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Finding the $k$: Shortest Simple Paths: Time and Space Trade-Offs*

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Abstract

The $k$ shortest simple path problem ($k$SSP) asks to compute a set of top-$k$ shortest simple paths from a source to a sink in a digraph. Yen (1971) proposed an algorithm with the best known polynomial time complexity for this problem. Since then, the problem has been widely studied from an algorithm engineering perspective. The most noticeable proposals are the node-classification (NC) algorithm (Feng, 2014) and the sidetracks-based (SB) algorithm (Kurz, Mutzel, 2016). The latest offers the best running time at the price of a significant memory consumption.

We first show how to speed up the SB algorithm using dynamic updates of shortest path trees resulting in a faster algorithm (SB*) with the same memory consumption. We then propose the parsimonious SB (PSB) algorithm that significantly reduces the memory consumption of SB at the cost of a small increase of the running time. Furthermore, we propose the postponed node-classification (PNC) algorithm that combines the best of the NC and the SB algorithms. It offers a significant speed up compared to the SB algorithm while using the same amount of memory as the NC algorithm.

Our experimental results on complex networks show that all the considered algorithms have low memory consumption, and that the PSB algorithm is the fastest. On road networks, the relative performances of the algorithms depend on the number $k$ of requested paths. Indeed, when the number $k$ of requested paths is small (i.e., $k \leq 20$ in our experiments), the SB* algorithm is the fastest among the considered algorithms, but it suffers from a large memory consumption and it offers very bad performances on some queries. When the number of requested paths is large (i.e., larger than 20 according to our experiments), the PNC algorithm is the fastest among the considered algorithms on road networks and it has a low memory footprint. The PNC algorithm is therefore a better choice on road networks.

Keywords: $k$ shortest simple paths; graph algorithm; space-time trade-off.

1 Introduction

The classical $k$ shortest paths problem ($k$SP) aims at finding the top-$k$ shortest paths between a pair of source and destination nodes in a graph. This problem has numerous applications in various kinds of networks (road and transportation networks, communications networks, social networks, etc.) and is also used as a building block for solving optimization problems. Let $D = (V, A)$ be a digraph. An $s$-$t$ path is a sequence $(s = v_0, v_1, \ldots, v_l = t)$ of vertices starting

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with \( s \) and ending with \( t \), such that \((v_i, v_{i+1}) \in A\) for all \( 0 \leq i < l \). It is called simple if it has no repeated vertices, \( i.e.\), \( v_i \neq v_j \) for all \( 0 \leq i < j \leq l \). The weight of a path is the sum of the weights of its arcs and the top-\( k \) shortest paths therefore comprise a set containing a shortest \( s\)-\( t \) path, a second shortest \( s\)-\( t \) path, etc., until a \( k \)th shortest \( s\)-\( t \) path.

Several algorithms for solving \( k \)SSP have been proposed. In particular, Eppstein [8] proposed an exact algorithm that computes \( k \) shortest paths (not necessarily simple) in time \( O(m + n \log n + k) \), where \( m \) is the number of arcs and \( n \) the number of vertices of the graph. An important variant of this problem is the \( k \) shortest simple paths problem (\( k \)SSP) introduced in 1963 by Clarke et al. [9] which adds the constraint that reported paths must be simple. This variant of the problem has various applications in transportation networks when paths with repeated vertices are not desired by the user. It is also a subproblem of other important problems like constrained shortest path problem, vehicle and transportation routing [16,19,35]. It can be applied successfully in bio-informatics [2], especially in biological sequence alignment [30] and in natural language processing [4]. For more applications, see Eppstein’s recent comprehensive survey on \( k \)-best enumeration [9].

The most famous algorithm for solving the \( k \)SSP problem has been proposed by Yen [36] and has time complexity \( O(kn(m + n \log n)) \). It has been proved that the \( k \)SSP problem can be solved in \( O(k) \) iterations of APSP (All Pairs Shortest Paths) [15]. This means that the \( k \)SSP problem can be solved in time \( O(kn(m + n \log \log n)) \) on sparse digraphs and in time \( O(kn^2(\log \log n)/\log^2 n) \) on dense digraphs using the fastest APSP algorithms [17,28]. Vassilevska Williams and Williams [33] show that a subcubic \( k \)SSP algorithm would also result in a subcubic algorithm for APSP, which seems unlikely at the moment. Recently, Eppstein and Kurz [10] proved that on digraphs with bounded treewidth, the \( k \)SSP problem can be solved in time \( O(n + k \log n) \).

On the other hand, several works have been proposed to improve the efficiency of the algorithm in practice [11,13,16,18,21,23,31]; they all feature the same worst-case running time as Yen’s algorithm, \( i.e.\), \( O(kn(m + n \log n)) \).

In particular, Feng [11] proposed an improvement of Yen’s algorithm called Node Classification (NC) algorithm. With the help of a pre-computed shortest path tree of the digraph, the NC algorithm classifies the vertices of the digraph in order to speed up the computation of shortest simple paths. An interesting fact about the NC algorithm is that its memory consumption is almost the same as Yen’s algorithm (only a shortest path tree is kept in the memory).

Recently, Kurz and Mutzel [22,23] obtained a tremendous improvement of the practical running time, designing an algorithm called Sidetrack Based (SB) with the same flavor as Eppstein’s algorithm. The key ideas are to define a path using a sequence of shortest path trees and deviations, and to postpone as much as possible the computation of shortest path trees. With this new algorithm, they were able to compute hundreds of paths in graphs with million nodes in about one second, while previous algorithms required an order of tens of seconds on the same instances. For instance, Kurz and Mutzel’s algorithm computed \( k = 1000 \) shortest simple paths in 50 milliseconds for the DC network [7] while it required about 5 seconds with Yen’s algorithm and about 0.3 seconds by the improvement proposed by Feng [11]. The drawback of the SB algorithm is the need for storing all computed shortest path trees, thus leading to a large usage of memory.

To sum up, the fastest algorithm with low memory consumption (\( i.e.\), the same memory consumption as Yen’s algorithm) is the Node Classification (NC) algorithm, proposed independently by [11] and [14]. With larger memory consumption, the Sidetrack Based algorithm (SB) [23] can achieve a tremendous speed up.

**Our contributions.** We propose a new algorithm with low memory consumption, called Postponed Node Classification (PNC), that combines the best of the NC and the SB algorithms.
More precisely, while the PNC algorithm has the same memory consumption than the NC algorithm, our experimental results show that it is the fastest $k$-SSP algorithm among all considered algorithms on road networks.

Considering large memory consumption, we show how to speed up the SB algorithm using dynamic updates of shortest path trees resulting with the SB* algorithm, that is, the fastest $k$-SSP algorithm (on median) with large memory consumption among the considered algorithms on road networks. Moreover, we propose a new algorithm, called Parsimonious Sidetrack Based (PSB), that is based on the SB algorithm. The PSB algorithm gives, on road networks, a space time trade-off between SB and NC algorithms. That is, on road networks, it significantly reduces the memory consumption of SB at the cost of an increase of the running time. Nonetheless, on complex networks, the PSB algorithm gives the best running time among all the considered algorithms. We further propose parameterized variants of PSB (called PSBv2 and PSBv3) that improve its performance, both in terms of memory consumption and of running time, on road networks.

We analyse the behavior of all of the aforementioned algorithms on different types of networks (road, biological, Internet and social networks). We have also investigated the relationships between the performances of the algorithms and the Dijkstra rank of the queries. Finally, we end up with a first empirical framework for selecting the most suitable $k$-SSP algorithm for a given use case.

This paper is organized as follows. We start recalling the Yen and NC algorithms in Section 2. Then, we present, in Section 3, the SB algorithm with our improvement SB*. In Section 4, we present the PSB algorithm and its two variants PSBv2 and PSBv3. Section 5 is devoted to the presentation of the PNC algorithm. We present our experimental evaluation of all these algorithms on various networks in Section 6. Finally, we conclude this paper in Section 7.

2 Preliminaries

2.1 Definition and Notation

Let $D = (V, A)$ be a directed graph (digraph for short) with vertex set $V$ and arc set $A$. Let $n = |V|$ be the number of vertices and $m = |A|$ be the number of arcs of $D$. Given a vertex $v \in V$, $N^+(v) = \{w \in V \mid vw \in A\}$ denotes the set of the out-neighbors of $v$ in $D$. Let $\omega_D : A \rightarrow \mathbb{R}^+$ be a weight function over the arcs. For every $u, v \in V$, a (directed) path from $u$ to $v$ in $D$ is a sequence $P = (u = v_0, v_1, \ldots, v_r = v)$ of vertices with $v_i, v_{i+1} \in A$ for all $0 \leq i < r$. Note that vertices may be repeated, i.e., paths are not necessarily simple. A path is simple if, moreover, $v_i \neq v_j$ for all $0 \leq i < j \leq r$. The weight of the path $P$ equals $\omega_D(P) = \sum_{0 \leq i < r} \omega_D(v_i, v_{i+1})$ (we will omit $D$ when there is no ambiguity). The distance $d_D(u,v)$ between two vertices $u, v \in V$ is the smallest weight of a $u$-$v$ path in $D$ (if any). Given two paths $P = (v_0, \ldots, v_r), Q = (w_0, \ldots, w_p)$ and $v_r, w_0 \in A$, we denote by $P.Q$ the $v_0$-$w_p$ path obtained by the concatenation of $P$ and $Q$. i.e., $P.Q = (v_0, \ldots, v_r, w_0, \ldots, w_p) = (v_0, \ldots, v_r, Q) = (P, w_1, \ldots, w_p)$.

Given $s, t \in V$, a top-$k$ set of shortest simple $s$-$t$ paths is any set $S$ of (pairwise distinct) simple $s$-$t$ paths such that $|S| = k$ and $\omega(P) \leq \omega(P')$ for every $s$-$t$ path $P \in S$ and $s$-$t$ path $P' \notin S$.

The $k$ shortest simple paths problem takes as input a weighted digraph $D = (V, A)$, $\omega_D : A \rightarrow \mathbb{R}^+$ and a pair of vertices $(s, t) \in V^2$ and asks to find a top-$k$ set of shortest simple $s$-$t$ paths (if they exist).

Let $t \in V$. An in-branching $T$ rooted at $t$ is any sub-digraph of $D$ that induces a (not necessarily spanning) tree containing $t$, such that every $u \in V(T) \setminus \{t\}$ has exactly one out-neighbor (that is, all paths go toward $t$). An in-branching $T$ is called a shortest path (SP)
in-branching rooted at \( t \) if, for every \( u \in V(T) \), the weight of the (unique) \( u \)-\( t \) path \( P^T_u \) in \( T \) equals \( d^T_{D}(u,t) \). Note that an SP in-branching is sometimes called reversed shortest path tree. Similarly, a shortest path (SP) out-branching \( T \) rooted at \( s \) is any sub-digraph of \( D \) inducing a tree containing \( s \), such that every \( u \in V(T) \setminus \{s\} \) has exactly one in-neighbor and the weight of the (unique) \( s \)-\( u \) path \( P^T_s \) in \( T \) equals \( d^T(s,u) \). Any algorithm able to compute an SP in-branching (or an SP out-branching) is called an SP algorithm. As we consider directed weighted digraphs, we use the algorithm of Dijkstra. However, it is possible to use any suitable shortest path algorithm instead.

In the forthcoming algorithms, the following procedure will often be used (and the key point when designing the algorithms is to limit the number of such calls and to optimize each of them). Given a sub-digraph \( H \) of \( D \) and \( u,t \in V(H) \), we use an SP algorithm to compute an SP in-branching rooted at \( t \) that contains a shortest \( u \)-\( t \) path in \( H \). Note that, the execution of an SP algorithm may be stopped as soon as a shortest \( u \)-\( t \) path has been computed (when \( u \) is reached), i.e., the in-branching may only be partial (not necessarily spanning \( H \)). The key point will be that this way to proceed not necessarily only returns a shortest \( u \)-\( t \) path in \( H \) (if any) but an SP in-branching rooted in \( t \), containing \( u \). Recall that any such call has worst-case time complexity \( O(m + n \log n) \).

Let \( P = (v_0, v_1, \ldots, v_r) \) be any path in \( D \) and \( i < r \). Any arc \( a = v_iv_i' \neq v_iv_{i+1} \) is called a deviation of \( P \) at \( v_i \). Moreover, any path \( P' = (v_0, \ldots, v_i, v_i', v_i'', \ldots, v_r) \) is called a detour of \( P \) at \( a \) (or at \( v_i \)). Note that neither \( P \) nor \( P' \) is required to be simple. However, if \( P' \) is simple, it will be called a simple detour of \( P \) at \( a \) (or at \( v_i \)). In addition, \( P' \) is called a shortest (simple) detour at \( v_i \) (or at \( a \)) if and only if \( P' \) is a detour with minimum weight among all (simple) detours of \( P \) at \( v_i \) (or at \( a \)).

### 2.2 Yen’s algorithm

We start by describing Yen’s algorithm \([36]\) trying to give its main properties and drawbacks.

All the algorithms described below start by computing a shortest \( s \)-\( t \) path \( P_0 = (s = v_0, v_1, \ldots, v_r = t) \), and assume that there is always at least one such path. This is done by applying an SP algorithm from \( s \). We clearly may assume that \( P_0 \) is simple since the weights are non-negative. A second shortest \( s \)-\( t \) simple path must be a shortest simple detour of \( P_0 \) at one of its vertices. Yen’s algorithm computes a shortest simple detour of \( P_0 \) at \( v_i \) for every vertex \( v_i \) in \( P_0 \) as follows. For every \( 0 \leq i < r \), let \( D_i(P_0) \) be the graph obtained from \( D \) by removing the vertices \( v_0, \ldots, v_{i-1} \) (this is to avoid non-simple detours) and the arc \( v_iv_{i+1} \) (to ensure that the computed path is a new one, i.e., different from \( P_0 \)). For every \( 0 \leq i < r \), an SP out-branching in \( D_i(P_0) \) rooted at \( v_i \) is computed using an SP algorithm until it reaches \( t \) and therefore returns a shortest path \( Q_i \) from \( v_i \) to \( t \). For every \( 0 \leq i < r \), the detour \( (v_0, \ldots, v_{i-1}, Q_i) \) of \( P_0 \) at \( v_i \) is added to a set \( \text{Candidate} \) (initially empty). Note that the index \( i \) (called below deviation-index) where the path \( (v_0, \ldots, v_{i-1}, Q_i) \) deviates from \( P_0 \) is kept explicit\(^1\) i.e., the path is stored with its deviation index. Once \( (v_0, \ldots, v_{i-1}, Q_i) \) has been added to \( \text{Candidate} \) for all \( 0 \leq i < r \), by remark above, a path with minimum weight in \( \text{Candidate} \) is a second shortest \( s \)-\( t \) simple path.

More generally, by induction on \( 0 < k' < k \), let us assume that a top-\( k' \) set \( S \) of shortest simple \( s \)-\( t \) paths has been computed and that \( \text{Candidate} \) is a set of simple \( s \)-\( t \) paths. Moreover, let us assume that there exists a shortest path \( Q \in \text{Candidate} \) such that \( S \cup \{Q\} \) is a top-(\( k' + 1 \)) set of shortest simple \( s \)-\( t \) paths. Moreover, let us assume by induction that, for every path \( R \) in \( \text{Candidate} \), with deviation index \( j \), all detours of \( R = (v_0, \ldots, v|R|) \) at vertices \( v_j \) for \( 0 \leq i < j \) have already been computed and added to \( \text{Candidate} \). Yen’s algorithm pursues as

\(^1\)The deviation-index is not kept explicitly in Yen’s algorithm. But, since it is a trivial improvement already existing in the literature \([24]\), we mention it here.
follows. Let \( Q = (v_0 = s, \ldots, v_r = t) \) be any shortest path in \textit{Candidate} and let \( 0 \leq j < r \) be its deviation-index. First, \( Q \) is extracted from \textit{Candidate} and it is added to \( S \) (as the \((k' + 1)\)th shortest \( s\)-\( t \) path). Then, for each vertex \( v \) in \( Q \), a shortest simple detour of \( Q \) at \( v \) is added to \textit{Candidate} (since potentially one of these detours is a next shortest \( s\)-\( t \) path). For this purpose, for every \( j \leq i < r \), let \( \pi_i = (v_0, \ldots, v_{i-1}) \) (\( \pi_i = \emptyset \) if \( i = 0 \)) and let \( D_i(Q) \) be a subdigraph of \( D \) containing a shortest \( v_i\)-\( t \) path \( Q_i \) in \( D \) such that \( Q_i \cap \pi_i = \emptyset \) and the path \( \pi_i, Q_i \) is new (\( \pi_i, Q_i \notin S \)). After the construction of \( D_i(Q) \) (described below), an SP out-branching of \( D_i(Q) \) rooted at \( v_i \) is computed using an SP algorithm until it reaches \( t \) and therefore returns a shortest path \( Q_i \) from \( v_i \) to \( t \) in \( D_i(Q) \). For every \( 0 \leq i < r \), the shortest simple detour \( \pi_i, Q_i \) of \( Q \) at \( v_i \) (together with its deviation index \( i \)) is added to the set \textit{Candidate}. This process is repeated until \( k \) paths have been found, \( i.e. \), when \( k' = k \). Indeed, the computed detours of \( Q \) are distinct from every previously computed paths as they have different prefixes (this is the reason to keep explicitly the deviation-index).

The procedure of constructing \( D_i(Q) \) is the following. First, to avoid non-simple detours, \( i.e. \), any intersection between \( Q_i \) and \( \pi_i \), the vertices \( v_0, \ldots, v_{i-1} \) (if \( i > 0 \)) are removed from \( D \). Second, to ensure that the computed path \((\pi_i, Q_i)\) is new (different from those in \( S \)), each arc \( v_i v' \) such that \( S \) already contains a path with prefix \((v_0, \ldots, v_i, v')\) is removed from \( D_i(Q) \).

Therefore, for each path \( Q \) that is extracted from \textit{Candidate}, \( O(|V(Q)|) \) calls of an SP algorithm are done. This gives an overall time-complexity of \( O(kn(m + n \log n)) \) which is the best theoretical (worst-case) time-complexity currently known (and of all algorithms described in this paper) to solve the \( k\)SSP problem.

### 2.3 A Node Classification algorithm

In this section, we present the Node Classification (NC) algorithm, an improvement of Yen’s algorithm proposed independently by Feng [11] and Gao et al. [14].

The most expensive part of Yen’s algorithm is its large number of calls to an SP algorithm. The NC algorithm aims at reducing the computing time of each of these calls, and possibly to avoid some of them.

Precisely, during the process of finding a detour, the search area of an SP algorithm is restricted to a digraph that is smaller than \( D \) with the help of a pre-computed shortest path in-branching. The NC algorithm starts by computing a shortest path in-branching of \( D \) rooted at \( t \) (used to extract a first shortest path \( P_0 \)). Then, when a path \( Q = (v_0, \ldots, v_r) \) with deviation-index \( j \) is extracted, its detours are computed from \( i = j \) to \( r - 1 \). The NC algorithm classifies the vertices as \textit{red}, \textit{yellow}, and \textit{green}: a vertex on the prefix \( i.e., \), the path \((v_0, \ldots, v_{i-1})\) is colored \textit{red}, a vertex \( u \) that can reach \( t \) through \( T \) without visiting a \textit{red} vertex \( i.e., \), \( P_{ut}^T \cap (v_0, \ldots, v_{i-1}) = \emptyset \) is colored \textit{green}, and all other vertices are colored \textit{yellow}. This coloring can be computed in linear time using a DFS in \( T \). Moreover, the coloring used to compute the detour at \( v_{i+1} \) can be obtained faster by updating the coloring for the detour at \( v_i \).

Another important ingredient of the NC algorithm is the notion of residual weight. For each arc \( e = uv \) not in \( T \), the residual weight of \( e \) is the cost of deviating from \( T \) at \( e \). Precisely, it is the weight of the path \( u, P_{ut}^T \) minus the weight of the path \( P_{ut}^T \). Formally, the residual weight of arc \( uv \) is \( \delta(u, v) = \omega(u, v) + \omega(P_{ut}^T) - \omega(P_{ut}^T) \). The residual weight is computed only once (after computing \( T \)) and remains valid till the end of the execution of the NC algorithm. Note that an arc in \( T \) has residual weight equals to 0, and so the residual weight of the path \( P_{ut}^T \) from any \textit{green} vertex \( u \) to \( t \) in \( T \) equals 0.

\(^2\)Actually \textit{Candidate} is implemented, using a heap, in such a way that extracting a shortest path in it takes logarithmic time and insertions are done in constant time.
Recall that to compute a detour of $Q$ at $v_i$, Yen’s algorithm executes an SP algorithm to compute a shortest path from $v_i$ to $t$ in $D_i(Q)$. In the case of the NC algorithm, Feng proved that it is sufficient to execute an SP algorithm using the residual weights and to stop its execution as soon as a green vertex is reached. This results in restricting the execution of the SP algorithm to the yellow subgraph, which is expected to be smaller than $D_i(Q)$, and so to speed up the computation of the detours.

In Section 3 we propose an adaptation of the NC algorithm (using ideas from the SB algorithm presented in the next section) that allows us to speed it up.

3 Sidetrack Based (SB) algorithm

We now present the Sidetrack Based (SB) algorithm, proposed by Kurz and Mutzel [23]. We start by describing the data structure used in the SB algorithm. Then, we explain it and provide a pseudo code (Algorithm 1). Finally we analyse a few of its aspects. Note that our contributions in Section 4 strongly rely on this algorithm and that is why we describe it in detail.

3.1 Compact representation of a path

The SB algorithm is based on a data structure generalizing the representation of a path proposed by Eppstein [8]. Such compact representation uses sequences of in-branchings $T_0, T_1, \ldots, T_h$ and deviations $e_0, e_1, \ldots, e_h$ (recall that a deviation of a path $P$ is any arc not in $P$ but with its tail in $P$).

Precisely, the sequence $\varepsilon = (T_0, e_0, T_1, e_1, \ldots, T_h, e_h, T_{h+1})$ with $e_i = v_i w_i$ for all $0 \leq i \leq h$, represents the path $P$ starting at $s$, following $T_0$ until the tail $v_0$ of $e_0$, then the deviation $e_0$, then $T_1$ from the head $w_0$ of $e_0$ until it reaches the tail $v_1$ of $e_1$, etc., until it reaches the head $w_h$ of $e_h$, plus (possibly) the path from $w_h$ to $t$ in $T_{h+1}$. That is, $P$ is the sequence of vertices of the paths $P_{w_0 h}^0 \subseteq P_{w_0 v_1}^1 \subseteq \cdots \subseteq P_{w_{h-1} h}^{h+1}$ followed by the vertices of $P_{w_h t}^{h+1}$ if this latter path exists. Two consecutive in-branchings $T_i$ and $T_{i+1}$ are not necessarily distinct. The SB algorithm ensures that, if $P$ is an s-t path (i.e., if $P_{w_h t}^{h+1}$ exists), then the subpath $\pi$ of $P$ going from $s$ to $w_h$ ($(v_0, \ldots, w_h)$ is always simple and $P$ is not simple only if $P_{w_h t}^{h+1}$ intersects $\pi$.

3.2 The SB algorithm

We are now ready to present the SB algorithm, whose pseudocode is presented in Algorithm 1. The main idea of the algorithm is to postpone the computation of the detours. That is, the in-branchings $T_{h+1}$ are computed only if needed. The subtlety is to know when to extract a path (without knowing its actual weight if $T_{h+1}$ has not been computed yet) from the set $\text{Candidate}$ (where the paths are encoded using the format presented above). For this purpose, a lower bound on the distance from $w_h$ to $t$ is used to get a lower bound on the weight of the path.

Roughly, the SB algorithm uses a set $\text{Candidate}$ to manage candidate paths that are encoded using the above data structure. Sequentially, it extracts a shortest element $\varepsilon$ from $\text{Candidate}$. If $\varepsilon$ represents a simple path, this path is added to the output and the representations of its detours (which will be found using the last tree in the representation of $\varepsilon$) are added to $\text{Candidate}$. Otherwise, the SB algorithm attempts to modify $\varepsilon$ by instantiating its last in-branching (see below). If this computation leads to a representation of a simple path, then it is added to $\text{Candidate}$. Otherwise, $\varepsilon$ is discarded. The SB algorithm goes on iteratively until it has found $k$ paths. The initialization consists in computing a first in-branching $T_0$ rooted at $t$ in $D$ (using
an SP algorithm) and so a shortest s-t (simple) path $P_{st}^{T_0}$ and adding its representation to Candidate.

More precisely, the set Candidate is a min-heap in which the weight (the key) of an element is a lower bound on the weight of the path it represents. Each element $\mu$ in Candidate has the form $\mu = (\varepsilon = (T_0, e_0, \ldots, e_h = (v_h, w_h), T_{h+1}), lb, \zeta)$ where each in-branching $T_{h'}$ (with $h' < h$) is already computed and $lb$ is a lower bound of the weight of the path represented by $\varepsilon$. The value $\zeta$ is a boolean indicating whether the path represented by $\varepsilon$ is known to be simple. If so, it will follow from the construction that $T_{h+1}$ has already been computed. Else $T_{h+1}$ must be first computed to know if $\varepsilon$ represents a simple path. For the initialization, the in-branching $T_0$ is computed and the element $(T_0, \omega(P_{st}^{T_0}), \zeta = 1)$ is inserted in Candidate.

The SB algorithm iteratively extracts elements from Candidate by minimum weight (with a priority to representation of simple paths to break ties) until $k$ paths are obtained or Candidate is empty. When an element $\mu = (\varepsilon = (T_0, e_0, \ldots, e_h = (v_h, w_h), T_{h+1}), lb, \zeta)$ is extracted from Candidate, two cases are distinguished. Let $i$ be the index of $w_h$ in the path $P$ represented by $\varepsilon$ (note that $i$ plays the same role as the deviation-index in Yen’s algorithm).

Case $\zeta = 1$. Then, $\varepsilon$ represents a simple path $P = (v_0 = s, \ldots, v_i = w_h, \ldots, v_r = t)$ and all of its in-branchings have already been computed. In this case, the path $P$ is added to the output. Then, the algorithm considers all deviations of $P$ starting from a vertex $v_j$ of the suffix $(v_i, \ldots, v_r = t)$ of $P$. More precisely, for every arc $e = (v_j, w)$, with $i \leq j < r$ and $w \not\in \{v_0 = s, v_1, \ldots, v_i, v_{j+1}\}$, let $P_{wt}^{T_{h+1}}$ be a shortest path from $w$ to $t$ in $T_{h+1}$ (if any) and let $Q(v_j, e) = (v_0, \ldots, v_j, P_{wt}^{T_{h+1}})$. If $Q(v_j, e)$ is simple, the representation $\mu' = ((T_0, e_0, \ldots, e_h, T_{h+1}, e = (v_j, w), T_{h+1}), lb(e), \zeta = 1)$ is added to Candidate with $lb(e) = \omega(Q(v_j, e))$ as a key (note that the computation of $lb(e)$ is done in constant time since, in particular, $T_{h+1}$ is already computed). Otherwise, i.e., $Q(v_j, e)$ is not simple, the representation $\mu'' = (\varepsilon' = (T_0, e_0, \ldots, e_h, T_{h+1}, e = (v_j, w), T'), lb(e), \zeta = 0)$ is added to Candidate, where $T'$ is the name of the in-branching of $D \setminus \{v_0, \ldots, v_j\}$ whose actual computation is postponed, and $lb(e) = \omega(Q(v_j, e))$ is a lower bound on the weight of the path represented by $\varepsilon''$.

Case $\zeta = 0$. In this case, the algorithm checks for the existence of a $w_h$-t path $P_{w_h t}^{T_{h+1}}$. To do so, the in-branching $T_{h+1}$ (whose computation had been postponed) is computed. Note that $T_{h+1}$ is an in-branching in $D \setminus \{v_0, \ldots, v_h\}$, which ensures that, if $P_{w_h t}^{T_{h+1}}$ is found, the path $P_{new} = (s = v_0, \ldots, v_h, P_{w_h t}^{T_{h+1}})$ is guaranteed to be simple. Moreover, $P_{new}$ has weight $\omega(P_{new}) = \omega((s = v_0, \ldots, v_h, w_h)) + \omega(P_{w_h t}^{T_{h+1}})$. Then, the representation $\mu' = (\varepsilon' = (T_0, e_0, \ldots, e_h = (v_h, w_h), T_{h+1}), \omega(P_{new}), \zeta = 1)$ is added to Candidate. Finally, if no $w_h$-t path can be found in $T_{h+1}$, $\mu$ is discarded.

Analysis. Let us first analyze the time complexity of the operations performed after the extraction of an element $\mu = (\varepsilon = (T_0, e_0, \ldots, e_h = (v_h, w_h), T_{h+1}), lb, \zeta)$ from Candidate. When $\zeta = 0$, the time complexity is dominated by the computation of the in-branching $T_{h+1}$, which is done in time $O(m + n \log n)$ using an SP algorithm. When $\zeta = 1$, the algorithm first extracts the path $P = (s = v_0, v_1, \ldots, v_i = w_h, \ldots, v_r = t)$ from the representation $\varepsilon$ in time $O(n)$. Then, it associates a label $\lambda(v)$ to every vertex $v \in V$ as follows. First, every vertex $v_j$ (for $i \leq j \leq r$) of $P$ is labelled with $j$. Then, each vertex $u \in V(T_{h+1})$ such that $v_j$ is the first vertex of $P$ reached from $u$ in $P_{st}^{T_{h+1}}$ is labelled with $j$. Finally, the vertices not in $V(T_{h+1}) \cup \{v_0, \ldots, v_i-1\}$ are labelled with 0 (these are the vertices for which all shortest paths to $t$ go through $(v_0, \ldots, v_i-1)$). This labeling operation can be done in overall time $O(n)$ using
Algorithm 1 Sidetrack Based (SB) algorithm for the $k$SSP

Require: A digraph $D = (V, A)$, a source $s \in V$, a sink $t \in V$, and an integer $k$

Ensure: $k$ shortest simple $s$-$t$ paths

1: Let $\text{Candidate} \leftarrow \emptyset$ and $\text{Output} \leftarrow \emptyset$
2: $T_0 \leftarrow$ an SP in-branching of $D$ rooted at $t$ containing $s$
3: Add $((T_0), \omega(P_{s,t}(T_0)), \zeta = 1)$ to $\text{Candidate}$
4: while $\text{Candidate} \neq \emptyset$ and $|\text{Output}| < k$ do
5: $\mu = (\varepsilon = ((T_0, e_0, \cdots, T_h, e_h = (u_h, v_h), T_{h+1}), lb, \zeta)) \leftarrow$ a shortest element in $\text{Candidate}$
6: if $\zeta = 1$ then
7: Extract $\mu$ from $\text{Candidate}$ and add $\varepsilon$ to $\text{Output}$
8: for every deviation $e = v_jv' \in P_{v_i, t}^{T_{h+1}}$ do
9: $\varepsilon' \leftarrow (T_0, e_0, \cdots, T_h, e_h, T_{h+1}, e, T_{h+1})$
10: $lb' \leftarrow lb - \omega(P_{v_j, t}^{T_{h+1}}) + \omega(e) + \omega(P_{v_i, t}^{T_{h+1}})$
11: if $\varepsilon'$ represents a simple path then
12: Add $\mu' = (\varepsilon', lb', \zeta = 1)$ to $\text{Candidate}$
13: else
14: $T' \leftarrow$ the name of an SP in-branching of $D_j(P)$ // $T'$ is not computed yet
15: Add $\mu'' = (\varepsilon'' = (T_0, e_0, \cdots, T_h, e_h, T_{h+1}, e, T'), lb', \zeta = 0)$ to $\text{Candidate}$
16: end
17: if $T_{h+1}$ has not been computed yet then
18: Compute $T_{h+1}$, an SP in-branching of $D_h(P)$
19: Let $\mu' = (\varepsilon' = (T_0, e_0, \cdots, T_h, e_h, T_{h+1}), lb + \omega(P_{v_i, t}^{T_{h+1}}) - \omega(P_{v_i, t}^{T_{h}}), \zeta = 1)$
20: Add $\mu'$ to $\text{Candidate}$
21: return $\text{Output}$

DFS on $T_{h+1}$ to propagate the labels from the vertices $v_i, \cdots, v_r$. Now, using the label $\lambda(u)$ associated to the vertex $u$, it is possible to check in constant time if the path $Q(v_j, e)$, with $i \leq j < r$ and $e = (v_j, w)$, is simple. Indeed, if $\lambda(w) \leq j$, the path $P_{v_i, t}^{T_{h+1}}$ intersects the subpath $(v_0, \cdots, v_j)$ of $P$ and so $Q(v_j, e)$ is not simple. Otherwise, the path $Q(v_j, e)$ is simple. Altogether, the operations performed when $\zeta = 1$ can be done in time $O(n + m)$.

In the worst case scenario, each first extraction of a path from $\text{Candidate}$ leads to a non-simple detour, and then a call of an SP algorithm. Note that no more than one call to an SP algorithm is done per vertex on a path, thanks to the test of line 17 of Algorithm 1. So, the complexity of the SB algorithm is bounded by $O(kn(m + n \log n))$ as the number of vertices of a simple path is bounded by $n$ and the algorithm stops once $k$ paths have been added to $\text{Output}$.

There are two key improvements for which the SB algorithm has, in practice, out-performed all other algorithms for solving the $k$SSP problem so far. First, it saves an SP algorithm call if the detour is simple. Second, if the detour is not simple it is inserted with a lower bound on its weight and the corresponding call to an SP algorithm is postponed. This way, if this detour leads to a long path (path with weight larger than the $k^{th}$ shortest path), the call to an SP algorithm will never be performed.

However, as the SB algorithm stores complete in-branchings in memory, it has the drawback of possibly consuming a large amount of memory (much more than the NC algorithm, which stores a single in-branching, while keeping the whole description of the paths it computes).
3.3 The SB* algorithm

Here, we propose the SB* algorithm, a variant of the SB algorithm that is a tiny modification of the SB algorithm but leading to a significant speed up (see Section 6.2).

In fact, each time a representation \((T_0, e_0, T_1, \cdots, e_{h-1}, T_h, e_h = (u_h, v_h), T_{h+1})\) is extracted from Candidate with \(\zeta = 0\) and \(T_{h+1}\) has not been computed yet (i.e., it is only a name), our algorithm does not compute \(T_{h+1}\) from scratch as the SB algorithm does. Instead, the SB* algorithm creates a copy \(T_h^{\prime}\) of \(T_h\), discards vertices of the path from \(v_{h-1}\) to \(u_h\) in \(T_h\), and updates the SP in-branching \(T\) using standard methods for updating a shortest path tree [13]. Then, the name \(T_{h+1}\) is associated with the new in-branching \(T\).

It is clear that the SB* algorithm computes (and stores) exactly the same number of in-branchings as the SB algorithm. The computational results presented in Section 6.2 show that this update procedure gives an average speed-up factor of 1.1 to 1.2 on road networks.

4 Space - time trade-offs

4.1 The Parsimonious Sidetrack Based algorithm

Here, we present the Parsimonious Sidetrack Based (PSB) algorithm, which is an adaptation of the SB algorithm allowing to reduce the memory consumption due to the storage of all in-branchings computed by the SB algorithm. Here, we only focus on the differences between the SB and the PSB algorithm.

The main difference is that the PSB algorithm stores two types of elements in Candidate. The first type, of the form \((\varepsilon = (T_0, e_0, T_1, e_1, \cdots, T_h, e_h, T_{h+1}), lb)\), represents a simple \(s\)-\(t\) path \(P\) of weight \(lb\). Contrary to the SB algorithm, the in-branching \(T_{h+1}\) has not necessarily been computed yet. The second type, of the form \((\varepsilon, Dev, lb)\), contains an extra field \(Dev\) (explained below) and, in this case, all of the in-branchings \(T_1, \cdots, T_{h+1}\) are already computed.

Let us start by considering a step of the PSB algorithm when an element \(\mu = (\varepsilon = (T_0, e_0, T_1, e_1, \cdots, T_h, e_h, T_{h+1}), lb)\) representing a simple path \(P\) is extracted from Candidate. \(T_{h+1}\) is computed at this step (if not already done) which allows to get \(P\) explicitly. Then, the PSB algorithm adds \(P = (s = v_0, \cdots, v_i = w_h, \cdots, v_r = t)\) to Output and (as for the SB algorithm), for every \(v \in \{v_i, \cdots, v_r\}\), and every deviation \(e\) with tail \(v\), the detour \(Q(v, e)\) of \(P\) at \(e\) is considered. If \(Q(v, e)\) is simple (this test is done as explained in the first paragraph of the analysis in Section 3.2), then \(\mu^\prime = ((T_0, e_0, T_1, e_1, \cdots, T_h, e_h, T_{h+1}, e, T_{h+1}), \omega(Q(v, e)))\) is added to Candidate. Otherwise, the deviation \(e\) is added to a list \(Dev\) (initially empty). Once all deviations have been considered, the (unique) element \((\varepsilon, Dev, lb^\prime)\) is added to Candidate, where \(lb^\prime = \min_{f_j = (u_j, u_j^\prime) \in Dev} \omega(Q(u_j, f_j))\). That is, \(Dev\) is the list of all “non-simple deviations” of \(P\) at the vertices between \(w_h\) and \(t\), ordered with respect to the index of their tail on \(P\), i.e., for two deviations \(f_i = (u_i, u_i^\prime), f_j = (u_j, u_j^\prime) \in Dev\), \(f_i \leq f_j\) if and only if \(i \leq j\). Finally, let \(lb^\prime\) be a lower bound on the weight of the detours at a deviation in \(Dev\). The important difference between the SB and the PSB algorithms comes from the fact that non-simple detours are considered as a unique object by the PSB algorithm.

Now, let us consider a step when the PSB algorithm extracts an element \(\mu = (\varepsilon = (T_0, e_0, T_1, e_1, \cdots, T_h, e_h, T_{h+1}), Dev = \{f_1, \cdots, f_j = (u_j, u_j^\prime), \cdots, f_l\}, lb)\) from Candidate. As mentioned above, in this case, \(\varepsilon\) encodes a simple \(s\)-\(t\) path \((v_0, \cdots, v_r)\). Let \(1 \leq \min \leq l\) be the smallest integer such that \(lb = \omega(Q(v_{\min}, f_{\min}))\).

Then, the PSB algorithm proceeds as follows. For \(j\) decreasing from \(l\) to \(\min\), an in-
branching $T_j$ in $D \setminus \{v_0, \ldots, v_{i_j} = u_j\}$ is computed (but not stored!) until a path $P_{u_j, t}^{T_j}$ from $u_j$ to $t$ is discovered (if no such path exists, $j$ is decreased by one). If $P_{u_j, t}^{T_j}$ exists, then $\varepsilon_j = (T_0, e_0, T_1, e_1, \ldots, T_h, e_h, T_{h+1}, f_j, T_j)$ represents a simple $s$-$t$ path of weight $lb_j = \omega((v_0, \ldots,v_i)) + \omega(f_j) + \omega(P_{u_j, t}^{T_j})$. Then, the element $\mu_j = (\varepsilon_j, lb_j)$ is added to Candidate, but $T_j$ is not stored (the PSB algorithm might have to recompute it later). A second key improvement is that to speed up the computation, $T_j$ is actually computed by updating $T_{j+1}^{T'}$, which is done using standard tools from [13]. Then, only when $j = min$, the in-branching $T_{min}^{T'}$ is stored and $\mu_{min} = (\varepsilon_{min}, lb_{min})$ is added to Candidate. The reason why $T_{min}^{T'}$ is stored (while other $T_j$ are not) is that $\mu_{min}$ is expected to be extracted soon from Candidate (because the path represented by $\varepsilon_{min}$ is expected to be short) and we want to avoid the recomputation of $T_{min}^{T'}$. Finally, the element $\mu' = (\varepsilon, Dev' = \{f_1, \ldots, f_{min-1}\}, lb')$ is added to Candidate, where $lb'$ is the minimum weight over the non-simple detours in Dev'.

The correctness follows from that of the SB algorithm. Moreover, since most of the computed in-branchings are not stored, the memory used by the PSB algorithm is significantly smaller than for the SB algorithm.

### 4.2 Special variants of the PSB algorithm

A better space and time trade-off than the PSB algorithm can be achieved if each computed and stored in-branching is going to be used in the future steps, i.e., it is used to extract a simple candidate path that is going to be extracted from Candidate (before the $k$th shortest path). Unfortunately, such information cannot be afforded as the weight of the $k$th path is not previously known. However, if computing an in-branching leads to constructing a path with weight relatively short, e.g., less than a threshold value times the weight of the currently considered path, then storing such in-branching is meaningful as the extraction of its corresponding element form Candidate is expected soon and storing it leads to save its redundant computation. Here, we present two variants of the PSB algorithm: PSBv2 and PSBv3. The PSBv2 algorithm is a tiny improvement of the PSB algorithm leading to consume less memory by storing less in-branching while the PSBv3 algorithm gives an adaptable trade-off depending on the value of the threshold.

Let us consider, again, a step when the PSB algorithm extracts an element $\mu = (\varepsilon, Dev = \{f_1, \ldots, f_j = (u_j, u'_j), \ldots, f_i\}, lb)$ from Candidate with $1 \leq min \leq l$ the smallest integer such that $lb = \omega(Q(\mu))$. The PSB algorithm algorithm iterates on $j$, decreasing from $l$ to $min$ as explained above, a corresponding in-branching $T_j'$ is computed for each $j$ (but not stored). Then, only when $j = min$, the in-branching $T_{min}^{T'}$ is stored.

The PSBv2 algorithm does not naively store $T_{min}^{T'}$. Instead, $T_{min}^{T'}$ is stored only if the weight of its corresponding detour is less than a threshold value $\theta$ times the weight of the shortest simple path in Candidate. That is, $lb_{min} = \omega((v_0, \ldots, v_{min})) + \omega(P_{u_{min}, t}^{T_{min}}) \leq \theta \omega(P_{next})$, where $P_{next}$ is the shortest simple path in Candidate. As a result, if the in-branching $T_{min}^{T'}$ leads to a (relatively) long path that is not expected to be extracted very soon from Candidate, then it is freed from the memory.

The PSBv3 algorithm behaves on every deviation in Dev between min and l the same way the PSBv2 algorithm behaves with $f_{min}$. Precisely, for each deviation $f_j$ with $min \leq j \leq l$, the PSBv3 algorithm computes its corresponding in-branching $T_j$. This in-branching $(T_j')$ is stored only if the weight of its corresponding detour is less than a threshold value $\theta$ times the weight of the shortest simple path in Candidate.

The value of the threshold $\theta$ could change dynamically during the execution. For instance,
it could be related to the ratio between the two upcoming paths, i.e., the two elements in Candidate with minimum weight.

5 Postponing the detours’ computation

In this section, we present the Postponed Node Classification (PNC) algorithm. The PNC algorithm has time complexity $O(kn(m + n \log n))$ with a memory consumption similar to the one of the NC algorithm. However, it is faster in practice than the NC algorithm.

Even though the NC algorithm consumes less time during each SP algorithm call than Yen’s algorithm, the total number of calls remains equal to Yen’s algorithm. Here, with the help of a lower bound on the weight of a simple detour, we propose (using an idea similar to the one of the SB algorithm) to postpone the calls in order to avoid some of them. We prove that such postponement does not hurt the correctness of the algorithm.

Let us describe our algorithm PNC.

As in the NC algorithm, our algorithm starts by computing an SP in-branching $T$ rooted at $t$ that will be used throughout the execution of the algorithm. Then, like the NC algorithm, the postponement does not hurt the correctness of the algorithm. Our algorithm differs from the NC algorithm in how and when it computes the detours of the paths but also in the structure of the elements in the heap Candidate.

Let us consider a phase when a $s$-$t$ path $P = (s = u_0, u_1, \cdots, u_i, \cdots, u_r = t)$ is extracted from Candidate. Let $0 \leq i < r$ and consider the step when a shortest simple detour of $P$ at $u_i$ is computed. Let $N$ be the set of neighbors $v$ of $u_i$ such that the paths with prefix $(u_0, \cdots, u_i, v)$ have already been added to Output. Let $D' = D \setminus \{(u_i, v) \mid v \in N\}$ and let $D_i(P) = D' \setminus (u_0, \cdots, u_{i-1})$.

Let us describe how the PNC algorithm finds a new shortest simple detour $P'$ of $P$ at $u_i$. Recall that a detour $P'$ is said new if $P'$ has not been added to the Output yet. Let $v_{LB} \in D_i(P)$ be a neighbor of $u_i$ (neither in $N$ nor in the prefix of $P$) such that the residual weight of $(u_i, v_{LB})$ is minimum, i.e., $\delta(u_i, v_{LB}) \leq \delta(u_i, v')$ for every $v' \in N^+_D(P)(u_i)$ (recall that $\delta(u, v) = \omega(u, v) + \omega(P^T_{vt}) - \omega(P^T_{ut})$ denotes the residual weight of arc $uv$ as defined in Section 2.2). Note that, by definition of the residual weight, the path $P_{LB} = (s, u_1, \cdots, u_i, P^T_{v_{LB}t})$ is a shortest new detour (not necessarily simple) of $P$ at $u_i$, so in particular:

Claim 1. $\omega(P_{LB}) \leq \omega(P')$ for any new simple detour $P'$ of $P$ at $u_i$

Similarly to the SB algorithm (and in contrast to the NC algorithm), the PNC algorithm may add non-simple paths to the set Candidate. Precisely, each element in Candidate has the form $(P = (s = u_0, \cdots, u_r = t), \omega(P), i, \zeta)$ where $i$ is its deviation index and $\zeta$ is a boolean flag indicating whether the path $P$ is simple or not.

The main idea of the PNC algorithm (Algorithm 2) is the following. Instead of computing naively all of the shortest simple detours of $P$, i.e., a shortest simple detour at $u_j$ for all $i \leq j < r$, the following procedure is used. We first label the vertices of the graph with respect to $P$ using the procedure presented in Section 3.2 for testing whether a deviation $Q(v_j, c)$ is simple or not. Then, for each vertex $u_j \in P$, with $i \leq j < r$, we find the neighbor $v_{LB}$ of $u_j$ (if any) such that the residual weight of $(u_j, v_{LB})$ is minimum and we check whether $\lambda(v_{LB}) > j$ or not. If this is the case, the detour $P_{LB} = (s, u_1, \cdots, u_j, P^T_{v_{LB}t})$ is simple and we add $(P_{LB}, \omega(P_{LB}), j, \zeta = 1)$ to the set Candidate. Otherwise, i.e., if $\lambda(v_{LB}) \leq j$, the detour $P_{LB}$ is added to the set Candidate (even though it is not simple) with its weight $\omega(P_{LB})$ as a key, i.e., the element $(P_{LB}, \omega(P_{LB}), j, \zeta = 0)$ is added to Candidate. The idea is that, in the
Algorithm 2 Postponed Node Classification (PNC)

1: **Input** A digraph \( D = (V, A) \), source \( s \in V \), sink \( t \in V \) and an integer \( k \)
2: **Output** \( k \) shortest simple \( s \)-\( t \) paths
3: Let \( \text{Candidate} \leftarrow \emptyset \) and \( \text{Output} \leftarrow \emptyset \)
4: \( T \leftarrow \) an SP in-branching of \( D \) rooted at \( t \)
5: Add \((P_{st}(T), \omega(P_{st}(T)), 0, 1)\) to \( \text{Candidate} \)
6: while \( \text{Candidate} \neq \emptyset \) and \( |\text{Output}| < k \) do
7: \((P = (s, u_1, \cdots, t), \omega(P), i, \zeta)\) \leftarrow\text{extract a shortest element from}\ \text{Candidate}
8: \( \pi \leftarrow (s, u_1, \cdots, u_{i-1}) \)
9: \( D_{\text{Devold}} = \{e = (u_i, v) \text{ s.t. there is a path in } \text{Output} \text{ having } \pi.e \text{ as prefix}\} \)
10: if \( \zeta = 1 \) (\( P \) is simple) then
11: Add \( P \) to \( \text{Output} \)
12: \( \lambda \leftarrow \text{labelling of the vertices of } D \text{ with respect to } P \text{ and } T \)
13: for each vertex \( u_j \) in \( (u_i, \cdots, t) \) do
14: \((u_j, v_{LB}) \leftarrow \) an arc in \( A \setminus D_{\text{Devold}} \) with minimum \( \delta \) among those with tail \( u_j \)
15: \( P_{LB} \leftarrow (s, \cdots, u_j, v_{LB}, \pi_{v_{LB}}) \)
16: \( \zeta' \leftarrow 0 \)
17: if \( \lambda(v_{LB}) > j \) (i.e., \( P_{LB} \) is simple) then
18: \( \zeta' \leftarrow 1 \)
19: Add \((P_{LB}, \omega(P_{LB}), j, \zeta')\) to \( \text{Candidate} \)
20: else
21: Compute a shortest \( u_i \)-\( t \) path \( Q \) in \( D' = (V \setminus \pi, A \setminus D_{\text{Devold}}) \)
22: if \( Q \) exists then
23: Add \((P' = \pi.Q, \omega(P'), i, 1)\) to \( \text{Candidate} \)
24: return \( \text{Output} \)

latter case, the non-simple path added to \( \text{Candidate} \) may never be extracted from \( \text{Candidate} \), in which case a call to an SP algorithm is saved.

When an element \((P, \omega(P), i, \zeta)\) is extracted from \( \text{Candidate} \). If \( \zeta = 1 \), the simple path \( P \) is added to the \( \text{Output} \) and its detours are added to \( \text{Candidate} \) as explained above. Otherwise, i.e., \( P \) is not simple, it will be “repaired” into a simple path and re-added to \( \text{Candidate} \). More precisely, after the extraction of \( P \) from \( \text{Candidate} \), an SP algorithm is called to find a shortest (simple) path \( Q \) from \( u_i \) to \( t \) in \( D_i(P) \) and \( P \) is replaced by \( P' = (s, u_1, \cdots, u_{i-1}, Q) \). Claim [I] ensures that the order of extraction of the simple paths from \( \text{Candidate} \) remains valid. And finally, such postponement of this SP algorithm call may end up by skipping it.

6 Experimental evaluation

In this section we describe our experimental evaluation of the algorithms presented in this paper. We start by describing our implementation and experimental settings (Section 6.1). Then, we discuss our experimental results on road in Section 6.2 and present in Section 6.3 a refined comparison of the PNC and the SB* algorithms on road networks with respect to the Dijkstra rank of the queries. We finally discuss our experimental results on complex networks in Section 6.4.
6.1 Experimental settings

Here, we specify the details of the implementation and the setting used in our experiments.

We have implemented all the algorithms presented in this paper (Yen, NC [11], PNC, SB [23], SB* and PSB) in C++ and our code is publicly available [1].

Following [23], we have implemented a pairing heap data structure [12] supporting the decrease key operation, and we use it for Dijkstra’s shortest path algorithm. Our implementation of the Dijkstra’s shortest path tree algorithm is lazy. That is, it stops the computation as soon as the distance from query node \( v \) to \( t \) is proved to be the smallest one. Further computations might be performed later for another node \( v' \) at a larger distance from \( t \) starting from this partial shortest path tree already computed. Our implementation of Dijkstra’s algorithm supports an update operation when a node or an arc is added to the graph. Moreover, we have implemented a special copy operation that updates the in-branching when a set of nodes is removed from the graph. This corresponds to the operations performed when creating an in-branching \( T_{h+1} \) from \( T_h \) in the SB* algorithm.

Observe that, in our implementations, the parameter \( k \) is not part of the input, and so the sets of candidates are simply implemented using pairing heaps. This choice enables us to use these methods as generators able to return the next shortest path as long as one exists. Note that, if \( k \) were part of the input, the data structure used to store the candidates could be changed in order to contain only the \( k \) best candidates, but the algorithm would only be able to return \( k \) paths even if more exist. Moreover, for the SB, SB*, PSB, PSBv2 and PSBv3 algorithms, following [23], we store the candidates in two heaps. The first heap stores the simple candidates (`Candidate\text{simple}`) and the second one stores the non-simple candidates (`Candidate\text{not-simple}`).

Then, we extract candidates from `Candidate\text{simple}` as long as the weight of a shortest simple path in `Candidate\text{simple}` is smaller or equal than the weight of a shortest non-simple path in `Candidate\text{not-simple}`. This way, we prioritize the extraction of simple paths.

Concerning the PSBv2 and PSBv3 algorithms, based on preliminary experiments, we choose to update the value of the threshold \( \theta \) dynamically as follows. Recall that, when looking for the \( i \)th path, a corresponding in-branching will be stored only if the weight of that path is at most \( \theta \) times the weight of the \((i-1)\)th path. Precisely, let \( \ell_s \) be the weight of a smallest element in `Candidate\text{simple}` and let \( \ell_{ns} \) be the weight of a smallest non-simple element in `Candidate\text{not-simple}`. If at least one of these sets is empty, \( \theta \) keeps its previous value. Otherwise, let \( c = \max(\frac{\ell_s}{\theta}, \frac{\ell_{ns}}{\theta}) \). The value of \( \theta \) is set to \( 1 + \alpha (c-1) \), for some constant \( \alpha > 0 \). The intuition is to store an in-branching only if it is expected to be used soon, that is, while extracting one of the upcoming paths. Observe that in our experiments we have set the factor \( \alpha \) in the formula for computing \( \theta \) to \( \alpha = 11 \), based on preliminary experiments.

**Test instances** We have evaluated the performances of our algorithms on some road networks from the 9th DIMACS implementation challenge [7] and on several complex networks.

A road network of a city is the digraph modeling its roads, i.e., a vertex is associated to each crossroad, and there is an arc of weight \( w \) between two vertices if and only if there is a road of physical length \( w \) (in km) between their corresponding crossroads. Road networks are known to be sparse, almost planar and to have a bounded degree [34]. The road networks ROME, DC and DE are referred to as small road networks, while the road networks NY, BAY and COL are referred to as big ones. The characteristics of these graphs are reported in Table 1.

On the other hand, complex networks model different types of networks. Generally, they are characterized by being small-world, i.e., they have a logarithmic diameter, a power-law degree distribution and a high clustering coefficient [5]. For instance, the BIOGRID network...
In this section, we describe and analyze our experimental results on road networks.

In Table 2 we have reported the average and the median of the algorithms’ running times in all considered road networks when \( k = 1000 \), and the number of stored in-branchings is reported represents mutation/deletion of genes resulting in lethality when combined in a same cell. The DIP network represents protein to protein interactions. The FB network represents social circles from Facebook. Likewise, the LOC network is a graph provided from Brightkite location-based social networking service provider.

### Table 1: Characteristics of the graphs used in kSSP experiments

<table>
<thead>
<tr>
<th>Network</th>
<th>( n )</th>
<th>( m )</th>
<th>( D )</th>
<th>( \langle d \rangle )</th>
<th>(-\alpha)</th>
<th>( \langle cc\rangle)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROME</td>
<td>3353</td>
<td>8870</td>
<td>57</td>
<td>5.2</td>
<td>-0.023</td>
<td></td>
<td>Road network of Rome</td>
</tr>
<tr>
<td>DC</td>
<td>9559</td>
<td>29682</td>
<td>140</td>
<td>6.2</td>
<td>-0.039</td>
<td></td>
<td>Road network of Washington DC</td>
</tr>
<tr>
<td>DE</td>
<td>49108</td>
<td>110520</td>
<td>573</td>
<td>4.8</td>
<td>-0.024</td>
<td></td>
<td>Road network of Delaware</td>
</tr>
<tr>
<td>NY</td>
<td>264346</td>
<td>735846</td>
<td>664</td>
<td>5.5</td>
<td>-0.02</td>
<td></td>
<td>Road network of New York</td>
</tr>
<tr>
<td>BAY</td>
<td>321270</td>
<td>800172</td>
<td>791</td>
<td>4.9</td>
<td>-0.016</td>
<td></td>
<td>Road network of San Francisco Bay area</td>
</tr>
<tr>
<td>COL</td>
<td>435606</td>
<td>1507066</td>
<td>1219</td>
<td>4.8</td>
<td>-0.017</td>
<td></td>
<td>Road network of Colorado area</td>
</tr>
<tr>
<td>BIOGRID</td>
<td>2318</td>
<td>12580</td>
<td>7</td>
<td>21.7</td>
<td>1.96</td>
<td>0.20</td>
<td>Mutation/deletion of genes resulting in cell lethality</td>
</tr>
<tr>
<td>FB</td>
<td>3698</td>
<td>85963</td>
<td>6</td>
<td>93.0</td>
<td>-0.61</td>
<td></td>
<td>Social circles from Facebook</td>
</tr>
<tr>
<td>P2P</td>
<td>5606</td>
<td>23510</td>
<td>8</td>
<td>16.7</td>
<td>-0.014</td>
<td></td>
<td>Peer-to-peer network of the Gnutella file sharing network</td>
</tr>
<tr>
<td>DIP</td>
<td>13969</td>
<td>60621</td>
<td>17</td>
<td>17.4</td>
<td>2.38</td>
<td>0.11</td>
<td>Protein-protein interaction network</td>
</tr>
<tr>
<td>CAIDA</td>
<td>29432</td>
<td>145000</td>
<td>9</td>
<td>19.1</td>
<td>2.06</td>
<td>0.42</td>
<td>Relationships between Autonomous Systems of the Internet</td>
</tr>
<tr>
<td>LOC</td>
<td>33187</td>
<td>188577</td>
<td>11</td>
<td>22.7</td>
<td>2.25</td>
<td>0.29</td>
<td>Brightkite location-based social networking service provider</td>
</tr>
</tbody>
</table>

For the complex networks, these statistics refer to the largest connected component.
<table>
<thead>
<tr>
<th></th>
<th>Rome</th>
<th>DC</th>
<th>DE</th>
<th>NY</th>
<th>BAY</th>
<th>COL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Y</strong></td>
<td>avg</td>
<td>1 053</td>
<td>4 470</td>
<td>73 727</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>388</td>
<td>423</td>
<td>9 434</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>16 382</td>
<td>67 434</td>
<td>1 611 504</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>NC</strong></td>
<td>avg</td>
<td>229</td>
<td>471</td>
<td>5 565</td>
<td>34 915</td>
<td>45 447</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>112</td>
<td>220</td>
<td>3 886</td>
<td>14 054</td>
<td>24 854</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>1 310</td>
<td>2 387</td>
<td>16 978</td>
<td>137 266</td>
<td>136 564</td>
</tr>
<tr>
<td><strong>PNC</strong></td>
<td>avg</td>
<td>120</td>
<td>238</td>
<td>1 924</td>
<td>12 699</td>
<td>16 082</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>89</td>
<td>165</td>
<td>1 216</td>
<td>6 677</td>
<td>9 260</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>690</td>
<td>2 011</td>
<td>10 251</td>
<td>91 333</td>
<td>108 320</td>
</tr>
<tr>
<td><strong>SB</strong></td>
<td>avg</td>
<td>339</td>
<td>325</td>
<td>8 574</td>
<td>43 394</td>
<td>79 667</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>286</td>
<td>166</td>
<td>3 808</td>
<td>24 006</td>
<td>36 014</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>1 324</td>
<td>2 011</td>
<td>74 172</td>
<td>137 266</td>
<td>136 564</td>
</tr>
<tr>
<td><strong>SB</strong>*</td>
<td>avg</td>
<td>269</td>
<td>298</td>
<td>7 208</td>
<td>36 433</td>
<td>68 001</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>218</td>
<td>150</td>
<td>3 381</td>
<td>18 290</td>
<td>31 930</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>1 275</td>
<td>2 011</td>
<td>58 293</td>
<td>663 995</td>
<td>1 453 920</td>
</tr>
<tr>
<td><strong>PSB</strong></td>
<td>avg</td>
<td>294</td>
<td>322</td>
<td>4 956</td>
<td>43 182</td>
<td>44 426</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>242</td>
<td>219</td>
<td>3 932</td>
<td>34 868</td>
<td>35 275</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>1 118</td>
<td>2 645</td>
<td>21 282</td>
<td>257 135</td>
<td>273 211</td>
</tr>
<tr>
<td><strong>PSBv2</strong></td>
<td>avg</td>
<td>302</td>
<td>332</td>
<td>4 845</td>
<td>42 003</td>
<td>44 426</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>258</td>
<td>230</td>
<td>3 888</td>
<td>34 481</td>
<td>32 966</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>1 134</td>
<td>2 538</td>
<td>22 811</td>
<td>228 794</td>
<td>257 373</td>
</tr>
<tr>
<td><strong>PSBv3</strong></td>
<td>avg</td>
<td>299</td>
<td>328</td>
<td>4 770</td>
<td>41 724</td>
<td>44 088</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>252</td>
<td>232</td>
<td>3 796</td>
<td>34 022</td>
<td>32 394</td>
</tr>
<tr>
<td></td>
<td>max</td>
<td>1 138</td>
<td>2 572</td>
<td>22 750</td>
<td>229 333</td>
<td>256 810</td>
</tr>
</tbody>
</table>

Table 2: Running time (ms) of the algorithms on road networks, \(k = 1,000\)

Moreover, in Figure 1, we report the evolution of the average and median running times of the algorithms in the COL and DC networks when the number \(k\) of reported paths increases (the results are similar for the other road networks). Finally, Figure 2 presents pairwise comparisons of the running times and number of stored in-branchings for all source-destination pairs on COL network.

<table>
<thead>
<tr>
<th></th>
<th>Rome</th>
<th>DC</th>
<th>DE</th>
<th>NY</th>
<th>BAY</th>
<th>COL</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NC, PNC and PNC</strong>*</td>
<td>avg</td>
<td>1 157</td>
<td>651</td>
<td>1 698</td>
<td>1 727</td>
<td>1 737</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>1 397</td>
<td>1 698</td>
<td>1 550</td>
<td>1 727</td>
<td>1 737</td>
</tr>
<tr>
<td><strong>SB and SB</strong>*</td>
<td>avg</td>
<td>827</td>
<td>447</td>
<td>633</td>
<td>700</td>
<td>629</td>
</tr>
<tr>
<td></td>
<td>med</td>
<td>766</td>
<td>454</td>
<td>578</td>
<td>604</td>
<td>569</td>
</tr>
<tr>
<td><strong>PSBv3</strong></td>
<td>avg</td>
<td>776</td>
<td>470</td>
<td>598</td>
<td>615</td>
<td>580</td>
</tr>
</tbody>
</table>

Table 3: Average number of stored trees using some \(kSSP\) algorithms on road networks, \(k = 1,000\)

We first observe from the results reported in Table 2 and Figures 1a and 1b that all algorithms are faster than Yen’s algorithm on small road networks (the average speed up is between one and two orders of magnitude). Similar experiments described in [11, 23] lead to the same conclusion on big road networks.

Next, the simulation results reported in Table 2 and Figure 1 confirm that the use of shortest
path tree update procedures in the SB* algorithm helps reducing the running time compared to the SB algorithm. More precisely, the average and median running times of the SB* algorithm are significantly smaller than the ones of the SB algorithm on all road networks. Note that, by design, the number of stored in-branchings is the same for both algorithms. Concerning the PSB algorithm, as shown in Table 2 and Figures 2c and 2d, it is faster than the NC algorithm and consumes less memory than the SB algorithm. It is also faster on average than the SB* algorithm on some networks (DE, BAY, COL) but its median running time is always larger than the one of the SB* algorithm. Moreover, from Tables 2 and 3 we observe that the two special variants of the PSB algorithm, namely the PSBv2 and the PSBv3 algorithms, lead to a small improvement of the running time (up to 5% time reduction) but to a significant reduction of the memory consumption (up to 15% memory reduction).

The most important observation from the results reported in Table 2 and Figures 1a and 1b is that the PNC algorithm outperforms all other algorithms on road networks when the number of requested paths is at least 20, except for the median running time on DC. When $k = 1,000$, the average and median running times of the PNC algorithm are between 2 and 5 times smaller than for the other algorithms. This is particularly interesting considering that the memory consumption of the PNC algorithm is very small. The conclusion is however different when the number of requested paths is less than 20. Indeed, we observe that the SB-like algorithms (SB, SB*, PSB, etc.) offer better performances for small values of $k$.

Another important observation from the results reported in Table 2 and Figures 1a and 1b is that, for all algorithms, there is a significant gap between the average and the median running times. To better analyze this observation, Figure 2 shows pairwise comparisons of the running times and number of stored in-branchings for all source-destination pairs on COL network when $k = 1,000$. In these scatter plots, we observe, for all algorithms, huge variations in the running times depending on the source-destination pairs. These variations are sufficient to explain the gap between the average and the median running times. Actually, Table 2 shows that the maximum running time for a source-destination pair can be up to 20 times larger than the average running time of an algorithm and up to 50 times larger than its median running time (e.g., SB on COL). To better understand this behavior, we analyse in Section 6.3 the impact of the Dijkstra rank of the queries on the running time of the algorithms.

To conclude, among the considered algorithms, the PNC and the SB* algorithms are the fastest on road networks. The PNC algorithm has a better average running time and is more stable than the SB* algorithm whose performances are seriously affected by a few outliers. Furthermore, the memory consumption of the PNC algorithm is very small compared to all SB-like algorithms. Hence, the PNC algorithm seems to be the best compromise on road networks.

### 6.3 Impact of the Dijkstra rank of the queries

As noticed in Section 6.2 some algorithms have different behaviors with respect to the structure of the network and the query’s properties. We therefore investigate in this section whether the Dijkstra rank of an $s$-$t$ query may explain the variations of the running time of the algorithms.

To this end, we have reported in Figure 3 the average and median running times of the algorithms on the COL road network for increasing values of the Dijkstra rank of the queries. We have also reported the average numbers of stored trees and Dijkstra calls. Observe also that the colors of the points in the scatter plots of Figure 2 correspond to the Dijkstra ranks of the queries.

We observe in Figures 3a and 3b that the Dijkstra rank of the queries has little impact on the running time of the PNC algorithm. This can also be concluded from the scatter plots of Figures 2a and 2c. On the other hand, we observe for the SB-like algorithms, and in particular for the SB* algorithm, that the running time decreases drastically when the Dijkstra rank is
large. More precisely, we observe almost one order of magnitude reduction on average and up to two orders of magnitude on median. To explain these behaviors, let us compare the operations performed by the PNC and the SB* algorithms after each extraction of a path \( P \) from \( Candidate \).

We start when a simple path \( P = (s = v_1, v_1, \cdots , v_r = t) \) with deviation index \( i \) is extracted. In this case, both algorithms start computing a labeling \( \lambda \) of the vertices that will be used to determine if a considered deviation leads to a simple detour or not. Then, while the SB* algorithm computes all its detours at once, the PNC algorithm considers only one deviation per vertex \( v_j \) with \( i \leq j < r \). More precisely, the SB* algorithm considers all the detours starting from deviation \((v_j, w)\), with \( i \leq j < r \) and \( w \in N^+(v_j) \), while the PNC algorithm considers for vertex \( v_j \) only one detour \((v_j, v_{LB})\) with minimum residual weight, where \( v_{LB} \in N^+(v_j) \). Overall, although the SB* algorithm do slightly more work than the PNC algorithm, the computational cost of the operations performed by both algorithms when a simple path is extracted from \( Candidate \) is similar and cannot be used to explain the differences of behaviors of the algorithms.

Let us now consider the case when a non-simple path \( P \) with deviation index \( i \) is extracted from \( Candidate \). In this case, the PNC algorithm searches for a shortest simple \( v_i\)-\( t \) path \( Q \) (line 21 of Algorithm 2) in the digraph using the residual weights. This is done using Dijkstra’s algorithm to compute an out-branching from \( v_i \) and computations are stopped as soon as \( t \) is extracted. Now, observe that, thanks to the use of the residual weights, after the extraction of a vertex \( v \) from the priority queue of Dijkstra’s algorithm, the next vertex \( v' \) to be extracted is likely to be an out-neighbor of \( v \) such that \( \delta(v, v') = 0 \). In other words, the algorithm tries
Figure 2: Comparison of the running time and the number of stores trees on COL. Each dot corresponds to one pair source/destination ($k = 1,000$).
to reach $t$ as early as possible while keeping the total number of extractions from the priority queue as small as possible. Hence, the running time of the operations performed by the PNC algorithm to “repair” a path is rather independent from the distance between $v_i$ and $t$ and also from the Dijkstra rank of the source-destination pair.

Concerning the SB* algorithm, it has to compute the SP in-branching $T_{h+1}$ of the sequence $\varepsilon = (T_0, e_0, T_1, e_1, \ldots, T_h, e_h, T_{h+1})$, representing the non-simple path $P_i$ until reaching the head $w_h$ of arc $e_h = (v_h, w_h)$. Clearly, the time needed to find the path $P^{T_{h+1}}_{w_h t}$ depends on the Dijkstra rank of $w_h$ in the in-branching $T_{h+1}$, which increases with the distance to $t$. Therefore, when considering a source-destination pair with large Dijkstra rank, the running time of the SB* algorithm will be very small if all computed detours involve deviations that are near $t$ (i.e., with small Dijkstra ranks) and very large if the deviations are far from $t$ (i.e., with large Dijkstra ranks). This explains the huge difference between the median and the average running times of the SB* algorithm for source-destination pairs with large Dijkstra rank observed in Figures 3a and 3b. This also explains the fact that, in the scatter plots of Figure 2c the outliers (i.e., the queries which require much more time for the SB* algorithm than for the PNC algorithm) are mostly of small Dijkstra ranks. To emphasize this latter fact, Table 4 represents the proportion of the queries, for each Dijkstra rank, that the SB* algorithm solves faster than the PNC algorithm (recall that there are exactly 100 queries per rank). Finally, the fact that the PNC algorithm is faster than the SB* algorithm on both average and median running times for queries with small Dijkstra rank is due to the use of the residual weights that speeds up the computation of the deviations.
Dijkstra rank | 2  | 10 | 100 | 1000 | 10000 | 100000 | 421096
---|---|---|---|---|---|---|---
SB* | 22.5 | 12.5 | 11.3 | 23.8 | 35.4 | 50.3 | 86.8
PNC | 77.5 | 87.5 | 88.7 | 76.2 | 64.6 | 49.7 | 13.2

Table 4: Percentage of the number of times one of the SB* or PNC algorithms is faster than the other on COL per Dijkstra ranks \((k = 1000)\)

### 6.4 Complex Networks

Here we analyse and try to explain the behavior of the algorithms on complex networks, except for Yen’s algorithm because of its excessive running time (as already mentioned above).

We have reported in Table 5 the average and the median of the algorithms’ running time in all considered complex networks when \(k = 10000\), and in Table 6 the number of stored in-branchings. Furthermore, we have reported in Figure 4 the evolution of the average and median running times of the algorithms in the CAIDA and LOC networks when the number \(k\) of reported paths increases (the results are similar for the other complex networks).

On complex networks, the PSB algorithm and its two variants PSBv2 and PSBv3 are the fastest \(kSSP\) algorithms among the considered algorithms (Table 5 and Figure 4). Considering the memory consumption, all of the \(kSSP\) algorithm have a small memory consumption even for \(k = 10,000\). Indeed, as shown in Table 6, the number of stored in-branchings does not exceed 59 and can be as small as 3 on FB. It is also shown in Table 5 that the running time of PSB and its two variants (PSBv2 and PSBv3) is similar.

In what follows, we give some qualitative arguments that may explain the fact that the PSB algorithm is the fastest among all considered algorithms on complex networks.

Suppose that \(P\) is a shortest simple path from \(s\) to \(t\). On complex networks, it is likely to have a vertex \(v\) on \(P\) with high degree. Let us study the behavior of the different variants of the PSB algorithm when they compute the detours of \(P\) at \(v\), in contrast with the other algorithms.

First, the Yen’s, NC and PNC algorithms may compute, independently, for each vertex \(v' \in N^+(v)\) neighbor of \(v\) a shortest path from \(v'\) to \(t\) resulting with \(|N^+(v)|\) shortest path algorithm calls to find the shortest simple detours at the neighbors of \(v\). On the other hand, the SB* and the PSB algorithms compute at most one shortest path in-branching \(T\) at \(v\), that works for each neighbor \(v'\) of \(v\). In other words, the SB* and the PSB algorithms are favorable to iterate on \(v\). Moreover, as \(v\) has a high degree, it is supposed that a large number of the neighbors \(v'\) of \(v\) leads to simple candidates, i.e., \(P_{v'^t} \cap (s, \ldots, v) = \emptyset\), where \((s, \ldots, v)\) is the prefix of the current path. So, the number of shortest path in-branchings computed and/or stored using the SB* and the PSB algorithms is expected to be small.

In addition, as the number of hops (i.e., number of edges) of a shortest path is “small” (remember that complex network are small-worlds and so have small diameters), the number of calls of the shortest path in-branching update is expected to be small for the PSB algorithm. As this procedure is faster for the PSB algorithm than for the SB* algorithm and that the number of calls is similar, the PSB algorithm is faster than the SB* algorithm on complex networks. This is not valid on road network because the number of hops of a shortest path may be big and the shortest path in-branching update is called many more times.

To conclude, on complex networks, the PSB algorithm is the fastest among the considered algorithms. It has a reasonable memory consumption and this seems to be related to the structural properties of complex networks.
Table 5: Running time (ms) of the algorithms on Complex networks, \((k = 10,000)\)

<table>
<thead>
<tr>
<th></th>
<th>BIOGRID</th>
<th>FB</th>
<th>P2P</th>
<th>DIP</th>
<th>CAIDA</th>
<th>LOC</th>
</tr>
</thead>
<tbody>
<tr>
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Table 6: Average number of stored trees using some \(kSSP\) algorithms on complex networks, \((k = 10,000)\)

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Figure 4: The running time of the $k$SSP algorithms on complex network with respect to the values of $k$. 

(a) Average running time of LOC
(b) Median running time on LOC
(c) Average running time of CAIDA
(d) Median running time on CAIDA
7 Conclusion

In this paper, we have presented several algorithms for the \(k\)SSP problem. In particular, we have proposed several new algorithms for this problem with the aim of reducing the running time and/or the memory consumption of the algorithms.

Our simulation results show that the best algorithm to be chosen for solving the \(k\)SSP problem depends on the use case. For instance, on the considered complex networks, the PSB algorithm achieves the best results. Indeed, it is the fastest among the considered algorithms, and, similarly to the other algorithms, it has low memory consumption. Besides, on road networks, the PNC algorithm seems very promising. Indeed, it is the fastest (on average, when \(k \geq 20\)) among the considered algorithms and it has a low memory consumption (it stores only one in-branching in memory). Furthermore, it is very stable as its performances are barely influenced by the Dijkstra rank of the source with respect to the destination. However, if we know that the Dijkstra rank of the queries is large compared to the number of vertices of the digraph, and if large memory consumption is allowed, then the SB* algorithm might be a good choice. Moreover, the SB* algorithm is faster than the PNC algorithm for reporting a small number of paths (e.g., less than 20).

An empirical framework for the selection of the most appropriate \(k\)SSP algorithm with respect to the use case is suggested in Figure 5.

An open problem is how to handle the \(k\)SSP problem on networks with arbitrary arc weights (including negative weights). Another interesting question is how to design a data structure enabling to quickly answer \(k\)SSP queries similarly to the data structures used by the hub labelling and contraction hierarchy schemes to answer distance queries [3]. A probably more difficult question would be to address dynamic networks, i.e., where the weights of the arcs evolve along time (e.g., in road networks where the traversal time of an arc may vary). Would it be possible to quickly update the solutions after a modification in the network?

![Figure 5: A framework of the appropriate \(k\)SSP algorithm with respect to the use case](image)

References


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