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Guaranteed Quantity of Interest Error Estimate Based on Equilibrated Flux Reconstruction

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The quality of a local physical quantity obtained by the numerical method, such as the finite element method (FEM), attracts more and more attention in computational electromagnetism. Inspired by the idea of goal-oriented error estimate given for the Laplace problem, this work is devoted to a guaranteed *a posteriori* error estimate adapted for the quantity of interest (QOI) linked to magnetostatic problems, in particular, to the value of the magnetic flux density. The development is principally based on an equilibrated flux construction, which ensures fully computable estimators without any unknown constant. The main steps of the mathematical development are given in detail with the physical interpretation. An academic example using an analytical solution is considered to illustrate the performance of the approach, and a discussion about different aspects related to the practical point of view is proposed.

Index Terms—A *a posteriori* error estimate, equilibrated flux reconstruction, finite element method (FEM), magnetic flux density, magnetostatic problem, quantity of interest (QOI).

I. INTRODUCTION

NOWADAYS, the finite element method (FEM) is widely used in the numerical modeling of low-frequency electromagnetic problems. In order to ensure the quality of the numerical solution, *a posteriori* error estimates play an important role, and different types of error estimates have been developed in the past 30 years. Among these estimators, in particular, in magnetostatics, the most used one is the equilibrated type since it can provide the estimate without any unknown constant, which verifies the constitutive laws of material by dual formulations [1]–[4] or dual meshes [5]–[8]. Some detailed comparisons with the residual type can be found in [9], which principally verifies the discontinuities of the physical quantity and the local equation in the strong sense. However, the abovementioned work relies on the energy norm of the error in a global sense, which is natural for the FEM.

In practical applications, people are frequently interested in the true value of a given local physical quantity, which is called the quantity of interest (QOI). In particular, while some semi-analytic methods are used in the coupled multi-physical computation, only the value of the magnetic flux density at a given point is useful, for example, when we compute the iron losses reported in [10] or the source term for the thermal part defined in [11]. In these cases, the accuracy of the local value becomes critical, which gives us the original motivation of this work.

Inspired by the work done in [12] for the Laplace problem, we adapt the error estimate in both potential formulations

in magnetostatics for the QOI and, in particular, the value of the magnetic flux density. Our main contribution includes two aspects: on the one hand, a complete mathematical development is given in detail with physical explanations; on the other hand, the performance of the proposed estimator is illustrated by an analytical example, with a discussion on different parameters, namely, the global mesh refinement level, the local mesh refinement level, the size of the region on which the QOI is investigated, and the high order finite element (FE) basis. It allows to provide a baseline for the utilization.

The organization of the article is given as follows. The mathematical development for the error estimate of the QOI is given in Section II. In Section III, the performance of the proposed estimator is evaluated through numerical examples. The conclusion is given in Section IV.

II. QOI FOR MAGNETOSTATIC PROBLEMS

Following the idea given in [12] for the classical Laplace problem, we adapt the development to magnetostatic problems for both formulations with a different definition of the QOI linked to the value of the magnetic flux density. To keep the completeness of the mathematical development, several necessary (sometimes well-known) definitions are briefly recalled in the following.

A. Potential Formulations

Considering a given divergence-free applied current density \mathbf{J}_s as the coil excitation, denoting \mathbf{B} and \mathbf{H} , respectively, the magnetic flux density and the magnetic field, μ the magnetic permeability, the magnetostatic problem reads

$$\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{H} = \mathbf{J}_s, \text{ and } \mathbf{B} = \mu \mathbf{H}. \quad (1)$$

Introducing the magnetic vector potential \mathbf{A} such that $\mathbf{B} = \nabla \times \mathbf{A}$ and the magnetic scalar potential Ω such

that $\mathbf{H} = \mathbf{H}_s - \nabla\Omega$ with \mathbf{H}_s the magnetic source satisfying $\nabla \times \mathbf{H}_s = \mathbf{J}_s$, the classic potential formulations are given as follows:

$$\nabla \times (\nu \nabla \times \mathbf{A}) = \mathbf{J}_s; \quad \nabla \cdot (\mu \nabla \Omega) = \nabla \cdot (\mu \mathbf{H}_s) \quad (2)$$

with ν being the reluctivity (inverse of the permeability).

Then, the corresponding variational formulations read

$$(\nu \nabla \times \mathbf{A}, \nabla \times \mathbf{A}') = (\mathbf{J}_s, \mathbf{A}'); \quad (\mu \nabla \Omega, \nabla \Omega') = (\mu \mathbf{H}_s, \nabla \Omega') \quad (3)$$

with \mathbf{A}' and Ω' the test functions in *ad hoc* continuous spaces. Let us denote by \mathbf{A}_h and Ω_h the numerical solutions obtained by FEM, which means that

$$(\nu \nabla \times \mathbf{A}_h, \nabla \times \mathbf{A}'_h) = (\mathbf{J}_s, \mathbf{A}'_h); \quad (\mu \nabla \Omega_h, \nabla \Omega'_h) = (\mu \mathbf{H}_s, \nabla \Omega'_h) \quad (4)$$

where \mathbf{A}'_h and Ω'_h are the test functions in *ad hoc* discrete spaces. Two different approximations of the magnetic flux density \mathbf{B} can be consequently deduced from these approximated potentials \mathbf{A}_h and Ω_h and are given by

$$\mathbf{B}_h = \nabla \times \mathbf{A}_h \text{ or } \mathbf{B}_h = \mu(\mathbf{H}_s - \nabla \Omega_h). \quad (5)$$

B. Quantity of Interest

In the general case, only the value of the magnetic flux density \mathbf{B} at certain points is relevant for some applications. However, for mathematical considerations, the value at a given point of the domain does not make sense since the FEM minimizes the error of the solution in the global energy norm. Let us denote D_{QOI} as the sub-region of the computational domain D , which contains the expected point. The QOI for the magnetic flux density \mathbf{B} in this sub-region D_{QOI} can be defined as

$$Q(\mathbf{B}) = \int_D \mathbf{q} \cdot \mathbf{B}, \quad \text{with } \mathbf{q} = \frac{1}{|D_{\text{QOI}}|} \delta_{D_{\text{QOI}}} \mathbf{I}_{\text{dir}} \quad (6)$$

where $\delta_{D_{\text{QOI}}}$ is the characteristic function of D_{QOI} and \mathbf{I}_{dir} is the given unit vector, which can be chosen as $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$. It should be mentioned that the definition of the QOI is still made in an energy sense, which can be seen as the average value of each component of \mathbf{B} in the sub-region D_{QOI} . In other words, for a 3-D computation, to evaluate the quality of \mathbf{B} , each component (as x , y , or z) will be evaluated separately by using different values of \mathbf{I}_{dir} . Besides, it is clear that this definition of the QOI depends on the sub-region size D_{QOI} . When the D_{QOI} region size decreases, its limit can be seen as the value at one point.

C. Adjoint Problems

With the definition of the QOI, in particular, the function \mathbf{q} defined in (6), the corresponding adjoint problems for the potential formulations (2) read

$$\nabla \times (\nu \nabla \times \tilde{\mathbf{A}}) = \nabla \times \mathbf{q}; \quad \nabla \cdot (\mu \tilde{\Omega}) = \nabla \cdot (\mu \mathbf{q}) \quad (7)$$

with their variational formulations given by

$$(\nu \nabla \times \tilde{\mathbf{A}}, \nabla \times \mathbf{A}') = (\mathbf{q}, \nabla \times \mathbf{A}'); \quad (\mu \nabla \tilde{\Omega}, \nabla \Omega') = (\mu \mathbf{q}, \nabla \Omega'). \quad (8)$$

These adjoint problems are similar to the corresponding primal ones [namely, potential formulations given in (2)] but with a different source term. It should be mentioned that the source terms of adjoint problems are well defined for both formulations. In particular, for the $\tilde{\mathbf{A}}$ formulation, there is no consistency problem with the source term [13].

D. Equilibrated Flux Reconstruction

It is clear that the numerical solutions \mathbf{A}_h and Ω_h obtained from (4) cannot satisfy (3) since the involved discrete spaces are only subspaces of the continuous ones. Then, the reconstructed equilibrated flux $\mathfrak{t}_h(\mathbf{A}_h)$ and $\mathfrak{t}_h(\Omega_h)$ can be defined as

$$(\mathfrak{t}_h(\mathbf{A}_h), \nabla \times \mathbf{A}') = (\pi_h^l(\mathbf{J}_s), \mathbf{A}') \quad (9)$$

$$(\mathfrak{t}_h(\Omega_h), \nabla \Omega') = (\pi_h^l(\mu \mathbf{H}_s), \nabla \Omega') \quad (10)$$

with π_h^l being an *ad hoc* projection operator from \mathbf{L}^2 space to the corresponding source term discretization spaces. Generally, the discretization error of the source between \mathbf{J}_s and $\pi_h^l(\mathbf{J}_s)$ [respectively, between \mathbf{H}_s and $\pi_h^l(\mathbf{H}_s)$] is considered negligible. Similar to the primal problems, the equilibrated flux for the adjoint problems can be constructed as

$$(\mathfrak{t}_h(\tilde{\mathbf{A}}_h), \nabla \times \mathbf{A}') = (\mathbf{q}, \nabla \times \mathbf{A}') \quad (11)$$

$$(\mathfrak{t}_h(\tilde{\Omega}_h), \nabla \Omega') = (\mu \mathbf{q}, \nabla \Omega'). \quad (12)$$

The reconstructed equilibrated flux can be seen as a correction of the obtained solution in the strong sense, which satisfies the continuous problems. There exist different ways to construct the equilibrated flux:

- 1) The dual formulations can naturally provide an equilibrated flux [1]–[4]. However, both formulations need to be solved simultaneously.
- 2) Only one formulation can be solved, but two different meshes are needed, as reported in [5] and [6].
- 3) This can also be done by some local flux reconstructions, as introduced in [7], [8], and [14].
- 4) In some particular cases with the open boundary conditions, dual formulations with the same matrix can be used, as reported in [15].

E. Error Estimate for the QOI

In the following part, we give a brief proof of the error estimate for the QOI. Only the \mathbf{A} formulation part is discussed, while the Ω part is similar to the derivation in [12].

From the definition of the QOI in (6), the obtained numerical solution of QOI is written as

$$Q(\mathbf{B}_h) = \int_D \mathbf{q} \cdot \mathbf{B}_h = \int_D \mathbf{q} \cdot \nabla \times \mathbf{A}_h. \quad (13)$$

Then, the error of the QOI is defined by

$$\varepsilon = Q(\mathbf{B}) - Q(\mathbf{B}_h). \quad (14)$$

For the first term $Q(\mathbf{B})$, replacing \mathbf{A}' by \mathbf{A} in (8), we get

$$Q(\mathbf{B}) = \int_D \mathbf{q} \cdot \nabla \times \mathbf{A} = \int_D \nu \nabla \times \tilde{\mathbf{A}} \cdot \nabla \times \mathbf{A}. \quad (15)$$

Replacing \mathbf{A}' by $\tilde{\mathbf{A}}$ in (3), we obtain

$$Q(\mathbf{B}) = \int_D \mathbf{J}_s \cdot \tilde{\mathbf{A}}. \quad (16)$$

For the second term $Q(\mathbf{B}_h)$, using (8), we have

$$Q(\mathbf{B}_h) = \int_D \nu \nabla \times \tilde{\mathbf{A}} \cdot \nabla \times \mathbf{A}_h. \quad (17)$$

Combining (16) and (17) with (14) yields

$$\varepsilon = \int_D \mathbf{J}_s \cdot \tilde{\mathbf{A}} - \int_D \nu \nabla \times \tilde{\mathbf{A}} \cdot \nabla \times \mathbf{A}_h. \quad (18)$$

By adding and subtracting $\int_D \mathbf{t}_h(\mathbf{A}_h) \cdot \nabla \times \tilde{\mathbf{A}}$, we get

$$\begin{aligned} \varepsilon = & \int_D \mathbf{J}_s \cdot \tilde{\mathbf{A}} - \mathbf{t}_h(\mathbf{A}_h) \cdot \nabla \times \tilde{\mathbf{A}} \\ & - \int_D \nabla \times \tilde{\mathbf{A}} \cdot (\nu \nabla \times \mathbf{A}_h - \mathbf{t}_h(\mathbf{A}_h)). \end{aligned} \quad (19)$$

Employing the definition of the equilibrated flux in (9), we have

$$\begin{aligned} \varepsilon = & \int_D (\mathbf{J}_s - \pi_h^l(\mathbf{J}_s)) \cdot (\tilde{\mathbf{A}} - \tilde{\mathbf{A}}_h + \tilde{\mathbf{A}}_h) \\ & - \int_D (\nabla \times \tilde{\mathbf{A}} - \mu \mathbf{t}_h(\tilde{\mathbf{A}}_h) + \mu \mathbf{t}_h(\tilde{\mathbf{A}}_h)) \\ & \cdot (\nu \nabla \times \mathbf{A}_h - \mathbf{t}_h(\mathbf{A}_h)) \\ = & \eta + \mathcal{R}. \end{aligned} \quad (20)$$

It leads to a fully computable estimator η given by

$$\begin{aligned} \eta = & \int_D (\mathbf{J}_s - \pi_h^l(\mathbf{J}_s)) \cdot \tilde{\mathbf{A}}_h \\ & - \int_D \mu \mathbf{t}_h(\tilde{\mathbf{A}}_h) \cdot (\nu \nabla \times \mathbf{A}_h - \mathbf{t}_h(\mathbf{A}_h)) \end{aligned} \quad (21)$$

as well as to the definition of \mathcal{R} given by

$$\begin{aligned} \mathcal{R} = & \int_D (\mathbf{J}_s - \pi_h^l(\mathbf{J}_s)) \cdot (\tilde{\mathbf{A}} - \tilde{\mathbf{A}}_h) \\ & - \int_D (\nabla \times \tilde{\mathbf{A}} - \mu \mathbf{t}_h(\tilde{\mathbf{A}}_h)) \cdot (\nu \nabla \times \mathbf{A}_h - \mathbf{t}_h(\mathbf{A}_h)). \end{aligned} \quad (22)$$

The estimator η given in (21) includes some important informations, such as the discretization error of the source for the primal problem, as well as the difference between the obtained numerical magnetic field and the reconstructed equilibrated flux always for the primal problem, weighted by the equilibrated flux of the adjoint problem. It ensures that the term has the same dimension as an energy. Besides, the term \mathcal{R} is negligible and corresponds to a high-order remainder [12].

III. NUMERICAL EXAMPLES

In this section, an academic example is considered. The unit cube $[0, 1]^3$ is crossed by a current density of 10 MA/m², as shown in Fig. 1. The analytic expression of the magnetic flux density can be found in [9]. The numerical implementation is realized by the free scientific computing software Freefem++ [16], which is easy to use for algorithm development in 3-D electromagnetic field computations [17]. The QOI is defined as the z -component of the magnetic flux density \mathbf{B}

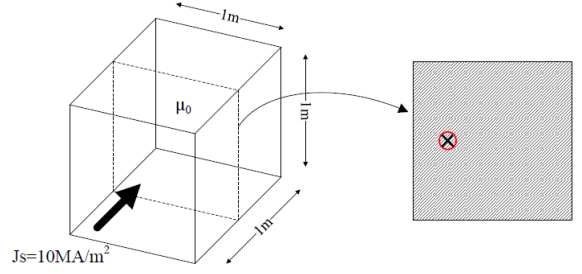


Fig. 1. Academic example with a known analytic solution.

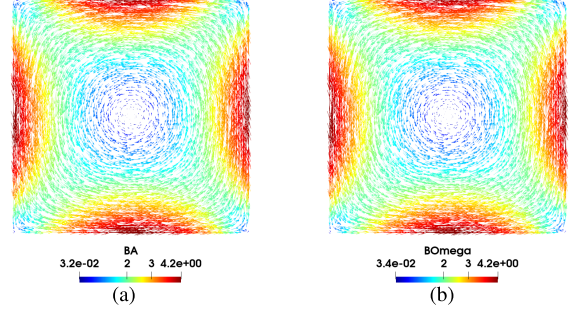


Fig. 2. Spatial distributions of the magnetic flux density \mathbf{B} (unit: T) obtained by the two formulations. (a) \mathbf{A} formulation. (b) Ω formulation.

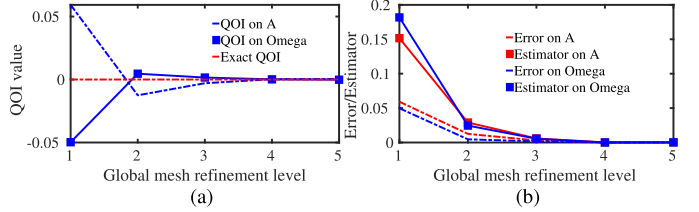


Fig. 3. Performance as a function of the global mesh refinement level. (a) QOI value. (b) Error/estimator.

at the point $(0.25, 0.5, 0.5)$ (the black cross shown in Fig. 1). The distribution of the magnetic flux density in the plane $y = 0.5$ is shown in Fig. 2 for both potential formulations. First, the performance of the proposed estimator as a function of the global mesh refinement level is evaluated. We fix the size of D_{QOI} , five meshes with different global refinement level are considered, and the QOI values for both formulations with these different meshes are plotted in Fig. 3(a). The corresponding exact error and estimator are shown in Fig. 3(b). It can be observed that the global mesh refinement can significantly reduce the error, and the proposed estimators provide a good agreement with the corresponding errors.

Besides, higher order FE basis (HOFEB) is used for solving the adjoint problems. The errors and estimators for both formulations are given in Fig. 4(a) and (b), respectively. It can be observed that the estimator with HOFEB is much close to the error with the \mathbf{A} formulation, but this observation is less clear for the Ω formulation. It seems to be not necessary to use HOFEB here since it introduces more computational cost.

Second, the performance of the proposed estimator as a function of the QOI sub-region size level is evaluated. This time, the global mesh refinement level is fixed, and only the

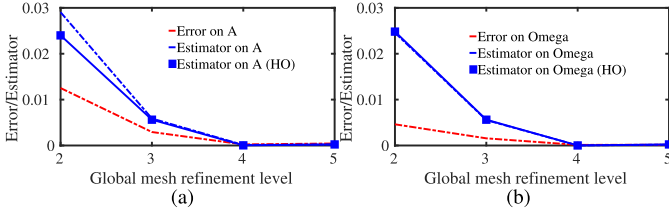


Fig. 4. Estimators as a function of the global mesh refinement level with HOFEB. (a) A formulation. (b) Ω formulation.

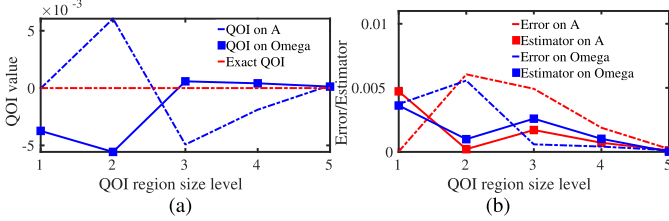


Fig. 5. Performance as a function of the QOI region size level. (a) QOI value. (b) Error/estimator.

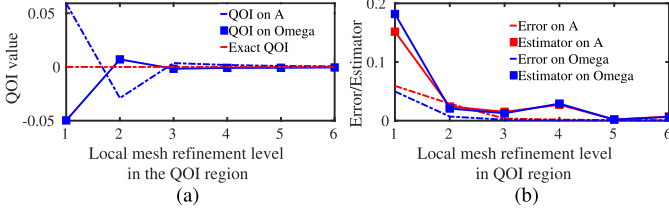


Fig. 6. Performance as a function of the local mesh refinement level in the QOI region. (a) QOI value. (b) Error/estimator.

sub-region size level changes with smaller and smaller sizes. It can be observed in Fig. 5(a) that, as expected, the QOI values converge to the exact value when the sub-region becomes smaller. However, the proposed estimator does not show good stable performances, as shown in Fig. 5(b). Indeed, when the global mesh refinement level is fixed, the source term in the adjoint problem becomes harder and harder to discretize accurately, while the area of D_{QOI} is decreasing. In other words, the adjoint problem is not more well solved. Third, the performance of the proposed estimator as a function of the local mesh refinement level in the QOI sub-region is evaluated. The size of the sub-region is fixed, and this time only the mesh in the sub-region is refined. As shown in Fig. 6(a) and (b), the QOI values obtained by the two formulations, as well as the errors and estimators, do not provide the same behavior as observed in the case of the global mesh refinement, as theoretically expected. This demonstrates that the quality of the QOI does not only depend on the local mesh in the QOI sub-region but also depends on its accuracy compared with the global mesh quality. The above different tests illustrate the fact that the accuracy of the QOI value depends on the global mesh refinement level, as well as on the local mesh refinement level, where the QOI is defined. The proposed estimate allows to provide satisfying information about the unknown error of the QOI. The influence of the high order FE basis is not significant

to improve the performance of the estimator, while it is still expensive, in particular, for the vector potential formulation. The chosen sub-region size should consider a tradeoff between the accuracy and the computation cost. Indeed, a small QOI sub-region is more relevant to approximate a point value but needs a finer mesh to ensure the performance of the estimator.

IV. CONCLUSION

In this article, *a posteriori* error estimate for the QOI linked to the value of magnetic flux density is developed for the potential formulations of magnetostatic problems. The proposed error estimate needs the information on the primal problems, as well as on their adjoint problems. Numerical examples show that the proposed estimator can provide a good agreement with the exact unknown QOI error.

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