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Complementarity relations: reduction of decision rules and informational representability

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1 Introduction

The main objectives of this paper are:

- (a) To show how reduction of decision rules and sets of decision rules can be based on an analysis of complementarity of objects defined by means of these rules,
- (b) To prove the informational representability of relational systems with complementarity relations.

In the paper we consider decision rules determined by a decision table. Each of the rules might be interpreted as a statement defining a decision that depends on properties of an object. We show that the complementarity relations derived from a decision table provide criteria for reduction of both individual rules and sets of rules. It follows that reasoning about complementarity might play an important role in development of decision support systems. Intuitively, two objects are complementary whenever the properties that they possess are negations of each other. For example, if x is green and y is not green, then x and y are complementary. We propose two kinds of structures to model complementarity phenomenon: abstract relational systems, referred to as complementarity frames, where the respective complementarity relations are determined by means of their properties; and 'concrete' structures, where the relations are derived directly from a decision table and are defined in terms of properties of objects. We prove that any abstract complementarity frame is informationally representable. The notion of informational representability of relational systems has been introduced and investigated in [1]. Let a similarity relation \bowtie in a class C of relational systems be given, for example a relation of 'being isomorphic'. Intuitively, a system \mathcal{K} from the class C is informationally \bowtie -representable if there is a decision table and a system \mathcal{K}' derived from the table such that \mathcal{K}' is in the class C and \mathcal{K} is \bowtie -related to \mathcal{K}' . The informational representability might be treated as a criterion of adequacy of a formal model in that it shows that the model captures the relevant features of an application domain (see for instance in [7, 8] representability results involving *similarity* structures).

2 Decision rules and their reduction

By an *information system* (see e.g. [6]) we mean a pair (OB, AT) where OB is a non-empty set of *objects* and AT is a non-empty set of *attributes*. Each attribute a is a mapping $a : OB \rightarrow \mathcal{P}(Val_a) \setminus \{\emptyset\}$. For each $a \in AT$, the non-empty set Val_a is the set of *values* of the attribute a . In that setting, two objects o_1, o_2 are said to be *indiscernible* with respect to a set of attributes $A \subseteq AT$ (in short $o_1 \text{ ind}(A) o_2$) iff for all $a \in A$, $a(o_1) = a(o_2)$. Various generalizations of the notion of information system and some other relations between the objects (similarity, weak indiscernibility, ...) can be found for instance in [4, 9].

An information system $S = (OB, AT, d)$, $d \notin AT$, with an additional attribute d , referred to as a decision attribute (or with a finite family of decision attributes) is called a *decision table*. In the rest of this section, we assume that $d(o)$ is a singleton set for each $o \in OB$. The values $d(o)$ of d for objects $o \in OB$ are called *decisions*. The attributes from AT are called *condition attributes*. Any decision table S with a finite set AT generates a set $D(S)$ of decision rules. By a rule determined by an object $o \in OB$ we mean the following statement. Let $AT = \{a_1, \dots, a_n\}$.

(r_o) If the value of a_1 for o is $a_1(o)$ and ... and the value of a_n for o is $a_n(o)$, then decide $d(o)$. We abbreviate such a rule as:

$$(a_1, a_1(o)) \ \& \ \dots \ \& \ (a_n, a_n(o)) \ \rightarrow \ d(o)$$

We say that a decision table is *inconsistent* if there are two objects o and o' such that $(o, o') \in \text{ind}(AT)$ and $d(o) \neq d(o')$. Otherwise, the table is *consistent*.

Example 1. Consider the decision table in Figure 1.

	colour	size	d (pick-up)
o_1	green	small	yes
o_2	not green	small	yes
o_3	blue	medium	no

Fig. 1. A decision table

In Figure 1, 'not green' stands for $Val_{colour} \setminus \{green\}$. This table generates the following set $D(S)$ of rules:

- r_{o_1} : (colour, green) & (size, small) \rightarrow pick it up
 r_{o_2} : (colour, not green) & (size, small) \rightarrow pick it up
 r_{o_3} : (colour, blue) & (size, medium) \rightarrow do not pick it up

Any decision rule can be treated in a natural way as a formula of the classical propositional calculus. For example the rule r_{o_1} from Example 1 corresponds to: if an object is green and small then pick it up. Clearly if S is consistent (respectively inconsistent) then $D(S)$ is consistent (respectively inconsistent) in the propositional calculus.

(comp) Two objects o_1 and o_2 are said to be in relation of *complementarity* with respect to $A \subseteq AT$ ($o_1 \text{ comp}(A) o_2$) iff for all $a \in A$, $a(o_1) = Val_a \setminus a(o_2)$. We say that a condition attribute $a \in AT$ is *redundant* in rule r_o if there is $o' \in OB$ such that the following conditions hold:

$$(red) \begin{cases} (o, o') \in \text{comp}(\{a\}), \\ (o, o') \in \text{ind}(AT \setminus \{a\}), \text{ and} \\ \forall z, \text{ if } d(z) \neq d(o) \text{ then } \exists b \in AT \setminus \{a\} \text{ such that } b(z) \neq b(o) \end{cases}$$

In other words, a is redundant in r_o whenever decision $d(o)$ does not depend on the condition attribute a . As a consequence, if a is redundant in r_o we may delete a part of the rule that refers to the attribute a . Moreover, we may delete from $D(S)$ all the rules $r_{o'}$ such that o and o' satisfy (red). It is easy to see that this reduction process preserves consistency/inconsistency of the set of decision rules. Traditionally, there are two groups of methods for reduction of decision rules and sets of decision rules: methods that reduce the number of condition attributes in the decision table (based on calculation of reducts) and methods of elimination of redundant conditions in a rule. As shown previously, an analysis of complementarity relations contributes to both kinds of methods.

Example 2. Consider the decision table from Figure 1. The object o_1 satisfies the condition (red). Hence we can reduce the rule r_{o_1} to:

$$r'_{o_1}: (\text{size, small}) \rightarrow \text{pick it up}$$

Moreover we remove the rule r_{o_2} from the set $D(S)$. Observe that if a row " o_4 | blue | small | no |" is added to Table 1 then the rule r_{o_1} would not be reducible. For otherwise, the reduced set of rules would be inconsistent, namely r'_{o_1} and r_{o_4} would be contradictory.

We conclude that reasoning about indiscernibility and complementarity of objects in a decision table S provides criteria for reduction of both individual decision rules and the set $D(S)$ of rules.

3 Complementarity frames

By a *complementarity frame* derived from an information system $S = (OB, AT)$ (or from a decision table $S = (OB, AT, d)$) we mean the system $(OB, \text{comp}(AT))$, where $\text{comp}(AT)$ is defined according to condition (comp) from Section 2. It can be observed that $\text{comp}(AT)$ is symmetric, irreflexive and intransitive. This is an example of a *concrete* structure (derived from user data). Moreover, when $o \text{ comp}(AT) o'$ holds, then for all $a \in AT$ the set Val_a is uniquely determined

by $a(o)$ and $a(o')$. Hence, a more subtle characterization of complementarity is needed. This problem has been open until now. The following definition provides hints to solve this problem.

A binary relation R over a set U is said to be a *complementarity* relation iff

- R is symmetric,
- R does not contain cycles of odd length and,
- for all $u, v, w, z \in U$, if $(u, v) \in R$, $(v, w) \in R$ and $(w, z) \in R$ then $(u, z) \in R$ (*3-transitivity*).

By an *abstract complementarity frame* we mean a system (U, R) such that U is a nonempty set and R is a complementarity relation over U . We write \mathcal{F} to denote the class of complementarity frames. Observe that the class \mathcal{F} is not closed under p-morphisms. It follows that \mathcal{F} is not *modally definable* (see [3]). Moreover, \mathcal{F} is *elementary* since the condition of being a complementarity relation can be expressed by a set Γ of formulas from the classical first-order logic. Hence modal logics characterized by complementarity frames might play a special role in modal logic theory. It is easy to see that $\text{comp}(AT)$ is a complementarity relation. In the following section we prove the informational representability of abstract complementarity frames.

4 Informational representability: a construction

For any binary relation R over the set U , for all $u \in U$ we write $C_{u,R}$ to denote the largest subset of U such that

- $u \in C_{u,R}$ and,
- for all $v \in C_{u,R} \setminus \{u\}$, there is a R' -path between u and v where R' is the symmetric closure of R .

Observe that $\{C_{u,R} : u \in U\}$ is a partition of U . For all $u \in U$, we write $C_{u,R}^0$ (resp. $C_{u,R}^1$) to denote the largest subset of $C_{u,R}$ such that for all $v \in C_{u,R}^0$, there is a R' -path of even (resp. odd) length between u and v . Moreover for all $U' \subseteq U$, we write $C_{u,R}^{U'}$ to denote $C_{u,R}^0$ if $u \in U'$, $C_{u,R}^1$ otherwise.

Observe that for all $u, v, w \in U$, and for a complementarity relation R ,

$$\text{if } \{v, w\} \subseteq C_{u,R} \text{ then } \{C_{v,R}^1, C_{v,R}^0\} = \{C_{w,R}^1, C_{w,R}^0\}.$$

In particular, if there is a R -path of odd length between v and w then $C_{v,R}^1 = C_{w,R}^0$ and $C_{v,R}^0 = C_{w,R}^1$ otherwise $C_{v,R}^0 = C_{w,R}^0$ and $C_{v,R}^1 = C_{w,R}^1$.

Lemma 1. *Let R be a complementarity relation over a set U . The set $\{C_{u,R}^i : u \in U, i \in \{0, 1\}\}$ is a partition of U .*

The proof of Lemma 1 is by an easy verification. It is worth mentioning that although $\{C_{u,R} : u \in U\}$ is a partition of U whatever is the binary relation R , the fact that R is a complementarity relation is a crucial hypothesis in Lemma 1.

Indeed if $(u, v) \in R^{2k}$ (resp. $(u, v) \in R^{2k+1}$) for some $k \geq 0$ then for all $k' \geq 0$, $(u, v) \notin R^{2k'+1}$ (resp. $(u, v) \notin R^{2k'}$), otherwise it would exist a cycle of odd length passing through u .

In order to show the informational representability, for all $(U, R) \in \mathcal{F}$ we construct an information system $S_{g(\mathcal{K})} = (U, \{at^{g(\mathcal{K})}\})$ such that

- $g(\mathcal{K}) \subseteq \mathcal{P}(U)$,
- $at^{g(\mathcal{K})}$ is a mapping from U into $\mathcal{P}(g(\mathcal{K}))$ and,
- $R = \text{comp}(\{at^{g(\mathcal{K})}\})$.

Let g be the mapping such that for all $\mathcal{K} = (U, R) \in \mathcal{F}$,

$$g(\mathcal{K}) = \left\{ \bigcup_{u \in Y} C_{u,R}^{U'} : \exists U' \subseteq U, \exists Y \in W \right\}$$

$$\text{with } W = \left\{ Y \subseteq U : \forall u, v \in Y, C_{u,R} \neq C_{v,R}, \bigcup_{u \in Y} C_{u,R} = U \right\}$$

The set W contains subsets of U with exactly one element for each set in $\{C_{u,R} : u \in U\}$. Moreover $g(\mathcal{K})$ can be roughly defined as the set of maximal subsets U' of U (with respect to inclusion) such that $R \cap U' \times U' = \emptyset$. Figure 2 illustrates the construction with a finite frame (the closure by 3-transitivity and symmetry is omitted in the picture for the sake of readability).

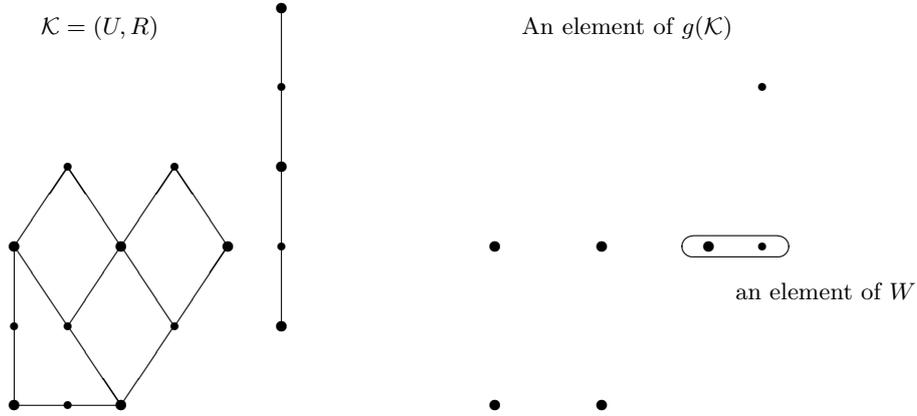


Fig. 2. Example

It is easy to show that for all $Y_0 \in W$, $g(\mathcal{K}) = \left\{ \bigcup_{u \in Y_0} C_{u,R}^{U'} : \exists U' \subseteq U \right\}$ and $\bigcup_{Y \in g(\mathcal{K})} Y = U$. For all $u \in U$, $C_{u,R}$ is not empty ($u \in C_{u,R}$). Let x^u be the *representative element* of $C_{u,R}$. Hence W is not empty since $\{x^u : u \in U\} \in W$.

Lemma 2. *Let $\mathcal{K} = (U, R)$ be a frame in \mathcal{F} . For all $u, v \in U$, $(u, v) \in R$ iff for all $Y \in g(\mathcal{K})$, either $(u \in Y \text{ and } v \notin Y)$ or $(u \notin Y \text{ and } v \in Y)$.*

Proof. (\rightarrow) Assume $(u, v) \in R$. Since $u \in C_{u,R}^0$ and $v \in C_{u,R}^1$, it follows that for all $U' \subseteq U$, $\{u, v\} \not\subseteq C_{u,R}^{U'}$. Hence there is no $Y \in g(\mathcal{K})$ such that $u \in Y$ and $v \in Y$. Moreover, for all $U' \subseteq U$, for all $Y' \in W$, $\{u, v\} \cap \bigcup_{u' \in Y'} C_{u',R}^{U'} \neq \emptyset$ and $\bigcup_{u' \in Y'} C_{u',R}^{U'} \in g(\mathcal{K})$. So for all $Y \in g(\mathcal{K})$, either $(u \in Y \text{ and } v \notin Y)$ or $(u \notin Y \text{ and } v \in Y)$.

(\leftarrow) Now assume that for all $Y \in g(\mathcal{K})$, $u \in Y$ iff $v \notin Y$. First suppose that $C_{u,R} \neq C_{v,R}$. It follows that there is $Y_0 \in W$ such that $\{u, v\} \subseteq Y_0$. In particular take $Y_0 = (\{x^w : w \in U\} \setminus \{x^u, x^v\}) \cup \{u, v\}$. Hence

$$\{u, v\} \subseteq \bigcup_{u' \in Y_0} C_{u',R}^{\{u,v\}} = X_0$$

So there is $X_0 \in g(\mathcal{K})$ such that $\{u, v\} \subseteq X_0$, which leads to a contradiction. Now assume $C_{u,R} = C_{v,R}$. Suppose there is a R -path of even length between u and v , say $(u_0, \dots, u_n, \dots, u_{2 \times n})$ with $u_0 = u$ and $v = u_{2 \times n}$ (recall that R is symmetric). There is $Y_0 \in W$ such that $\{u_n\} \subseteq Y_0$. In particular take $Y_0 = (\{x^w : w \in U\} \setminus \{x^{u_n}\}) \cup \{u_n\}$. Hence

$$\{u, v\} \subseteq \bigcup_{u' \in Y_0} C_{u',R}^{U'} = X_0$$

where $U' = U$ if n is even, otherwise $U' = \emptyset$. So there is $X_0 \in g(\mathcal{K})$ such that $\{u, v\} \subseteq X_0$, which leads to a contradiction. So there is a R -path of odd length between u and v , say $(u_0, \dots, u_{(2 \times n)+1})$ with $u_0 = u$ and $v = u_{(2 \times n)+1}$.

- If $n = 0$, then $(u, v) \in R$.
- If $n = 1$, then by the 3-transitivity condition $(u, v) \in R$.
- Now assume $n > 1$. By the 3-transitivity condition, $(u_{(2 \times (n-1))}, u_{(2 \times n)+1}) \in R$. So there is a path of length $(2 \times (n-1)) + 1$ between u and v . By induction on n one can therefore prove that $(u, v) \in R$.

We write $at^{g(\mathcal{K})}$ to denote the mapping $U \rightarrow \mathcal{P}(\mathcal{P}(U))$ such that for all $x \in U$, $at^{g(\mathcal{K})}(x) = \{Z : x \in Z \in g(\mathcal{K})\}$. Let $S_{g(\mathcal{K})}$ be the structure $(U, \{at^{g(\mathcal{K})}\})$. For all $x \in U$, $at^{g(\mathcal{K})}(x) \neq \emptyset$ since $\bigcup_{Z \in g(\mathcal{K})} Z = U$. It follows that $S_{g(\mathcal{K})}$ is an information system. It is worth mentioning that for all $u, v \in U$, $(\forall Y \in g(\mathcal{K}), \text{ either } u \in Y \text{ and } v \notin Y \text{ or } u \notin Y \text{ and } v \in Y)$ iff $(u, v) \in comp(\{at^{g(\mathcal{K})}\})$. Indeed,

$$\begin{aligned} (u, v) \in comp(\{at^{g(\mathcal{K})}\}) &\text{ iff } at^{g(\mathcal{K})}(u) = g(\mathcal{K}) \setminus at^{g(\mathcal{K})}(v) \\ &\text{ (by definition of } comp) \\ &\text{ iff } \{Z : u \in Z \in g(\mathcal{K})\} = g(\mathcal{K}) \setminus \{Z : v \in Z \in g(\mathcal{K})\} \\ &\text{ (by definition of } at^{g(\mathcal{K})}) \\ &\text{ iff } \{Z : u \in Z \in g(\mathcal{K})\} = \{Z : v \notin Z \in g(\mathcal{K})\} \\ &\text{ iff } \forall Z \in g(\mathcal{K}), u \in Z \text{ iff } v \notin Z \end{aligned}$$

Hence,

Lemma 3. $comp(\{at^{g(\mathcal{K})}\}) = R$.

So for any complementarity frame $\mathcal{K} = (U, R)$, there exists an information system (U, AT) such that $comp(AT) = R$. Moreover, for any information system (OB, AT) , the structure $(OB, comp(AT))$ is a complementarity frame. Hence,

Corollary 4. $\{(OB, comp(AT)) : (OB, AT, d) \text{ is an information system}\}$ provides an informational representation of the class \mathcal{F} .

This informational representability result relates two classes of structures, namely information systems and abstract complementarity frames. This abstract characterization of the relations $comp(AT)$ might also be useful in mechanizing the reduction process of sets of decision rules that has been intuitively described in Section 2. Namely, the result assures that any inference method designed for a modal logic determined by the class of complementarity frames ([5]) guarantees a logically adequate transformations of sets of decision rules.

Besides, as observed in [2], in the definition for the complementarity relations, the condition on cycles of odd length can be equivalently replaced by the condition that the relation is irreflexive. However it is an open problem whether the representability construction can be carried on without referring to the cycles property.

5 Concluding remarks

We discussed a formal framework for the development of a method of reduction of decision rules derived from a decision table. We have shown that an analysis of complementarity is useful for that purpose. We formulated and proved the informational representability of relational systems with complementarity relations. Our construction can also be used to characterize the incompleteness relations between objects. Let $S = (OB, AT)$ be an information system. Two objects o_1 and o_2 are said to be in relation of *incompleteness with respect to* $A \subseteq AT$ (in short $o_1 \text{ incomp}(A) o_2$) iff for all $a \in A$, $(Val_a \setminus a(o_1)) \neq a(o_2)$. Let \mathcal{F}' be the class of frames (U, R) such that the complement of R is a complementarity relation. For any frame $\mathcal{K} = (U, R)$ in \mathcal{F}' we define $g'(\mathcal{K}) = g((U, -R))$ where $'-'$ is the complement with respect to $U \times U$. We can easily show that $incomp(\{at^{g(\mathcal{K})}\}) = R$. Hence $\{(OB, incomp(AT)) : (OB, AT) \text{ is an information system}\}$ provides an informational representation of the class \mathcal{F}' .

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