



HAL
open science

HOLOMORPHIC $sl(2, \mathbb{C})$ -SYSTEMS WITH FUCHSIAN MONODROMY (WITH AN APPENDIX BY TAKURO MOCHIZUKI)

Indranil Biswas, Sorin Dumitrescu, Lynn Heller, Sebastian Heller

► **To cite this version:**

Indranil Biswas, Sorin Dumitrescu, Lynn Heller, Sebastian Heller. HOLOMORPHIC $sl(2, \mathbb{C})$ -SYSTEMS WITH FUCHSIAN MONODROMY (WITH AN APPENDIX BY TAKURO MOCHIZUKI). 2021. hal-03196444

HAL Id: hal-03196444

<https://hal.science/hal-03196444>

Preprint submitted on 12 Apr 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HOLOMORPHIC $\mathfrak{sl}(2, \mathbb{C})$ -SYSTEMS WITH FUCHSIAN MONODROMY (WITH AN APPENDIX BY TAKURO MOCHIZUKI)

INDRANIL BISWAS, SORIN DUMITRESCU, LYNN HELLER, AND SEBASTIAN HELLER

ABSTRACT. For every integer $g \geq 2$ we show the existence of a compact Riemann surface Σ of genus g such that the rank two trivial holomorphic vector bundle $\mathcal{O}_{\Sigma}^{\oplus 2}$ admits holomorphic connections with $\mathrm{SL}(2, \mathbb{R})$ monodromy and maximal Euler class. Such a monodromy representation is known to coincide with the Fuchsian uniformizing representation for some Riemann surface of genus g . The construction carries over to all very stable and compatible real holomorphic structures for the topologically trivial rank two bundle over Σ and gives the existence of holomorphic connections with Fuchsian monodromy in these cases as well.

CONTENTS

Introduction	1
Strategy	3
1. Preliminaries: Logarithmic connections and parabolic bundles	4
2. Logarithmic connections on \mathcal{S}_4	6
3. Pullback to hyperelliptic Riemann surfaces	10
4. Logarithmic connections on the square torus with one marked point	19
5. Proofs	28
Appendix A. A result on WKB approximation	31
Acknowledgements	35
References	36

INTRODUCTION

For a compact Riemann surface Σ the holomorphic Riemann-Hilbert correspondence associates to every pair (V, ∇) , consisting of a (flat) holomorphic connection ∇ on a holomorphic vector bundle V over Σ , its monodromy homomorphism. This is an equivalence of categories (see for instance [De] or [Ka, p. 544]). For surfaces with nonabelian fundamental group finding holomorphic connections with prescribed monodromy behavior is notoriously difficult and an obstacle to a deeper understanding of various mathematical

Date: April 10, 2021.

1991 Mathematics Subject Classification. 34M03, 34M56, 14H15, 53A55.

Key words and phrases. Fuchsian representation, holomorphic connection, parabolic bundle, abelianization, WKB analysis.

problems ranging from algebraic geometry and number theory, over geometric structures on manifolds [AQ, DM, Th], to constructions in quantum field theories and mirror symmetry [AGM, W, GMN, FGUTe] and to the theory of harmonic maps and minimal surfaces [Hi2, Wo, HHS, Tr].

In this paper we restrict to the case of $\mathrm{SL}(2, \mathbb{C})$ -connections over compact Riemann surfaces of genus $g \geq 2$. This case is of particular interest as it is deeply linked to the geometry of the underlying surface. Starting from the XIXth century mathematicians have investigated group representations appearing as monodromy of solutions to algebraic differential equations on the complex domain. The relationship to geometry stems from the fact that the inverse of solutions to certain linear differential equations parametrize the Riemann surface. As discovered by Poincaré and Klein (see [StG] for a historical survey of the subject), every Riemann surface can be realized as a quotient of the hyperbolic plane \mathcal{H}^2 by a Fuchsian group (a torsion-free, discrete, and cocompact subgroup of $\mathrm{PSL}(2, \mathbb{R})$) identifying the space of Fuchsian representations with the Teichmüller space. Lifting Fuchsian representations from $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{R})$, they can be considered as monodromy representations of holomorphic $\mathrm{SL}(2, \mathbb{C})$ -connections on a fixed Riemann surface Σ via Riemann-Hilbert correspondence. The holomorphic structure on the rank two vector bundle given by the uniformization of Σ is the unique nontrivial extension of L^{-1} by L , where L is a theta characteristic on Σ [Gu1]. This bundle will be referred to as the uniformization bundle. Note that Fuchsian representations are $\mathrm{SL}(2, \mathbb{R})$ -representations with maximal Euler class $g - 1$. This gives 2^{2g} connected components with maximal Euler class in the space of $\mathrm{SL}(2, \mathbb{R})$ -representations corresponding to the different choices of the theta characteristic [Mi, Hi1, Go2].

In this context it is natural to ask which holomorphic rank two bundles over a given Riemann surface Σ admit holomorphic connections with Fuchsian monodromy representations. Indeed, this question was first raised by Katz in [Ka, p. 555–556] (where the question is attributed to Bers) in 1978 and is still unsolved. On the other hand, the analogue to Bers' question for the compact group $\mathrm{SU}(2)$ is fully understood. The celebrated Narasimhan-Seshadri Theorem shows that every stable holomorphic structure admits a unique compatible flat connection with irreducible unitary monodromy and vice versa. Motivated by problems in algebraic geometry and number theory, e.g., Weil conjecture, a related question of realizing Fuchsian representations as the monodromy homomorphism of regular singular $\mathrm{SL}(2, \mathbb{C})$ -connections on the uniformization bundle over (marked) Riemann surfaces was addressed by Faltings [Fa]. Remarkably, even when restricting to the trivial rank two holomorphic bundle, it was previously unknown whether a holomorphic connection ∇ with Fuchsian monodromy representation exists. This is the main question to be addressed in the present article. We prove

Theorem 1 (Main Theorem). *For every $k \in \mathbb{N}^{\geq 3}$ there exists a (hyperelliptic) Riemann surface Σ_k of genus $k - 1$ such that the trivial holomorphic rank two bundle admits infinitely many holomorphic connections with Fuchsian monodromy representation.*

A major difference to Narasimhan-Seshadri Theorem when considering the split real group $\mathrm{SL}(2, \mathbb{R})$ is that uniqueness fails, e.g., our Main Theorem shows the existence of infinitely many holomorphic connections with Fuchsian monodromy on the trivial holomorphic bundle. Likewise, for the holomorphic structure given by the uniformization bundle, the infinitely many holomorphic connections with Fuchsian monodromy correspond to integral graftings, see [Mas, Hej, SuT, Fa, Go1]. Although other holomorphic bundles with holomorphic connections with Fuchsian monodromy do exist, no explicit example other than the uniformization bundle itself were found.

Our Main Theorem 1 is in fact a consequence of an additional real symmetry of the considered Riemann surface Σ_k . Therefore, the proof carries over verbatim to all very stable holomorphic structures – i.e., their (non-zero) Higgs fields are not nilpotent – on the topologically trivial rank two bundle compatible with the construction and with the real symmetry of the Riemann surface (as specified in Lemma 4.5). The space of these real holomorphic structures can be identified with a circle with a single point removed in a projective line. An immediate corollary is

Corollary 1. *For every $k \in \mathbb{N}^{\geq 3}$ there exists a (hyperelliptic) Riemann surface Σ_k of genus $k - 1$ such that all very stable and compatible real holomorphic structures of the topologically trivial rank two bundle over Σ_k admit infinitely many holomorphic connections with Fuchsian monodromy representation.*

In a similar vein, Ghys raised the question about whether there is a pair (Σ, ∇) consisting of a compact Riemann surface of genus $g \geq 2$ and an irreducible holomorphic connection ∇ on the rank two trivial holomorphic vector bundle such that the image of the monodromy homomorphism of ∇ lies in a cocompact lattice of $\mathrm{SL}(2, \mathbb{C})$. Such a pair would give rise to a nontrivial holomorphic map from the Riemann surface Σ to the compact quotient of $\mathrm{SL}(2, \mathbb{C})$ by that cocompact lattice. Constructing such holomorphic maps is also known as the Margulis problem (see [CDHL] for the discussion about Ghys question and Margulis problem).

Motivated by the above question of Ghys, the authors of [CDHL] initiated a study of the Riemann-Hilbert correspondence for genus two surfaces and $\mathrm{SL}(2, \mathbb{C})$ -connections. Their main result asserts that the Riemann-Hilbert monodromy mapping, which associates to an irreducible holomorphic differential system its monodromy representation, is a local biholomorphism. Then Theorem 1 and the result of [CDHL] gives

Corollary 2. *There exists a nonempty open subset \mathcal{U} of the Teichmüller space of compact curves of genus $g = 2$ such that every $\Sigma \in \mathcal{U}$ possesses a holomorphic connection $\nabla(\Sigma)$ on $\mathcal{O}_{\Sigma}^{\oplus 2}$ with quasi-Fuchsian¹ monodromy representation.*

Every curve $\Sigma \in \mathcal{U}$ therefore admits a nontrivial holomorphic map into the quotient of $\mathrm{SL}(2, \mathbb{C})$ by a quasi-Fuchsian group as the image of the monodromy homomorphism of $\nabla(\Sigma)$.

Theorem 1 and Corollary 2 are geometrization results through holomorphic $\mathrm{SL}(2, \mathbb{C})$ -connections on the trivial bundle instead of the usual hyperbolic or Bers simultaneous uniformization for quasi-Fuchsian representations. It should be mentioned that, in higher Teichmüller spaces, geometrization results for representations of fundamental group of surfaces into Lie groups is currently a very lively and dynamic field of research (see for instance [BuIW, GuiW, La] and references therein).

STRATEGY

We show the existence of holomorphic connections with Fuchsian monodromy representation for particular hyperelliptic surfaces Σ_k of genus $(k - 1)$ given by a totally branched k -fold covering f_k of \mathcal{S}_4 – the complex projective line with four marked points $\pm 1, \pm\sqrt{-1}$. On Σ_k there are two connections of particular interest; the trivial de Rham differential d and the uniformizing connection ∇^U of Σ_k . Both connections can be realized, modulo

¹A representation of a surface group is called quasi-Fuchsian, if the monodromy homomorphism is faithful and has discrete image in $\mathrm{SL}(2, \mathbb{C})$ admitting a Jordan curve as limit set for its action on $\mathbb{C}P^1$.

singular gauge transformations, as the pull-back of the logarithmic connections D (Proposition 3.1) and $\tilde{\nabla}$ (Proposition 3.4) by f_k . Our aim is to deform D by a parabolic Higgs field such that the new connection has real monodromy, lies in the connected component of $\tilde{\nabla}$ and pulls back to Σ_k as a holomorphic connection (without singularities).

The moduli space of logarithmic connections on \mathcal{S}_4 has a natural set of coordinates given by the abelianization procedure [HH]. These coordinates determine logarithmic connections on \mathcal{S}_4 as a twisted push forward of flat line bundle connections on the torus Σ_2 obtained by the branched double cover f_2 of \mathcal{S}_4 . The twist is given by some meromorphic off-diagonal 1-forms determined by the flat line bundle and the eigenvalues of the residues. We restrict to the most symmetric case, where the behavior of the logarithmic connection at every marked point of Σ_2 is the same. More precisely, we consider connections on the torus Σ_2 with four marked points that descend to connections on the torus T^2 with only one marked point by taking the quotient with respect to its half-lattice. In this way Theorem 4.8 identifies the moduli space of logarithmic connections on T^2 with the moduli space of logarithmic connections on \mathcal{S}_4 . Moreover, D is identified with a connection \tilde{D} on the torus T^2 with one marked point in Lemma 4.10.

The crucial idea is to consider the asymptotic behavior of the family of connections

$$\tilde{D} + t\tilde{\Phi},$$

where $t \in \mathbb{R}$ and $\tilde{\Phi}$ is a specific parabolic Higgs field of \tilde{D} . By Theorem 4.8, this family corresponds to $\nabla^t = D + t\Phi$ on \mathcal{S}_4 , where Φ is the corresponding parabolic Higgs field of D . By construction all connections $f_k^* \nabla^t$ have the same underlying holomorphic structure, namely the trivial one induced by the de Rham differential d . For t large we then use WKB analysis and an additional real involution of the torus (Lemma 4.6) to ensure the existence of a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that ∇^{t_n} has real monodromy (Corollary 4.7). The necessary WKB analysis result is proved by Takuro Mochizuki in the Appendix.

Since the pull-back under f_k preserves the connected components of real representations, it remains to show that ∇^{t_n} lies in the same connected component as $\tilde{\nabla}$ on \mathcal{S}_4 . To do so, we compute that $\tilde{\nabla}$ is also induced by a singular connection ∇^F on the one-punctured torus in Lemma 4.11. The claim then follows from the fact that the four components of logarithmic connections with $\mathrm{SL}(2, \mathbb{R})$ -monodromy on the one-punctured torus are mapped into the same real component of the moduli space on \mathcal{S}_4 via Theorem 4.8. Therefore, the pull-back $f_k^* \nabla^{t_n}$ to Σ_k is Fuchsian and has trivial holomorphic structure.

In fact, it is necessary to consider singular connections on the one-punctured torus, since there exists 3 other components of irreducible $\mathrm{SL}(2, \mathbb{R})$ -representations on the four-punctured sphere, whose boundary contain reducible connections and do not lift to the Fuchsian component on Σ_k . Related examples of irreducible holomorphic $\mathrm{SL}(2, \mathbb{C})$ -connections with real monodromy on the trivial holomorphic rank two bundle were constructed in [BDH]. However, these connections are never of maximal Euler class.

1. PRELIMINARIES: LOGARITHMIC CONNECTIONS AND PARABOLIC BUNDLES

Let Σ be a compact connected Riemann surface; its holomorphic cotangent bundle is denoted by K_Σ . An $\mathrm{SL}(2, \mathbb{C})$ -bundle on Σ is a holomorphic rank two vector bundle V over Σ with trivial determinant, i.e., the line bundle $\det V = \bigwedge^2 V$ is holomorphically trivial.

Let $\mathbf{D} = p_1 + \dots + p_n$ be a divisor on Σ with pairwise distinct points $p_i \in \Sigma$. Consider a holomorphic $\mathrm{SL}(2, \mathbb{C})$ -bundle V on Σ together with its sheaf \mathcal{V} of holomorphic sections and its Dolbeault operator $\bar{\partial}$. A *logarithmic* $\mathrm{SL}(2, \mathbb{C})$ -connection $\nabla = \bar{\partial} + \partial^\nabla$ on V with polar part contained in \mathbf{D} is given by a holomorphic differential operator

$$\partial^\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes K_\Sigma \otimes \mathcal{O}_\Sigma(\mathbf{D})$$

satisfying the Leibniz rule

$$\partial^\nabla(fs) = f\nabla(s) + s \otimes df$$

for all locally defined holomorphic sections s of V and locally defined holomorphic functions f on Σ , such that the induced differential operator on $\det V$ coincides with the de Rham differential d on \mathcal{O}_Σ .

Since Σ is of complex dimension one, all logarithmic connections over Σ are flat. Moreover, at every singular point p_j , $1 \leq j \leq n$, of a logarithmic $\mathrm{SL}(2, \mathbb{C})$ -connection ∇ on V the residue

$$\mathrm{Res}_{p_j}(\nabla) \in \mathrm{End}(V_{p_j})$$

is tracefree.

If the two eigenvalues $\lambda_{j,1}$, $\lambda_{j,2}$ of the residue $\mathrm{Res}_{p_j}(\nabla)$ do not differ by an integer (this is known as the non-resonancy condition), then the local monodromy of ∇ around p_j is conjugate to the diagonal matrix with entries $\exp(-2\pi\sqrt{-1}\lambda_{j,1})$ and $\exp(-2\pi\sqrt{-1}\lambda_{j,2})$ (see [De, p. 53, Théorème 1.17]). If $\frac{1}{n_j}$ is an eigenvalue of the residue, with $n_j \geq 2$ an integer, the local monodromy of ∇ at p_j is a rational rotation on the eigenlines.

Let V be a holomorphic $\mathrm{SL}(2, \mathbb{C})$ -bundle on Σ . A parabolic structure \mathcal{P} on V with parabolic divisor $\mathbf{D} = p_1 + \dots + p_n$ consists of quasiparabolic lines $L_j \subset V_{p_j}$ together with weights $\rho_j \in]0, \frac{1}{2}[$ for every $1 \leq j \leq n$. For a holomorphic line subbundle $W \subset V$ the parabolic degree is given by

$$\mathrm{par}\text{-deg}(W) := \mathrm{degree}(W) + \sum_{j=1}^n \rho_j^W,$$

where $\rho_j^W = \rho_j$ if $W_{p_j} = L_j$ and $\rho_j^W = -\rho_j$ if $W_{p_j} \neq L_j$; see [MS], [MY].

Definition 1.1. A parabolic bundle (V, \mathcal{P}) is called *stable* (respectively, *semistable*) if $\mathrm{par}\text{-deg}(W) < 0$ (respectively, $\mathrm{par}\text{-deg}(W) \leq 0$) for every holomorphic line subbundle $W \subset V$. A parabolic bundle will be called *unstable* if it is not semistable.

Take a non-resonant logarithmic $\mathrm{SL}(2, \mathbb{C})$ -connection ∇ such that the eigenvalues of the residues lie in $]-\frac{1}{2}, \frac{1}{2}[$. It induces a parabolic structure on the underlying holomorphic vector bundle V . The parabolic divisor is $\mathbf{D} = p_1 + \dots + p_n$, where p_j are the singular points of the connection. The parabolic weight ρ_j at p_j is the positive eigenvalue of $\mathrm{Res}_{p_j}(\nabla)$, and the quasiparabolic line at p_j is the eigenline of $\mathrm{Res}_{p_j}(\nabla)$ for the eigenvalue ρ_j .

Two non-resonant $\mathrm{SL}(2, \mathbb{C})$ -connection on V with same weights $\rho_j \in]0, \frac{1}{2}[$ induce the same parabolic structure \mathcal{P} if and only if they differ by a *strongly parabolic Higgs field* on (V, \mathcal{P}) . Recall that a strongly parabolic Higgs field on (V, \mathcal{P}) is a trace free holomorphic section

$$\Theta \in H^0(\Sigma, \mathrm{End}(V) \otimes K_\Sigma \otimes \mathcal{O}_\Sigma(\mathbf{D}))$$

such that

$$\Theta(p_j)(V_{p_j}) \subset L_j \otimes (K_\Sigma \otimes \mathcal{O}_\Sigma(\mathbf{D}))_{p_j}$$

for all $1 \leq j \leq n$. These conditions imply that $\Theta(p_j)$ is nilpotent and the quasiparabolic line L_j lies in the kernel of $\Theta(p_j)$, for all $1 \leq j \leq n$.

2. LOGARITHMIC CONNECTIONS ON \mathcal{S}_4

Consider the Riemann sphere $\mathbb{C}P^1$ with three unordered marked points $\{0, 1, \infty\}$

$$\mathcal{S}_3 = (\mathbb{C}P^1, \{0, 1, \infty\})$$

and let

$$\mathcal{S}_3 := \mathbb{C}P^1 \setminus \{0, 1, \infty\}$$

be the three-punctured sphere. Fix a base point $p \in \mathcal{S}_3$ and elements

$$\gamma_0, \gamma_1 \in \pi_1(\mathcal{S}_3, p)$$

such that γ_0 (respectively, γ_1) is the free homotopy class of the oriented loop around the puncture 0 (respectively, 1). Then

$$\gamma_\infty := (\gamma_1 \gamma_0)^{-1}$$

is the free homotopy class of the oriented loop around the puncture ∞ .

2.1. Hyperbolic triangle and uniformization of the orbifold sphere.

Consider \mathcal{S}_3 equipped with an orbifold structure, i.e., we assign to each marked point an angle $\alpha_i = \frac{2\pi}{k_i}$, $i \in \{0, 1, \infty\}$, where $k_i > 1$ are integers. Assume that $\frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_\infty} < 1$. A hyperbolic uniformization of \mathcal{S}_3 equipped with the above orbifold structure is given by the following construction which goes back to the work of Schwarz, Klein and Poincaré (see [StG, Chapter VI]).

The group $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{C})$ acts by Möbius transformations on the upper half plane $\mathcal{H}^2 := \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$. By viewing the upper half plane as the hyperbolic plane, $\mathrm{PSL}(2, \mathbb{R})$ is in fact the group of orientation preserving isometries of \mathcal{H}^2 . Up to orientation preserving isometries, there exists a unique hyperbolic triangle T in \mathcal{H}^2 with prescribed angles $(\frac{\pi}{k_0}, \frac{\pi}{k_1}, \frac{\pi}{k_\infty})$ [StG, Proposition IX.2.6]. Denote by $p_0, p_1, p_\infty \in \mathcal{H}^2$ the corresponding (ordered) vertices of T .

Denote by $\sigma_0, \sigma_1, \sigma_\infty$ the hyperbolic reflections across the geodesic arcs (p_1, p_∞) , (p_0, p_∞) and (p_0, p_1) respectively. They generate a discrete subgroup of isometries of \mathcal{H}^2 . Consider its index two subgroup Γ generated by $m_0 = \sigma_\infty \circ \sigma_1$, $m_1 = \sigma_0 \circ \sigma_\infty$ and $m_\infty = \sigma_1 \circ \sigma_0$. Geometrically, Γ is generated by an even number of reflections across every geodesic edge of a hyperbolic geodesic triangle T ; it is called a hyperbolic triangle group. It is classical that such $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a Fuchsian subgroup with a fundamental quadrilateral in \mathcal{H}^2 given by $P = T \cup \sigma_1(T)$. The vertices of P are the points p_0, p_1, p_∞, p_2 with $p_2 := \sigma_1(p_1)$ (see [StG, Theorem VI.1.10 and Section VI.2.1]).

The oriented geodesic edges of P satisfy

$$m_0 \cdot (p_0, p_1) = (p_0, p_2) \quad \text{and} \quad m_\infty \cdot (p_\infty, p_2) = (p_\infty, p_1).$$

The maps m_0, m_1 and m_∞ are of order k_0, k_1 and k_∞ , respectively, and $m_\infty \circ m_1 \circ m_0 = \mathrm{Id}$ by construction. Therefore, the hyperbolic triangle group Γ generated by m_0, m_1 and m_∞ satisfies

$$m_\infty \circ m_1 \circ m_0 = \mathrm{Id} \quad \text{and} \quad m_0^{k_0} = m_1^{k_1} = m_\infty^{k_\infty} = \mathrm{Id}.$$

The quotient of \mathcal{H}^2 by the above Fuchsian hyperbolic triangle group Γ endows \mathcal{S}_3 , equipped with the orbifold structure $(\frac{2\pi}{k_0}, \frac{2\pi}{k_1}, \frac{2\pi}{k_\infty})$ at the points $\{0, 1, \infty\}$ respectively,

with a compatible hyperbolic structure [StG, Chapter VI and Section VI.2.1]. In particular, the monodromy around the punctures p_0, p_1, p_∞ of this uniformizing hyperbolic structure coincides with the rotations by the angles $\frac{2\pi}{k_0}, \frac{2\pi}{k_1}, \frac{2\pi}{k_\infty}$, respectively.

2.2. Logarithmic connection on trivial bundle with Fuchsian monodromy.

For $\tilde{\rho} \in]0, \frac{1}{2}[$ fixed, consider the logarithmic connection on the trivial holomorphic vector bundle $\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$

$$\nabla = d + \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & -\frac{1}{8} \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} -4\tilde{\rho}^2 & 1 \\ \tilde{\rho}^2 - 16\tilde{\rho}^4 & 4\tilde{\rho}^2 \end{pmatrix} \frac{dz}{z-1}. \quad (2.1)$$

Since the singular locus of ∇ is $\{0, 1, \infty\}$, we consider ∇ as a logarithmic connection on \mathcal{S}_3 . Throughout the paper we will use the convention that the marked points of a Riemann surface are the singular points of a logarithmic connection and branch points of coverings. Further, let

$$M_{\tilde{\rho}} : \pi_1(\mathcal{S}_3, p) \longrightarrow \mathrm{SL}(2, \mathbb{C})$$

be the monodromy representation of the flat connection ∇ in (2.1).

Lemma 2.1. *With the above notation the monodromy representation $M_{\tilde{\rho}}$ of ∇ in (2.1) is conjugate to an irreducible $\mathrm{SU}(2)$ representation for $\tilde{\rho} < \frac{1}{4}$, and to an irreducible $\mathrm{SL}(2, \mathbb{R})$ representation for $\frac{1}{4} < \tilde{\rho} < \frac{1}{2}$.*

If $\tilde{\rho} = \frac{k-1}{2k} \in (\frac{1}{4}, \frac{1}{2})$, with $k \in \mathbb{N}^{>2}$, the monodromy representation $M_{\tilde{\rho}}$ is conjugated to the monodromy of a hyperbolic structure uniformizing \mathcal{S}_3 equipped with the orbifold structure $(\frac{\pi}{2}, \frac{2\pi}{k}, \frac{\pi}{2})$ at points $\{0, 1, \infty\}$ respectively. In particular, the image of the monodromy representation is the Fuchsian group generated by an even number of reflections across the geodesic edges of the hyperbolic geodesic triangle with angles $(\frac{\pi}{4}, \frac{\pi}{k}, \frac{\pi}{4})$.

Proof. Let X_0, X_1 and X_∞ denote the elements $M_{\tilde{\rho}}(\gamma_0), M_{\tilde{\rho}}(\gamma_1)$ and $M_{\tilde{\rho}}(\gamma_\infty)$ of $\mathrm{SL}(2, \mathbb{C})$ respectively. Moreover, let R_0, R_1, R_∞ denote the respective residues of ∇ at $0, 1, \infty$. Note that for $\tilde{\rho} \in]0, \frac{1}{2}[$ none of the eigenvalues of R_0, R_1, R_∞ lies in $\frac{1}{2}\mathbb{Z}$; in other words, ∇ is non-resonant. Consequently, the conjugacy class of X_i is given by

$$\exp(-2\pi\sqrt{-1}R_i) \quad (2.2)$$

for $i = 0, 1, \infty$ (see [De, p. 53, Théorème 1.17]). For ∇ in (2.1) we therefore compute

$$\mathrm{tr}(X_0) = \sqrt{2} = \mathrm{tr}(X_\infty), \quad \mathrm{tr}(X_1) = 2\cos(2\pi\tilde{\rho}). \quad (2.3)$$

This gives that the representation $M_{\tilde{\rho}}$ is irreducible for

$$0 < \tilde{\rho} < \frac{1}{4} \quad \text{and} \quad \frac{1}{4} < \tilde{\rho} < \frac{1}{2},$$

see [Go2, p. 574, Proposition 4.1 (iii)]. It also follows that the three equations in (2.3) determine $M_{\tilde{\rho}}$ uniquely up to the conjugation by an element of $\mathrm{SL}(2, \mathbb{C})$ [Go2, p. 574, Proposition 4.1 (iv & v)].

Consider the matrices

$$\begin{aligned} \tilde{X}_0 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ \tilde{X}_1 &= \begin{pmatrix} \cos 2\pi\tilde{\rho} - \sqrt{(-1 + \cos 2\pi\tilde{\rho}) \cos 2\pi\tilde{\rho}} & 1 - \cos 2\pi\tilde{\rho} + \sqrt{(-1 + \cos 2\pi\tilde{\rho}) \cos 2\pi\tilde{\rho}} \\ -1 + \cos 2\pi\tilde{\rho} + \sqrt{(-1 + \cos 2\pi\tilde{\rho}) \cos 2\pi\tilde{\rho}} & \cos 2\pi\tilde{\rho} + \sqrt{(-1 + \cos 2\pi\tilde{\rho}) \cos 2\pi\tilde{\rho}} \end{pmatrix} \\ \tilde{X}_\infty &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1 + 2\cos 2\pi\tilde{\rho} - 2\sqrt{(-1 + \cos 2\pi\tilde{\rho}) \cos 2\pi\tilde{\rho}}}{\sqrt{2}} \\ \frac{1 - 2\cos 2\pi\tilde{\rho} - 2\sqrt{(-1 + \cos 2\pi\tilde{\rho}) \cos 2\pi\tilde{\rho}}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$

These determine a monodromy homomorphism

$$M' : \pi_1(S_3, p) \longrightarrow \mathrm{SL}(2, \mathbb{C})$$

that takes γ_0, γ_1 and γ_∞ to \tilde{X}_0, \tilde{X}_1 and \tilde{X}_∞ respectively. Since the three equations in (2.3) determine $M_{\tilde{\rho}}$ uniquely up to conjugation by some element of $\mathrm{SL}(2, \mathbb{C})$, we conclude that $M_{\tilde{\rho}}$ and M' are conjugate to each other. Evidently, the image of M' lies in $\mathrm{SU}(2)$ if $0 < \tilde{\rho} < \frac{1}{4}$, and it lies in $\mathrm{SL}(2, \mathbb{R})$ if $\frac{1}{4} < \tilde{\rho} < \frac{1}{2}$, proving the first part of the lemma.

To prove the second part, fix $k \in \mathbb{N}^{\geq 3}$ and consider the special case of $\tilde{\rho} = \frac{k-1}{2k} \in (\frac{1}{4}, \frac{1}{2})$. In this case the corresponding $\mathrm{SL}(2, \mathbb{R})$ matrices generating the monodromy group Λ for ∇ specialize to

$$\begin{aligned} \tilde{X}_0 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ \tilde{X}_1 &= \begin{pmatrix} -\cos \frac{\pi}{k} - \sqrt{(1 + \cos \frac{\pi}{k}) \cos \frac{\pi}{k}} & 1 + \cos \frac{\pi}{k} + \sqrt{(1 + \cos \frac{\pi}{k}) \cos \frac{\pi}{k}} \\ -1 - \cos \frac{\pi}{k} + \sqrt{(1 + \cos \frac{\pi}{k}) \cos \frac{\pi}{k}} & -\cos \frac{\pi}{k} + \sqrt{(1 + \cos \frac{\pi}{k}) \cos \frac{\pi}{k}} \end{pmatrix} \\ \tilde{X}_\infty &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1 - 2 \cos \frac{\pi}{k} - 2 \sqrt{(1 + \cos \frac{\pi}{k}) \cos \frac{\pi}{k}}}{\sqrt{2}} \\ \frac{1 + 2 \cos \frac{\pi}{k} - 2 \sqrt{(1 + \cos \frac{\pi}{k}) \cos \frac{\pi}{k}}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$

Let $m(\tilde{X}_0), m(\tilde{X}_1)$ and $m(\tilde{X}_\infty)$ be the automorphisms of the upper half plane \mathcal{H}^2 given by \tilde{X}_0, \tilde{X}_1 and \tilde{X}_∞ respectively. The points of \mathcal{H}^2

$$\begin{aligned} p_0 &= \sqrt{-1} \\ p_1 &= \frac{1 + \cos \frac{\pi}{k} + \sqrt{\cos \frac{\pi}{k} (1 + \cos \frac{\pi}{k})}}{\sqrt{\cos \frac{\pi}{k} (1 + \cos \frac{\pi}{k})} - \sqrt{-1} \sin \frac{\pi}{k}} \\ p_\infty &= \sqrt{-1} (1 + 2 \cos \frac{\pi}{k} + 2 \sqrt{\cos \frac{\pi}{k} (1 + \cos \frac{\pi}{k})}) \end{aligned}$$

are fixed by $m(\tilde{X}_0), m(\tilde{X}_1)$ and $m(\tilde{X}_\infty)$ respectively. Recall that an element of $\mathrm{PSL}(2, \mathbb{R})$ is completely determined by a fixed point in \mathcal{H}^2 together with the differential at the fixed point. The differentials of $m(\tilde{X}_0), m(\tilde{X}_1)$ and $m(\tilde{X}_\infty)$ at p_0, p_1 and p_∞ , respectively, are rotations and a short computation shows that these are given by

$$\begin{aligned} D_{p_0} m(\tilde{X}_0) &= -\sqrt{-1} \\ D_{p_1} m(\tilde{X}_1) &= e^{-\frac{2\pi\sqrt{-1}}{k}} \\ D_{p_\infty} m(\tilde{X}_\infty) &= -\sqrt{-1}. \end{aligned} \tag{2.4}$$

Therefore Λ is conjugated in $\mathrm{PSL}(2, \mathbb{R})$ to the Fuchsian hyperbolic triangle group associated to the hyperbolic triangle (p_0, p_1, p_∞) . The transformations $m(\tilde{X}_0), m(\tilde{X}_1)$ and $m(\tilde{X}_\infty)$ coincide with m_0, m_1 and m_∞ defined in Section 2.1, respectively (see also [StG, Chapter VI]).

From (2.4), the internal angles of the hyperbolic triangle are

$$\alpha_0 = \frac{\pi}{4} \quad \alpha_1 = \frac{\pi}{k} \quad \text{and} \quad \alpha_\infty = \frac{\pi}{4}.$$

It follows from Section 2.1 that the monodromy homomorphism of ∇ is conjugated in $\mathrm{PSL}(2, \mathbb{R})$ to the monodromy homomorphism of the uniformizing hyperbolic structure of the orbifold \mathcal{S}_3 with angles $(\frac{\pi}{2}, \frac{2\pi}{k}, \frac{\pi}{2})$ at the points $\{0, 1, \infty\}$ respectively. \square

Let \mathcal{S}_4 denote the Riemann sphere $\mathbb{C}P^1$ with unordered four marked points

$$\{1, \sqrt{-1}, -1, -\sqrt{-1}\}$$

and let

$$S_4 := \mathbb{C}P^1 \setminus \{1, \sqrt{-1}, -1, -\sqrt{-1}\} \quad (2.5)$$

be the four-punctured sphere. Similarly, denote by \mathcal{S}_6 the Riemann sphere $\mathbb{C}P^1$ with six unordered marked points $\{0, 1, \sqrt{-1}, -1, -\sqrt{-1}, \infty\}$, and define $S_6 := \mathcal{S}_4 \setminus \{0, \infty\}$. Consider the map

$$f : \mathcal{S}_6 \longrightarrow \mathcal{S}_3; \quad z \longmapsto z^4.$$

For the logarithmic connection ∇ in (2.1), let

$$\nabla^1 := f^*\nabla \quad (2.6)$$

be the logarithmic connection on the trivial holomorphic bundle $\mathcal{O}_{S_6}^{\oplus 2}$ whose singular points coincide with the marked points. We will construct a logarithmic connection on $\mathcal{O}_{S_4}^{\oplus 2}$ using ∇^1 .

Let X denote $\mathbb{C}P^1$ with the ten unordered marked point a_1, \dots, a_{10} such that

$$a_i^2 \in \{0, 1, \sqrt{-1}, -1, -\sqrt{-1}, \infty\}.$$

Let

$$\varpi : X \longrightarrow S_6, \quad w \longmapsto w^2 \quad (2.7)$$

be the ramified covering map. We have the logarithmic connection $\varpi^*\nabla^1$ on $\varpi^*\mathcal{O}_{S_6}^{\oplus 2} = \mathcal{O}_X^{\oplus 2}$, where ∇^1 is defined in (2.6). The Galois group of the ramified covering map $\text{Gal}(\varpi) = \mathbb{Z}/2\mathbb{Z}$ in (2.7) acts on the vector bundle $\mathcal{O}_X^{\oplus 2}$; this action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{O}_X^{\oplus 2}$ evidently preserves the logarithmic connection $\varpi^*\nabla^1$.

Let z denote the standard holomorphic coordinate on $S_6 \subset \mathbb{C}P^1 \setminus \{0, \infty\}$, so $\sqrt{z} := z \circ \varpi$ is a nowhere vanishing holomorphic function on $\varpi^{-1}(S_6) \subset X$. For notational convenience, we denote the subset $\varpi^{-1}(S_6) \subset X$ by X' . Consider the holomorphic automorphism (= gauge transformation)

$$G := \begin{pmatrix} \frac{1}{\sqrt{z}} & \frac{4}{-1+16\tilde{\rho}^2}\sqrt{z} \\ 0 & \sqrt{z} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\tilde{\rho}(1-16\tilde{\rho}^2)}} & 0 \\ 0 & \sqrt{\tilde{\rho}(1-16\tilde{\rho}^2)} \end{pmatrix}$$

of $\mathcal{O}_{X'}^{\oplus 2}$ and let

$$\nabla^2 := ((\varpi^*\nabla^1)|_{X'}) \cdot G = ((\varpi^*f^*\nabla)|_{X'}) \cdot G \quad (2.8)$$

be the holomorphic connection on $\mathcal{O}_{X'}^{\oplus 2}$ given by the action of the automorphism G on the connection $\varpi^*\nabla^1|_{X'}$ (the connection ∇^1 is defined in (2.6)).

Although the above mentioned action $\text{Gal}(\varpi) = \mathbb{Z}/2\mathbb{Z}$ on $\varpi^*\mathcal{O}_{S_6}^{\oplus 2} = \mathcal{O}_{X'}^{\oplus 2}$ does not preserve G , it is straightforward to check that the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{O}_{X'}^{\oplus 2}$ actually preserves the connection ∇^2 defined in (2.8). Indeed, the action of the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ takes G to $-G$. On the other hand, the action of $-I \in \text{SL}(2, \mathbb{C})$ fixes every connection on the trivial bundle $\mathcal{O}_{X'}^{\oplus 2}$. These imply that the action of $\mathbb{Z}/2\mathbb{Z}$ preserves the connection ∇^2 . Hence there is a unique holomorphic connection on $\mathcal{O}_{S_6}^{\oplus 2}$ whose pull-back, by ϖ , is the connection ∇^2 on $\mathcal{O}_{X'}^{\oplus 2} = \varpi^*\mathcal{O}_{S_6}^{\oplus 2}$. Let $\tilde{\nabla}$ be the unique holomorphic connection on $\mathcal{O}_{S_6}^{\oplus 2}$ such that

$$\varpi^*\tilde{\nabla} = \nabla^2 = ((\varpi^*f^*\nabla)|_{X'}) \cdot G.$$

A computation shows that

$$\tilde{\nabla} = d + \begin{pmatrix} 0 & 4\tilde{\rho} \\ 4\tilde{\rho}z^2 & 0 \end{pmatrix} \frac{dz}{z^4 - 1} \quad (2.9)$$

on $\mathcal{O}_{\mathcal{S}_6}^{\oplus 2} = \mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$. In particular, $\tilde{\nabla}$ is a logarithmic connection on $\mathcal{O}_{\mathcal{S}_4}^{\oplus 2}$, because $0, \infty$ are regular points of $\tilde{\nabla}$.

Lemma 2.2. *The monodromy representation of the flat connection $\tilde{\nabla}$ on $\mathcal{O}_{\mathcal{S}_4}^{\oplus 2}$ in (2.9) is conjugate to a $SU(2)$ representation if $0 < \tilde{\rho} < \frac{1}{4}$, and it is conjugate to a $SL(2, \mathbb{R})$ representation if $\frac{1}{4} < \tilde{\rho} < \frac{1}{2}$.*

Moreover, if $\tilde{\rho} = \frac{k-1}{2k}$ with $k \in \mathbb{N}^{>2}$, then the monodromy representation for $\tilde{\nabla}$ is conjugated to the monodromy of a hyperbolic structure uniformizing \mathcal{S}_4 equipped with the orbifold structure $(\frac{2\pi}{k}, \frac{2\pi}{k}, \frac{2\pi}{k}, \frac{2\pi}{k})$ at the four marked points. In particular, the image of the monodromy representation is an index 4 subgroup in the Fuchsian group generated by an even number of reflections across the geodesic edges of the hyperbolic geodesic triangle with angles $(\frac{\pi}{4}, \frac{\pi}{k}, \frac{\pi}{4})$.

Proof. The monodromy representation of a pulled-back flat connection is the pull-back of the monodromy representation. Further, a gauge transformation of a flat connection does not change the conjugacy class of the monodromy representation. Let $\mathcal{I} \subset SL(2, \mathbb{C})$ be the image of the monodromy homomorphism for the connection ∇^1 in (2.6) (the conjugacy class of this subgroup is unique). Let

$$\tilde{\mathcal{I}} \subset SL(2, \mathbb{C})$$

be the subgroup generated by I and $-I$. Since the action of the nontrivial element of the structure group $\mathbb{Z}/2\mathbb{Z}$ of the principal bundle in (2.7) takes G to $-G$, the image of the monodromy homomorphism of $\tilde{\nabla}$ is contained in $\tilde{\mathcal{I}}$ by construction. Then the first assertion of the lemma follows from Lemma 2.1.

For the second statement, let $\tilde{\rho} = \frac{k-1}{2k} \in (\frac{1}{4}, \frac{1}{2})$, with $k \in \mathbb{N}^{>2}$. It was shown in Lemma 2.1 that the monodromy homomorphism for ∇ is conjugated to the monodromy homomorphism of the uniformizing hyperbolic structure of the orbifold \mathcal{S}_3 , with angles $(\frac{\pi}{2}, \frac{2\pi}{k}, \frac{\pi}{2})$ at points $\{0, 1, \infty\}$ respectively.

The monodromy homomorphism for $\tilde{\nabla}$ is the pull-back of the monodromy homomorphism for ∇ through a 4-fold covering totally branched over the marked points $0, \infty \in \mathcal{S}_3$. Therefore, the monodromy homomorphism for $\tilde{\nabla}$ is conjugated to the monodromy homomorphism of the uniformizing hyperbolic structure of the orbifold \mathcal{S}_4 with angles $(\frac{2\pi}{k}, \frac{2\pi}{k}, \frac{2\pi}{k}, \frac{2\pi}{k})$ (i.e., the orbifold structures at the four preimages of 1 are same). The image of the monodromy homomorphism of $\tilde{\nabla}$ is an index 4 subgroup in the Fuchsian triangle group Λ defined in the proof of Lemma 2.1. \square

3. PULLBACK TO HYPERELLIPTIC RIEMANN SURFACES

Let Σ_k be the compact Riemann surface of genus $k-1$ defined by the algebraic equation

$$Y^k = \frac{Z^2 - 1}{Z^2 + 1}. \quad (3.1)$$

It has the projection of degree k

$$f_k : \Sigma_k \longrightarrow \mathbb{C}P^1, \quad (Y, Z) \longmapsto Z. \quad (3.2)$$

The hyperelliptic involution is given by $(Y, Z) \mapsto (Y, -Z)$. Note that for $k = 2$, the elliptic curve Σ_2 is of square conformal type and we identify $\Sigma_2 = \mathbb{C}/(2\mathbb{Z} + 2\sqrt{-1}\mathbb{Z})$.

For $k \in \mathbb{N}^{\geq 3}$ let $\tilde{\rho} = \frac{k-1}{2k} \in (\frac{1}{4}, \frac{1}{2})$ and consider the logarithmic connection

$$D = d + \begin{pmatrix} \tilde{\rho} & 0 \\ 0 & -\tilde{\rho} \end{pmatrix} \left(\frac{dz}{z-1} - \frac{dz}{z+\sqrt{-1}} + \frac{dz}{z+1} - \frac{dz}{z-\sqrt{-1}} \right) \quad (3.3)$$

on $\mathcal{O}_{\mathcal{S}_4}^{\oplus 2}$ over \mathcal{S}_4 . Then D can be pulled back to the logarithmic connection

$$(f_k^* \mathcal{O}_{\mathcal{S}_4}^{\oplus 2}, f_k^* D) = (\mathcal{O}_{\Sigma_k}^{\oplus 2}, f_k^* D) \quad (3.4)$$

by the map f_k in (3.2). The singular points of $f_k^* D$ are

$$p_1 = (0, 1), p_2 = (\infty, \sqrt{-1}), p_3 = (0, -1), p_4 = (\infty, -\sqrt{-1}) \quad (3.5)$$

in terms of the above pair of coordinate functions (Y, Z) on Σ_k . Let

$$\Sigma'_k := \Sigma_k \setminus \{p_1, p_2, p_3, p_4\} \quad (3.6)$$

be the complement of the points in (3.5). Then the following proposition holds.

Proposition 3.1. *Let $k \in \mathbb{N}^{\geq 3}$ and $\tilde{\rho} = \frac{k-1}{2k} \in (\frac{1}{4}, \frac{1}{2})$.*

- (1) *If k is odd, then there is a meromorphic automorphism G of $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ such that*
 - *G is nonsingular on Σ'_k ,*
 - *G gauges the holomorphic connection $(f_k^* D)|_{\Sigma'_k}$ to the trivial holomorphic connection on $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$. In particular, $f_k^* D$ has trivial monodromy.*
- (2) *If k is even, then there is a holomorphic line bundle \mathcal{L} over Σ_k with a logarithmic connection $\nabla^{\mathcal{L}}$ with polar part contained in $\mathbf{D} = p_1 + p_2 + p_3 + p_4$ such that*
 - *the image of the monodromy homomorphism of $\nabla^{\mathcal{L}}$ is $\{\pm 1\} \subset \mathbb{C}^*$,*
 - *there is a meromorphic isomorphism*

$$G : \mathcal{O}_{\Sigma_k}^{\oplus 2} \longrightarrow \mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L} = \mathcal{L}^{\oplus 2},$$

singular at \mathbf{D} , which gauges the holomorphic connection $(f_k^ D) \otimes \nabla^{\mathcal{L}}$ on $\mathcal{L}^{\oplus 2}|_{\Sigma'_k}$ to the trivial holomorphic connection on $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$. In particular, the monodromy of $(f_k^* D) \otimes \nabla^{\mathcal{L}}$ is trivial.*

Remark 3.2. Throughout the paper we use the convention that the tensor product of two connections ∇^1 on V^1 and ∇^2 on V^2 is the connection on $V^1 \otimes V^2$ given by the operator

$$\nabla^1 \otimes \nabla^2 := \nabla^1 \otimes \text{Id} + \text{Id} \otimes \nabla^2.$$

Proof. Equation (3.1) gives that

$$d \log Y = \frac{1}{k} d \log \frac{Z^2 - 1}{Z^2 + 1}.$$

Thus, for k odd,

$$\Psi = \begin{pmatrix} Y^{-\frac{k-1}{2}} & 0 \\ 0 & Y^{\frac{k-1}{2}} \end{pmatrix} \quad (3.7)$$

is a well-defined global meromorphic frame of $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ that satisfies the following:

- (1) the restriction $\Psi|_{\Sigma'_k}$ is a holomorphic frame of $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$,
- (2) $\Psi|_{\Sigma'_k}$ is a parallel frame for the holomorphic connection $(f_k^* D)|_{\Sigma'_k}$.

To show that $\Psi|_{\Sigma'_k}$ is indeed parallel note that $\frac{4zdz}{z^4-1} = \frac{dz}{z-1} - \frac{dz}{z+\sqrt{-1}} + \frac{dz}{z+1} - \frac{dz}{z-\sqrt{-1}}$. Let G be the automorphism of $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$ that takes the standard frame to the frame $\Psi|_{\Sigma'_k}$. Then G gauges $(f_k^*D)|_{\Sigma'_k}$ to the trivial connection on $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$, because $\Psi|_{\Sigma'_k}$ is a parallel frame for $(f_k^*D)|_{\Sigma'_k}$. This proves the proposition for odd k .

If k is even, Ψ in (3.7) is no longer single valued. Nevertheless, we can still recover the trivial connection on $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$ by twisting the pull-back of D to Σ_k by an appropriate line bundle connection. The construction goes as follows. The values of $Y^{\frac{k-1}{2}}$ produce a nontrivial double covering

$$\delta : \tilde{\Sigma} \longrightarrow \Sigma_k$$

branched over the subset $\{p_1, p_2, p_3, p_4\}$ in (3.5). Let

$$\tilde{\Sigma}' := \delta^{-1}(\Sigma'_k) \subset \tilde{\Sigma}.$$

So $\delta|_{\tilde{\Sigma}'} : \tilde{\Sigma}' \longrightarrow \Sigma'_k$ is an unramified double covering. Now Ψ produces a meromorphic frame $\tilde{\Psi}$ of $\delta^*\mathcal{O}_{\Sigma_k}^{\oplus 2} = \mathcal{O}_{\tilde{\Sigma}}^{\oplus 2}$ such that the restriction of $\tilde{\Psi}$ to $\tilde{\Sigma}'$ is a holomorphic frame of $\mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2}$. This frame $\tilde{\Psi}|_{\tilde{\Sigma}'}$ is parallel for the flat connection $(\delta^*f_k^*D)|_{\tilde{\Sigma}'}$ on $\mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2}$.

The Galois group $\text{Gal}(\delta) = \mathbb{Z}/2\mathbb{Z}$ for δ has a natural action on $\delta^*\mathcal{O}_{\Sigma_k}^{\oplus 2} = \mathcal{O}_{\tilde{\Sigma}}^{\oplus 2}$. The action of the nontrivial element of $\text{Gal}(\delta) = \mathbb{Z}/2\mathbb{Z}$ evidently takes the frame $\tilde{\Psi}$ to $-\tilde{\Psi}$. Therefore, the holomorphic frame $\tilde{\Psi}|_{\tilde{\Sigma}'}$ does not descend to a holomorphic frame of $\mathcal{O}_{\Sigma'_k}^{\oplus 2}$. Note that the action of $\text{Gal}(\delta)$ on $\mathcal{O}_{\tilde{\Sigma}}^{\oplus 2}$ preserves the logarithmic connection $\delta^*f_k^*D$. We will now construct a suitable twist of $\tilde{\Psi}$ that descends.

Consider the holomorphic line bundle

$$\tilde{\mathcal{L}} := \mathcal{O}_{\tilde{\Sigma}}$$

equipped with the following action of $\text{Gal}(\delta)$: the nontrivial element $\alpha \in \text{Gal}(\delta)$ acts as multiplication by -1 over the involution α , meaning $f \mapsto -f \circ \alpha$, for any locally defined holomorphic function f on $\tilde{\Sigma}$. (The notation $\tilde{\mathcal{L}}$ is used for emphasizing the nontrivial action of $\text{Gal}(\delta)$.) It has a holomorphic connection defined by the de Rham differential; this connection, which will be denoted by $\nabla^{\tilde{\mathcal{L}}}$, is preserved by the action of $\text{Gal}(\delta)$ on $\tilde{\mathcal{L}}$.

Now consider the holomorphic vector bundle

$$\mathcal{F} := (\delta^*\mathcal{O}_{\Sigma_k}^{\oplus 2}) \otimes \tilde{\mathcal{L}} = \mathcal{O}_{\tilde{\Sigma}}^{\oplus 2} \otimes \tilde{\mathcal{L}} \quad (3.8)$$

on $\tilde{\Sigma}$. It has the meromorphic frame $\tilde{\Psi} \otimes \mathbf{1}$, where $\mathbf{1}$ denotes the constant function 1. This frame $\tilde{\Psi} \otimes \mathbf{1}$ is holomorphic over $\tilde{\Sigma}'$ and it is preserved by the action of $\text{Gal}(\delta)$ on \mathcal{F} (recall that $\alpha \in \text{Gal}(\delta)$ acts as multiplication by -1 on both $\tilde{\Psi}$ and $\mathbf{1}$). With respect to the product connection

$$\nabla^{\mathcal{F}} := (\delta^*f_k^*D) \otimes \nabla^{\tilde{\mathcal{L}}} \quad (3.9)$$

on \mathcal{F} , the holomorphic frame $(\tilde{\Psi} \otimes \mathbf{1})|_{\tilde{\Sigma}'}$ is in fact parallel on $\mathcal{F}|_{\tilde{\Sigma}'}$. The actions of $\text{Gal}(\delta)$ on $\tilde{\mathcal{L}}$ and $\delta^*\mathcal{O}_{\Sigma_k}^{\oplus 2}$ together produce an action of $\text{Gal}(\delta)$ on the vector bundle \mathcal{F} in (3.8). The logarithmic connection $\nabla^{\mathcal{F}}$ in (3.9) is evidently invariant under this action of $\text{Gal}(\delta)$ on \mathcal{F} .

Define the invariant direct image

$$\mathcal{L} := (\delta_*\tilde{\mathcal{L}})^{\text{Gal}(\delta)} \subset \delta_*\tilde{\mathcal{L}}$$

for the action of $\text{Gal}(\delta)$ on $\delta_*\tilde{\mathcal{L}}$. It is a holomorphic line bundle on Σ_k such that $\delta^*\mathcal{L} = \mathcal{O}_{\tilde{\Sigma}}(-q_1 - q_2 - q_3 - q_4)$, where $q_i \in \tilde{\Sigma}$ satisfies $\delta(q_i) = p_i$. The connection $\nabla^{\tilde{\mathcal{L}}}$ on $\tilde{\mathcal{L}}$, being preserved by the action of $\text{Gal}(\delta)$ on $\tilde{\mathcal{L}}$, produces a logarithmic connection $\nabla^{\mathcal{L}}$ on \mathcal{L} ; its residue is $\frac{1}{2}$ at each marked point p_i . Since the logarithmic connection $\nabla^{\tilde{\mathcal{L}}}$ has trivial monodromy representation, and the residues of $\nabla^{\mathcal{L}}$ are $\frac{1}{2}$, it follows that the image of the monodromy homomorphism for the above logarithmic connection $\nabla^{\mathcal{L}}$ on \mathcal{L} is exactly $\{\pm 1\} \subset \mathbb{C}^*$.

The above construction of \mathcal{L} from $\tilde{\mathcal{L}}$ shows that the pull-back bundle $\delta^*(\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L}) = \delta^*\mathcal{L}^{\oplus 2}$ is holomorphically isomorphic to $\mathcal{F} \otimes \mathcal{O}_{\tilde{\Sigma}}(-q_1 - q_2 - q_3 - q_4)$ (see (3.8)) by a $\text{Gal}(\delta)$ -equivariant holomorphic isomorphism.

The logarithmic connection $\nabla^{\mathcal{F}}$ in (3.9) descends to a logarithmic connection on $\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L}$, because $\nabla^{\mathcal{F}}$ is preserved by the action $\text{Gal}(\delta)$ on \mathcal{F} . This descended logarithmic connection on $\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L}$ clearly coincides with $f_k^*D \otimes \nabla^{\mathcal{L}}$.

The meromorphic frame $\tilde{\Psi} \otimes \mathbf{1}$ of \mathcal{F} descends to a holomorphic frame of $(\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L})|_{\Sigma'_k}$ because $\tilde{\Psi} \otimes \mathbf{1}$ is preserved by the action of $\text{Gal}(\delta)$. It was observed above that the holomorphic frame $(\tilde{\Psi} \otimes \mathbf{1})|_{\tilde{\Sigma}'}$ of $(\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L})|_{\tilde{\Sigma}'}$ is parallel with respect to the holomorphic connection $\nabla^{\mathcal{F}}|_{\tilde{\Sigma}'}$ in (3.9). Consequently, the holomorphic frame of $(\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L})|_{\Sigma'_k}$ given by $\tilde{\Psi} \otimes \mathbf{1}$ is parallel with respect to the holomorphic connection $f_k^*D \otimes \nabla^{\mathcal{L}}$ on $(\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L})|_{\Sigma'_k}$, completing the proof. \square

The parabolic structure on $\mathcal{O}_{S_4}^{\oplus 2}$ induced by the logarithmic connection D in (3.3) admits the strongly parabolic Higgs field

$$\Phi = \begin{pmatrix} 0 & \frac{dz}{z-1} - \frac{dz}{z+1} \\ \frac{dz}{z-\sqrt{-1}} - \frac{dz}{z+\sqrt{-1}} & 0 \end{pmatrix}. \quad (3.10)$$

The following lemma states that the singularities of Φ have the same behavior under pull-back and gauge transformation as the connection D itself.

Lemma 3.3. *Let Φ be the strongly parabolic Higgs field defined in (3.10), f_k the projection from Σ_k to S_4 in (3.2) and G the gauge transformation in Proposition 3.1. Then*

$$G^{-1} \circ f_k^* \Phi \circ G$$

extends to a holomorphic Higgs field on the trivial holomorphic bundle $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ over Σ_k .

Proof. As before we have to distinguish between even k and odd k . For odd k it is evident that $G^{-1} \circ f_k^* \Phi \circ G$ is a holomorphic Higgs field on Σ'_k with respect to the trivial holomorphic structure induced by $d = (f_k^*D).G$. We have to show that $G^{-1} \circ f_k^* \Phi \circ G$ is holomorphic at the branch points of f_k . Consider $p_1 = f_k^{-1}(1)$. Then the pull-back of Φ , considered as an endomorphism-valued 1-form, is meromorphic and of the form

$$f_k^* \Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

The diagonal entries of the pull-back vanish identically, while the lower left entry c has a zero of order $k-1$ at p_1 as f_k is totally branched. The upper right entry b has a pole of order 1 at p_1 . Since the meromorphic function Y (of degree 2) on Σ_k (see (3.1)) has a

zero of order 1 at p_1 , and $k > 1$,

$$G^{-1} \circ f_k^* \Phi \circ G = \begin{pmatrix} 0 & bY^{k-1} \\ cY^{-k+1} & 0 \end{pmatrix}$$

is holomorphic at p_1 . The same argument works for the other branch points p_2, p_3, p_4 of f_k showing that $G^{-1} \circ f_k^* \Phi \circ G$ is a holomorphic Higgs field on the trivial holomorphic bundle.

When k is even, we consider

$$f_k^* \Phi \cong f_k^* \Phi \otimes \mathbf{1}$$

as an endomorphism-valued 1-form on the vector bundle $\mathcal{O}_{\Sigma_k}^{\oplus 2} \otimes \mathcal{L}$ over Σ'_k . It is holomorphic with respect to the holomorphic structure induced by the connection $(f_k^* D) \otimes \nabla^{\mathcal{L}}$. The same arguments as for k odd then show that $G^{-1} \circ f_k^* \Phi \circ G$ extends to a holomorphic endomorphism-valued 1-form on the trivial bundle $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ over Σ_k . \square

The following proposition and its proof are similar to some results about symmetric minimal surfaces in the 3-sphere [HHS, Section 3.3].

Proposition 3.4. *Let $k \in \mathbb{N}^{>2}$ and $\tilde{\rho} = \frac{k-1}{2k}$. Consider the logarithmic connection $\tilde{\nabla}$ on $\mathcal{O}_{\mathcal{S}_4}^{\oplus 2}$ given in (2.9) and its pull-back $f_k^* \tilde{\nabla}$ on $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ with polar part in $\mathbf{D} = p_1 + p_2 + p_3 + p_4$. Then the parabolic structure associated to $\tilde{\nabla}$ is unstable. Furthermore,*

- (1) if k is odd
 - there exists a flat C^∞ connection on $\mathcal{O}_{\Sigma_k}^{\oplus 2} \rightarrow \Sigma_k$ which is C^∞ gauge equivalent to $(f_k^* \tilde{\nabla})|_{\Sigma'_k}$ over Σ'_k . In particular, $f_k^* \tilde{\nabla}$ has trivial local monodromy around the singular points $\{p_1, p_2, p_3, p_4\}$;
 - the monodromy homomorphism of $f_k^* \tilde{\nabla}$ is the one of the uniformizing hyperbolic structure of Σ_k , in particular it is Fuchsian.
- (2) If k is even, then there is a holomorphic line bundle \mathcal{L} over Σ_k with a logarithmic connection $\nabla^{\mathcal{L}}$ with polar part in $\mathbf{D} = p_1 + p_2 + p_3 + p_4$ such that
 - the image of the monodromy homomorphism for $\nabla^{\mathcal{L}}$ is $\{\pm 1\} \subset \mathbb{C}^*$;
 - there exists a C^∞ vector bundle isomorphism

$$G : \mathcal{O}_{\Sigma'_k}^{\oplus 2} \rightarrow \mathcal{O}_{\Sigma'_k}^{\oplus 2} \otimes \mathcal{L} = \mathcal{L}^{\oplus 2}$$

over $\Sigma'_k \subset \Sigma_k$ which gauges $((f_k^* \tilde{\nabla}) \otimes \nabla^{\mathcal{L}})|_{\Sigma'_k}$ to a C^∞ flat connection $\hat{\nabla}$ with Fuchsian monodromy on the trivial bundle over Σ_k ;

- $f_k^* \tilde{\nabla} \otimes \nabla^{\mathcal{L}}$ on $\mathcal{L}^{\oplus 2}$ has trivial local monodromy around the singular points $\{p_1, p_2, p_3, p_4\}$, and its monodromy representation coincides with the monodromy homomorphism of the uniformizing hyperbolic structure of Σ_k .

Remark 3.5. The holomorphic line bundle \mathcal{L} and the logarithmic connection $\nabla^{\mathcal{L}}$ in the statement of Proposition 3.4 (2) are the same as in the statement of Proposition 3.1 (2).

Proof of Proposition 3.4. The logarithmic connection $\tilde{\nabla}$ on \mathcal{S}_4 in (2.9) is

$$\tilde{\nabla} = d + \begin{pmatrix} 0 & \tilde{\rho}z^{-1} \\ \tilde{\rho}z & 0 \end{pmatrix} \left(\frac{dz}{z-1} - \frac{dz}{z+\sqrt{-1}} + \frac{dz}{z+1} - \frac{dz}{z-\sqrt{-1}} \right). \quad (3.11)$$

At each point of the singular locus $\{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ the eigenvalues of the residue of $\tilde{\nabla}$ are $\tilde{\rho}$ and $-\tilde{\rho}$. Using (3.11) we compute the eigenlines for the positive eigenvalue $\tilde{\rho}$ of the residues of $\tilde{\nabla}$ at $x \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ to be:

$$l_x = \mathbb{C} \cdot (x, 1) \subset \mathbb{C}^2.$$

Recall from Section 1 that the quasiparabolic structures at $\{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ are given by the eigenlines for the eigenvalue $\tilde{\rho}$. Let

$$\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2} \supset \mathbb{L} \longrightarrow \mathbb{C}P^1 \quad (3.12)$$

be the tautological subbundle whose fiber over any $z \in \mathbb{C}$ is $\mathbb{C} \cdot (z, 1)$ and the fiber over ∞ is $\mathbb{C} \cdot (1, 0)$. Therefore, at each point x of the singular locus $\{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ the subspace $\mathbb{L}_x \subset (\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2})_x = \mathbb{C}^2$ coincides with the eigenline of $\text{Res}_x(\tilde{\nabla})$ with respect to the eigenvalue $\tilde{\rho}$. Consequently, the parabolic degree of the line subbundle $\mathbb{L} \subset \mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$ in (3.12), with respect to the parabolic structure induced by $\tilde{\nabla}$ is

$$\text{par-deg}(\mathbb{L}) = \text{degree}(\mathbb{L}) + 4\tilde{\rho} = 4\tilde{\rho} - 1 > 0 = \text{par-deg}(\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}). \quad (3.13)$$

Therefore, $\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$ equipped with the parabolic structure given by $\tilde{\nabla}$ is unstable.

Consider the standard inner product on \mathbb{C}^2 . It produces a constant Hermitian structure on $\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$ which is flat with respect to the trivial holomorphic connection on $\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$. Let \mathbb{L}^\perp denote the orthogonal complement of the line subbundle \mathbb{L} in (3.12), so we have the C^∞ decomposition

$$\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2} = \mathbb{L} \oplus \mathbb{L}^\perp. \quad (3.14)$$

Note that \mathbb{L}^\perp is identified with $\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}/\mathbb{L} = \mathbb{L}^*$, because $\wedge^2 \mathcal{O}_{\mathbb{C}P^1}^{\oplus 2} = \mathcal{O}_{\mathbb{C}P^1}$. With respect to the decomposition in (3.14), the holomorphic structure of $\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$, which is the same as the $(0, 1)$ -part of the flat connection for $\tilde{\nabla}$, is

$$\bar{\partial}^{\tilde{\nabla}} = \begin{pmatrix} \bar{\partial}^{\mathbb{L}} & \psi \\ 0 & \bar{\partial}^{\mathbb{L}^*} \end{pmatrix}$$

for some non-trivial C^∞ section ψ of $\overline{K}_{\mathbb{C}P^1} \otimes \mathbb{L}^{\otimes 2}$ over $\mathbb{C}P^1$, where $\bar{\partial}^{\mathbb{L}}$ and $\bar{\partial}^{\mathbb{L}^*}$ are the Dolbeault operators for \mathbb{L} and \mathbb{L}^* respectively. The $(1, 0)$ -part $\partial^{\tilde{\nabla}}$ of $\tilde{\nabla}$ is

$$\partial^{\tilde{\nabla}} = \begin{pmatrix} \partial^{\mathbb{L}} & \alpha \\ \varphi & \partial^{\mathbb{L}^*} \end{pmatrix}, \quad (3.15)$$

where $\partial^{\mathbb{L}}$ is a C^∞ $(1, 0)$ -connection on the holomorphic line bundle \mathbb{L} over S_4 (defined in (2.5)), and $\partial^{\mathbb{L}^*}$ is the dual $(1, 0)$ -connection on $\mathbb{L}^*|_{S_4}$. Furthermore, in (3.15) α is a C^∞ section of $K_{\mathbb{C}P^1} \otimes \mathbb{L}^{\otimes 2}$ over S_4 , and φ is a holomorphic section of $K_{\mathbb{C}P^1} \otimes (\mathbb{L}^*)^{\otimes 2}$. In fact φ is the second fundamental form of the holomorphic subbundle $\mathbb{L} \subset \mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}$ for the logarithmic connection $\tilde{\nabla}$. We note that φ is holomorphic over the entire $\mathbb{C}P^1$ because at every singular point q_l of $\tilde{\nabla}$, the fiber $\mathbb{L}_{q_l} \subset (\mathcal{O}_{\mathbb{C}P^1}^{\oplus 2})_{q_l}$ is an eigenline of the residue of $\tilde{\nabla}$.

If $\varphi = 0$, then the line subbundle \mathbb{L} is preserved by $\tilde{\nabla}$ which gives a contradiction since the parabolic degree \mathbb{L} with respect to the induced parabolic structure is nonzero (see (3.13) and [Oh]). Hence we conclude that $\varphi \neq 0$ and, by choosing a suitable holomorphic isomorphism between $K_{\mathbb{C}P^1}$ and $\mathbb{L}^{\otimes 2}$, we can normalize φ to be the constant function 1.

Consider the pulled back logarithmic connection $f_k^* \tilde{\nabla}$ on the trivial holomorphic vector bundle $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ over Σ_k , where f_k is the map in (3.2). It is singular over the four branched points p_1, \dots, p_4 in (3.5).

Case 1: k is odd.

We desingularize $f_k^* \tilde{\nabla}$ at p_l , $1 \leq l \leq 4$, as follows. Take a holomorphic coordinate function z defined on an open neighborhood of $f_k(p_l) \in \mathbb{C}P^1$ with $z(f_k(p_l)) = 0$. Let y be a holomorphic coordinate function defined on an open subset $U_l \subset \Sigma_k$ containing p_l such that $y^k = z \circ f_k$. Consider the meromorphic endomorphism

$$h_l = \begin{pmatrix} y^{-\frac{k-1}{2}} & 0 \\ 0 & y^{\frac{k-1}{2}} \end{pmatrix} \quad (3.16)$$

of $\mathcal{O}_{U_l}^{\oplus 2} = \mathcal{O}_{\Sigma_k}^{\oplus 2}|_{U_l}$. It is a holomorphic automorphism over $U'_l := U_l \setminus \{p_l\}$. Let $(f_k^* \tilde{\nabla})|_{U'_l} \cdot (h_l|_{U'_l})$ be the holomorphic connection on $\mathcal{O}_{U'_l}^{\oplus 2}$ produced by the action of the gauge transformation $h_l|_{U'_l}$ on the connection $(f_k^* \tilde{\nabla})|_{U'_l}$.

We claim that $(f_k^* \tilde{\nabla})|_{U'_l} \cdot (h_l|_{U'_l})$ extends to a C^∞ connection on $\mathcal{O}_{U_l}^{\oplus 2}$. To prove the above claim, first note that the upper right entry of the connection $(f_k^* \tilde{\nabla})|_{U'_l} \cdot (h_l|_{U'_l})$ (with respect to the splitting $\mathbb{L} \oplus \mathbb{L}^\perp$) is multiplied with the function y^{k-1} and is therefore smooth at p_l (it vanishes at p_l with some higher order). Moreover, the pull-back $f_k^* \varphi$ of the non-vanishing 1-form φ with values in $\mathcal{O}_{\mathbb{C}P^1}(2)$ has vanishing order $k-1$ at p_l . Hence, the lower left entry of $(f_k^* \tilde{\nabla})|_{U'_l} \cdot (h_l|_{U'_l})$ with respect to (3.14), which becomes

$$y^{1-k} f_k^* \varphi,$$

extends smoothly and non-vanishingly to p_l . This proves the claim.

Since $(f_k^* \tilde{\nabla})|_{U'_l} \cdot (h_l|_{U'_l})$ extends to a C^∞ connection on $\mathcal{O}_{U_l}^{\oplus 2}$, the local monodromy of $f_k^* \tilde{\nabla}$ at each p_l is trivial.

Now fix a global C^∞ automorphism

$$h \in C^\infty(\Sigma'_k, \text{Aut}(\mathcal{O}_{\Sigma'_k}^{\oplus 2})) \quad (3.17)$$

such that $\det h = 1$ and, for each $1 \leq l \leq 4$, it coincides with h_l (see (3.16)) on a neighborhood of p_l ; such a global gauge h does exist. From the above observation that $(f_k^* \tilde{\nabla})|_{U'_l} \cdot (h_l|_{U'_l})$ extends to a C^∞ connection on $\mathcal{O}_{U_l}^{\oplus 2}$ it follows immediately that $(f_k^* \tilde{\nabla}) \cdot h$ is a C^∞ flat connection on the trivial C^∞ vector bundle

$$\Sigma_k \times \mathbb{C}^2 =: E^0.$$

The holomorphic structure on E^0 given by the flat connection $(f_k^* \tilde{\nabla}) \cdot h$ is not the trivial holomorphic structure on $\mathcal{O}_{\Sigma_k}^{\oplus 2}$, as h in (3.17) is not holomorphic. In fact, we claim that it is a uniformization bundle on Σ_k .

Let \mathcal{E}^0 denote the holomorphic vector bundle over Σ_k given by the holomorphic structure of $(f_k^* \tilde{\nabla}) \cdot h$. Since $\det h = 1$, it follows that \mathcal{E}^0 is a holomorphic $\text{SL}(2, \mathbb{C})$ -bundle with $(f_k^* \tilde{\nabla}) \cdot h$ being a holomorphic $\text{SL}(2, \mathbb{C})$ -connection on it.

Consider the pulled back line bundle

$$f_k^* \mathbb{L} \subset f_k^* \mathcal{O}_{\mathbb{C}P^1}^{\oplus 2} = \mathcal{O}_{\Sigma_k}^{\oplus 2},$$

where \mathbb{L} is the tautological bundle constructed in (3.12). Note that $h(f_k^* \mathbb{L}) \subset \mathcal{E}^0|_{\Sigma'_k}$ is a holomorphic line subbundle (recall that h in (3.17) is defined only on Σ'_k). Since h is meromorphic near each p_l (as h_l in (3.16) is meromorphic around p_l and h coincides with h_l around p_l), we conclude that $h(f_k^* \mathbb{L}) =: \tilde{\mathbb{L}}$ extends to a holomorphic subbundle of \mathcal{E}^0 over the entire Σ_k .

For $1 \leq l \leq 4$ fixed, let s be a non-vanishing holomorphic section of \mathbb{L} defined on an open subset $\tilde{U}_l \subset \mathbb{C}P^1$ around $f_k(p_l)$. Then the holomorphic section $f_k^*(s|_{\tilde{U}_l \setminus \{f_k(p_l)\}})$ of $\tilde{\mathbb{L}}|_{f_k^{-1}(\tilde{U}_l \setminus \{f_k(p_l)\})}$ extends to a holomorphic section of $\tilde{\mathbb{L}}|_{f_k^{-1}(\tilde{U}_l)}$ vanishing at p_l with order $(k-1)/2$. Indeed, this follows immediately from the expression of h_l in (3.16). From this we conclude that

$$\text{degree}(\tilde{\mathbb{L}}) = \text{degree}(f_k) \cdot \text{degree}(\mathbb{L}) + 4 \frac{k-1}{2} = -k + 2k - 2 = k - 2 = \text{genus}(\Sigma_k) - 1. \quad (3.18)$$

Lemma 2.2 then shows that the monodromy representation of $(f_k^* \tilde{\nabla}).h$ is conjugate to $\text{SL}(2, \mathbb{R})$ and its Euler class is maximal by (3.18). More precisely, since the map f_k in (3.2) is a k -fold covering of \mathcal{S}_4 totally branched over the 4 marked points, Lemma 2.2 gives that the monodromy representation for $(f_k^* \tilde{\nabla}).h$ coincides with the monodromy of the uniformizing hyperbolic structure for Σ_k . Therefore, the monodromy homomorphism of the connection $(f_k^* \tilde{\nabla}).h$ coincides with the one given by the hyperbolic uniformization of Σ_k .

Case 2: k is even.

Following the same desingularization procedure as in the previous case, consider the local gauge transformation

$$h_l = \begin{pmatrix} y^{-\frac{k-1}{2}} & 0 \\ 0 & y^{\frac{k-1}{2}} \end{pmatrix} \quad (3.19)$$

with respect to the pull-back by f_k of the C^∞ decomposition of the rank 2 bundle in (3.14). As in the proof of point (2) of Proposition 3.1, the values of $y^{\frac{k-1}{2}}$ produce a ramified double covering of Σ_k

$$\delta : \tilde{\Sigma} \longrightarrow \Sigma_k$$

which is ramified exactly over the subset $\{p_1, p_2, p_3, p_4\}$ in (3.5). As before let

$$\tilde{\Sigma}' := \delta^{-1}(\Sigma'_k) \subset \tilde{\Sigma}$$

be the largest open subset such that $\delta|_{\tilde{\Sigma}'} : \tilde{\Sigma}' \longrightarrow \Sigma'_k$ is an unramified double covering. Let $q_l \in \tilde{\Sigma}$, $1 \leq l \leq 4$, be the points such that $\delta(q_l) = p_l$. As in the proof of part (1), fix a C^∞ automorphism

$$h : \mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2} \longrightarrow \mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2}$$

such that

- $\det h = 1$,
- the action of $\text{Gal}(\delta) = \mathbb{Z}/2\mathbb{Z}$ on $\delta^* \mathcal{O}_{\Sigma'_k}^{\oplus 2} = \mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2}$ takes h to $-h$, and
- the restriction of h near each marked point q_l coincides with

$$h_l = \begin{pmatrix} \tilde{y}^{1-k} & 0 \\ 0 & \tilde{y}^{k-1} \end{pmatrix},$$

where $\tilde{y}^{2k} = z \circ f_k \circ \delta$ with z being a holomorphic coordinate function around $f_k(p_l) \in \mathbb{C}P^1$ with $z(f_k(p_l)) = 0$.

The C^∞ connection $((\delta^* f_k^* \tilde{\nabla})|_{\tilde{\Sigma}'}) . h$ on $\mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2}$ (considered as the trivial C^∞ vector bundle) extends to a flat connection on $\delta^* \mathcal{O}_{\Sigma'_k}^{\oplus 2} = \mathcal{O}_{\tilde{\Sigma}'}^{\oplus 2}$ preserved by the action of $\text{Gal}(\delta)$ on $\delta^* \mathcal{O}_{\Sigma'_k}^{\oplus 2}$. Hence it induces a C^∞ flat connection on the trivial bundle $\mathcal{O}_{\Sigma_k}^{\oplus 2}$; this flat connection is $\hat{\nabla}$ in the statement of the proposition. As before we emphasize that the

holomorphic structure given by $\widehat{\nabla}$ does not coincide with the natural holomorphic structure of $\mathcal{O}_{\Sigma_k}^{\oplus 2}$ but gives a uniformization bundle.

In order to see how exactly $\widehat{\nabla}$ and $f_k^* \widetilde{\nabla}$ correspond to each other on Σ_k , we consider the holomorphic line bundle $\mathcal{L} \rightarrow \Sigma_k$ equipped with the logarithmic connection $\nabla^{\mathcal{L}}$ as in the proof of point (2) in Proposition 3.1. It is straightforward to check that $\widehat{\nabla}$ and $(\mathcal{L}, \nabla^{\mathcal{L}})$ satisfy all the properties stated in the proposition. The homomorphism ψ in the proposition is given by $h \otimes \mathbf{1}$. \square

Remark 3.6. The reason why we have to use a 2-valued gauge transformation for even k (and hence the flat line bundle $(\mathcal{L}, \nabla^{\mathcal{L}})$) is that a hyperbolic isometric rotation by an angle $\frac{2\pi}{k}$ for $\tilde{k} \in 2\mathbb{Z}$ cannot be represented by an $\mathrm{SL}(2, \mathbb{R})$ -matrix of order \tilde{k} but only by a $\mathrm{SL}(2, \mathbb{R})$ -matrix of order $2\tilde{k}$. See also [BoHS, Section 4] for the related case of symmetric minimal surfaces in \mathbb{S}^3 .

Proposition 3.7. *Let $k \in \mathbb{N}^{\geq 3}$ and $\tilde{\rho} = \frac{k-1}{2k}$. Fix base points $p_0 \in S_4$ and $p \in f_k^{-1}(p_0) \subset \Sigma_k \setminus \{p_1, p_2, p_3, p_4\}$. Consider two logarithmic connections D_1 and D_2 on $\mathcal{O}_{S_4}^{\oplus 2}$ such that the two monodromy homomorphisms lie in the same connected component of $\mathrm{Hom}(\pi_1(S_4, p_0), \mathrm{SL}(2, \mathbb{R}))$, with the same prescribed local conjugacy classes determined by the parabolic weight $\tilde{\rho}$. Then the following hold:*

- (1) *If k is odd, the pull-back through f_k in (3.2) of the monodromy representations of D_1 and D_2 lie in the same connected component of $\mathrm{Hom}(\pi_1(\Sigma_k, p), \mathrm{SL}(2, \mathbb{R}))$.*
- (2) *If k is even, then the monodromy representations of $(f_k^* D_j) \otimes \nabla^{\mathcal{L}}$, $j = 1, 2$, lie in the same connected component of $\mathrm{Hom}(\pi_1(\Sigma_k, p), \mathrm{SL}(2, \mathbb{R}))$, where $\nabla^{\mathcal{L}}$ is the logarithmic connection defined in Proposition 3.4.*

Proof. We prove the statement only for odd k ; the even case works analogously. The (unbranched) covering $f_k : \Sigma'_k \rightarrow S_4$ induces a covering-monodromy

$$\pi_1(S_4, p_0) \longrightarrow \mathcal{S}(k)$$

into the symmetric group $\mathcal{S}(k) \cong \mathcal{S}(f_k^{-1}(p_0))$. Therefore, its first fundamental group $\pi_1(\Sigma_k \setminus \{p_1, \dots, p_4\}, p)$ can be identified with the subgroup of $\pi_1(S_4, p_0)$ which is given by the kernel of the covering-monodromy. Moreover, the inclusion map

$$\Sigma_k \setminus \{p_1, p_2, p_3, p_4\} \hookrightarrow \Sigma_k$$

induces a surjective homomorphism of fundamental groups

$$\pi_1(\Sigma_k \setminus \{p_1, p_2, p_3, p_4\}, p) \longrightarrow \pi_1(\Sigma_k, p).$$

The monodromy morphism commutes with the pull-back by f_k . Moreover, since D_1 and D_2 have real monodromy representation and parabolic weights $\frac{k-1}{2k}$, the monodromy representation of the flat connection $f_k^* D_j$ on $\Sigma_k \setminus \{p_1, p_2, p_3, p_4\}$ factors through a representation of $\pi_1(\Sigma_k, p)$, for $j = 1, 2$ (as the local monodromy at the marked points is trivial for both the connections). By hypothesis, the monodromy homomorphisms for D_1 and D_2 are in the same connected component of $\mathrm{SL}(2, \mathbb{R})$ -representations, and hence their monodromy representations can be joined by a continuous path inside the space of $\mathrm{SL}(2, \mathbb{R})$ -representations of $\pi_1(S_4, p_0)$ with fixed local monodromies. The pull-back of this path to the subspace of $\mathrm{SL}(2, \mathbb{R})$ -representations of $\pi_1(\Sigma_k \setminus \{p_1, \dots, p_4\}, p)$, lying in the kernel of the covering-monodromy, is continuous as well. Moreover, by the same arguments as above, all these representations (determined by the path) factor through representations of $\pi_1(\Sigma_k, p)$. Recall that $\pi_1(\Sigma_k \setminus \{p_1, p_2, p_3, p_4\}, p) \rightarrow \pi_1(\Sigma_k, p)$ is

surjective. Therefore, $f_k^* D_1$ and $f_k^* D_2$ are in the same connected component of $\mathrm{SL}(2, \mathbb{R})$ -representations. \square

4. LOGARITHMIC CONNECTIONS ON THE SQUARE TORUS WITH ONE MARKED POINT

We consider the square torus

$$T^2 := \mathbb{C}/\Gamma \quad (4.1)$$

with lattice

$$\Gamma = \mathbb{Z} + \sqrt{-1}\mathbb{Z} \subset \mathbb{C}$$

and one marked point $o = [0] \in T^2$. The point $\frac{1+\sqrt{-1}}{4} \in T^2$ will be denoted by p_0 .

Recall that the fundamental group $\pi_1(T^2 \setminus \{o\}, p_0)$ of the one-punctured torus $T^2 \setminus \{o\}$ is a free group of two generators; it is generated by $\gamma_x, \gamma_y \in \pi_1(T^2 \setminus \{o\})$, where

$$\gamma_x : [0, 1] \longrightarrow T^2 \setminus \{o\}; \quad s \longmapsto s + \frac{1 + \sqrt{-1}}{4} \quad (4.2)$$

and

$$\gamma_y : [0, 1] \longrightarrow T^2 \setminus \{o\}; \quad s \longmapsto \sqrt{-1}s + \frac{1 + \sqrt{-1}}{4}.$$

The commutator $\gamma_y^{-1}\gamma_x^{-1}\gamma_y\gamma_x \in \pi_1(T^2 \setminus \{o\})$ corresponds to a simple loop going around the marked point o .

4.1. The character variety of the one-punctured torus.

For $\rho \in]0, \frac{1}{2}[$, let $\mathcal{M}_{1,1}^\rho$ be the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ -connections on the one-punctured torus $T^2 \setminus \{o\}$ (defined in (4.1)) with local monodromy around the puncture o lying in the conjugacy class of the element

$$\begin{pmatrix} e^{-2\pi\sqrt{-1}\rho} & 0 \\ 0 & e^{2\pi\sqrt{-1}\rho} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \quad (4.3)$$

The above de Rham moduli space $\mathcal{M}_{1,1}^\rho$ depends only on the topology of $T^2 \setminus \{o\}$; in particular, it does not depend on the complex structure of T^2 . The conjugacy class of the element in (4.3) is determined by its trace, which is $2 \cos(2\pi\rho)$; see [Go4].

For a flat $\mathrm{SL}(2, \mathbb{C})$ -connection ∇ on $T^2 \setminus \{o\}$, let X, Y denote its monodromies along $\gamma_x, \gamma_y \in \pi_1(T^2 \setminus \{o\}, p_0)$ (defined in (4.2)) respectively. Let

$$x = \mathrm{tr}(X), \quad y = \mathrm{tr}(Y), \quad z = \mathrm{tr}(YX). \quad (4.4)$$

The moduli space $\mathcal{M}_{1,1}^\rho$ is diffeomorphic (via the monodromy mapping) to the character variety of the one-punctured torus for which the conjugacy class of the local monodromy at the puncture is the one in (4.3); this character variety is given by the equation

$$x^2 + y^2 + z^2 - xyz - 2 - 2 \cos(2\pi\rho) = 0, \quad (4.5)$$

where $x, y, z \in \mathbb{C}$. Equivalently, for a fixed $\rho \in]0, \frac{1}{2}[$, any triple $(x, y, z) \in \mathbb{C}^3$ satisfying (4.5) determines, up to conjugacy, a unique representation of $\pi_1(T^2 \setminus \{o\}, p_0)$ into $\mathrm{SL}(2, \mathbb{C})$ such that the local monodromy around the puncture is conjugate to (4.3), and x, y, z are as in (4.4); see [Go4]. Note that the character variety is smooth for $\rho \in]0, \frac{1}{2}[$. The next lemma gives a characterization of the real points in this character variety.

Lemma 4.1. *Take $\Theta \in \mathrm{Hom}(\pi_1(T^2 \setminus \{o\}, p_0), \mathrm{SL}(2, \mathbb{C}))$, and denote $X = \Theta(\gamma_x)$, $Y = \Theta(\gamma_y)$. Assume that $x = \mathrm{tr}(X)$, $z_1 = \mathrm{tr}(YX)$ and $z_2 = \mathrm{tr}(Y^{-1}X)$ are real. Then either $y = \mathrm{tr}(Y) \in \mathbb{R}$ or $x = 0$.*

Proof. A short computation (see also [Go4]) shows that up to conjugation we can choose

$$X = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\zeta \\ \zeta^{-1} & y \end{pmatrix} \quad (4.6)$$

with

$$z_1 = \zeta^{-1} + \zeta.$$

For given x, y the traces z_1 and z_2 are solutions of the quadratic equation in (4.5). Using (4.6) we compute that

$$z_2 = xy - z_1.$$

If $x, z_1, z_2 \in \mathbb{R}$, then either $y \in \mathbb{R}$ or $x = 0$. \square

The following theorem proved in [Go4, Section 2.6 & Section 3.3] describes the connected components of the real points in the character variety.

Theorem 4.2 (Goldman [Go4]). *For $\rho \in]0, \frac{1}{2}[$, the set of real points of the character variety defined by (4.5) has 5 connected components. There is one compact component which is characterized by $x, y, z \in [-2, 2]$, and there are 4 non-compact components which are all diffeomorphic to each other. The compact component consists of $\mathrm{SU}(2)$ -representations and the non-compact components consist of $\mathrm{SL}(2, \mathbb{R})$ -representations.*

Remark 4.3. The four non-compact components of the character variety are interchanged by the group of sign-change automorphisms [Go4, Section 2.2.1 & Section 2.6]. This means, that the coordinates (x, y, z) are mapped to $((-1)^{\epsilon_1}x, (-1)^{\epsilon_2}y, (-1)^{\epsilon_3}z)$ where $\epsilon_l \in \{0, 1\}$ for $l = 1, 2, 3$, such that $\epsilon_1 + \epsilon_2 + \epsilon_3 \in \{0, 2\}$. In terms of the Hitchin-Kobayashi correspondence, these four components correspond to the four distinct spin structures on a torus.

4.2. The de Rham moduli space of the one-punctured torus.

Let w be the global coordinate on the universal covering \mathbb{C} of T^2 in (4.1). Since T^2 is a square torus, there exists an anti-holomorphic involution

$$\eta: T^2 \longrightarrow T^2, \quad [w] \longmapsto [-\sqrt{-1}\bar{w}] \quad (4.7)$$

on T^2 corresponding to the reflection along a diagonal of the square. Note that the marked point $o \in T^2$ is fixed by the map η . The induced real involution of the de Rham moduli space

$$\mathcal{M}_{1,1}^\rho \longrightarrow \mathcal{M}_{1,1}^\rho, \quad [\nabla] \longmapsto [\eta^*\bar{\nabla}]$$

is well-defined as ρ is real.

For notational convenience we denote by L the trivial C^∞ bundle $T^2 \times \mathbb{C} \longrightarrow T^2$. Let $a, \chi \in \mathbb{C}$ be coordinates of $\mathcal{M}_{1,1}^\rho$ obtained from abelianization (see [BDH, (2.3)], or [HH]). For this purpose recall from [BDH, (2.3)] that any element in $\mathcal{M}_{1,1}^\rho$ (with $\rho \in]0, \frac{1}{2}[$) is represented by a logarithmic flat connection on $L \oplus L^*$ with a unique pole at o

$$\nabla = \nabla^{a,\chi,\rho} = \begin{pmatrix} \nabla^L & \gamma_\chi^- \\ \gamma_\chi^+ & \nabla^{L^*} \end{pmatrix}, \quad (4.8)$$

where ∇^L is the flat connection on L defined by

$$\nabla^L = d + adw + \chi d\bar{w}; \quad (4.9)$$

w being the above global holomorphic coordinate of T^2 and $a, \chi \in \mathbb{C}$. Moreover ∇^{L^*} is its dual connection on L^* , while γ_χ^+ and γ_χ^- are meromorphic sections with respect to the holomorphic structure given by the Dolbeault operators $\bar{\partial}^0 - 2\chi d\bar{w}$ and $\bar{\partial}^0 + 2\chi d\bar{w}$ respectively, with simple poles at $o \in T^2$ and residues determined by ρ . Here $\bar{\partial}^0 = d''$ is

the $(0, 1)$ -part of the de Rham differential operator d ; in particular, there is a holomorphic structure induced by ∇ in (4.8) on L , the one given given by the Dolbeault operator $\bar{\partial}^0 + \chi d\bar{w}$.

Remark 4.4. Note that the parabolic weight at o of the logarithmic connection $\nabla^{a, \chi, \rho}$ is ρ . The parabolic line is determined, up to a holomorphic automorphism of $L \oplus L^*$, by the condition that it is neither the line L_o , nor the line L_o^* .

Lemma 4.5. *Let $\Gamma^* := \pi\mathbb{Z} + \sqrt{-1}\pi\mathbb{Z}$. The gauge class of the connection $\nabla = \nabla^{a, \chi, \rho}$ as in (4.8) on the one-punctured torus is fixed by the involution η defined in (4.7) if*

$$\chi \in (1 - \sqrt{-1})\mathbb{R} \setminus \frac{1}{2}\Gamma^* \quad \text{and} \quad a \in (1 + \sqrt{-1})\mathbb{R},$$

or

$$\chi \in (1 + \sqrt{-1})\mathbb{R} \setminus \frac{1}{2}\Gamma^* \quad \text{and} \quad a \in (1 - \sqrt{-1})\mathbb{R}.$$

Proof. We have

$$\eta^* dw = -\sqrt{-1}d\bar{w} \quad \text{and} \quad \eta^* d\bar{w} = \sqrt{-1}dw.$$

Hence, for $\chi \in (1 - \sqrt{-1})\mathbb{R}$ and $a \in (1 + \sqrt{-1})\mathbb{R}$,

$$\eta^* \overline{\nabla^L} = \nabla^L$$

with ∇^L given by (4.9). By [BDH, Proposition 2.5] the meromorphic sections γ_χ^\pm in (4.8), described above, are unique, up to scaling, under the given condition that the quadratic residue at $o \in T^2$ of the meromorphic quadratic differential

$$\gamma_\chi^+ \gamma_\chi^- (dw)^2$$

is ρ^2 . Thus, we obtain constants $c^+, c^- \in \mathbb{C}^*$, with $c^+ c^- = 1$, such that

$$\eta^* \overline{\gamma_\chi^\pm} dw = c^\pm \gamma_\chi^\pm dw.$$

In particular, ∇ and $\eta^* \overline{\nabla}$ are gauge equivalent. If $\chi \in -(1 - \sqrt{-1})\mathbb{R} \setminus \frac{1}{2}\Gamma^*$ and $a \in -(1 + \sqrt{-1})\mathbb{R}$ then $\eta^* \overline{\nabla^L} = (\nabla^L)^*$, and the proof works analogously. \square

Lemma 4.6. *Let $\nabla = \nabla^{a, \chi, \rho}$ be a connection on $T^2 \setminus \{o\}$ as in (4.8) with $[\eta^* \overline{\nabla}] = [\nabla]$. Then*

$$z_1 = \text{tr}(YX) \in \mathbb{R} \quad \text{and} \quad z_2 = \text{tr}(Y^{-1}X) \in \mathbb{R}.$$

Proof. Consider the p_0 -based loops γ^{z_1} and γ^{z_2} on $T^2 \setminus \{o\}$ which are the concatenations of the loops γ_x and γ_y (defined in 4.2) and of the loops γ_x and γ_y^{-1} respectively. Their corresponding elements in $\pi_1(T^2 \setminus \{o\}, p_0)$ (for which we use the same notation) satisfy $\gamma^{z_1} = \gamma_y \gamma_x$ and $\gamma^{z_2} = \gamma_y^{-1} \gamma_x$. By definition z_1 and z_2 are the traces of the monodromy of ∇ along the loops γ^{z_1} and γ^{z_2} respectively.

Note that the real involution η in (4.7) maps the closed curve γ^{z_2} to a curve $\eta(\gamma^{z_2})$ which is free homotopic (i.e., without fixed base point) to γ^{z_2} ; see Figure 1. Since by hypothesis $\eta^* \overline{\nabla} \cong \nabla$, we thus obtain that $z_2 = \bar{z}_2$. Similarly, the closed curve γ^{z_1} is mapped by η to a curve $\eta(\gamma^{z_1})$ which is free homotopic to $(\gamma^{z_1})^{-1}$; see Figure 1. As $\text{tr}(M) = \text{tr}(M^{-1})$ for every $M \in \text{SL}(2, \mathbb{C})$, we obtain that $z_1 = \bar{z}_1$. \square

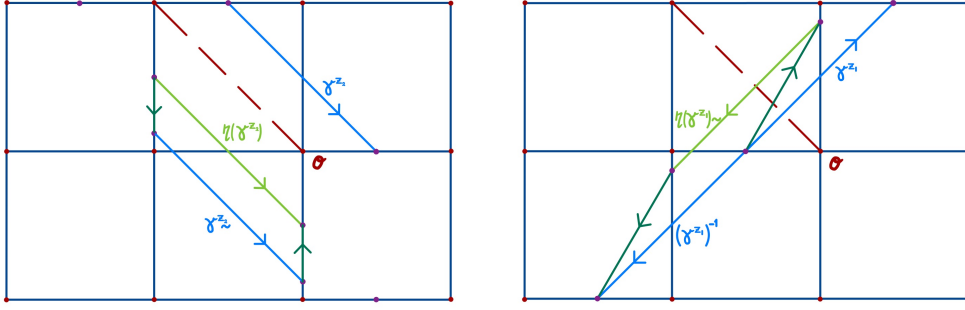


FIGURE 1. The curves γ^{z_2} and γ^{z_1} : their monodromy traces are z_2 and z_1 .

4.3. A consequence of WKB analysis.

Fix $\rho \in]0, \frac{1}{2}[$ and

$$\chi^0 = \frac{\pi}{4}(1 - \sqrt{-1}), \quad a^0 = \frac{\pi}{4}(1 + \sqrt{-1}). \quad (4.10)$$

Consider the family of flat connections, parametrized by $t \in \mathbb{R}$ on $T^2 \setminus \{o\}$ (defined in (4.1)) given by

$$\nabla^t := \nabla^{(1-t)a^0, \chi^0, \rho} = \nabla^{a^0, \chi^0, \rho} + t \frac{\pi}{4}(1 + \sqrt{-1}) \begin{pmatrix} -dw & 0 \\ 0 & dw \end{pmatrix}.$$

In this section we study the behavior of

$$x(t) := \text{tr}(X(t)),$$

where $X(t)$ is the monodromy of ∇^t along $\gamma_x \in \pi_1(T^2 \setminus \{o\}, p_0)$. By Lemma 4.5, the connection ∇^t is compatible with the involution η (see (4.7)), in the sense that $[\nabla^t] = \eta^*[\overline{\nabla^t}]$ for all $t \in \mathbb{R}$. In particular, the traces $z_1(t)$, $z_2(t)$ defined in Lemma 4.6 are real for all ∇^t , with $t \in \mathbb{R}$.

From the definition of γ_x in (4.2) we have

$$\gamma'_x(s) = 1 \quad \forall s \in [0, 1].$$

For the vector $v = 1 \in \mathbb{C}$ we have

$$\text{Re}(-\frac{\pi}{4}(1 + \sqrt{-1})dw(v)) = -1 > 0.$$

Hence, the curve γ_x is a WKB curve (see Section A.1 in the Appendix) for the 1-form

$$-\frac{\pi}{4}(1 + \sqrt{-1})dw.$$

From Corollary A.4 of the Appendix (compare also with [GMN, Appendix 4]) we get a non-zero constant $C \in \mathbb{C}^*$ such that

$$\lim_{t \in \mathbb{R}^{>0}, t \rightarrow \infty} x(t) \exp(-t\pi \frac{1 + \sqrt{-1}}{4}) = C. \quad (4.11)$$

From this the following corollary is obtained.

Corollary 4.7. *There exist a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $x(t_n)$ is real and non-zero for every $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} t_n = \infty$. In particular, the monodromy representation of ∇^{t_n} is conjugate to a $\text{SL}(2, \mathbb{R})$ -representation for all n .*

Proof. Equation (4.11) with $C \neq 0$ yields a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that

$$x(t_n) \in \mathbb{R} \setminus [-2, 2]$$

for all $n \in \mathbb{N}$. From Lemma 4.6 we know that $z_1 = z_1(t_n)$ and $z_2 = z_2(t_n)$ are both real. Recall that

$$z_2 = xy - z_1$$

and $x(t_n) \neq 0$. Therefore, Lemma 4.1 shows that $y(t_n) \in \mathbb{R}$ for all n , and hence the representation is given by a real point in the character variety. Since $x(t_n) \in \mathbb{R} \setminus [-2, 2]$, Goldman's result (Theorem 4.2) implies that the monodromy representation of ∇^{t_n} is conjugated to an $\mathrm{SL}(2, \mathbb{R})$ representation for all $n \in \mathbb{N}$. \square

Corollary 4.7 shows the existence of logarithmic connections ∇^{t_n} on the one-punctured torus T^2 with real monodromy. Recall that the Dolbeault operator $\bar{\partial}^0 + \chi d\bar{w}$ is gauge equivalent with $\bar{\partial}^0$ (and hence defines the trivial holomorphic line bundle structure on L) if and only if $2\chi \in \Gamma^*$, with Γ^* defined in Lemma 4.5. Hence, it follows that the holomorphic structure of L induced by ∇^{t_n} (for which $\chi^0 = \frac{\pi}{4}(1 - \sqrt{-1})$) is that of a holomorphic line bundle of order 4 on T^2 . In order to lift ∇^{t_n} , for an appropriate ρ , to the Riemann surface Σ_k , we first need to relate the moduli space $\mathcal{M}_{1,1}^\rho$ in Section 4.1 with the moduli space of flat connections $\mathcal{M}_{0,4}^{\tilde{\rho}}$ on S_4 .

4.4. Abelianization and connection.

In [HH], logarithmic $\mathfrak{sl}(2, \mathbb{C})$ -connections $d + \xi$ on the rank two trivial holomorphic bundle on $\mathbb{C}P^1$ with four marked points $\{\pm 1, \pm\sqrt{-1}\}$ are studied by an abelianization procedure. We need to recall (and adapt to our situation) some of the results of [HH]. We restrict hereby to logarithmic connection on $\mathcal{O}_{S_4}^{\oplus 2}$ such that all residues have the same eigenvalues

$$\pm \tilde{\rho} \quad \text{for some} \quad \tilde{\rho} \in]\frac{1}{4}, \frac{1}{2}[. \quad (4.12)$$

4.4.1. *The character variety of a four-punctured sphere.* As before S_4 denotes the complex projective line $\mathbb{C}P^1$ with punctures at the points

$$x_l := e^{(l-1)\sqrt{-1}\frac{\pi}{2}} \quad (4.13)$$

for $l = 1, \dots, 4$ and $p_0 \in S_4$ a base point. For any $l = 1, \dots, 4$, consider a simple oriented p_0 -based loop γ_{x_l} going around the puncture x_l . The fundamental group $\pi_1(S_4, p_0)$ is generated by γ_{x_l} , with $l = 1, \dots, 4$; the generators satisfy the relation $\gamma_{x_4}\gamma_{x_3}\gamma_{x_2}\gamma_{x_1} = \mathrm{Id}$. The following is a well-known result dating back to Fricke; see [Go2].

Any $\mathrm{SL}(2, \mathbb{C})$ -representation of $\pi_1(S_4, p_0)$ is determined by the images $M_l \in \mathrm{SL}(2, \mathbb{C})$ of the generators $\gamma_{x_l} \in \pi_1(S_4, p_0)$, for $l = 1, \dots, 4$. We have

$$M_4 M_3 M_2 M_1 = \mathrm{Id}.$$

Let

$$\mu = 2 \cos(2\pi\tilde{\rho}).$$

We restrict to the case

$$\mathrm{tr}(M_l) = \mu \quad \forall \quad l = 1, \dots, 4.$$

If the representation is irreducible or totally reducible, the traces

$$\tilde{x} = \mathrm{tr}(M_2 M_1), \quad \tilde{y} = \mathrm{tr}(M_3 M_2), \quad \tilde{z} = \mathrm{tr}(M_3 M_1)$$

determine the representation uniquely up to conjugation. Moreover, these affine coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ satisfy the equation

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 + \tilde{x}\tilde{y}\tilde{z} - 2\mu^2(\tilde{x} + \tilde{y} + \tilde{z}) + 4(\mu^2 - 1) + \mu^4 = 0. \quad (4.14)$$

Furthermore, a totally reducible representation is conjugate to a $SU(2)$ -representation if and only if $\tilde{x}, \tilde{y}, \tilde{z} \in [-2, 2]$, while it is conjugate to an $SL(2, \mathbb{R})$ -representation if $\tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{R}$ but not

$$\tilde{x} \in [-2, 2] \quad \text{and} \quad \tilde{y} \in [-2, 2] \quad \text{and} \quad \tilde{z} \in [-2, 2].$$

4.4.2. Abelianization. We consider logarithmic connections $d + \xi$ on the rank two trivial holomorphic bundle over \mathcal{S}_4 which are symmetric, in the sense that all four residues have eigenvalues $\pm\tilde{\rho}$. As explained in Section 1, a logarithmic connection induces a parabolic bundle E . The parabolic weights are hereby $\tilde{\rho}$ at each of the four singular points. The generic underlying holomorphic vector bundle for parabolic bundles is trivial. So once the parabolic weight is fixed, the parabolic structure E is essentially determined by the lines defining the quasiparabolic structures, or in other words, the cross-ratio of the 4 quasiparabolic lines in the trivial vector space \mathbb{C}^2 ; see [LoSa] or [HH].

It can be shown (see [HH, Proposition 2.1]) that for a generic parabolic structure E , i.e., for a generic cross-ratio of the 4 parabolic lines, the space of strongly parabolic Higgs fields is complex 1-dimensional. Moreover, for a generic parabolic structure E , the determinant of a non-zero strongly parabolic Higgs field θ is a non-zero constant multiple of

$$\frac{(dz)^2}{z^4 - 1}. \quad (4.15)$$

Take a strongly parabolic Higgs bundle (E, θ) such that $\det \theta$ is non-zero constant multiple of (4.15). Let

$$f : \Sigma_2 \longrightarrow \mathbb{C}P^1$$

be the spectral curve and $\mathbf{L} \longrightarrow \Sigma_2$ the holomorphic line bundle corresponding to (E, θ) ; see also [Hi1] for the smooth case. We recall that Σ_2 is contained in the total space of $K_{\mathbb{C}P^1} \otimes \mathcal{O}_{\mathbb{C}P^1}(x_1 + x_2 + x_3 + x_4)$, where the x_i 's are the fourth roots of unity as in (4.13), and $f = f_2$ (as in (3.2) for $k = 2$) is the ramified double cover of $\mathbb{C}P^1$ branched over the singular points x_1, x_2, x_3, x_4 ; the holomorphic line bundle \mathbf{L} is the subbundle of $f^*E = \mathcal{O}_{\Sigma_2}^{\oplus 2}$ given by the eigenline bundle of θ . We have $\text{genus}(\Sigma_2) = 1$ and $\text{degree}(\mathbf{L}) = -2$. As before, denote the point $f^{-1}(x_i)$ by p_i . Let $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ be the nontrivial element of the Galois group $\text{Gal}(f)$. Then

$$\mathbf{L} \otimes \sigma^*\mathbf{L} = \mathcal{O}_{\Sigma_2}(-p_1 - p_2 - p_3 - p_4);$$

see [HH, Section 3]. When p_1 is chosen as the identity element of the addition law, p_2, p_3, p_4 become the nontrivial order two points of the elliptic curve. So

$$-3p_1 + p_2 + p_3 + p_4$$

is a principal divisor (associated to the derivative \wp' of the Weierstrass \wp -function), and therefore

$$(\mathcal{O}_{\Sigma_2}(-2p_1))^{\otimes 2} = \mathbf{L} \otimes \sigma^*\mathbf{L}.$$

Thus, there is $L_0 \in \text{Jac}(\Sigma_2)$ with

$$\mathcal{O}_{\Sigma_2}(-2p_1) \otimes L_0 = \mathbf{L} \quad \text{and} \quad \mathcal{O}_{\Sigma_2}(-2p_1) \otimes L_0^* = \sigma^*\mathbf{L}. \quad (4.16)$$

Consider the logarithmic connection on $\mathcal{O}_{\Sigma_2}(-p_1 - p_2 - p_3 - p_4)$ given by the de Rham differential. It produces a logarithmic connection on $\mathcal{O}_{\Sigma_2}(-4p_1)$ once an isomorphism of $\mathcal{O}_{\Sigma_2}(-p_1 - p_2 - p_3 - p_4)$ with $\mathcal{O}_{\Sigma_2}(-4p_1)$ is chosen (for instance, the isomorphism defined by the multiplication with \wp'); this connection on $\mathcal{O}_{\Sigma_2}(-4p_1)$ does not depend on the choice of the isomorphism. A connection on $\mathcal{O}_{\Sigma_2}(-4p_1)$ produces a connection on $\mathcal{O}_{\Sigma_2}(-2p_1)$. Let

$$D^s \quad (4.17)$$

be the logarithmic connection on $\mathcal{O}_{\Sigma_2}(-2p_1)$ obtained this way. It satisfies the equation

$$D^s s_{-2p_1} = -\frac{d\wp'}{2\wp'} \otimes s_{-2p_1}, \quad (4.18)$$

where s_{-2p_1} is the meromorphic section with double pole at p_1 .

In particular, D^s is singular at p_1, \dots, p_4 , all residues being equal to $\frac{1}{2}$ (and hence the monodromy around the singular points being -1); for more details see (the proof of) [HHS, Theorem 3.2]. Denote by $(D^s)^*$ the dual connection of D^s on $\mathcal{O}_{\Sigma_2}(2p_1)$.

The holomorphic bundle underlying the pull-back f^*E of the parabolic bundle E is the rank two trivial holomorphic bundle over Σ_2 . Recall that both \mathbf{L} and $\sigma^*\mathbf{L}$ are holomorphic subbundles of the rank two trivial holomorphic bundle over Σ_2 . This inclusion map defines a holomorphic vector bundle map

$$\mathbf{L} \oplus \sigma^*\mathbf{L} \rightarrow \mathcal{O}_{\Sigma_2} \oplus \mathcal{O}_{\Sigma_2}$$

which is an isomorphism away from the divisor $p_1 + p_2 + p_3 + p_4$. Consider now the holomorphic isomorphism

$$L_0 \oplus L_0^* \rightarrow (\mathbf{L} \oplus \sigma^*\mathbf{L}) \otimes \mathcal{O}_{\Sigma_2}(2p_1).$$

It is shown in [HH, Section 3] that the induced logarithmic connection $(f_2^*(d + \xi)) \otimes (D^s)^*$ on

$$L_0 \oplus L_0^*$$

is given by

$$d + \begin{pmatrix} \nabla^{L_0} & \beta^- \\ \beta^+ & (\nabla^{L_0})^* \end{pmatrix}.$$

Here, ∇^{L_0} and $(\nabla^{L_0})^*$ are dual holomorphic line bundle connections on L_0 respectively L_0^* . Moreover, the second fundamental forms β^+ and β^- are meromorphic sections of

$$K_{\Sigma_2} \otimes L_0^{-2} \quad \text{and} \quad K_{\Sigma_2} \otimes L_0^2$$

respectively; they can be explicitly determined in terms of ϑ -functions [HH, Proposition 3.2]. Moreover, the eigenvalues of the residues of $(f_2^*(d + \xi)) \otimes (D^s)^*$ are

$$\pm(2\tilde{\rho} - \frac{1}{2})$$

which implies that the quadratic residue of the meromorphic quadratic differential $\beta^+\beta^-$ is $(2\tilde{\rho} - \frac{1}{2})^2$.

The relationship between the abelianization of symmetric logarithmic connections on \mathcal{S}_4 and flat connections on the one-punctured torus is given as follows. Consider the 4-fold covering induced by the identity map on \mathbb{C}

$$\pi_4 : \Sigma_2 = \mathbb{C}/(2\mathbb{Z} + 2\sqrt{-1}\mathbb{Z}) \longrightarrow T^2 = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}).$$

The pull-back of topologically trivial holomorphic line bundles defines a 4-fold covering

$$\text{Jac}(T^2) \longrightarrow \text{Jac}(\Sigma_2).$$

Spin bundles on T_2 are mapped to the trivial holomorphic line bundle on Σ_2 . Further, holomorphic line bundles of order 4 on T^2 are mapped to nontrivial spin bundles on Σ_2 .

As shown in [HHS, Section 3.1] (see also [HH, Remark 3.3] and [H, Section 4]), for a symmetric logarithmic connection $d + \xi$ with local weights $\tilde{\rho}$ on \mathcal{S}_4 with underlying parabolic bundle admitting a strongly parabolic Higgs field of non-vanishing determinant, there exists $a, \chi \in \mathbb{C}$, $\chi \notin \frac{1}{2}\Gamma^*$, such that $f_2^*(d + \xi) \otimes (D^s)^*$ and $\pi_4^*\nabla^{a, \chi, \rho}$ are gauge equivalent (with the connection $\nabla^{a, \chi, \rho}$ as in (4.8)) and

$$\pi_4^*\gamma^\pm = \beta^\pm.$$

The above abelianization-procedure leads to the following theorem.

Theorem 4.8. *Let $\rho \in]0, \frac{1}{2}[$ and $\tilde{\rho} = \frac{2\rho+1}{4}$. There is a degree 4 birational map*

$$\mathcal{M}_{1,1}^\rho \longrightarrow \mathcal{M}_{0,4}^{\tilde{\rho}}$$

compatible with the underlying parabolic structures. On the character variety this map is given by

$$(x, y, z) \longmapsto (\tilde{x}, \tilde{y}, \tilde{z}) = (2 - x^2, 2 - y^2, 2 - z^2).$$

Remark 4.9. In our symmetric case, where the parabolic weight $\tilde{\rho} \in]\frac{1}{4}, \frac{1}{2}[$ is the same at every marked point of \mathcal{S}_4 , there are only two polystable parabolic structures which admit a compatible logarithmic connection (as defined in Section 1), but no strongly parabolic Higgs field with non-zero determinant. The first of the two exceptions is induced by $\tilde{\nabla}$ constructed in (2.9), and the second is a stable parabolic structure defined on $\mathcal{O}_{\mathcal{S}_4}(1) \oplus \mathcal{O}_{\mathcal{S}_4}(-1)$.

There are exactly three totally reducible connections having semistable parabolic structure on \mathcal{S}_4 , (see [HH]); one of them being D as defined in (3.3). The semistable parabolic structures of these three totally reducible connections admit strongly parabolic Higgs fields with non-zero determinant. In particular, the parabolic bundle induced by D has the strongly parabolic Higgs field defined in (3.10). The corresponding line bundles L_0 in (4.16) of these semistable parabolic structures are exactly the non-trivial spin bundles of Σ_2 . They correspond to holomorphic line bundles of order 4 on T^2 .

Proof of Theorem 4.8. The birational map $\mathcal{M}_{1,1}^\rho \longrightarrow \mathcal{M}_{0,4}^{\tilde{\rho}}$ is given via abelianization. Note that for a nontrivial Zariski open set in $\mathcal{M}_{0,4}^{\tilde{\rho}}$, the parabolic bundle induced via the Riemann-Hilbert correspondence is defined on the rank two trivial holomorphic bundle over \mathcal{S}_4 and admits a parabolic Higgs field of non-vanishing determinant, see for example [LoSa]. As explained above, there exists $a, \chi \in \mathbb{C}$, $\chi \notin \frac{1}{2}\Gamma^*$, such that $f_2^*(d + \xi) \otimes (D^s)^*$ and $\pi_4^* \nabla^{a, \chi, \rho}$ are gauge equivalent by [HH]. The connection $\nabla^{a, \chi, \rho}$ is a preimage of $d + \xi$ through our birational map.

There are four preimages, because the pull-backs of two connections $\pi_4^* \nabla^{a_1, \chi_1, \rho}$ and $\pi_4^* \nabla^{a_2, \chi_2, \rho}$ from the one-punctured to the four-punctured torus are gauge equivalent if and only if they differ by a spin-connection, i.e.,

$$(a_2 - a_1, \chi_2 - \chi_1) = (-\bar{\nu}, \nu), \quad \nu \in \frac{1}{2}\Gamma^*.$$

Recall also that the elements in $\mathcal{M}_{1,1}^\rho$ admitting a representative of the form $\nabla^{a, \chi, \rho}$, with $a, \chi \in \mathbb{C}$, $\chi \notin \frac{1}{2}\Gamma^*$, form a nontrivial Zariski open set (see [HH, Theorem 1] or [BDH, Section 2.3]).

It remains to determine the relationship between the character varieties. First observe that Equation (4.14) for $(\tilde{x}, \tilde{y}, \tilde{z}) = (2 - x^2, 2 - y^2, 2 - z^2)$ factors as

$$(x^2 + y^2 + z^2 - xyz - 4 + \mu^2)(x^2 + y^2 + z^2 + xyz - 4 + \mu^2) = 0$$

with $\mu = 2 \cos(2\pi\tilde{\rho})$. Replacing

$$2 - \mu^2 = \kappa := 2 \cos(2\pi\rho)$$

then gives

$$(x^2 + y^2 + z^2 - xyz - 2 + \kappa)(x^2 + y^2 + z^2 + xyz - 2 + \kappa) = 0.$$

The first factor coincides with Equation (4.5) for the one-punctured torus with parabolic weight ρ . Hence, the map between the character varieties is well-defined.

We need to show that the above map

$$(x, y, z) \mapsto (\tilde{x}, \tilde{y}, \tilde{z}) = (2 - x^2, 2 - y^2, 2 - z^2)$$

is compatible with the birational map between the moduli space. Consider an element $[\nabla] \in \mathcal{M}_{1,1}^\rho$ determined by the monodromy representation Θ . Let $X, Y \in \mathrm{SL}(2, \mathbb{C})$ be the monodromies along the loops γ_x and γ_y on the one-punctured torus.

Recall that the monodromy of the connection D^s on $\mathcal{O}_{\Sigma_2}(-2p_1)$ is given by -1 around the singularities p_l . The generators $2, 2\sqrt{-1} \in 2\mathbb{Z} + 2\sqrt{-1}2\mathbb{Z}$ of the lattice defining the torus Σ_2 define two generators of the fundamental group of Σ_2 . The monodromy of D^s along these two generators of $2\mathbb{Z} + 2\sqrt{-1}2\mathbb{Z}$ is also -1 .

Let Θ' be the monodromy representation corresponding to the image in $M_{0,4}^{\tilde{\rho}}$ of $[\nabla] \in \mathcal{M}_{1,1}^\rho$ through the birational map in the statement of the Theorem. Denote by $M_l = \Theta'(\gamma_l)$, $l = 1, \dots, 4$ the (local) monodromies of Θ' along the simple oriented loops γ_l on S_4 going around the 4 punctures $x_l := e^{(l-1)\sqrt{-1}\frac{\pi}{2}} \in \mathbb{C}P^1$.

Consider the loops $\gamma_2\gamma_1$, $\gamma_3\gamma_2$ and $\gamma_3\gamma_1$ on S_4 . Their images through the monodromy homomorphism Θ' are M_2M_1 , M_3M_2 and M_3M_1 . Lifting these curves to the four-punctured torus Σ_2 together with the above properties of the monodromy of D^s shows

$$\begin{aligned} M_2M_1 &\equiv -X^2 \\ M_3M_2 &\equiv -Y^2 \\ M_3M_1 &\equiv -(YX)^2, \end{aligned} \tag{4.19}$$

where \equiv is the equivalence relation of lying in the same conjugacy class. Taking traces yields the claimed map between the character varieties. Moreover, the local monodromies around the 4 singular points p_l in Σ_2 are given by

$$M_l^2 \equiv -Y^{-1}X^{-1}YX$$

Taking the trace gives $2 - (2\cos(2\pi\tilde{\rho}))^2 = 2\cos(2\pi\rho)$ corresponding to $\rho = 2\tilde{\rho} - \frac{1}{2}$. \square

Lemma 4.10. *Let D and $\tilde{\rho}$ be as in Proposition 3.1. Then,*

$$f_2^*D \otimes (D^s)^*$$

is given by $\pi_4^*\nabla^{a^0, \chi^0, \rho}$ with $\chi^0 = \pi\frac{1-\sqrt{-1}}{4}$, $a^0 = \pi\frac{1+\sqrt{-1}}{4}$ and $\rho = 2\tilde{\rho} - \frac{1}{2}$.

Proof. This assertion follows from the proof of [HHS, Theorem 3.2]. In the geometric context of [HHS] the connection $\nabla^{a^0, \chi^0, \rho}$ with $\chi^0 = \pi\frac{1-\sqrt{-1}}{4}$ and $a^0 = \pi\frac{1+\sqrt{-1}}{4}$ solves the extrinsic closing condition of a compact CMC surface in the 3-sphere \mathbb{S}^3 . Particular instances of minimal surfaces are given by the famous Lawson surfaces [L]. \square

Lemma 4.11. *There exists a flat $\mathrm{SL}(2, \mathbb{R})$ -connection ∇^F in $\mathcal{M}_{1,1}^\rho$ such that $\pi_4^*\nabla^F$ and $f_2^*\tilde{\nabla} \otimes (D^s)^*$ are gauge equivalent on the four-punctured torus Σ_2 . The connections $\tilde{\nabla}$ and D^s are defined in (2.9) and (4.17) respectively.*

Proof. Let $k \in \mathbb{N}^{\geq 3}$, $\tilde{\rho} = \frac{k-1}{2k}$, and consider the associated connection $\tilde{\nabla}$ in Lemma 2.2. Using Lemma 2.1, the monodromy representation for $\tilde{\nabla}$ is determined by the following

characters

$$\begin{aligned}\tilde{x} &= -2 - 4 \cos \frac{\pi}{k} \\ \tilde{y} &= -2 - 4 \cos \frac{\pi}{k} \\ \tilde{z} &= -2(2 + 4 \cos \frac{\pi}{k} + \cos \frac{2\pi}{k})\end{aligned}\tag{4.20}$$

with $\mu = 2 \cos \pi \frac{k-1}{k}$.

Consider the flat connection ∇^F on the one-punctured torus determined by the following element of the character variety of the one-punctured torus

$$\begin{aligned}x &= 2\sqrt{1 + \cos \frac{\pi}{k}} \\ y &= 2\sqrt{1 + \cos \frac{\pi}{k}} \\ z &= 4(\cos \frac{\pi}{2k})^2 \\ \kappa &= 2 \cos 2\pi \frac{k-2}{2k} = -2 \cos \frac{2\pi}{2k}.\end{aligned}\tag{4.21}$$

Here $\rho = 2\tilde{\rho} - \frac{1}{2} = \frac{k-2}{2k}$ and $\nabla^F \in \mathcal{M}_{1,1}^\rho$. The proof of Theorem 4.8 shows that $\pi_4^* \nabla^F$ and $f_2^* \tilde{\nabla} \otimes (D^s)^*$ define the same element in the character variety of the four-punctured torus Σ_2 . This implies that $\pi_4^* \nabla^F$ and $f_2^* \tilde{\nabla} \otimes (D^s)^*$ are gauge equivalent on the four-punctured torus Σ_2 . \square

Remark 4.12. The connection $\tilde{\nabla}$ in (2.9) does not admit a strongly parabolic Higgs field with non-zero determinant, and the abelianization-procedure does not apply directly. But Lemma 4.11 shows that it is possible to determine a connection ∇^F on T^2 such that $\pi_4^* \nabla^F$ and $f_2^* \tilde{\nabla} \otimes (D^s)^*$ are gauge equivalent on the four-punctured torus Σ_2 . In [HH, Theorem 3.5], the connection $\pi_4^* \nabla^F$ is written as a limit of connections of the form in (4.8). It can be shown that the underlying holomorphic bundle of ∇^F is a non-trivial extension of the spin bundle by itself.

5. PROOFS

Proof of Theorem 1.

Let $k \in \mathbb{N}^{\geq 3}$, $\tilde{\rho} = \frac{k-1}{2k}$ and $\rho = 2\tilde{\rho} - \frac{1}{2} = \frac{k-2}{2k}$. Consider a sequence of distinct connections ∇^{t_n} with real monodromy, as constructed in Corollary 4.7. By Theorem 4.8, the connection ∇^{t_n} induces a logarithmic connection on \mathcal{S}_4 with real monodromy. This connection is given by

$$D^{\tau_n} := D + \tau_n \Phi,$$

where Φ is the strongly parabolic Higgs field for D given in (3.10), and $\tau_n \in \mathbb{C} \setminus \{0\}$ is determined by t_n . To be more explicit, the holomorphic quadratic differential $\det(\Phi) = \frac{4\sqrt{-1}(dz^2)}{z^4-1}$ pulls back to $c^2(dw)^2$ on Σ_2 for some $c \in \mathbb{R}^{>0}$. Also note that $D + h\Psi$ is gauge equivalent to $D - h\Psi$ for every $h \in \mathbb{C}$. Thus, we have

$$\tau_n = \pi \frac{1 + \sqrt{-1}}{4c} t_n.$$

By Proposition 3.1, the pull-back of D to Σ_k , through the map f_k in (3.2), is gauge equivalent to the de Rham differential. The same gauge transformation sends Φ to a

holomorphic Higgs field with respect to the trivial holomorphic structure by Lemma 3.3. Since ∇^{t_n} and ∇^F (constructed in Lemma 4.11) are both $\mathrm{SL}(2, \mathbb{R})$ -connections on the one-punctured torus $T^2 \setminus \{o\}$, the map given in Theorem 4.8 sends them into the same real component of connections on \mathcal{S}_4 (that of $\tilde{\nabla}$ in (2.9)); see Remark 4.3. By Proposition 3.4 and Proposition 3.7 we obtain that the pull back of $D + \tau_n \Phi$ to Σ_k is in the connected component with maximal Euler class $g - 1 = k - 2$. \square

Proof of Corollary 1.

Consider for $\rho = \frac{k-2}{2k}$ the connections

$$\nabla_\chi^t = \nabla^{(1-t)a, \chi, \rho}$$

with

$$\chi \in (1 \mp \sqrt{-1})\mathbb{R} \setminus \frac{1}{2}\Gamma^* \quad \text{and} \quad a \in (1 \pm \sqrt{-1})\mathbb{R}$$

such that ∇^t is equivariant under the real involution η , see Lemma 4.5. Recall that the moduli space of S-equivalence classes of rank two stable bundles with trivial determinant over Σ_k is a projective variety. The subspace of (semistable) equivariant holomorphic bundles over Σ_k identifies with the moduli space of corresponding parabolic structures on \mathcal{S}_4 by pull-back and desingularization. As such it is a projective line as explained in Section 4.4.2, see also [LoSa]. The two lines $\chi \in (1 \mp \sqrt{-1})\mathbb{R} \setminus \frac{1}{2}\Gamma^*$ in the Jacobian are mapped onto two semicircles constituting a circle in the aforementioned projective line. The trivial holomorphic structure corresponding to χ^0 is the only point contained in the intersection of the semicircles (as the holomorphic line bundles determined by $\overline{\chi^0}$ and χ^0 only differ by a spin bundle on T^2). We refer to these as the *compatible real holomorphic structures* on Σ_k . The only missing point in the circle is given by a wobbly bundle, where our method does not apply. The proof of Corollary 1 works verbatim using ∇_χ^t instead of ∇^t .

Proof of Corollary 2.

By Theorem 1 there exists a compact curve Σ_3 of genus $g = 2$ and a holomorphic connection $\nabla(\Sigma_3)$ on the rank two trivial holomorphic bundle over Σ_3 such that the monodromy homomorphism of $\nabla(\Sigma_3)$ is Fuchsian.

Consider an open neighborhood \mathcal{V} of the monodromy of $\nabla(\Sigma_3)$ in the space of conjugacy classes of group homomorphisms $\pi_1(\Sigma_3) \rightarrow \mathrm{SL}(2, \mathbb{C})$ formed by quasi-Fuchsian representations. Recall that quasi-Fuchsian representations are faithful and their image in $\mathrm{SL}(2, \mathbb{C})$ is a discrete group whose canonical action on $\mathbb{C}P^1$ has a Jordan curve as limit set and preserves each component of the domain of discontinuity. By Bers' simultaneous uniformization each conjugacy class of a quasi-Fuchsian representation is determined by the pair of elements in the Teichmüller space given by the quotient of the two connected components of the discontinuity domain by the image of the representation.

The main result in [CDHL] gives an open neighborhood \mathcal{W} of $(\Sigma_3, \nabla(\Sigma_3))$ in the space of irreducible holomorphic differential systems (i.e., pairs of the form (Σ, ∇) where Σ is an element in the Teichmüller space of compact curves of genus $g = 2$ and ∇ is an irreducible holomorphic $\mathrm{SL}(2, \mathbb{C})$ -connections on $\mathcal{O}_\Sigma^{\oplus 2}$) such that the restriction of the Hilbert-Riemann monodromy mapping to \mathcal{W} is a biholomorphism between \mathcal{W} and \mathcal{V} . This proves the first statement in the Corollary.

Consider now the open set \mathcal{U} in the Teichmüller space of compact curves of genus $g = 2$ which is the image of \mathcal{W} through the natural forgetful projection. Take $\Sigma \in \mathcal{U}$ and $\nabla(\Sigma)$ a holomorphic connection on rank two holomorphic trivial bundle $\mathcal{O}_\Sigma^{\oplus 2}$ with

quasi-Fuchsian monodromy representation. Denote by $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ the image of the monodromy homomorphism for $\nabla(\Sigma)$.

Let $\tilde{\Sigma} \rightarrow \Sigma$ be the universal cover of Σ and let $\tilde{\nabla}(\tilde{\Sigma})$ be the pull-back of $\nabla(\Sigma)$ to the rank two trivial holomorphic bundle $\mathcal{O}_{\tilde{\Sigma}}^{\oplus 2}$ over $\tilde{\Sigma}$ through the covering map.

Since $\tilde{\nabla}(\tilde{\Sigma})$ is flat and $\tilde{\Sigma}$ is simply connected, there exists a global $\tilde{\nabla}(\tilde{\Sigma})$ -parallel frame of the rank two trivial bundle over $\tilde{\Sigma}$. Such a parallel frame on the holomorphically trivial bundle $\mathcal{O}_{\tilde{\Sigma}}^{\oplus 2}$ is determined by a holomorphic map $\tilde{\Sigma} \rightarrow \mathrm{SL}(2, \mathbb{C})$ which is equivariant with respect to two actions of the fundamental group of Σ , namely by deck transformations on $\tilde{\Sigma}$ and through the monodromy morphism of $\nabla(\Sigma)$ on $\mathrm{SL}(2, \mathbb{C})$. This provides a holomorphic map $\Sigma \rightarrow \mathrm{SL}(2, \mathbb{C})/\Gamma$, with Γ being the image of the monodromy homomorphism for $\nabla(\Sigma)$. Here, we make use of the holomorphic trivialization of $\mathcal{O}_{\tilde{\Sigma}}^{\oplus 2}$ which is the pull-back of the holomorphic trivialization of $\mathcal{O}_{\Sigma}^{\oplus 2}$.

Since $\nabla(\Sigma)$ is irreducible (and therefore nontrivial), the above map is non-constant. Notice that, up to a finite index subgroup (and an associated finite cover of the target), we can assume that Γ is torsion free and hence $\mathrm{SL}(2, \mathbb{C})/\Gamma$ is a complex threefold (without orbifold points).

Moreover, such quotients of $\mathrm{SL}(2, \mathbb{C})$ are diffeomorphic to the orthonormal frame bundle of the associated quasi-Fuchsian hyperbolic 3-manifold (which is known to be isometric to the quotient of a convex set in the hyperbolic 3-space by the quasi-Fuchsian group of hyperbolic isometries). Note that the boundary of the quasi-Fuchsian manifold has two connected components that are conformally equivalent to the pair of points in the Teichmüller space given by Bers' simultaneous uniformization; the complex structure on the oriented orthonormal frame bundle of the quasi-Fuchsian manifold comes from the identification of the orientation preserving isometry group $\mathrm{PSL}(2, \mathbb{C})$ with the oriented orthonormal frame bundle of the hyperbolic 3-space [Gh]. \square

We would like to formulate a general problem similar to that of Ghys and to the questions asked in [CDHL, Ka]. Consider a compact orientable surface S_g of genus $g \geq 2$. Characterize the conjugacy classes of $\mathrm{SL}(2, \mathbb{C})$ -representations of the fundamental group of S_g such that the associated rank two flat vector bundle over S_g is holomorphically trivial with respect to some point in the Teichmüller space of S_g .

The analogous question for the uniformization bundle has been answered completely in [GaKaMa]. Note that a holomorphic $\mathrm{SL}(2, \mathbb{C})$ -connection on the uniformization bundle gives rise to a complex projective structure on the Riemann surface and vice versa after the choice of a theta characteristic. For the case of the trivial rank one bundle this question was answered in [Ha] (see also [Ka] where this result was rediscovered).

APPENDIX A. A RESULT ON WKB APPROXIMATION

By Takuro Mochizuki

A.1. Limiting behavior of a family of flat connections. Let X be a Riemann surface, which is not necessarily compact. Let V be a vector bundle on X equipped with a flat $\mathrm{SL}_2(\mathbb{C})$ -connection ∇ . Let $\bar{\partial}_V$ denote the induced holomorphic structure of V . Let Φ be a Higgs field of the holomorphic vector bundle $(V, \bar{\partial}_V)$ such that $\mathrm{tr} \Phi = 0$. We obtain the family of flat connections $\nabla^t = \nabla + t\Phi$ on V ($t \geq 0$).

Assumption A.1. We assume that there exist a holomorphic one form ω and a decomposition $V = V_+ \oplus V_-$ such that $\Phi = \omega(\pi_{V_+} - \pi_{V_-})$, where π_{V_\pm} denote the projections of V onto V_\pm with respect to the decomposition. \square

Note that there exists a unique decomposition $\nabla = \nabla^\circ + f$, where ∇° is the direct sum of connections ∇_{V_\pm} of V_\pm , and f is a holomorphic section of $(\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)) \otimes \Omega^1$.

We set $[0, 1] := \{0 \leq u \leq 1\}$. Let $\gamma : [0, 1] \rightarrow X$ be a C^∞ -path which is a WKB-curve with respect to ω , i.e.,

$$\mathrm{Re}(\gamma^*(\omega)(\partial_u)) < 0$$

at any point of $[0, 1]$. Let $\mathcal{P}_\gamma^t : V_{|\gamma(0)} \simeq V_{|\gamma(1)}$ denote the isomorphism obtained as the parallel transport of $\nabla + t\Phi$ along γ . Similarly, let $\mathcal{P}_{\pm, \gamma}$ denote the isomorphisms $V_{\pm|\gamma(0)} \simeq V_{\pm|\gamma(1)}$ obtained as the parallel transport of ∇_{V_\pm} along γ .

We shall explain a proof of the following proposition in §A.4 after preliminaries in §A.2–§A.3.

Proposition A.2. For $(w_+, w_-) \in V_{|\gamma(0)} = V_{+|\gamma(0)} \oplus V_{-|\gamma(0)}$, we have

$$\lim_{t \rightarrow \infty} e^{t \int_\gamma \omega} \cdot \mathcal{P}_\gamma^t(w_+, w_-) = (\mathcal{P}_{+, \gamma}(w_+), 0) \in V_{+|\gamma(1)} \oplus V_{-|\gamma(1)}.$$

Remark A.3. Proposition A.2 and its proof are essentially explained in [GMN, Appendix C]. Hopefully, a more detailed explanation in this appendix would be useful. It is also closely related to the Riemann-Hilbert WKB problem studied in [KNPSi]. \square

We obtain the following corollary as an immediate consequence of Proposition A.2.

Corollary A.4. If γ is closed, i.e., $\gamma(0) = \gamma(1)$, we obtain

$$\lim_{t \rightarrow \infty} \mathrm{tr}(\mathcal{P}_\gamma^t) e^{t \int_\gamma \omega} = \mathrm{tr}(\mathcal{P}_{+, \gamma}) \neq 0.$$

 \square

A.2. An elementary lemma. For any $s_1 < s_2$, we set $[s_1, s_2] := \{s_1 \leq s \leq s_2\}$. For any non-negative integer ℓ , let $C^\ell([s_1, s_2])$ denote the space of \mathbb{C} -valued C^ℓ -functions on $[s_1, s_2]$. For any $f \in C^0([s_1, s_2])$, we set $\|f\|_{C^0([s_1, s_2])} := \max_{s \in [s_1, s_2]} |f(s)|$.

Fix $\rho > 0$, $\epsilon > 0$ and $C_0 > 0$. Suppose that $\alpha \in C^0([0, 1])$ satisfies $\mathrm{Re}(\alpha(s)) > \rho$ for any $s \in [0, 1]$. Let $\beta \in C^0([0, 1])$ such that $\|\beta\|_{C^0([0, 1])} \leq C_0$. Suppose that $f^t \in C^1([0, 1])$ ($t \geq 0$) satisfies

$$\|f^t\|_{C^0([0, 1])} + \|\partial_s f^t + (t\alpha + \beta)f^t\|_{C^0([0, 1])} \leq \epsilon.$$

Take $0 < \delta < 1$. We recall the following standard and elementary lemma, which we prove just for the convenience of the reader.

Lemma A.5. *There exist $C_1 > 0$ and $t_1 > 0$, depending only on C_0 , ρ and δ such that the following holds for any $t \geq t_1$:*

$$\|f^t\|_{C^0([\delta,1])} \leq C_1 \epsilon (1+t)^{-1}$$

Proof. We set $\tilde{\alpha}(s) := \int_0^s \alpha(u) du$ and $\tilde{\beta}(s) := \int_0^s \beta(u) du$. We have $\|\tilde{\beta}\|_{C^0([0,1])} \leq C_0$. For any $0 \leq s_1 \leq s_2$, we have

$$\operatorname{Re} \tilde{\alpha}(s_2) - \operatorname{Re} \tilde{\alpha}(s_1) = \int_{s_1}^{s_2} \operatorname{Re} \alpha(u) du > \rho(s_2 - s_1).$$

We set $g^t := \partial_s f^t + (t\alpha + \beta)f^t$. Because $\partial_s(e^{t\tilde{\alpha} + \tilde{\beta}} f^t) = e^{t\tilde{\alpha} + \tilde{\beta}} g^t$, we obtain

$$f^t = e^{-t\tilde{\alpha}(s) - \tilde{\beta}(s)} \int_0^s e^{t\tilde{\alpha}(u) + \tilde{\beta}(u)} g^t(u) du + e^{-t\tilde{\alpha}(s) - \tilde{\beta}(s)} f^t(0).$$

We have $\|e^{-t\tilde{\alpha}(s) - \tilde{\beta}(s)} f^t(0)\|_{C^0([\delta,1])} \leq \epsilon e^{-t\rho\delta + C_0}$. We also have the following inequalities for $s \in [0, 1]$:

$$\left| e^{-t\tilde{\alpha}(s) - \tilde{\beta}(s)} \int_0^s e^{t\tilde{\alpha}(u) + \tilde{\beta}(u)} g^t(u) du \right| \leq \int_0^s e^{-t\rho(s-u) + 2C_0} \epsilon du \leq \frac{\epsilon e^{2C_0}}{\rho t}.$$

Then, we obtain the claim of the lemma. \square

Let us state a variant. Suppose that $\alpha_1 \in C^0([0, 1])$ satisfies $\operatorname{Re}(\alpha_1(s)) < -\rho$ for any $s \in [0, 1]$. Let $\beta_1 \in C^0([0, 1])$ such that $\|\beta_1\|_{C^0([0,1])} \leq C_0$. Suppose that $f_1^t \in C^1([0, 1])$ ($t \geq 0$) satisfies

$$\|f_1^t\|_{C^0([0,1])} + \|\partial_s f_1^t + (t\alpha_1 + \beta_1)f_1^t\|_{C^0([0,1])} \leq \epsilon.$$

Lemma A.6. *The following inequality holds for any $t \geq t_1$:*

$$\|f_1^t\|_{C^0([0,1-\delta])} \leq C_1 \epsilon (1+t)^{-1}.$$

Here, C_1 and t_1 are positive constants in Lemma A.5.

Proof. It is enough to apply Lemma A.5 to the function $f_1^t(1-s)$. \square

A.3. A singular perturbation theory. We recall some results from [Mo, §2.4] with a complementary estimate for the convenience of the reader.

A.3.1. Notation. Let r be a positive integer. Let $M_r(\mathbb{C})$ denote the space of $r \times r$ complex matrices. Let $M_r(\mathbb{C})_0 \subset M_r(\mathbb{C})$ denote the subspace of diagonal matrices, and let $M_r(\mathbb{C})_1 \subset M_r(\mathbb{C})$ denote the subspace of off-diagonal matrices, i.e.,

$$M_r(\mathbb{C})_0 = \{(a_{i,j}) \in M_r(\mathbb{C}) \mid a_{i,j} = 0 \ (i \neq j)\},$$

$$M_r(\mathbb{C})_1 = \{(a_{i,j}) \in M_r(\mathbb{C}) \mid a_{i,j} = 0 \ (i = j)\}.$$

For any non-negative integer ℓ , let $C^\ell([s_1, s_2], M_r(\mathbb{C}))$ denote the space of $M_r(\mathbb{C})$ -valued C^ℓ -functions on $[s_1, s_2]$. Similarly, let $C^\ell([s_1, s_2], M_r(\mathbb{C})_\kappa)$ ($\kappa = 0, 1$) denote the spaces of $M_r(\mathbb{C})_\kappa$ -valued C^ℓ -functions on $[s_1, s_2]$. We set $\|Y\|_{C^0([s_1, s_2])} := \max_{i,j} \|Y_{i,j}\|_{C^0([s_1, s_2])}$ for any $Y \in C^0([s_1, s_2], M_r(\mathbb{C}))$.

A.3.2. Gauge transformations. Fix $C_0 > 0$. Suppose that $a_j, b_j \in C^0([0, 1])$ ($j = 1, \dots, r$) satisfy the following conditions.

- $\operatorname{Re} a_1(s) < \operatorname{Re} a_2(s) < \dots < \operatorname{Re} a_r(s)$ for any $s \in [0, 1]$.
- $\|b_j\|_{C^0([0,1])} \leq C_0$.

For $t \geq 0$, let $A^t(s)$ denote the $M_r(\mathbb{C})_0$ -valued function whose (i, i) -entries are $ta_i(s) + b_i(s)$. The following proposition is proved in [Mo, Proposition 2.18].

Proposition A.7. *There exist $C_1 > 0$ and $\epsilon_1 > 0$, depending only on C_0 , such that the following holds:*

- For any $t \geq 0$ and any $B \in C^0([0, 1], M_r(\mathbb{C})_1)$ satisfying $\|B\|_{C^0([0,1])} \leq \epsilon_1$, there exist $G^t \in C^1([0, 1], M_r(\mathbb{C})_1)$ and $H^t \in C^0([0, 1], M_r(\mathbb{C})_0)$ satisfying

$$\|G^t\|_{C^0([0,1])} + \|\partial_s G^t + [A^t, G^t]\|_{C^0([0,1])} + \|H^t\|_{C^0([0,1])} \leq C_1 \|B\|_{C^0([0,1])}, \quad (\text{A.1})$$

$$A^t + B = (I + G^t)^{-1}(A^t + H^t)(I + G^t) + (I + G^t)^{-1} \partial_s G^t. \quad (\text{A.2})$$

Here, $I \in M_r(\mathbb{C})$ denote the identity matrix. \square

Remark A.8. In Proposition A.7, we assume that $C_1 \epsilon_1$ is sufficiently small so that $I + G^t$ is invertible. \square

Let us add a complementary estimate to Proposition A.7. There exist $C_2 > 0$ and $C_3 > 0$ such that (i) $\operatorname{Re}(a_{i+1}(s) - a_i(s)) > C_2$ for any $s \in [0, 1]$ and $i = 1, \dots, r - 1$, (ii) $\|a_i\|_{C^0([0,1])} \leq C_3$ for any i . Take $0 < \delta < \frac{1}{2}$.

Lemma A.9. *There exist $C_4 > 0$ and $t_4 > 0$, depending only on C_0, C_2, C_3 and δ , such that the following holds on $[\delta, 1 - \delta]$ for $t \geq t_4$:*

- Let G^t and H^t be as in Proposition A.7. Then, we have

$$\|G^t\|_{C^0([\delta,1-\delta])} + \|H^t\|_{C^0([\delta,1-\delta])} \leq C_4(1+t)^{-1} \|B\|_{C^0([0,1])}.$$

Proof. Note that $G^t_{i,i} = 0$ for any i . For $i \neq j$, we have

$$\|G^t_{i,j}\|_{C^0([0,1])} + \|\partial_s G^t_{i,j} + (t(a_i - a_j) + b_i - b_j)G^t_{i,j}\|_{C^0([0,1])} \leq C_1 \|B\|_{C^0([0,1])}. \quad (\text{A.3})$$

By Lemma A.5 and Lemma A.6, there exist $C_{10} > 0$ and $t_{10} > 0$, depending only on C_0, C_2 and δ such that the following holds for any $t \geq t_{10}$:

$$\|G^t\|_{C^0([\delta,1-\delta])} \leq \frac{C_{10}}{1+t} \|B\|_{C^0([0,1])}. \quad (\text{A.4})$$

By (A.3) and (A.4), there exist $C_{11} > 0$, depending only on C_0, C_2, C_3 and δ such that the following holds for any $t \geq t_{10}$:

$$\|\partial_s G^t\|_{C^0([\delta,1-\delta])} \leq C_{11} \|B\|_{C^0([0,1])}. \quad (\text{A.5})$$

By (A.2), we have

$$(I + G^t)(A^t + B)(I + G^t)^{-1} = A^t + H^t + \partial_s(G^t) \cdot (I + G^t)^{-1}. \quad (\text{A.6})$$

Note that the diagonal entries of B, G and $\partial_s G^t$ are 0. By (A.4), (A.5) and (A.6), there exist $C_{12} > 0$, depending only on C_0, C_2, C_3 and δ such that the following holds for any $t \geq t_{10}$:

$$\|H^t\|_{C^0([\delta,1-\delta])} \leq C_{12}(1+t)^{-1} \|B\|_{C^0([0,1])}.$$

Thus, we obtain the claim of the lemma. \square

A.3.3. Reformulation. Let us recall the reformulation of Proposition A.7 with a complementary estimate, as in [Mo, Corollary 2.19]. Let A^t , C_i ($i = 0, 1, 2, 3$) and ϵ_1 be as in §A.3.2. Let E be a C^1 -vector bundle on $[0, 1]$ with a frame $\mathbf{v} = (v_1, \dots, v_r)$. Let $B \in C^0([0, 1], M_r(\mathbb{C})_1)$ satisfying $\|B\|_{C^0([0,1])} \leq \epsilon_1$. For $t \geq 0$, let ∇^t denote the connection of E determined by $\nabla^t \mathbf{v} = \mathbf{v} \cdot (A^t + B) ds$. We obtain the following corollary from Proposition A.7 and Lemma A.9.

Corollary A.10. *There exist matrix valued functions $G^t \in C^1([0, 1], M_r(\mathbb{C})_1)$ and $H^t \in C^0([0, 1], M_r(\mathbb{C})_0)$ such that the following holds.*

- $\|G^t\|_{C^0([0,1])} + \|\partial_s G^t + [A^t, G^t]\|_{C^0([0,1])} + \|H^t\|_{C^0([0,1])} \leq C_1 \|B\|_{C^0([0,1])}$.
- For the frame $\mathbf{u}^t = \mathbf{v} \cdot (I + G^t)^{-1}$, we have $\nabla^t \mathbf{u}^t = \mathbf{u}^t \cdot (A^t + H^t) ds$.

Moreover, there exist $C_{20} > 0$ and $t_{20} > 0$ depending only on C_i ($i = 0, 2, 3$) such that the following holds for any $t \geq t_{20}$:

$$\|G^t\|_{C^0([1/4, 3/4])} + \|H^t\|_{C^0([1/4, 3/4])} \leq \frac{C_{20}}{1+t} \|B\|_{C^0([0,1])}$$

□

For each t , ∇^t induces an isomorphism $\Psi^t : E|_{s=1/4} \simeq E|_{s=3/4}$. It is represented by the diagonal matrix with respect to the bases $\mathbf{u}^t|_{s=1/4}$ and $\mathbf{u}^t|_{s=3/4}$, whose (j, j) -entries are

$$\exp\left(-\int_{1/4}^{3/4} (ta_j(s) + b_j(s) + H_{jj}^t(s)) ds\right).$$

A.4. Proof of Proposition A.2. Let us return to the setting in §A.1. We extend γ to a C^∞ -map $\tilde{\gamma} : [-1, 2] \rightarrow X$ such that $\operatorname{Re} \tilde{\gamma}^* \omega(\partial_u) < 0$ at any point of $[-1, 2]$. There exists a C^∞ -frame v_\pm of $\tilde{\gamma}^* V_\pm$. We have $\tilde{\gamma}^*(\Phi)(v_\pm) = \pm \tilde{\gamma}^*(\omega)v_\pm$. We obtain a C^∞ -map $\mathcal{B} : [-1, 2] \rightarrow M_2(\mathbb{C})$ determined by

$$\tilde{\gamma}^*(\nabla)(v_+, v_-) = (v_+, v_-) \cdot \mathcal{B} du.$$

We have $\tilde{\gamma}^*(\nabla_{V_+})v_+ = \mathcal{B}_{11}v_+ du$ and $\tilde{\gamma}^*(\nabla_{V_-})v_- = \mathcal{B}_{22}v_- du$.

We obtain $\alpha \in C^\infty([-1, 2])$ by $\tilde{\gamma}^* \omega = \alpha du$. We have $\operatorname{Re}(\alpha) < 0$ at any point of $[-1, 2]$. Let $\mathcal{A} : [-1, 2] \rightarrow M_2(\mathbb{C})_0$ be the C^∞ -map determined by $\mathcal{A}_{11} = \alpha$ and $\mathcal{A}_{22} = -\alpha$. We have

$$\tilde{\gamma}^*(\nabla^t)(v_+, v_-) = (v_+, v_-) \cdot (t\mathcal{A} + \mathcal{B}) du.$$

There exists $C_0 > 0$ such that $\|\mathcal{B}_{j,j}\|_{C^0([-1,2])} \leq C_0$ for $j = 1, 2$. Let C_1 and ϵ_1 be positive constants as in Proposition A.7, depending on C_0 . There exists a positive integer $N > 10$ such that

$$\|\mathcal{B}_{1,2}\|_{C^0([-1,2])} + \|\mathcal{B}_{2,1}\|_{C^0([-1,2])} \leq \frac{N}{10} \epsilon_1.$$

We set $u(i) := \frac{i}{N}$ for $i = -N, \dots, 2N$. We obtain the decomposition $[-1, 2] = \bigcup_{i=-N}^{2N-1} [u(i), u(i+1)]$. Let $\Pi_i^t : \tilde{\gamma}^*(V)|_{u(i)} \simeq \tilde{\gamma}^*(V)|_{u(i+1)}$ denote the isomorphisms obtained as the parallel transport of $\tilde{\gamma}^* \nabla^t$.

Lemma A.11. *There exist constants $C_{30} > 0$ and $t_{30} > 0$, a family of 2×2 -matrices $G_{i,0}^t, G_{i,1}^t \in M_2(\mathbb{C})_1$ for $t \geq 0$ and $-N \leq i \leq 2N-1$, and families of continuous functions $H_{i,+}^t, H_{i,-}^t \in C^0([u(i), u(i+1)])$ for $-N \leq i \leq 2N-1$, such that the following holds.*

- $C_{30}(1+t_{30})^{-1} \leq 1/10$.

- $|G_{i,0}^t| + |G_{i,1}^t| \leq C_{30}(1+t)^{-1}$ for any $t \geq t_{30}$. Note that we obtain the bases $(v_+, v_-)|_{u(i)}(I+G_{i,0}^t)^{-1}$ and $(v_+, v_-)|_{u(i+1)}(I+G_{i,1}^t)^{-1}$ of $\tilde{\gamma}^*(V)|_{u(i)}$ and $\tilde{\gamma}^*(V)|_{u(i+1)}$, respectively.
- $\|H_{i,\pm}^t\|_{C^0([u(i), u(i+1)])} \leq C_{30}(1+t)^{-1}$ for any $t \geq t_{30}$.
- For each (i, t) , Π_i^t is represented by a diagonal matrix \mathcal{C}_i^t with respect to the bases $(v_+, v_-)|_{u(i)}(I+G_{i,0}^t)^{-1}$ and $(v_+, v_-)|_{u(i+1)}(I+G_{i,1}^t)^{-1}$. Moreover, we obtain

$$(\mathcal{C}_i^t)_{1,1} = \exp\left(-\int_{u(i)}^{u(i+1)} (t\alpha + \mathcal{B}_{1,1} + H_{i,+}^t) du\right),$$

$$(\mathcal{C}_i^t)_{2,2} = \exp\left(-\int_{u(i)}^{u(i+1)} (-t\alpha + \mathcal{B}_{2,2} + H_{i,-}^t) du\right).$$

Proof. Let $F_i : [0, 1] \simeq [\frac{2i-1}{2N}, \frac{2i+3}{2N}]$ be the affine isomorphism given by $F_i(s) = \frac{1}{2N}(2i-1+4s)$. Note that F_i induces $[\frac{1}{4}, \frac{3}{4}] \simeq [u(i), u(i+1)]$. We obtain the bundle $F_i^*(\tilde{\gamma}^*V)$, equipped with the frame $F_i^*(v_+, v_-)$ and the family of connections $F_i^*(\tilde{\gamma}^*(\nabla^t))$. Because $F_i^*(du) = \frac{2}{N}ds$, we obtain $\|F_i^*\mathcal{B}_{j,j} \cdot \partial_s F_i^*(u)\|_{C^0([0,1])} \leq C_0$ and

$$\|F_i^*\mathcal{B}_{1,2} \cdot \partial_s F_i^*(u)\|_{C^0([0,1])} + \|F_i^*\mathcal{B}_{2,1} \cdot \partial_s F_i^*(u)\|_{C^0([0,1])} \leq \epsilon_1.$$

By applying Corollary A.10 to $F_i^*(\tilde{\gamma}^*V)$ with $F_i^*(v_+, v_-)$ and $F_i^*(\tilde{\gamma}^*(\nabla^t))$, we obtain Lemma A.11. \square

We set $\tilde{\mathcal{C}}_i^t := (I+G_{i,1}^t)^{-1}\mathcal{C}_i^t \cdot (I+G_{i,0}^t)$. Note that Π_i^t is represented by $\tilde{\mathcal{C}}_i^t$ with respect to the bases $(v_+, v_-)|_{u(i)}$ and $(v_+, v_-)|_{u(i+1)}$. We set

$$\mathcal{D}^t := \tilde{\mathcal{C}}_{N-1}^t \cdot \tilde{\mathcal{C}}_{N-2}^t \cdots \tilde{\mathcal{C}}_1^t \cdot \tilde{\mathcal{C}}_0^t.$$

Let Π^t be the isomorphism $\tilde{\gamma}^*(V)|_0 \simeq \tilde{\gamma}^*(V)|_1$ obtained as the parallel transport of $\tilde{\gamma}^*\nabla^t$. Because $\Pi^t = \Pi_{N-1}^t \circ \Pi_{N-2}^t \circ \cdots \circ \Pi_1^t \circ \Pi_0^t$, the isomorphism Π^t is represented by \mathcal{D}^t with respect to the bases $(v_+, v_-)|_0$ and $(v_+, v_-)|_1$. For any $1 \leq k, \ell \leq 2$, we have

$$\lim_{t \rightarrow \infty} e^{t \int_{u(i)}^{u(i+1)} \tilde{\gamma}^* \omega} \cdot \tilde{\mathcal{C}}_{k,\ell}^t = \begin{cases} \exp(-\int_{u(i)}^{u(i+1)} \mathcal{B}_{1,1} du) & ((k, \ell) = (1, 1)) \\ 0 & (\text{otherwise}). \end{cases}$$

We obtain

$$\lim_{t \rightarrow \infty} e^{t \int_0^1 \tilde{\gamma}^* \omega} \mathcal{D}_{k,\ell}^t = \begin{cases} \exp(-\int_0^1 \mathcal{B}_{1,1} du) & ((k, \ell) = (1, 1)) \\ 0 & (\text{otherwise}). \end{cases}$$

Thus, we obtain the claim of Proposition A.2. \square

ACKNOWLEDGEMENTS

IB is partially supported by a J. C. Bose Fellowship. SD was partially supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR2152IDEX201. LH is supported by the DFG grant HE 7914/2-1 of the DFG priority program SPP 2026 Geometry at Infinity. SH is supported by the DFG grant HE 6829/3-1 of the DFG priority program SPP 2026 Geometry at Infinity. TM is partially supported by the Grant-in-Aid for Scientific Research (S) (No. 17H06127), the Grant-in-Aid for Scientific Research (S) (No. 16H06335), and the Grant-in-Aid for Scientific Research (C) (No. 20K03609),

Japan Society for the Promotion of Science. We would also like to thank Andrew Neitzke, Sebastian Schulz and Carlos Simpson for very helpful comments on WKB analysis.

REFERENCES

- [AGM] L. F. Alday, D. Gaiotto and J. Maldacena, Thermodynamic bubble ansatz, *J. High Energ. Phys.* **32** (2011).
- [AQ] D. Alessandrini and Q. LI, ADS 3-manifolds and Higgs bundles, *Proc. AMS* **146** (2018), 845–860.
- [B] I. Biswas, Parabolic bundles as orbifold bundles, *Duke Math. Jour.* **88** (1997), 305–326.
- [BDH] I. Biswas and S. Dumitrescu and S. Heller, Irreducible flat $SL(2, \mathbb{R})$ -connections on the trivial holomorphic bundle, *Jour. Math. Pures. Appl.*, arxiv.org/abs/2003.06997.
- [BuIW] M. Burger, A. Iozzi and A. Wienhard, Surface group representations with maximal Toledo invariant, *Ann. of Math.* **172** (2010), 517–566.
- [BoHS] A. Bobenko, S. Heller and N. Schmitt, Constant mean curvature surfaces based on fundamental quadrilaterals, arXiv:2102.03153.
- [CDHL] G. Calsamiglia, B. Deroin, V. Heu and F. Loray, The Riemann-Hilbert mapping for $sl(2)$ -systems over genus two curves, *Bull. Soc. Math. Fr.* **147** (2019), 159–195.
- [CoPoTe] I. Coman, E. Pomoni, J. Teschner, From quantum curves to topological string partition functions, arXiv:1811.01978.
- [De] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.
- [DM] P. Deligne et G. Mostow, *Commensurabilities among lattices in $PU(n, 1)$* , Annals of Mathematics Studies, Number 132, Princeton University Press, Princeton, N.J., (1993).
- [Fa] G. Faltings, Real projective structures on Riemann surfaces, *Composit. Math.* **48** (1983), 223–269.
- [FGuTe] E. Frenkel, S. Gukov and J. Teschner, Surface operators and separation of variables, *J. High Energ. Phys.*, 179 (2016).
- [GMN] D. Gaiotto, G. Moore and A. Neitzke, Wall-crossing, Hitchin systems, and the WKB approximation, *Adv. Math.* **234** (2013), 239–403.
- [GaKaMa] D. Gallo, M. Kapovich, A. Marden, The monodromy groups of Schwarzian equations on closed Riemann surfaces, *Ann. of Math.*, (2) 151 (2000), no. 2, 625–704.
- [Gh] E. Ghys, Déformations des structures complexes sur les espaces homogènes de $SL(2, \mathbb{C})$, *Jour. Reine Angew. Math.* **468** (1995), 113–138.
- [Go1] W. M. Goldman, Projective structures with Fuchsian holonomy, *J. Differential Geom.* **25** (1987), no. 3, 297–326.
- [Go2] W. M. Goldman, Topological components of spaces of representations, *Invent. Math.* **93** (1988), 557–607.
- [Go3] W. M. Goldman, Ergodic theory on moduli spaces, *Ann. of Math.* **146** (1997), 475–507.
- [Go4] W. M. Goldman, The modular group action on real $SL(2)$ -characters of a one-holed torus, *Geom. Top.* **7** (2003), 443–486.
- [Gu1] R. C. Gunning, *Lectures on vector bundles over Riemann surfaces*, University of Tokyo Press, Tokyo; Princeton University Press, Princeton, (1967).
- [Gu2] R.C. Gunning, Special coordinate coverings of Riemann surfaces, *Math. Ann.* **170** (1967), 67–86.
- [GuiW] O. Guichard and A. Wienhard, Anosov representations, domains of discontinuity and applications, *Invent. Math.* **190** (2012), 357–438.
- [Ha] O. Haupt, Ein Satz über die Abelschen Integrale 1. Gattung, *Math Z.* **6** (1920), 219–237.
- [Hej] D. A. Hejhal, Monodromy groups and linearly polymorphic functions, *Acta Math.* **135** (1975), no. 1, 1–55.
- [H] S. Heller, A spectral curve approach to Lawson symmetric surfaces of genus 2, *Math. Ann.* (2014).
- [HH] L. Heller and S. Heller, Abelianization of Fuchsian systems and applications, *Jour. Symp. Geom.* **14** (2016), 1059–1088.
- [HHS] L. Heller, S. Heller and N. Schmitt, Navigating the space of symmetric cmc surfaces, *Jour. Diff. Geom.* **110** (2018), 413–455.
- [Hi1] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* **55** (1987), 59–126.
- [Hi2] N. J. Hitchin, Harmonic maps from a 2-torus to the 3-sphere, *J. Differential Geom.* **31** (1990), no. 3, 627–710.

- [Ka] N. M. Katz, An overview of Deligne’s work on Hilbert’s twenty-first problem, *Mathematical developments arising from Hilbert problems* (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), pp. 537–557. Amer. Math. Soc., Providence, R. I., 1976.
- [Kap1] M. Kapovich, On monodromy of complex projective structures, *Invent Math* **119**, 243–265 (1995).
- [KNPSi] L. Katzarkov, A. Noll, P. Pandit and C. Simpson, Harmonic maps to buildings and singular perturbation theory, *Comm. Math. Phys.* **336** (2015), 853–903.
- [La] F. Labourie, Anosov flows, surface groups and curves in the projective space, *Invent. Math.* **165** (2006), 51–114.
- [L] H. B. Lawson, Complete minimal surfaces in S^3 , *Ann. of Math. (2)* **92** (1970), 335–374 .
- [LoSa] F. Loray and M.-H. Saito, Lagrangian fibration in duality on moduli space of rank two logarithmic connections over the projective line, *Int. Math. Res. Not.*, Volume: 2015, Issue: 4, 2015.
- [MY] M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves, *Math. Ann.* **293** (1992), 77–99.
- [Mas] B. Maskit, On a class of Kleinian groups, *Ann. Acad. Sci. Fenn.*, Ser. A I No. 442 (1969).
- [MS] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980), 205–239.
- [Mi] J. Milnor, On the existence of a connection with curvature zero, *Comment. Math. Helv.* **21** (1958), 215–223.
- [Mo] T. Mochizuki, Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces, *Jour. Topol.* **9** (2016), 1021–1073.
- [Oh] M. Ohtsuki, A residue formula for Chern classes associated with logarithmic connections, *Tokyo Jour. Math.* **5** (1982), 13–21.
- [Si] C. T. Simpson, Harmonic bundles on noncompact curves, *Jour. Amer. Math. Soc.* **3** (1990), 713–770.
- [StG] H. P. de Saint Gervais, *Uniformization of Riemann Surfaces*. Revisiting a hundred year old theorem. Translated from the 2010 French original by Robert G. Burns, Heritage of European Mathematics. European Mathematical Society (EMS), Zrich, 2016.
- [SuT] D. Sullivan and W. Thurston, Manifolds with canonical coordinate charts: some examples, *Enseign. Math. (2)* **29** (1983), no. 1-2, 15–25.
- [Th] W. P. Thurston, *Three-dimensional geometry and topology*, ed: Silvio Levy, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, (1997).
- [Tr] M. Traizet, Gluing Delaunay ends to minimal n-noids using the DPW method, *Mathematische Annalen* **377** (3), 1481–1508 (2020).
- [W] E. Witten, Mirror Symmetry, Hitchin’s Equations, and Langlands Duality, *The Many Facets of Geometry: A Tribute to Nigel Hitchin*, eds: Oscar Garcia-Prada, Jean Pierre Bourguignon, and Simon Salamon (2010).
- [Wo] M. Wolf, The Teichmüller Theory of Harmonic Maps, *J. Differential Geom.* **29** (1989), 449–479.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

E-mail address: `indranil@math.tifr.res.in`

UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE

E-mail address: `dumitres@unice.fr`

INSTITUTE OF DIFFERENTIAL GEOMETRY, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER

E-mail address: `lynn.heller@math.uni-hannover.de`

INSTITUTE OF DIFFERENTIAL GEOMETRY, LEIBNIZ UNIVERSITÄT HANNOVER, WELFENGARTEN 1, 30167 HANNOVER

E-mail address: `seb.heller@gmail.com`

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: `takuro@kurims.kyoto-u.ac.jp`