$

The main identity of Zagier then establishes a remarkable closed formula for the generating series

$$C_\tau(X,Y,T) := \frac{(X+Y)(XY-1)}{X^2Y^2T^2} + \sum_{k=2}^{\infty} (C_k^{\mathrm{cusp}} + C_k^{\mathrm{Eis}}) T^{k-2},$$

which factorizes as a product of two Kronecker theta functions :

Theorem 3.3. *[Za1] In $(XYT)^{-2}\mathbb{Q}[X,Y][[q,T]]$ we have*

$$C_\tau(X,Y,T) = \Theta(XT', YT')\, \Theta(T', -XYT')/\omega^2, \tag{10}$$

where $\omega = 2\pi i, T' = T/\omega$.

Equation (10) shows that complete information on Hecke eigenforms of any desired weight for $SL_2(\mathbb{Z})$ and their period polynomial is encoded in the Laurent expansion at $T = 0$ of the right side, that consists of the product of two building blocks Kronecker theta functions. To further support that claim, Zagier explains in the sequel paper [Za2] how to deduce from Equation (10) an elementary proof of the Eichler-Selberg formula for traces of Hecke operators on $SL_2(\mathbb{Z})$.

The period polynomials satisfy a collection of linear relations under the action of $SL_2(\mathbb{Z})$. These cocycle relations are reflected using Equation (10) by relations satisfied by Kronecker's theta functions, e.g. [Za1, p. 461]. A typical example is

$$C_\tau(X,Y,T) + C_\tau(1-\frac{1}{X}, Y, TX) + C_\tau(\frac{1}{1-X}, Y, T(1-X)) = 0, \tag{11}$$

which is the counterpart of the classical relation for the period polynomial

$$r_f \mid 1 + U + U^2 = 0,\ U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the remainder of this paper, we explain how the method of Eisenstein-Schellbach, when properly modulated, is the adequate tool to produce systematically general $(n-1)$-cocycle relations for $GL_n(\mathbb{Z})$ involving products of n Kronecker theta functions, including Equation (11) as a very special case.

3.3 Trigonometric cocycles on GL_n.

Let $0 < a < 1, 0 < b < 1$ be rational numbers, and $x, y \in \mathbb{C} - \mathbb{Z}$ be complex parameters. We use the shorthand $0 < \{t\} < 1$ for the fractional part of a non-integral real number t. From a direct computation or a mild generalisation of Lemma 2.1, one deduces the following relation. It amounts to the addition formula for the function $\phi(u,\xi)$:

$$\frac{\mathbf{e}(xa)\mathbf{e}(yb)}{(\mathbf{e}(x)-1)(\mathbf{e}(y)-1)} - \frac{\mathbf{e}((x+y)a)\mathbf{e}(y\{b-a\})}{(\mathbf{e}(x+y)-1)(\mathbf{e}(y)-1)} - \frac{\mathbf{e}((x+y)b)\mathbf{e}(x\{a-b\})}{(\mathbf{e}(x+y)-1)(\mathbf{e}(x)-1)} = 0. \qquad (12)$$

As pointed out by the second author in [Scz2], this identity can naturally be recasted in terms of the cohomology of the group $SL_2(\mathbb{Z})$. The building blocks are products of two copies of the trigonometric function $\phi(u,\xi)$. Given to pairs $u=(u_1,u_2)$ and $\xi=(\xi_1,\xi_2)^t$, we set

$$\mathbf{\Phi}(u,\xi) := \phi(u_1,\xi_1)\phi(u_2,\xi_2).$$

To any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ is associated $\sigma = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \in M_2(\mathbb{Z})$. If $c \neq 0$ we define

$$\Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(u,\xi) = \mathrm{sign}(c) \sum_{\mu\in\mathbb{Z}^2/\sigma\mathbb{Z}^2} \mathbf{\Phi}(u\sigma, \sigma^{-1}(\mu+\xi)).$$

If $c=0$ then $\Psi(A)(u,\xi)=0$ by definition. The next proposition stands for the case $n=2$ of a more general $(n-1)$-cocycle relation for the group $GL_n(\mathbb{Z})$ obtained in [Scz3, Cor. p. 598].

Proposition 3.4. *Let $A, B \in SL_2(\mathbb{Z})$ be two matrices. For any u in a dense open domain in $\mathbb{C}^2$ and any non zero $\xi \in \mathbb{Q}^2$, the following inhomogenous 1-cocycle relation holds :*

$$\Psi(AB)(u,\xi) - \Psi(A)(u,\xi) - \Psi(B)(uA, A^{-1}\xi) = 0.$$

The addition law (12) corresponds to the choice of matrices $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ in this proposition.

The coefficients of the Laurent expansion at $u=0$ of the general $(n-1)$-cocycle Ψ carry a great deal of arithmetic information. According to the main result of [Scz3, Th. 1], the pairing of Ψ with a $(n-1)$-cycle built out of the abelian group generated by the fundamental units of a totally real number field F of degree n produces the values $\zeta_F(k)$ at non-positive integers k of the Dedekind zeta function of F, and thereby establishes their rationality. This provides a new proof, deeply rooted in the Eisenstein-Schellbach method, of the Klingen-Siegel rationality result.

An integral avatar of the cocycle Ψ has later been introduced by Charollois and Dasgupta to study the integrality properties of the values $\zeta_F(k)$, enabling them to deduce a new construction the p-adic L-functions of Cassou-Noguès and Deligne-Ribet.

Combining their cohomological construction with the recent work of Spiess, they also show in [CD] that the order of vanishing of these p-adic L-functions at $s=0$ is at least equal to the expected one, as conjectured by Gross. This result was already known from Wiles' proof of the Iwasawa Main Conjecture.

3.4 On elliptic functions depending on four pairs of real variables.

Our goal in this last section is to propose a q-deformation of the trigonometric cocycle Ψ to construct an elliptic cocycle, where the role of the function $\phi(u,\xi)$ is now played by Kronecker's theta function. It will make it clear that the trigonometric relations we have encountered so far are specialization of elliptic ones when $q \to 0$, i.e. $\tau \to i\infty$.

Kronecker has not been given the opportunity to implement the Eisenstein-Schellbach method in his own elliptic investigations. Let's proceed now by fixing τ in the upper half-plane and first perform in Equation (12) the change of variables

$$x \leftarrow n\tau + x,$$
$$y \leftarrow n'\tau + y.$$

We also aim to twist that equation by a pair of roots of unity $\mathbf{e}(nr)\mathbf{e}(n's)$ with $r, s \in \mathbb{Q}$, and then sum over $n, n' \in \mathbb{Z}$. We write $x_0 = a\tau + r, y_0 = b\tau + s, q = \mathbf{e}(\tau)$, so that the real and imaginary parts of x, x_0, y, y_0 make up four pair of real variables. The first summand becomes

$$S := \mathbf{e}(xa)\mathbf{e}(yb) \sum_{n,n' \in \mathbb{Z}} \frac{\mathbf{e}(nr) q^{na} q^{n'b} \mathbf{e}(n's)}{(q^n \mathbf{e}(x) - 1)(q^{n'} \mathbf{e}(y) - 1)}.$$

The assumption $0 < a < 1$ ensures that $1 > |q^a| > |q|$ and similarly for q^b, so that this double series is absolutely convergent. The sum S naturally splits as a product, whose value is deduced from two consecutive uses Kronecker's Theorem 2.4 :

$$S = \mathbf{e}(xa)\mathbf{e}(yb)\, \Theta(x, x_0)\, \Theta(y, y_0).$$

A similar resummation process can be performed on the second and the third term of Eq. (12). After simplification by the common factor $\mathbf{e}(xa + yb)$, we obtain the identity

$$\Theta(x, x_0)\Theta(y, y_0) - \Theta(x + y, x_0)\Theta(y, y_0 - x_0) - \Theta(x + y, y_0)\Theta(x, x_0 - y_0) = 0, \qquad (13)$$

which can be extended analytically to remove the restrictive assumptions made on the parameters. Equation (13) is just another form of the Riemann theta addition relation, also known as Fay trisecant identity. It implies simultaneously Eq. (12) when $\tau \to i\infty$ and Eq. (11) under the choice of parameters $x = XT', x_0 = YT', y = -T', y_0 = XYT'$.

More generally, the resummation of the trigonometric $(n-1)$-cocycle Ψ gives rise to an elliptic $(n-1)$-cocycle which we name the Eisenstein-Kronecker cocycle κ on $GL_n(\mathbb{Z})$. One recovers Ψ from κ by letting $\tau \to i\infty$. The coefficients of the Laurent expansion of κ at zero now display modular forms, essentially sums of products of n Eisenstein series of various weights and levels, that are members of a compatible p-adic family.

When paired with a $(n-1)$-cycle built out of the fundamental units of a totally real number field F of degree n, the elliptic cocycle κ produces a generating series for the pullbacks of Hecke-Eisenstein series over the Hilbert modular group $SL_2(\mathcal{O}_F)$, whose constant terms are the values $\zeta_F(k), k \leq 0$. These classical modular forms already played a prominent role in the original proof by Siegel of the Klingen-Siegel theorem. More details on the construction of the Eisenstein-Kronecker cocycle κ and its properties will be given in [Ch].

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