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INFORMATIONAL REPRESENTABILITY OF MODELS  
FOR INFORMATION LOGICS

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**Abstract**

The purpose of this paper is to elaborate a formal framework for expressing and proving informational representability of abstract frames. The property of informational representability can be meaningful in investigations of nonclassical logics. We introduce a general notion of informational representability, we develop a method of proving informational representability and we give examples of informational representability and non-representability of frames.

## 1 Introduction

Any abstract mathematical structure, e.g. an abstract algebra or an abstract relational system, is intended to serve as a pattern of a class of 'concrete' structures. In the concrete structures their components, e.g. the operations or the relations, are defined directly, while in the abstract structures they are defined in terms of a set of conditions (axioms). Consider an example of Boolean algebras. In the algebras of sets the operations of union, intersection and complement of sets are defined in the well known way, these are the direct definitions saying how the respective compound sets are made out of the component sets. An abstract Boolean algebra is defined in an axiomatic way. Join, meet and complement of elements of any algebra from the class are assumed to satisfy some conditions.

The adequacy of an abstract structure for providing a general scheme of concrete structures is typically expressed as a representation theorem of the following form: For every abstract structure  $S$  from a certain class  $C$  of structures there is a concrete structure  $S'$  that is a member of  $C$  and  $S$  and  $S'$  are isomorphic in a suitable sense. For example, the Stone representation theorem for Boolean algebras establishes an isomorphism between any abstract Boolean algebra and an algebra of sets.

In this paper we put forward an idea of representability for the structures that arise in connection with reasoning about incomplete information. The abstract structures dealt with in the paper are relational systems that are extensions of frames of standard multimodal logics. The concrete structures are frames derived from information systems. The relations in those frames are defined in terms of the informational resources of an information system.

## 1.1 Information systems

Different application areas use different conceptual means for representation of information about the domains they deal with. It can be observed that a great variety of informational structures and data employ the notions of 'object' and 'property' as the most elementary pieces of information. Similarly, it is well established in philosophical logic that an analysis of concepts is based on these notions. Concepts are determined by defining their extension or denotation and intension or connotation. The extension of a concept consists of the objects that are instances of this concept and the intension of a concept consists of the properties that are characteristic for the objects to which this concept applies. For example, to define the concept 'organism' we should list the earmarks of organism and the typical species of organisms. Very often properties of objects are expressed in terms of attributes and values of the attributes. The concrete information structures considered in this paper are based on information systems that are collections of (names of) objects together with their descriptions in terms of the values of some attributes meaningful for these objects. We use the notion of information system that evolved from the developments presented in [25, 26, 36, 29].

Let a set  $OB$  of objects be given, and suppose that properties of those objects are articulated in terms of attributes from a set  $AT$  and values of these attributes. For example, property of 'being green' is represented as a pair (colour, green), where 'colour' is an attribute, and 'green' is one of its values. In general, a description of an object is of the form (attribute, a subset of values). Information of that kind has several intuitive meanings. For instance, if the age of a person is known approximately, say between 20 and 25, then this information is represented as a pair ( $age, \{20, \dots, 25\}$ ). With another interpretation a pair (attribute, a subset of values) can be viewed as a many-valued information discussed, among others, in [29]. For example, if a person speaks French, German, Polish, then this information is represented as the pair (languages spoken,  $\{F, D, Pl\}$ ) with the intuition that a given object assumes all the values  $F, D$ , and  $Pl$  of the attribute 'languages spoken'. In [8] and [14] knowledge representation languages are discussed where symbolic objects of the form (attribute, a subset of values) are also used. A simple form of description of objects in terms of binary attributes having values 'yes' and 'no' is a basis for knowledge representation within the framework of formal concept analysis, see e.g. [44].

By an information system we mean a triple  $(OB, AT, \{VAL_a : a \in AT\})$  where  $OB$  is a nonempty set of objects and  $AT$  is a nonempty set of attributes,  $VAL_a$  is a nonempty set of values of the attribute  $a$  and each attribute is understood as a mapping  $a : OB \rightarrow P(VAL_a) \setminus \emptyset$ . We shall also use the more concise  $(OB, AT)$  to represent  $(OB, AT, \{VAL_a : a \in AT\})$ . We write  $\mathcal{IS}$  to denote the class of information systems. An information system  $S' = (OB', AT')$  is said to be a *subsystem* of the information system  $S = (OB, AT)$  iff  $OB' \subseteq OB$  and  $\{a_{OB'} : a \in AT\} = AT'$  where  $a_{OB'}$  denotes the restriction of  $a$  to  $OB'$ .

## 1.2 Information frames

Any information system contains an implicit information about relationships among the objects. Let an information system  $(OB, AT, \{VAL_a : a \in AT\})$  be given and let  $A$  be a subset of  $AT$ . A classical relationship among objects that has been a starting point of the notion of 'rough set' is indiscernibility:

Strong (weak) indiscernibility:  $(x, y) \in ind(A)$  ( $wind(A)$ ) iff  $a(x) = a(y)$  for all  $a \in A$  (for some  $a \in A$ ).

Intuitively, two objects are  $A$ -indiscernible whenever all the  $A$ -properties that they possess are the same, in other words up to discriminative resources from  $A$  these objects are the same. Objects are weakly  $A$ -indiscernible whenever some of their  $A$ -properties are the same. Indiscernibility relations derived from an information system were introduced in [23].

Important applications of indiscernibility relations are concerned with representation of approximations of subsets of objects in information systems. Let  $A$  be a subset of  $AT$  and  $X$  be a subset of  $OB$ , then the lower  $ind(A)$ -approximation of  $X$  ( $L(ind(A))X$ ) and the upper  $ind(A)$ -approximation of  $X$  ( $U(ind(A))X$ ) are defined as follows:

$$L(ind(A))X = \{x \in OB : \text{for all } y \in OB \text{ if } (x, y) \in ind(A), \text{ then } y \in X\}$$

$$U(ind(A))X = \{x \in OB : \text{there is } y \in OB \text{ such that } (x, y) \in ind(A) \text{ and } y \in X\}$$

In the classical rough set theory setting, if  $ind(A)$  is a strong indiscernibility relation, then we obtain the following hierarchy of definability of sets. A subset  $X$  of  $OB$  is said to be:

$A$ -definable if  $L(ind(A))X = X = U(ind(A))X$ ;

roughly  $A$ -definable if  $L(ind(A))X \neq \emptyset$  and  $U(ind(A))X \neq OB$ ;

internally  $A$ -indefinable if  $L(ind(A))X = \emptyset$ ;

externally  $A$ -indefinable if  $U(ind(A))X = OB$ ;

totally  $A$ -indefinable if internally  $A$ -indefinable and externally  $A$ -indefinable.

It is easy to see that operations of lower and upper approximation coincide with modal operators of necessity and possibility, respectively.

The other application of indiscernibility relations is concerned with modeling uncertain knowledge acquired from information about objects that is given in an information system [30, 31, 33, 5, 7].

Let  $X$  be a subset of  $OB$ , we define the sets of  $A$ -positive ( $POS(A)X$ ),  $A$ -borderline ( $BOR(A)X$ ) and  $A$ -negative ( $NEG(A)X$ ) instances of  $X$  as follows:

$$POS(A)X = L(ind(A))X;$$

$$BOR(A)X = U(ind(A))X \setminus L(ind(A))X;$$

$$NEG(A)X = OB \setminus U(ind(A))X.$$

Knowledge about a set  $X$  of objects that can be discovered from information provided in an information system can be modelled in the following way:

$$K(A)X = POS(A)X \cup NEG(A)X$$

Intuitively,  $A$ -knowledge about  $X$  consists of those objects that are either  $A$ -positive instances of  $X$  (they are members of  $X$  up to properties from  $A$ ) or  $A$ -negative instances of  $X$  (they are not members of  $X$  up to properties from  $A$ ). We say that  $A$ -knowledge about  $X$  is:

complete if  $K(A)X = OB$ , otherwise incomplete;

rough if  $POS(A)X, BOR(A)X, NEG(A)X \neq \emptyset$ ;

pos-empty if  $POS(A)X = \emptyset$ ;

neg-empty if  $NEG(A)X = \emptyset$ ;

empty if pos-empty and neg-empty.

Epistemic operators  $K(A)$  are extensions of operators introduced in [19]. The above examples suggest that modal logics are an adequate formal tool for representation of and reasoning about information provided in the form of an information system.

Many other relationships among objects can be derived in a natural way from any information system. In sections 5 and 6 several of them are discussed, see also [35]. All these relationships are referred to as information relations. By an information frame derived from an information system  $S = (OB, AT, \{VAL_a : a \in AT\})$  we mean a relational system  $(OB, \{R(A) : A \subseteq AT\})$  where  $\{R(A) : A \subseteq AT\}$  is a family of information relations derived from  $S$ . In a more general setting we also admit frames with several families of information relations. Frames derived from information systems are the concrete structures. Abstract information frames are relational systems that, however, differ from the standard Kripke frames in that their accessibility relations are relative to subsets of a set. This set is intuitively interpreted as the set of attributes that are meaningful for the objects from the universe of the frame [32, 1]. The relations are assumed to satisfy various properties, for example strong indiscernibility relations are equivalence relations. All these frames determine models of the respective modal logics. Modal operators in these logics are determined by information relations. The logics are referred to as information logics. A sample of recent results on information logics can be found in [1, 2, 11, 22, 21, 35, 28].

The purpose of this paper is to elaborate a formal framework for expressing and proving informational representability of abstract information frames. Let a similarity relation  $\triangleright$  in a class  $C$  of frames be given, for example the relation of 'being isomorphic' or 'being modally equivalent'. Intuitively, a frame  $\mathcal{K}$  from the class  $C$  is informationally  $\triangleright$ -representable if there is an information system  $S$  and a frame  $\mathcal{K}'$  derived from this system  $S$  such that  $\mathcal{K}'$  is in the class  $C$  and, moreover,  $\mathcal{K}$  is  $\triangleright$ -related to  $\mathcal{K}'$ . The first theorem of this kind has been proved in [37, 38]. In [34] it has been observed that a property of informational representability might be meaningful in investigations of nonclassical logics, and a notion of informational representability has been proposed. In the present paper we introduce a general notion of informational representability, we develop a method of proving informational representability and we give examples of informational representability and non-representability of frames.

The paper is organized as follows. In section 2 we introduce a general notion of an abstract frame and a scheme of modal languages whose modal operators are determined by relations from those frames. A great variety of information logics considered in the literature are defined using some instances of these notions. In section 3 we present a construction of concrete frames derived from an information system. We present a language determined by information systems. The expressions of this language enable us to express definitions of relations in the derived frames. In section 4 a method of proving informational representability

of abstract frames is presented. The method is a generalization of the method developed in [37]. In sections 5 and 6 many examples of representability and non-representability results are presented for classes of frames of information logics.

The present paper is an extended version of the two conference papers [10, 9].

## 2 Frames with parameterised accessibility relations

In this section we introduce a general notion of frame that captures most of the types of frames involved in the semantics of information logics. This notion is an extension of the notion of frame used in the theory of modal logics (see e.g. [24, 3, 43, 18, 16]). We shall consider frames with several (finitely many) families of accessibility relations of different arities, and moreover, each of these families will be indexed with subsets (and not individual elements) of a set, referred to as the set of parameters. Parameters are intended to be abstract counterparts of entities that determine relations. For example, if we are interested in information relations of an information system, then we should take the attributes of the system as the parameters. If we deal with a logic of knowledge, then the parameters are knowledge agents. Instead of ordinary frames of multimodal logics that contain just several relations, we will be dealing with frames with families of relations. Intuitively, each family consists of relations of the same type, that is all the relations in a family satisfy the same conditions e.g. they are equivalence relations, and in general there are several relations in every such family, each of which is determined by a subset of parameters. For example, the family of indiscernibility relations of an information system consists of relations that reflect indiscernibility of objects with respect to any subset of attributes in that information system. Each of these relations is an equivalence relation. Classes of frames indexing the relations by sets of parameters have been intensively studied in the past (see e.g. [32, 20, 1]).

A *signature*  $\Sigma$  is a pair  $\langle \mathbf{P}, \langle n_1, \dots, n_k \rangle \rangle$  where  $\mathbf{P}$  is a non-empty set of *parameters* and  $\langle n_1, \dots, n_k \rangle$  is a non-empty sequence of natural numbers greater than 2. Let  $\Sigma = \langle \mathbf{P}, \langle n_1, \dots, n_k \rangle \rangle$  be a signature. By a  $\Sigma$ -*frame* we understand a structure

$$(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$$

where  $U$  is a non-empty set and for all  $P \subseteq \mathbf{P}$ , for all  $l \in \{1, \dots, k\}$ ,  $\mathcal{R}_l(P)$  is a  $n_l$ -ary relation on  $U$ . We write  $\mathcal{F}_\Sigma$  to denote the class of  $\Sigma$ -frames.

The *modal language*  $\mathbf{LM}$  is determined by the non-empty set  $\mathbf{F}_0$  of *propositional variables* and by the set  $\mathbf{OP} = \{\neg, \wedge\}$  of logical propositional operators. The set  $\mathbf{F}$  of *LM-formulae* is the smallest set that satisfies the following conditions:  $\mathbf{F}_0 \subseteq \mathbf{F}$ , if  $c \in \mathbf{OP}$  (of arity  $n$ ) then  $c(F_1, \dots, F_n) \in \mathbf{F}$  and if  $P \subseteq \mathbf{P}$ ,  $l \in \{1, \dots, k\}$ , and  $F_1, \dots, F_{n_l-1} \in \mathbf{F}$  then  $\langle P \rangle_l(F_1, \dots, F_{n_l-1}) \in \mathbf{F}$ . As usual, by a *LM-model* we understand a triple  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\}, V)$  such that  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$  is a  $\Sigma$ -frame and  $V$  is a mapping  $\mathbf{F}_0 \rightarrow \mathcal{P}(U)$ . Let  $\mathcal{M} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\}, V)$  be a  $\mathbf{LM}$ -model. We say that a formula  $F$  is *satisfied by*  $u \in U$  in  $\mathcal{M}$  (written  $\mathcal{M}, u \models F$ ) when the following conditions are satisfied.

- $\mathcal{M}, u \models p$  iff  $u \in V(p)$ , for all  $p \in \mathbf{F}_0$ ;
- $\mathcal{M}, u \models \neg F$  iff not  $\mathcal{M}, u \models F$ ;
- $\mathcal{M}, u \models F \wedge G$  iff  $\mathcal{M}, u \models F$  and  $\mathcal{M}, u \models G$ ;

- $\mathcal{M}, u \models \langle P \rangle_l(F_1, \dots, F_{n_l-1})$  iff there are  $v_1, \dots, v_{n_l-1} \in U$  such that for all  $i \in \{1, \dots, n_l - 1\}$ ,  $\mathcal{M}, v_i \models F_i$  and  $\mathcal{R}_l(P)(u, v_1, \dots, v_{n_l-1})$ .

Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\})$  be a  $\Sigma$ -frame. We write  $\mathcal{K} \models F$  to denote the fact that for all models  $\mathcal{M} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\}, V)$  and for all  $u \in U$ ,  $\mathcal{M}, u \models F$ . The Kripke-style semantics of formulae with  $n$ -ary modal operators ( $n \geq 2$ ) can be found for instance in [27].

DEFINITION 2.1. Let  $\mathcal{K}_i = (U_i, \{\mathcal{R}_l^i(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\})$  be  $\Sigma$ -frames for  $i = 1, 2$ .

- (1)  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are said to be *isomorphic* (written  $\mathcal{K}_1 \equiv \mathcal{K}_2$ ) iff there is a 1-1 mapping  $g : U_1 \rightarrow U_2$  such that for all  $l \in \{1, \dots, k\}$ , for all  $P \subseteq \mathbb{P}$  and for all  $u_1, \dots, u_{n_l} \in U_1$ ,  $\mathcal{R}_l^1(P)(u_1, \dots, u_{n_l})$  iff  $\mathcal{R}_l^2(P)(g(u_1), \dots, g(u_{n_l}))$ .
- (2)  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are said to be *modally equivalent* (written  $\mathcal{K}_1 \equiv_m \mathcal{K}_2$ ) iff for all  $F \in \mathbb{F}$ ,  $\mathcal{K}_1 \models F$  iff  $\mathcal{K}_2 \models F$ .

▽

It is known that  $\equiv \subseteq \equiv_m$ .

EXAMPLE 2.1. Let  $S = (OB, AT)$  be an information system. Consider the signature  $\Sigma_0 = \langle AT, \langle 2 \rangle \rangle$ . Two objects  $o_1$  and  $o_2$  are said to be *indiscernible with respect to*  $A \subseteq AT$  (in short  $o_1 \text{ ind}(A) o_2$ ) iff for all  $a \in A$ ,  $a(o_1) = a(o_2)$ . In the  $\Sigma_0$ -frame  $(OB, \{\text{ind}(A) : A \subseteq AT\})$ ,  $\text{ind}(A)$  is an equivalence relation on  $OB$  for every  $A \subseteq AT$  and for any  $A, A' \subseteq AT$ ,  $\text{ind}(A \cup A') = \text{ind}(A) \cap \text{ind}(A')$ .

As usual, a  $\Sigma$ -frame  $\mathcal{K}' = (U', \{\mathcal{R}_l'(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\})$  is said to be a *subframe* of the  $\Sigma$ -frame  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\})$  iff  $U' \subseteq U$  and for all  $l \in \{1, \dots, k\}$ ,  $P \subseteq \mathbb{P}$ ,  $\mathcal{R}_l(P) \cap (U' \times U') = \mathcal{R}_l'(P)$ .

### 3 Frames derived from information systems

In order to derive  $\Sigma$ -frames from information systems, a first task consists in relating the set of parameters with a given set of attributes. That is why, any derivation of frames shall be defined modulo a *contribution function*.

DEFINITION 3.1. Let  $S = (OB, AT) \in \mathcal{IS}$  and  $\Sigma$  be a signature  $\langle \mathbb{P}, \langle n_1, \dots, n_k \rangle \rangle$ . A *contribution function*  $I$  for  $S$  is a mapping  $I : AT \rightarrow \mathcal{P}(\mathbb{P})$  such that  $\bigcup_{a \in AT} I(a) = \mathbb{P}$ .  
▽

For any  $P \subseteq \mathbb{P}$ , we write  $I^{-1}(P)$  to denote the set  $\{a \in AT : I(a) \cap P \neq \emptyset\}$ . The intended meaning of  $I$  is the following: every attribute  $a$  contributes to the construction of relations involving some parameters in  $I(a)$ . Moreover, every parameter  $\mathbf{p}$  has at least one attribute that contributes to  $\mathbf{p}$ .

#### 3.1 A language for information systems

The language  $\mathcal{LS}$  is determined by seven sets which are supposed to be pairwise disjoint: the set  $\mathbf{CONS} = \{0, 1\}$  of constants, the non-empty countable set  $\mathbf{VAR}$  of variables, the non-empty set  $\mathbf{FUN}$  of unary function symbols, the set  $\mathbf{OR} = \{\cap, \cup, -\}$  of constructors (of respective arity 2,2 and 1), the set  $\mathbf{PRE} = \{=, \subseteq\}$  of predicate

symbols, the set  $\{\forall, \exists\}$  of quantifiers and the set  $\text{OP} = \{\neg, \wedge\}$  of propositional logical operators. Function symbols are intended to represent the attributes and variables will range over the set of objects of an information system. The set  $\text{T}$  of *terms* is the smallest set that satisfies the following conditions:

- (1)  $\text{CONS} \subseteq \text{T}$ ;
- (2) if  $f \in \text{FUN}$  and  $x \in \text{VAR}$ , then  $f(x), -f(x) \in \text{T}$  and
- (3) for all  $\oplus \in \text{OR} \setminus \{-\}$  if  $t_1, t_2 \in \text{T}$ , then  $\oplus(t_1, t_2) \in \text{T}$ .

The set  $\text{F}_0$  of atomic formulae is the set of expressions  $\{\oplus(t_1, t_2) : t_1, t_2 \in \text{T}, \oplus \in \text{PRE}\}$ . The set  $\text{F}$  of formulae is the smallest set that satisfies the following conditions:  $\text{F}_0 \subseteq \text{F}$  and if  $c$  is any  $n$ -ary propositional operator and  $F_1, \dots, F_n \in \text{F}$ , then  $c(F_1, \dots, F_n) \in \text{F}$ . An *extended formula* is an expression of the form  $q_1 f_1 \dots q_n f_n F$  (also written  $\mathcal{Q}F$ ) with  $F \in \text{F}$ ,  $\{q_1, \dots, q_n\} \subseteq \{\exists, \forall\}$  and  $\{f_1, \dots, f_n\} \subseteq \text{FUN}$ . The set of extended formulae is written  $\text{F}^e$ . The extended formula  $q_1 f_1 \dots q_n f_n F$  is said to be *weak* (resp. *strong*) iff  $\{q_1, \dots, q_n\} = \{\exists\}$  (resp.  $\{q_1, \dots, q_n\} = \{\forall\}$ ).

For any syntactic set  $\text{X}$ , and for any syntactic object  $\text{O}$ , we write  $\text{X}(\text{O})$  to denote the set consisting of elements of  $\text{X}$  occurring in  $\text{O}$ .

An extended formula  $\mathcal{Q}F$  is said to be *well-closed* iff for all  $f \in \text{FUN}(F)$ ,  $f$  occurs exactly once in  $\mathcal{Q}$ . We shall adopt the convention  $\text{F} \subseteq \text{F}^e$  by considering that a formula in  $\text{F}$  is an extended formula with an empty string of quantifications.

Let  $S = (\text{OB}, \text{AT})$  be an information system. A *function interpretation*  $m$  in  $S$  is a mapping  $m : \text{FUN} \rightarrow \text{AT}$ . An *object interpretation*  $v$  in  $S$  is a mapping  $v : \text{VAR} \rightarrow \text{OB}$ . The interpretation of terms generated by  $m$  and  $v$  is the mapping  $I_{m,v} : \text{T} \rightarrow \mathcal{P}(\bigcup_{a \in \text{AT}} \text{Val}_a)$  such that:

- $I_{m,v}(f(x)) = m(f)(v(x))$ ,  $I_{m,v}(-f(x)) = \text{Val}_{m(f)} \setminus I_{m,v}(f(x))$ ,  $I_{m,v}(0) = \emptyset$ ,
- $I_{m,v}(1) = \bigcup_{a \in \text{AT}} \text{Val}_a$ ,  $I_{m,v}(\oplus(t_1, t_2)) = \oplus(I_{m,v}(t_1), I_{m,v}(t_2))$  when  $\oplus \in \text{OR} \setminus \{-\}$ .

By abusing our notation, as usual  $\cap, \cup, -$  denote the Boolean operations on sets. It follows that the terms represent sets of values of attributes.

EXAMPLE 3.1. The expressions below are in  $\text{F}^e$ :

$$\neg(f(x)=0); \forall f (f(x)=1) \Rightarrow (-f(x)=0); \forall f \exists f' f(x)=f'(x)$$

By contrast, the following ones do not belong to  $\text{F}^e$ :

$$-(-f(x)=f(x)); \forall f (f(x) \subseteq 1 \wedge \exists f' f'(x) \subseteq 1); \forall x \forall f f(x) \subseteq 1$$

This underlines the fact that the choice of a language of information systems does not have to be unique and LS is sufficient for the forthcoming developments.

Let  $S = (\text{OB}, \text{AT})$  be an information system,  $m$  be a function interpretation and  $v$  be an object interpretation. We say that an extended formula  $F$  is *satisfied in  $S$  under the interpretation  $I_{m,v}$*  (written  $S, I_{m,v} \models F$ ) when the following conditions are satisfied.

- $S, I_{m,v} \models \oplus(t_1, t_2)$  iff  $\oplus(I_{m,v}(t_1), I_{m,v}(t_2))$  when  $\oplus \in \text{PRE}$ ;
- $S, I_{m,v} \models \neg F$  iff not  $S, I_{m,v} \models F$ ;



- $S, I_{m,v} \models F \wedge G$  iff  $S, I_{m,v} \models F$  and  $S, I_{m,v} \models G$ ;
- $S, I_{m,v} \models \forall f F$  iff for all  $a \in AT$ ,  $S, I_{m'_a,v} \models F$  where  $m'_a$  is defined as follows:  $m'_a(f) = a$  and for all  $f' \neq f$ ,  $m'_a(f') = m(f')$ ;
- $S, I_{m,v} \models \exists f F$  iff there exists  $a \in AT$  such that  $S, I_{m'_a,v} \models F$  where  $m'_a$  is defined as above.

Let  $S = (OB, AT)$  be an information system and  $F \in \mathbf{F}^e$  such that the variables occurring in  $F$  are  $x_1, \dots, x_n$  (in the order of enumeration). For all function interpretations  $m$  in  $S$ , we write  $m_S(F)$  to denote the set

$$m_S(F) = \{\langle v(x_1), \dots, v(x_n) \rangle : \exists v : \mathbf{VAR} \rightarrow OB, S, I_{m,v} \models F\}$$

If  $F$  has the form  $\forall f G$  (resp.  $\exists f G$ ), then  $m_S(F) = \bigcap_{a \in AT} (m'_a)_S(G)$  (resp.  $m_S(F) = \bigcup_{a \in AT} (m'_a)_S(G)$ ).

EXAMPLE 3.2. (Example 2.1 continued) Consider the formula  $f(x_1)=f(x_2)$ , say  $F_0$ . Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. It is easy to show that for all  $A \subseteq AT$ ,  $o_1, o_2 \in OB$ ,  $o_1 \text{ ind}(A) o_2$  iff for all  $a \in A$ ,  $(o_1, o_2) \in m_{(OB, \{a\})}(F_0)$  where  $m$  is the *unique* function interpretation in  $(OB, \{a\})$ .

### 3.2 $\Sigma$ -specification

The language LS enables us to express definitions of relations derived from information systems. These definitions will be referred to as *specifications*.

DEFINITION 3.2. Let  $\Sigma$  be the signature  $\langle \mathbf{P}, \langle n_1, \dots, n_k \rangle \rangle$ . A  $\Sigma$ -specification  $\mathbf{S}$  is a sequence of  $k$  well-cosed extended formulas, say  $\langle F_1, \dots, F_k \rangle$ , such that for all  $l \in \{1, \dots, k\}$ ,  $\text{card}(\mathbf{VAR}(F_l)) = n_l$ .  $\nabla$

A  $\Sigma$ -specification  $\mathbf{S}$  is said to be *strong* (resp. *weak*) iff  $\mathbf{S}$  is a sequence of strong extended formulae (resp.  $\mathbf{S}$  is a sequence of either strong or weak extended formulae). Since every extended formula occurring in a  $\Sigma$ -specification  $\langle Q_1 F_1, \dots, Q_k F_k \rangle$  is *closed* with respect to the function symbols, for all  $l \in \{1, \dots, k\}$ , for all information systems  $S = (OB, AT)$  and for all function interpretations  $m, m'$  in  $S$ ,  $m_S(Q_l F_l) = m'_S(Q_l F_l)$ .

For any information system  $S = (OB, AT)$  and for any contribution function  $I$  for  $S$  if  $\text{card}(\mathbf{FUN}(F)) = 1$ , then for all  $P, P' \in \mathcal{P}(\mathbf{P}) \setminus \{\emptyset\}$ , if  $q_1 f_1 F$  is strong (resp. weak), then

$$m_{(OB, I^{-1}(P \cup P'))}(q_1 f_1 F) = \bigcap_{Q \in \{P, P'\}} m_{(OB, I^{-1}(Q))}(q_1 f_1 F)$$

$$\text{(resp. } m_{(OB, I^{-1}(P \cup P'))}(q_1 f_1 F) = \bigcup_{Q \in \{P, P'\}} m_{(OB, I^{-1}(Q))}(q_1 f_1 F)\text{)}.$$

Besides, if  $\text{card}(AT) = 1$ , then for all extended formulae  $Q_1 F$  and  $Q_2 F$  and for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,

$$m_{(OB, I^{-1}(P))}(Q_1 F) = m_{(OB, I^{-1}(P))}(Q_2 F) = m_{(OB, I^{-1}(P))}(F)$$

We write  $\text{spec}_{\Sigma}^1$  to denote the set of  $\Sigma$ -specifications  $\langle F_1, \dots, F_k \rangle$  such that for all  $l \in \{1, \dots, k\}$ ,  $\text{card}(\mathbf{FUN}(F_l)) = 1$ . It means that in each  $F_i$ , there occurs exactly one function symbol representing an attribute.

### 3.3 A frame derived from an information system

We are now in position to define a family of derivation functions parameterised by the signatures and specifications.

DEFINITION 3.3. Let  $\Sigma = \langle \mathbb{P}, \langle n_1, \dots, n_k \rangle \rangle$  be a signature,  $S = (OB, AT) \in \mathcal{IS}$  be an information system,  $I$  be a contribution function for  $S$  and  $\mathbf{S} = \langle F_1, \dots, F_k \rangle$  be a  $\Sigma$ -specification. We write  $D_{\Sigma, \mathbf{S}}(S, I)$  to denote the  $\Sigma$ -frame  $(OB, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\})$  such that

$$\forall l \in \{1, \dots, k\}, \forall \emptyset \neq P \subseteq \mathbb{P}, \mathcal{R}_l(P) = m_{(OB, I^{-1}(P))}(F_l)$$

By convention, for all  $l \in \{1, \dots, k\}$ , if  $F_l$  is weak, then  $\mathcal{R}_l(\emptyset) = \emptyset$  otherwise  $\mathcal{R}_l(\emptyset) = OB \times OB$   $\nabla$

$D_{\Sigma, \mathbf{S}}$  will be referred to as a derivation function. It is parameterised by a signature  $\Sigma$  and a specification  $\mathbf{S}$ . Given an information system  $S$  and a contribution function  $I$  for  $S$ , the derivation function returns a frame whose signature is  $\Sigma$  and whose relations are defined according to the specification  $\mathbf{S}$  using the attributes of the system  $S$ .

It is important to observe that the correctness of Definition 3.3 rests on the fact that each  $F_l$  is closed with respect to the function symbols.

## 4 Informational representability of $\Sigma$ -frames

In this section we present a notion of informational representability of a class of frames and a general method of proving representability. Next, the method will be applied to some particular classes of frames.

DEFINITION 4.1. Let  $\Sigma$  be a signature,  $X \subseteq \mathcal{F}_\Sigma$  be a set of  $\Sigma$ -frames,  $Y \subseteq \mathcal{IS}$  be a set of information systems, and  $\mathbf{S} = \langle F_1, \dots, F_k \rangle$  be a  $\Sigma$ -specification. The class of  $\Sigma$ -frames  $X$  is said to be  $(\triangleright, \mathbf{S})$ -*inf-representable* ( $\triangleright \in \{=, \equiv, \equiv_m\}$ ) in  $Y$  iff

- (1) (soundness) for all information systems  $S \in Y$  and for all contribution functions  $I$  for  $S$ , the  $\Sigma$ -frame  $D_{\Sigma, \mathbf{S}}(S, I)$  belongs to  $X$ ,
- (2) (completeness) for all  $\Sigma$ -frames  $\mathcal{K} \in X$ , there is an information system  $S \in Y$  and a contribution function  $I$  for  $S$  such that  $D_{\Sigma, \mathbf{S}}(S, I) \triangleright \mathcal{K}$ .

$\nabla$

The notion of representability introduced in Definition 4.1 depends on the derivation functions  $D_{\Sigma, \mathbf{S}}$ . A more general concept of representability can be introduced but Definition 4.1 shall be sufficient in the forthcoming developments. For instance, the condition (2) in Definition 4.1 states that for any  $\Sigma$ -frame  $\mathcal{K}$  belonging to the set  $X$  of  $\Sigma$ -frames, there is an information system from which a  $\Sigma$ -frame *similar* to  $\mathcal{K}$  can be derived.

### 4.1 Nice pair proof technique

This technique has been originally introduced in [37] for the information logic NIL. We show that this technique can be extended to a more general case of an arbitrary information frame considered in this paper.

For all non-empty sets  $X$ , for all  $\langle \mathbf{p}, Y \rangle \in \mathbf{P} \times \mathcal{P}(\mathcal{P}(X))$ , we write  $at^{\langle \mathbf{p}, Y \rangle}$  to denote the mapping

$$at^{\langle \mathbf{p}, Y \rangle} : X \rightarrow \mathcal{P}(\{\mathbf{p}\} \times Y)$$

such that

$$\text{for all } x \in X, at^{\langle \mathbf{p}, Y \rangle}(x) = \{\langle \mathbf{p}, Z \rangle : x \in Z \in Y\}.$$

Hence for all  $Z \in Y$ ,  $x \in X$ , we have  $x \in Z$  iff  $\langle \mathbf{p}, Z \rangle \in at^{\langle \mathbf{p}, Y \rangle}(x)$ . Roughly speaking, the second component of  $at^{\langle \mathbf{p}, Y \rangle}(x)$  contains the sets of  $Y$  having  $x$  as an element.

DEFINITION 4.2. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\}) \in \mathcal{F}_\Sigma$  be a  $\Sigma$ -frame and  $\mathbf{S} = \langle F_1, \dots, F_k \rangle \in \mathbf{spec}_\Sigma^1$  be a  $\Sigma$ -specification such that in each formula  $F_i$  there occurs exactly one function symbol representing an attribute. A *nice pair with respect to  $\mathcal{K}$  and  $\mathbf{S}$* , say  $\mathcal{N} = \langle \mathbf{p}, X \rangle$ , is a member of  $\mathbf{P} \times \mathcal{P}(\mathcal{P}(U))$  such that

- (1)  $\bigcup\{Y : Y \in X\} = U$  and
- (2) for all  $l \in \{1, \dots, k\}$ ,  $\mathcal{R}_l(\{\mathbf{p}\}) \subseteq m_{(U, \{at^{\langle \mathbf{p}, X \rangle})}(F_l)$  where  $m$  is the *unique* function interpretation in the information system  $(U, \{at^{\langle \mathbf{p}, X \rangle})$ .

▽

Condition (1) in Definition 4.2 guarantees that  $(U, \{at^{\langle \mathbf{p}, X \rangle})$  is an information system. A nice pair can be viewed as an encoding of some information system from which can be partly defined some  $\Sigma$ -frame. Definition 4.3 below presents different kinds of nice pairs and sets of nice pairs having strong properties.

DEFINITION 4.3. With the notations of Definition 4.2, a nice pair  $\mathcal{N}$  is  $(\mathcal{K}, \mathbf{S})$ -complete with respect to the parameter  $\mathbf{p}$  iff the inclusion in Definition 4.2(2) is replaced by an equality. A set of nice pairs  $X$  is said to be  $(\mathcal{K}, \mathbf{S})$ -complete iff for all  $\mathbf{p} \in \mathbf{P}$ , there is  $\langle \mathbf{p}, Y \rangle \in X$  such that  $\langle \mathbf{p}, Y \rangle$  is  $(\mathcal{K}, \mathbf{S})$ -complete with respect to  $\mathbf{p}$ . A set of nice pairs  $X$  is said to be  $\mathbf{P}$ -full iff for all  $\mathbf{p} \in \mathbf{P}$ ,  $\{\langle \mathbf{p}', Y \rangle \in X : \mathbf{p} = \mathbf{p}'\} \neq \emptyset$ . A set of nice pairs  $X$  is *minimally  $(\mathcal{K}, \mathbf{S})$ -complete* iff  $X$  is  $(\mathcal{K}, \mathbf{S})$ -complete and for all  $Y \subset X$  (strict inclusion),  $Y$  is not  $(\mathcal{K}, \mathbf{S})$ -complete. ▽

The crucial notion in Definition 4.3 is that of  $(\mathcal{K}, \mathbf{S})$ -complete set  $X$  of nice pairs. Indeed, the relations in  $\mathcal{K}$  parameterised by a singleton can be defined from  $\mathbf{S}$  and  $X$  only. Definition 4.4 shall be mainly needed in Section 4.2.

DEFINITION 4.4. Let  $\Sigma$  be a signature,  $\mathbf{S} = \langle F_1, \dots, F_k \rangle \in \mathbf{spec}_\Sigma^1$  be a  $\Sigma$ -specification and  $X \subseteq \mathcal{F}_\Sigma$  be a set of  $\Sigma$ -frames. A (*resp. minimal*) *nice pair function* with respect to  $\mathbf{S}$  and  $X$  is a mapping  $\mathbf{truc}$  such that for all  $\mathcal{K} \in X$ ,  $\mathbf{truc}(\mathcal{K})$  is a (*resp. minimally*)  $(\mathcal{K}, \mathbf{S})$ -complete set of nice pairs. ▽

EXAMPLE 4.1. (Example 3.2 continued) Consider the  $\Sigma_0$ -specification  $\mathbf{S}_0 = \langle \forall f F_0 \rangle$  with  $\Sigma_0 = \langle \mathbf{P}, \langle 2 \rangle \rangle$  for some set  $\mathbf{P} = \{\mathbf{p}\}$  of parameters. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \{\mathbf{p}\}, l \in \{1\}\})$  be a  $\Sigma_0$ -frame such that  $\mathcal{R}_1(\emptyset) = U \times U$  and  $\mathcal{R}_1(\{\mathbf{p}\})$  is an equivalence relation. Consider the pair  $\langle \mathbf{p}, \{\mathcal{R}_1(\{\mathbf{p}\})(u) : u \in U\} \rangle = \langle \mathbf{p}, X \rangle$  ( $X$  is merely the set of equivalence classes). Since  $\mathcal{R}_1(\{\mathbf{p}\})$  is reflexive, then  $\bigcup_{Y \in X} Y = U$ . Moreover, for all  $u, v \in U$

$$\begin{aligned}
(u, v) \in \mathcal{R}_1(\{\mathbf{p}\}) &\text{ iff } \mathcal{R}_1(\{\mathbf{p}\})(u) = \mathcal{R}_1(\{\mathbf{p}\})(v) \\
&\text{ iff } \{Y : u \in Y \in X\} = \{Y : v \in Y \in X\} \\
&\text{ iff } \{\langle \mathbf{p}, Y \rangle : u \in Y \in X\} = \{\langle \mathbf{p}, Y \rangle : v \in Y \in X\} \\
&\text{ iff } at^{\langle \mathbf{p}, X \rangle}(u) = at^{\langle \mathbf{p}, X \rangle}(v) \\
&\text{ iff } (u, v) \in m_{(U, \{at^{\langle \mathbf{p}, X \rangle}\})}(F_0)
\end{aligned}$$

So  $\langle \mathbf{p}, X \rangle$  is  $(\mathcal{K}, \mathbf{S}_0)$ -complete with respect to  $\mathbf{p}$  and  $\{\langle \mathbf{p}, X \rangle\}$  is a minimally  $(\mathcal{K}, \mathbf{S}_0)$ -complete set of nice pairs.

PROPOSITION 4.1. With the notations of Definition 4.2, let  $X$  be a  $\mathbf{P}$ -full set of nice pairs. Then,

- (1)  $S_X = (U, \{at^{\langle \mathbf{p}, Y \rangle} : \langle \mathbf{p}, Y \rangle \in X\})$  is an information system and
- (2)  $I_X : \{at^{\langle \mathbf{p}, Y \rangle} : \langle \mathbf{p}, Y \rangle \in X\} \rightarrow \mathcal{P}(\mathbf{P})$  with  $I_X(at^{\langle \mathbf{p}, Y \rangle}) = \{\mathbf{p}\}$ , is a contribution function for  $S_X$ .

Observe that for any  $\langle \mathbf{p}, Y \rangle, \langle \mathbf{p}', Y' \rangle \in X$  with  $\mathbf{p} \neq \mathbf{p}'$ ,  $at^{\langle \mathbf{p}, Y \rangle} \neq at^{\langle \mathbf{p}', Y' \rangle}$ .  $S_X$  (in Proposition 4.1) is a *set-theoretical* information system following Vakarelov's terminology (see e.g. [41, 42]). Lemma 4.2 below can be seen as the main technical result of the paper since it establishes correspondences between  $\Sigma$ -frames and set-theoretical information systems obtained from complete set of nice pairs (using the language **LS**). Proposition 4.3 below states some consequences for the informational representability.

LEMMA 4.2. Let  $\Sigma$  be a signature,  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\}) \in \mathcal{F}_\Sigma$  be a  $\Sigma$ -frame,  $\mathbf{S} = \langle F_1, \dots, F_k \rangle$  be a weak  $\Sigma$ -specification in  $\mathbf{spec}_\Sigma^1$  and  $X$  be a set of nice pairs with respect to  $\mathcal{K}$  and  $\mathbf{S}$ . Moreover, assume that for all  $l \in \{1, \dots, k\}$ , for all  $\emptyset \neq P \subseteq \mathbf{P}$ , if  $F_l$  is strong, then  $\mathcal{R}_l(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})$  otherwise  $\mathcal{R}_l(P) = \bigcup_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})$ .

- (1) If  $\mathbf{S}$  is strong and  $X$  is  $(\mathcal{K}, \mathbf{S})$ -complete, then
  - ( $\star$ )  $\forall l \in \{1, \dots, k\}, \forall \emptyset \neq P \subseteq \mathbf{P}, \mathcal{R}_l(P) = m_{(U, I_X^{-1}(P))}(F_l)$
- (2) If  $X$  is minimally  $(\mathcal{K}, \mathbf{S})$ -complete, then ( $\star$ ).

PROOF: We first consider the case  $P = \{\mathbf{p}_0\}$  for some  $\mathbf{p}_0 \in \mathbf{P}$ . Assume that  $\mathcal{R}_l(P)(o_1, \dots, o_{n_l})$  for some  $o_1, \dots, o_{n_l} \in U$ . By Definition 4.2(2), for any  $\langle \mathbf{p}_0, Y \rangle \in X$ ,  $m_{(U, \{at^{\langle \mathbf{p}_0, Y \rangle}\})}(F_l)(o_1, \dots, o_{n_l})$ . If  $F_l$  is strong, then

$$m_{(U, I_X^{-1}(P))}(F_l) = \bigcap_{a \in I_X^{-1}(P)} m_{(U, \{a\})}(F_l)$$

and  $m_{(U, I_X^{-1}(P))}(F_l)(o_1, \dots, o_{n_l})$  since

$$I_X^{-1}(P) = I_X^{-1}(\{\mathbf{p}_0\}) = \{at^{\langle \mathbf{p}_0, Y \rangle} : \langle \mathbf{p}_0, Y \rangle \in X\}$$

If  $F_l$  is weak, then  $I_X^{-1}(P)$  is a singleton (by hypothesis), say  $\{\langle \mathbf{p}_0, Y_0 \rangle\}$ . Hence

$$m_{(U, \{at^{\langle \mathbf{p}_0, Y_0 \rangle}\})}(F_l) = m_{(U, I_X^{-1}(P))}(F_l)$$

Hence  $m_{(U, I_X^{-1}(P))}(F_l)(o_1, \dots, o_{n_l})$ . Now assume  $m_{(U, I_X^{-1}(P))}(F_l)(o_1, \dots, o_{n_l})$ . Since  $X$  is  $(\mathcal{K}, \mathbf{S})$ -complete, there is  $\langle \mathbf{p}_0, Y_0 \rangle \in X$  such that

$$m_{(U, \{at^{\langle \mathbf{p}_0, Y_0 \rangle}\})}(F_l)(o_1, \dots, o_{n_l}) \text{ iff } \mathcal{R}_l(P)(o_1, \dots, o_{n_l})$$

If  $F_l$  is strong, then  $m_{(U, I_X^{-1}(P))}(F_l) \subseteq m_{(U, \{at(\mathbf{p}_0, Y_0)\})}(F_l)$ . Hence  $\mathcal{R}_l(P)(o_1, \dots, o_{n_l})$ . In the case when  $F_l$  is weak and  $X$  is minimally  $(\mathcal{K}, \mathbf{S})$ -complete, we have  $I_X^{-1}(P) = \{at(\mathbf{p}_0, Y_0)\}$ . Hence,  $m_{(U, \{at(\mathbf{p}_0, Y_0)\})}(F_l) = m_{(U, I_X^{-1}(P))}(F_l)$  and therefore  $\mathcal{R}_l(P)(o_1, \dots, o_{n_l})$ . Now take any  $\emptyset \neq P \subseteq \mathbb{P}$  and assume  $F_l$  is weak (resp. strong).

$\mathcal{R}_l(P)(o_1, \dots, o_{n_l})$  iff  $\bigcup_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l})$   
 (resp.  $\bigcap_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l})$ )  
 (by hypothesis)  
 iff  $\exists \mathbf{p} \in P$  such that (resp.  $\forall \mathbf{p} \in P$ )  $\mathcal{R}_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l})$   
 (by definition of  $\cap$  and  $\cup$ )  
 iff  $\exists \mathbf{p} \in P$  such that (resp.  $\forall \mathbf{p} \in P$ )  
 $m_{(U, I_X^{-1}(\{\mathbf{p}\}))}(F_l)(o_1, \dots, o_{n_l})$   
 (by the previous cases with singletons)  
 iff  $\exists \mathbf{p} \in P, \exists a \in I_X^{-1}(\{\mathbf{p}\})$  such that  
 (resp.  $\forall \mathbf{p} \in P, \forall a \in I_X^{-1}(\{\mathbf{p}\})$ )  $m_{(U, \{a\})}(F_l)(o_1, \dots, o_{n_l})$   
 (by construction of  $m_S(F)$ )  
 iff  $\exists a \in \{at(\mathbf{p}, Y) : \langle \mathbf{p}, Y \rangle \in X, \mathbf{p} \in P\}$  such that (resp.  
 $\forall a \in \{at(\mathbf{p}, Y) : \langle \mathbf{p}, Y \rangle \in X, \mathbf{p} \in P\}$ )  $m_{(U, \{a\})}(F_l)(o_1, \dots, o_{n_l})$   
 (by construction of  $I_X$ )  
 iff  $\exists a \in I_X^{-1}(P)$  such that  
 (resp.  $\forall a \in I_X^{-1}(P)$ )  $m_{(U, \{a\})}(F_l)(o_1, \dots, o_{n_l})$   
 iff  $m_{(U, I_X^{-1}(P))}(F_l)(o_1, \dots, o_{n_l})$   
 (by definition of  $m$ )

Q.E.D.

Proposition 4.3 below states sufficient conditions to establish informational representability.

PROPOSITION 4.3. Let  $\Sigma$  be a signature,  $\mathbf{S} = \langle F_1, \dots, F_k \rangle$  be a strong (resp. weak)  $\Sigma$ -specification in  $\text{spec}_{\Sigma}^1$ . Let  $X \subseteq \mathcal{F}_{\Sigma}$  be a set of  $\Sigma$ -frames such that for all  $l \in \{1, \dots, k\}$ , for all  $\emptyset \neq P \subseteq \mathbb{P}$ , for all  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\}) \in X$ , if  $F_l$  is strong, then  $\mathcal{R}_l(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})$  and  $\mathcal{R}_l(\emptyset) = U \times U$  otherwise  $\mathcal{R}_l(P) = \bigcup_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})$  and  $\mathcal{R}_l(\emptyset) = \emptyset$ . Let  $Y \subseteq \mathcal{IS}$  and  $\triangleright \in \{=, \equiv, \equiv_m\}$  be such that:

- (1) for all information systems  $S \in Y$ , for all contribution functions  $I$ , we have  $D_{\Sigma, \mathbf{S}}(S, I) \in X$ .
- (2) there is a (resp. minimal) nice pair function  $\text{truc}$  with respect to  $\mathbf{S}$  and  $X$  such that  $\{S_{\text{truc}(\mathcal{K})} : \exists \mathcal{K} \in X\} \subseteq Y$ .  $S_{\text{truc}(\mathcal{K})}$  is defined as in Proposition 4.1.

Then  $X$  is  $(\triangleright, \mathbf{S})$ -inf-representable in  $Y$ .

PROOF: By hypothesis (1), it is immediate that the condition (1) in Definition 4.1 holds. Now take some  $\Sigma$ -frame  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\}) \in X$  and  $X' = \text{truc}(\mathcal{K})$ . We write  $\mathcal{K}' = (U, \{\mathcal{R}'_l(P) : P \subseteq \mathbb{P}, l \in \{1, \dots, k\}\})$  to denote the  $\Sigma$ -frame  $D_{\Sigma, \mathbf{S}}(S_{X'}, I_{X'})$  (remember the information system  $S_{X'}$  belongs to  $Y$  by hypothesis). By Lemma 4.2, for all  $l \in \{1, \dots, k\}$ , for all  $\emptyset \neq P \subseteq \mathbb{P}$ ,  $\mathcal{R}_l(P) = \mathcal{R}'_l(P)$ . Moreover, for all  $l \in \{1, \dots, k\}$ , if  $F_l$  is strong (resp. weak), then  $\mathcal{R}'_l(\emptyset) = U \times U = \mathcal{R}_l(\emptyset)$  (resp.  $\mathcal{R}'_l(\emptyset) = \emptyset = \mathcal{R}_l(\emptyset)$ ). Hence  $\mathcal{K} = \mathcal{K}'$ . The condition (2) in Definition 4.1 holds which terminates the proof. Q.E.D.

EXAMPLE 4.2. (Example 4.1 continued) We write  $X_0$  to denote the set of  $\Sigma_0$ -frames such that  $\mathcal{R}_1(\emptyset) = U \times U$  and  $\mathcal{R}_1(\{\mathbf{p}\})$  is an equivalence relation. We show that  $X_0$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ . For any information system  $S$ , and for any contribution function  $I$  (actually there is only one), it is easy to check that  $D_{\Sigma_0, \mathbf{S}_0}(S, I) \in X_0$ . For each  $\mathcal{K} \in X_0$ , we build the pair  $\mathbf{truc}(\mathcal{K}) = \langle \mathbf{p}, X \rangle$  as in Example 4.1. From Example 4.1, for each  $\mathcal{K} \in X_0$ ,  $\{\mathbf{truc}(\mathcal{K})\}$  is a minimally  $(\mathcal{K}, \mathbf{S}_0)$ -complete set of nice pairs.  $\{\mathbf{S}_{\mathbf{truc}(\mathcal{K})} : \mathcal{K} \in X\} \subseteq \mathcal{IS}$  by Proposition 4.1 and by Proposition 4.3,  $X_0$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

## 4.2 Negation and reordering lemmas

In this section, we present two ways to obtain (minimal) nice pair functions from existing ones by relating adequately the specifications and the classes of frames. In that way, we facilitate the application of Proposition 4.3.

Let  $F \in \mathbf{F}^e$  be an extended formula such that  $\mathbf{VAR}(F) = \{x_1, \dots, x_n\}$ ,  $x_1, \dots, x_n$  being in the order of enumeration. For any permutation  $\sigma$  of the set  $\{1, \dots, n\}$  we write  $F\sigma$  to denote the formula obtained from  $F$  by substituting simultaneously in  $F$ ,  $x_i$  by  $x_{\sigma(i)}$  for all  $i \in \{1, \dots, n\}$ . Moreover, for any  $n$ -ary relation  $R$  and for any permutation  $\sigma$  of the set  $\{1, \dots, n\}$  we write  $R\sigma$  to denote the following  $n$ -ary relation:

$$R\sigma = \{(o_{\sigma(1)}, \dots, o_{\sigma(n)}) : (o_1, \dots, o_n) \in R\}$$

Observe that for any information system  $S = (OB, AT)$ , for any function interpretation  $m$  in  $S$ ,  $m_S(F)\sigma = m_S(F\sigma)$ . For any set of permutations  $\{\sigma_l : \{1, \dots, n_l\} \rightarrow \{1, \dots, n_l\}, l \in \{1, \dots, k\}\}$ , for any  $\Sigma$ -frame  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$  we write  $\mathcal{K}\sigma_1 \dots \sigma_k = (U, \{\mathcal{R}'_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$  to denote the  $\Sigma$ -frame such that for any  $l \in \{1, \dots, k\}$ , for any  $P \subseteq \mathbf{P}$ ,  $\mathcal{R}'_l(P) = \mathcal{R}_l(P)\sigma_l$ .

LEMMA 4.4. (Reordering) Let  $\Sigma$  be a signature,  $\mathbf{S} = \langle F_1, \dots, F_k \rangle$  be a weak  $\Sigma$ -specification in  $\mathbf{spec}_{\Sigma}^1$ ,  $X \subseteq \mathcal{F}_{\Sigma}$  be a set of  $\Sigma$ -frames, and  $\mathbf{truc}$  be a minimal nice pair function with respect to  $\mathbf{S}$  and  $X$ . Let  $\sigma_l$  be a permutation of the set  $\{1, \dots, n_l\}$  for  $l \in \{1, \dots, k\}$ . Then  $\mathbf{truc}'$  is a minimal nice pair function with respect to  $\mathbf{S}' = \langle F_1\sigma_1, \dots, F_k\sigma_k \rangle$  and  $\{\mathcal{K}\sigma_1 \dots \sigma_k : \exists \mathcal{K} \in X\}$  where  $\mathbf{truc}'(\mathcal{K}\sigma_1 \dots \sigma_k) = \mathbf{truc}(\mathcal{K})$  for all  $\mathcal{K} \in X$ .

PROOF: Let  $\mathbf{p} \in \mathbf{P}$ ,  $l \in \{1, \dots, k\}$ ,  $o_1, \dots, o_{n_l} \in U$  and  $\langle \mathbf{p}, Y \rangle \in \mathbf{truc}'(\mathcal{K}\sigma_1 \dots \sigma_k)$ . We have

$$\begin{aligned} \mathcal{R}'_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l}) &\text{ iff } \mathcal{R}_l(\{\mathbf{p}\})(o_{\sigma_l(1)}, \dots, o_{\sigma_l(n_l)}) \text{ (by definition of } \mathcal{K}\sigma_1 \dots \sigma_k) \\ &\text{ iff } m_{(U, \{at(\mathbf{p}, Y)\})}(F_l)(o_{\sigma_l(1)}, \dots, o_{\sigma_l(n_l)}) \\ &\text{ (}\langle \mathbf{p}, Y \rangle \text{ is } (\mathcal{K}, \mathbf{S})\text{-complete)} \\ &\text{ iff } (m_{(U, \{at(\mathbf{p}, Y)\})}(F_l)\sigma_l)(o_1, \dots, o_{n_l}) \text{ (by definition of } R\sigma) \\ &\text{ iff } m_{(U, \{at(\mathbf{p}, Y)\})}(F_l\sigma_l)(o_1, \dots, o_{n_l}) \text{ (property of } m) \end{aligned}$$

Moreover,  $\bigcup_{Z \in Y} Z = U$  since  $\langle \mathbf{p}, Y \rangle$  is a  $(\mathcal{K}, \mathbf{S})$ -complete set of nice pairs. Hence  $\{\langle \mathbf{p}, Y \rangle\}$  is a  $(\mathcal{K}\sigma_1 \dots \sigma_k, \mathbf{S}')$ -complete set of nice pairs. The minimality of  $\mathbf{truc}$  entails the minimality of  $\mathbf{truc}'$ . Q.E.D.

The starting point of Lemma 4.5 below rests on the fact that for any  $S = (OB, AT) \in \mathcal{IS}$ , for any function interpretation  $m$  in  $S$ ,

$$m_S(\neg F) = OB^{\text{card}(\mathbf{VAR}(F))} \setminus m_S(F)$$

with  $F \in \mathbf{F}$ . For any  $\Sigma$ -specification  $\mathbf{S} = \langle F_1, \dots, F_k \rangle$ , for any  $\Sigma$ -frame  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$  we write  $\mathcal{K}^\neg = (U, \{\mathcal{R}'_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$  to denote the  $\Sigma$ -frame such that

- if  $F_l$  is weak, then  $\mathcal{R}'_l(\emptyset) = \emptyset$  otherwise  $\mathcal{R}'_l(\emptyset) = U \times U$ ;
- for all  $\emptyset \neq P \subseteq \mathbf{P}$ , if  $F_l$  is strong, then  $\mathcal{R}'_l(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}'_l(\{\mathbf{p}\})$  otherwise  $\mathcal{R}'_l(P) = \bigcup_{\mathbf{p} \in P} \mathcal{R}'_l(\{\mathbf{p}\})$ ;
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}'_l(\{\mathbf{p}\}) = \neg \mathcal{R}_l(\{\mathbf{p}\})$ .

LEMMA 4.5. (Negation) Let  $\Sigma$  be a signature,  $\mathbf{S} = \langle q_1 f_1 F_1, \dots, q_k f_k F_k \rangle$  be a weak  $\Sigma$ -specification in  $\text{spec}_\Sigma^1$ ,  $X \subseteq \mathcal{F}_\Sigma$  be a set of  $\Sigma$ -frames and  $\text{truc}$  be a minimal nice pair function with respect to  $\mathbf{S}$  and  $X$ . Then  $\text{truc}'$  is a minimal nice pair function with respect to  $\mathbf{S}' = \langle q_1 f_1 \neg F_1, \dots, q_k f_k \neg F_k \rangle$  and  $\{\mathcal{K}^\neg : \exists \mathcal{K} \in X\}$  where  $\text{truc}'(\mathcal{K}^\neg) = \text{truc}(\mathcal{K})$  for all  $\mathcal{K} \in X$ .

PROOF: Let  $\mathbf{p} \in \mathbf{P}$ ,  $l \in \{1, \dots, k\}$ ,  $o_1, \dots, o_{n_l} \in U$  and  $\langle \mathbf{p}, Y \rangle \in \text{truc}'(\mathcal{K}^\neg)$ . We have

$$\begin{aligned}
\mathcal{R}'_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l}) &\text{ iff not } \neg \mathcal{R}'_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l}) \\
&\text{ (by definition of the complement relation '}\neg\text{'}) \\
&\text{ iff not } \mathcal{R}_l(\{\mathbf{p}\})(o_1, \dots, o_{n_l}) \text{ (by definition of } \mathcal{K}^\neg\text{)} \\
&\text{ iff not } m_{(U, \{at^{\langle \mathbf{p}, Y \rangle}\})}(q_l f_l F_l)(o_1, \dots, o_{n_l}) \\
&\text{ (}\langle \mathbf{p}, Y \rangle \text{ is } (\mathcal{K}, \mathbf{S})\text{-complete)} \\
&\text{ iff not } m_{(U, \{at^{\langle \mathbf{p}, Y \rangle}\})}(F_l)(o_1, \dots, o_{n_l}) \text{ (card}(\{at^{\langle \mathbf{p}, Y \rangle}\}) = 1\text{)} \\
&\text{ iff } m_{(U, \{at^{\langle \mathbf{p}, Y \rangle}\})}(\neg F_l)(o_1, \dots, o_{n_l}) \text{ (property of } m\text{)} \\
&\text{ iff } m_{(U, \{at^{\langle \mathbf{p}, Y \rangle}\})}(q_l f_l \neg F_l)(o_1, \dots, o_{n_l}) \text{ (card}(\{at^{\langle \mathbf{p}, Y \rangle}\}) = 1\text{)}
\end{aligned}$$

The rest of the proof follows the lines of the second part of the proof of Lemma 4.4. Q.E.D.

## 5 Examples of informational representability

We present examples of informational representability theorems for frames with information relations from the indiscernibility group and the orthogonality group. The analogous representability results for many other classes of frames can be obtained using the method developed in the paper.

### 5.1 $\Sigma$ -frames with a single family of binary relations

In the rest of the section,  $\Sigma$  is assumed to be a signature  $\langle \mathbf{P}, \langle 2 \rangle \rangle$  for some non-empty set  $\mathbf{P}$  of parameters.

#### 5.1.1 Indiscernibility

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. In the language  $\mathbf{LS}$ , the relation  $\text{ind}(AT)$  is defined as follows:  $\text{ind}(AT) = m_S(F_0)$  where  $F_0 = \forall f f(x_1) = f(x_2)$  (see Example 2.1). Observe that  $\text{ind}(AT)$  is an equivalence relation. We define a class of abstract  $\Sigma$ -frames related to the  $\Sigma$ -specification  $\mathbf{S}_0 = \langle F_0 \rangle$ . Let  $\text{truc}$  be the mapping such that for all  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_\Sigma$ ,

$$\text{truc}(\mathcal{K}) = \{\langle \mathbf{p}, \{\mathcal{R}_1(\{\mathbf{p}\})(u) : u \in U \rangle \rangle : \mathbf{p} \in \mathbf{P}\}$$

We write  $\mathcal{F}_{\mathbf{S}_0}$  to denote the set of  $\Sigma$ -frames  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\})$  such that,

- $\mathcal{R}_1(\emptyset) = U \times U$ ; for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_1(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_1(\{\mathbf{p}\})$ ;
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  is an equivalence relation.

The rest of the section is devoted to showing that  $\mathbf{truc}$  is a minimal nice pair function with respect to  $\mathbf{S}_0$  and  $\mathcal{F}_{\mathbf{S}_0}$ . Hence  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ . It is easy to check that for all  $\langle \mathbf{p}, X \rangle \in \mathbf{truc}(\mathcal{K})$ ,  $\bigcup_{Y \in X} Y = U$ .

LEMMA 5.1. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_{\mathbf{S}_0}$  be a  $\Sigma$ -frame. For all  $\langle \mathbf{p}, X \rangle \in \mathbf{truc}(\mathcal{K})$ , for all  $u, v \in U$ ,  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$  iff (for all  $Y \in X$ ,  $u \in Y$  iff  $v \in Y$ ).

PROOF: See Example 4.1. Q.E.D.

LEMMA 5.2. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_{\mathbf{S}_0}$  be a  $\Sigma$ -frame.  $\mathbf{truc}(\mathcal{K})$  is a minimally  $(\mathcal{K}, \mathbf{S}_0)$ -complete set of nice pairs.

PROOF: For all  $\langle \mathbf{p}, X \rangle \in \mathbf{truc}(\mathcal{K})$ ,  $u, v \in U$ ,  $(u, v) \in m_{(U, \{\text{at}(\mathbf{p}, X)\})}(F_0)$  iff for all  $Y \in X$ ,  $u \in Y$  iff  $v \in Y$ . So  $\mathbf{truc}(\mathcal{K})$  is a minimally  $(\mathcal{K}, \mathbf{S}_0)$ -complete set of nice pairs. Q.E.D.

COROLLARY 5.3.  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *diversity with respect to*  $A \subseteq AT$  (in short  $o_1 \text{ div}(A) o_2$ ) iff for all  $a \in A$ ,  $a(o_1) \neq a(o_2)$ . In the language  $\mathbf{LS}$ , the relation  $\text{div}(AT)$  is defined as follows:  $\text{div}(AT) = m_S(F'_0)$  with  $F'_0 = \forall f \neg(f(x_1)=f(x_2))$ . Using Lemma 4.5 and the above construction,  $\{\mathcal{K}^\neg : \exists \mathcal{K} \in \mathcal{F}_{\mathbf{S}_0}\}$  is  $(=, \langle F'_0 \rangle)$ -inf-representable in  $\mathcal{IS}$ .

### 5.1.2 Complementarity

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *complementarity with respect to*  $A \subseteq AT$  (in short  $o_1 \text{ comp}(A) o_2$ ) iff for all  $a \in A$ ,  $a(o_1) = \text{Val}_a \setminus a(o_2)$ . In the language  $\mathbf{LS}$ , the relation  $\text{comp}(AT)$  is defined as follows:  $\text{comp}(AT) = m_{(OB, AT)}(F_0)$  with  $F_0 = \forall f f(x_1) = \neg f(x_2)$ . Observe that  $\text{comp}(A)$  is symmetrical, irreflexive and intransitive. When  $o_1 \text{ comp}(A) o_2$  holds, for all  $a \in A$ ,  $\text{Val}_a$  is uniquely determined by  $a(o_1)$  and  $a(o_2)$ , that is  $\text{Val}_a = a(o_1) \cup a(o_2)$ . This may explain why the representation of complementarity relations has been an open problem until now [42] (see also in [12], their use for the reduction of decision rules). We define a class of abstract  $\Sigma$ -frames related to the  $\Sigma$ -specification  $\mathbf{S}_0 = \langle F_0 \rangle$ . First, some preliminary definitions are needed.

DEFINITION 5.1. Let  $R$  be a binary relation over the set  $U$ ,  $R$  is said to be *complementarity* iff  $R$  is symmetrical,  $R$  does not contain cycles of odd length and for all  $u, v, w, z \in U$ , if  $(u, v) \in R$ ,  $(v, w) \in R$  and  $(w, z) \in R$ , then  $(u, z) \in R$  ( $\beta$ -transitivity).  $\nabla$

It can be shown that  $\text{comp}(AT)$  is a complementarity relation. We write  $\mathcal{F}_{\mathbf{S}_0}$  to denote the set of  $\Sigma$ -frames  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\})$  such that,



$\mathcal{R}_1(\emptyset) = U \times U$ , for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_1(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_1(\{\mathbf{p}\})$  and for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  is complementarity. The rest of the section is devoted to showing that  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

For any binary relation  $R$  over the set  $U$ , for all  $u \in U$  we write  $C_{u,R}$  to denote the largest subset of  $U$  such that  $u \in C_{u,R}$  and for all  $v \in C_{u,R} \setminus \{u\}$ , there is a  $R'$ -path between  $u$  and  $v$  where  $R'$  is the symmetrical closure of  $R$ . Observe that  $\{C_{u,R} : u \in U\}$  is a partition of  $U$ . For all  $u \in U$ , we write  $C_{u,R}^0$  (resp.  $C_{u,R}^1$ ) to denote the largest subset of  $C_{u,R}$  such that for all  $v \in C_{u,R}^0$ , there is a  $R'$ -path of even (resp. odd) length between  $u$  and  $v$  where  $R'$  is the symmetrical closure of  $R$ . Moreover, for all  $U' \subseteq U$ , we write  $C_{u,R}^{U'}$  to denote  $C_{u,R}^0$  if  $u \in U'$ ,  $C_{u,R}^1$  otherwise. Observe that for all  $u, v, w \in U$ , when  $R$  is complementarity, if  $\{v, w\} \subseteq C_{u,R}$ , then  $\{C_{v,R}^1, C_{v,R}^0\} = \{C_{w,R}^1, C_{w,R}^0\}$ .

LEMMA 5.4. Let  $R$  be a complementarity relation over the set  $U$ . The set  $\{C_{u,R}^i : u \in U, i \in \{0, 1\}\}$  is a partition of  $U$ .

Let  $\text{truc}$  be the mapping such that for all  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_{\mathbf{S}_0}$ ,

$$\text{truc}(\mathcal{K}) = \{\langle \mathbf{p}, \{\bigcup_{u \in Y} C_{u, \mathcal{R}_1(\{\mathbf{p}\})}^{U'} : \exists U' \subseteq U, \exists Y \in W_{\mathbf{p}}\} \rangle : \mathbf{p} \in \mathbf{P}\}$$

with  $W_{\mathbf{p}} = \{Y \subseteq U : \forall u, v \in Y, C_{u, \mathcal{R}_1(\{\mathbf{p}\})} \neq C_{v, \mathcal{R}_1(\{\mathbf{p}\})}, \bigcup_{u \in Y} C_{u, \mathcal{R}_1(\{\mathbf{p}\})} = U\}$ . The set  $W_{\mathbf{p}}$  contains the subsets of  $U$  with exactly one element for each set in  $\{C_{u, \mathcal{R}_1(\{\mathbf{p}\})} : u \in U\}$ . Moreover,  $\text{truc}(\mathcal{K})$  can be roughly defined as the set of pairs  $\langle \mathbf{p}, U' \rangle$  such that  $U' \subseteq U$  is a maximal (with respect to set inclusion) subset of  $U$  such that

$$\mathcal{R}_1(\{\mathbf{p}\}) \cap U' \times U' = \emptyset$$

It is easy to show that for all  $\langle \mathbf{p}, X \rangle \in \text{truc}(\mathcal{K})$ , for all  $Y_0 \in W_{\mathbf{p}}$ ,

$$X = \{\bigcup_{u \in Y_0} C_{u, \mathcal{R}_1(\{\mathbf{p}\})}^{U'} : \exists U' \subseteq U\}$$

It is worth mentioning that, for all  $u, v \in U$ ,  $(\forall Y \in X, \text{either } u \in Y \text{ and } v \notin Y \text{ or } u \notin Y \text{ and } v \in Y)$  iff  $(u, v) \in m_{(U, \{\text{at}(\mathbf{p}, X)\})}(F_0)$ . Moreover, for all  $\langle \mathbf{p}, X \rangle \in \text{truc}(\mathcal{K})$ ,  $\bigcup_{Y \in X} Y = U$ .

LEMMA 5.5. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_{\mathbf{S}_0}$ . For all  $\langle \mathbf{p}, X \rangle \in \text{truc}(\mathcal{K})$ , for all  $u, v \in U$ ,  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$  iff for all  $Y \in X$ , either  $(u \in Y \text{ and } v \notin Y)$  or  $(u \notin Y \text{ and } v \in Y)$ .

PROOF:  $(\rightarrow)$  Assume  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$ . Since  $u \in C_{u, \mathcal{R}_1(\{\mathbf{p}\})}^0$  and  $v \in C_{u, \mathcal{R}_1(\{\mathbf{p}\})}^1$ , it follows that for all  $U' \subseteq U$ ,  $\{u, v\} \not\subseteq C_{u, \mathcal{R}_1(\{\mathbf{p}\})}^{U'}$ . Hence there is no  $Y \in X$  such that  $u \in Y$  and  $v \in Y$ . Moreover, for all  $U' \subseteq U$ , for all  $Y' \in W_{\mathbf{p}}$ ,

$$\{u, v\} \cap \bigcup_{u' \in Y'} C_{u', \mathcal{R}_1(\{\mathbf{p}\})}^{U'} \neq \emptyset$$

and

$$\bigcup_{u' \in Y'} C_{u', \mathcal{R}_1(\{\mathbf{p}\})}^{U'} \in X$$

So for all  $Y \in X$ , either  $(u \in Y \text{ and } v \notin Y)$  or  $(u \notin Y \text{ and } v \in Y)$ .

( $\leftarrow$ ) Now assume for all  $Y \in X$ ,  $u \in Y$  iff  $v \notin Y$ . First suppose that  $C_{u, \mathcal{R}_1(\{\mathbf{p}\})} \neq C_{v, \mathcal{R}_1(\{\mathbf{p}\})}$ . It follows that there is  $Y_0 \in W_{\mathbf{p}}$  such that  $\{u, v\} \subseteq Y_0$ . In particular take  $Y_0 = (\{x^w : w \in U\} \setminus \{x^u, x^v\}) \cup \{u, v\}$  where  $x^w$  is some *representative element* of  $C_{w, \mathcal{R}_1(\{\mathbf{p}\})}$ . Hence

$$\{u, v\} \subseteq \bigcup_{u' \in Y_0} C_{u', \mathcal{R}_1(\{\mathbf{p}\})}^{\{u, v\}} = X_0$$

So there is  $X_0 \in X$  such that  $\{u, v\} \subseteq X_0$ , which leads to a contradiction. So  $C_{u, \mathcal{R}_1(\{\mathbf{p}\})} = C_{v, \mathcal{R}_1(\{\mathbf{p}\})}$ . Suppose there is a  $\mathcal{R}_1(\{\mathbf{p}\})$ -path of even length between  $u$  and  $v$ , say  $(u_0, \dots, u_n, \dots, u_{2 \times n})$  with  $u_0 = u$  and  $v = u_{2 \times n}$  (recall that  $\mathcal{R}_1(\{\mathbf{p}\})$  is symmetrical). There is  $Y_0 \in W_{\mathbf{p}}$  such that  $\{u_n\} \subseteq Y_0$ . Hence

$$\{u, v\} \subseteq \bigcup_{u' \in Y_0} C_{u', \mathcal{R}_1(\{\mathbf{p}\})}^{U'} = X_0$$

where  $U' = U$  if  $n$  is even otherwise  $U' = \emptyset$ . So there is  $X_0 \in X$  such that  $\{u, v\} \subseteq X_0$ , which leads to a contradiction. So there is a  $\mathcal{R}_1(\{\mathbf{p}\})$ -path of odd length between  $u$  and  $v$ , say  $(u_0, \dots, u_{(2 \times n) + 1})$  with  $u_0 = u$  and  $v = u_{(2 \times n) + 1}$ . If  $n = 0$ , then  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$ . If  $n = 1$ , then by the *3-transitivity condition*  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$ . Now assume  $n > 1$ . By the *3-transitivity condition*,  $(u_{(2 \times (n-1))}, u_{(2 \times n) + 1}) \in \mathcal{R}_1(\{\mathbf{p}\})$ . So there is a path of length  $(2 \times (n-1)) + 1$  between  $u$  and  $v$ . By induction on  $n$  we can therefore prove that  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$ . Q.E.D.

LEMMA 5.6. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_{\mathbf{S}_0}$  be a  $\Sigma$ -frame.  $\text{truc}(\mathcal{K})$  is a minimally  $(\mathcal{K}, \mathbf{S}_0)$ -complete set of nice pairs.

COROLLARY 5.7.  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

For any  $\mathbf{p} \in \mathbf{P}$  the class of frames  $\mathcal{F}_{\mathbf{p}}$  (in the standard sense for modal logics) defined by  $\mathcal{F}_{\mathbf{p}} = \{(U, \mathcal{R}_1(\{\mathbf{p}\})) : (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_{\mathbf{S}_0}\}$  is not closed by the *p-morphism* construction. It follows that  $\mathcal{F}_{\mathbf{p}}$  is not *modally definable* (see [17]). The condition of being a complementarity relation can be expressed by a set  $\Gamma$  of formulas from the classical first-order logic.

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in relation of *incomplementarity with respect to*  $A \subseteq AT$  (in short  $o_1 \text{ incomp}(A) o_2$ ) iff for all  $a \in A$ ,  $(\text{Val}_a \setminus a(o_1)) \neq a(o_2)$ . In the language  $\text{LS}$ , the relation  $\text{incomp}(AT)$  is defined as follows:  $\text{incomp}(AT) = m_S(F'_0)$  with  $F'_0 = \forall f \neg(-f(x_1) = f(x_2))$ . Using Lemma 4.5 and the above construction,  $\{\mathcal{K}^\neg : \exists \mathcal{K} \in \mathcal{F}_{\mathbf{S}_0}\}$  is  $(=, \langle F'_0 \rangle)$ -inf-representable in  $\mathcal{IS}$ .

### 5.1.3 Positive and negative similarity

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *positive similarity (resp. negative similarity) with respect to*  $A \subseteq AT$  (in short  $o_1 \text{ psim}(A) o_2$  -resp.  $o_1 \text{ nsim}(A) o_2$ ) iff for all  $a \in A$ ,  $a(o_1) \cap a(o_2) \neq \emptyset$  (resp.  $(\text{Val}_a \setminus a(o_1)) \cap (\text{Val}_a \setminus a(o_2)) \neq \emptyset$ ). In the language  $\text{LS}$ , the relation  $\text{psim}(AT)$  is defined as follows:  $\text{psim}(AT) = m_{(OB, AT)}(F_0)$  (resp.  $\text{nsim}(AT) = m_{(OB, AT)}(F_0)$ ) with

$$F_0 = \forall f \neg(f(x_1) \cap f(x_2) = 0)$$

(resp.  $F_0 = \forall f \neg(-f(x_1) \cap -f(x_2)=0)$ )

Let  $\mathbf{truc}$  be the mapping such that for all  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_\Sigma$ ,

$$\mathbf{truc}(\mathcal{K}) = \{\langle \mathbf{p}, \{\{u, v\} : (u, v) \in \mathcal{R}_1(\{\mathbf{p}\})\} \rangle : \mathbf{p} \in \mathbf{P}\}$$

$$(\text{resp. } \mathbf{truc}(\mathcal{K}) = \{\langle \mathbf{p}, \{U \setminus \{u, v\} : (u, v) \in \mathcal{R}_1(\{\mathbf{p}\})\} \cup \{U\} \rangle : \mathbf{p} \in \mathbf{P}\})$$

We write  $\mathcal{F}_{\mathbf{S}_0}$  to denote the set of  $\Sigma$ -frames  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\})$  such that,  $\mathcal{R}_1(\emptyset) = U \times U$ , for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_1(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_1(\{\mathbf{p}\})$  and for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  is (resp. weakly) reflexive<sup>1</sup> and symmetrical. Following the lines of the previous sections, it can be shown that  $\mathbf{truc}$  is a minimal nice pair function with respect to  $\mathbf{S}_0$  and  $\mathcal{F}_{\mathbf{S}_0}$ . Hence  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

Moreover, let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *right orthogonality* (resp. *left orthogonality*) with respect to  $A \subseteq AT$  (in short  $o_1$  *rorth*( $A$ )  $o_2$  -resp.  $o_1$  *lorth*( $A$ )  $o_2$ ) iff for all  $a \in A$ ,  $a(o_1) \subseteq (Val_a \setminus a(o_2))$  (resp.  $(Val_a \setminus a(o_1)) \subseteq a(o_2)$ ). In the language  $\mathbf{LS}$ , the relation *rorth*( $AT$ ) (*lorth*( $AT$ )) is defined as follows: *rorth*( $AT$ ) =  $m_{(OB, AT)}(F'_0)$  (resp. *lorth*( $AT$ ) =  $m_{(OB, AT)}(F'_0)$ ) with  $F'_0 = \forall f f(x_1) \subseteq -f(x_2)$  (resp.  $F'_0 = \forall f -f(x_1) \subseteq f(x_2)$ ). Using Lemma 4.5 and the above construction,  $\{\mathcal{K}^\top : \exists \mathcal{K} \in \mathcal{F}_{\mathbf{S}_0}\}$  is  $(=, \langle F'_0 \rangle)$ -inf-representable in  $\mathcal{IS}$ .

#### 5.1.4 Backward and forward inclusion

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *backward inclusion* with respect to  $A \subseteq AT$  (in short  $o_1$  *bin*( $A$ )  $o_2$ ) iff for all  $a \in A$ ,  $a(o_1) \subseteq a(o_2)$ . In the language  $\mathbf{LS}$ , the relation *ind*( $AT$ ) is defined as follows: *ind*( $AT$ ) =  $m_{(OB, AT)}(F_0)$  with  $F_0 = \forall f f(x_1) \subseteq f(x_2)$ . Let  $\mathbf{truc}$  be the mapping such that for all  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\}) \in \mathcal{F}_\Sigma$ ,

$$\mathbf{truc}(\mathcal{K}) = \{\langle \mathbf{p}, \{\mathcal{R}_1(\{\mathbf{p}\})(u) : u \in U\} \rangle : \mathbf{p} \in \mathbf{P}\}$$

We write  $\mathcal{F}_{\mathbf{S}_0}$  to denote the set of  $\Sigma$ -frames  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\})$  such that,

- $\mathcal{R}_1(\emptyset) = U \times U$ , for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_1(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_1(\{\mathbf{p}\})$ ;
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  is reflexive and transitive.

Following the lines of the previous sections, it can be shown that  $\mathbf{truc}$  is a minimal nice pair function with respect to  $\mathbf{S}_0$  and  $\mathcal{F}_{\mathbf{S}_0}$ . Hence  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *forward inclusion* with respect to  $A \subseteq AT$  (in short  $o_1$  *fin*( $A$ )  $o_2$ ) iff for all  $a \in A$ ,  $a(o_2) \subseteq a(o_1)$ . In the language  $\mathbf{LS}$ , the relation *fin*( $AT$ ) is defined as follows: *fin*( $AT$ ) =  $m_{(OB, AT)}(F'_0)$  with  $F'_0 = \forall f f(x_2) \subseteq f(x_1)$ . Using Lemma 4.4 and the above construction,  $\mathcal{F}' = \{\mathcal{K}\sigma : \exists \mathcal{K} \in \mathcal{F}_{\mathbf{S}_0}\}$  is  $(=, \langle F'_0 \rangle)$ -inf-representable in  $\mathcal{IS}$  with  $\sigma$  the 1-1 mapping  $\{1, 2\} \rightarrow \{1, 2\}$  such that  $\sigma(1) = 2$ .

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *left negative similarity with respect to*  $A \subseteq AT$  (in

<sup>1</sup>A binary relation  $R$  on the set  $U$  is said to be *weakly reflexive* iff for all  $u \in U$ , if for some  $v \in U$ ,  $(u, v) \in R$ , then  $(u, u) \in R$ .

short  $o_1 \text{ lnsim}(A) o_2$  iff for all  $a \in A$ ,  $(\text{Val}_a \setminus a(o_1)) \cap a(o_2) \neq \emptyset$ . In the language  $\mathcal{LS}$ , the relation  $\text{lnsim}(AT)$  is defined as follows:  $\text{lnsim}(AT) = m_S(G_0)$  with

$$G_0 = \forall f \neg(-f(x_1) \cap f(x_2)=0)$$

Using Lemma 4.5 and the above construction,  $\mathcal{F}'' = \{\mathcal{K}^\neg : \exists \mathcal{K} \in \mathcal{F}'\}$  is  $(=, \langle G_0 \rangle)$ -inf-representable in  $\mathcal{IS}$ .

Let  $S = (OB, AT) \in \mathcal{IS}$  be an information system. Two objects  $o_1$  and  $o_2$  are said to be in the relation of *right negative similarity with respect to*  $A \subseteq AT$  (in short  $o_1 \text{ rnsim}(A) o_2$ ) iff for all  $a \in A$ ,  $(\text{Val}_a \setminus a(o_2)) \cap a(o_1) \neq \emptyset$ . In the language  $\mathcal{LS}$ , the relation  $\text{rnsim}(AT)$  is defined as follows:  $\text{rnsim}(AT) = m_S(G'_0)$  with

$$G'_0 = \forall f \neg(f(x_1) \cap -f(x_2)=0)$$

Using Lemma 4.4 and the above construction,  $\{\mathcal{K}\sigma : \exists \mathcal{K} \in \mathcal{F}''\}$  is  $(=, \langle G'_0 \rangle)$ -inf-representable in  $\mathcal{IS}$  ( $\sigma$  as above).

## 5.2 $\Sigma$ -frames with a single family of $n$ -ary relations

Some of the constructions previously presented can be generalized to  $n$ -ary relations. In the rest of the section  $\Sigma = \langle \mathbb{P}, \langle n \rangle \rangle$  for some  $n \geq 2$ . We focus our attention to the generalized version of the forward inclusion, positive similarity and indiscernibility relations. For any  $n$ -ary relation  $R$  on  $U$ ,

- $R$  is *n-reflexive* iff for all  $x \in U$ ,  $R(x, \dots, x)$ ;
- $R$  is *n-symmetrical* iff for all  $x_1, \dots, x_n \in U$  if  $R(x_1, \dots, x_n)$ , then for any set  $\{y_1, \dots, y_n\} \subseteq \{x_1, \dots, x_n\}$   $R(y_1, \dots, y_n)$ ;
- $R$  is *n-transitive* iff for all  $x_1, \dots, x_n, y_1, \dots, y_n \in U$  such that  $R(x_1, \dots, x_n)$  and  $R(y_1, \dots, y_n)$ , if there exist  $i, j \in \{1, \dots, n\}$  such that  $x_i = y_j$ , then for all  $j' \geq j$ ,  $R(x_1, \dots, x_{i-1}, y_{j'}, \dots, y_{j'})$ .

The classes of frames with the binary relations (see Section 5.1.1, 5.1.3, 5.1.4) are extended to  $n$ -ary relations where reflexivity (resp. symmetry, transitivity) is extended to  $n$ -reflexivity (resp.  $n$ -symmetry,  $n$ -transitivity). For any frame  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbb{P}, l \in \{1\}\})$  of the respective extended classes, the nice pair function  $\text{truc}$  is defined by,

$$(\text{pos. sim.}) \text{truc}(\mathcal{K}) = \{\langle \mathbb{p}, \{\{u_1, \dots, u_n\} : (u_1, \dots, u_n) \in \mathcal{R}_1(\{\mathbb{p}\})\} \rangle : \mathbb{p} \in \mathbb{P}\}$$

$$(\text{for. incl.}) \text{truc}(\mathcal{K}) = \{\langle \mathbb{p}, \{\{v : \mathcal{R}_1(\{\mathbb{p}\})(\dots, u, \dots, v, \dots)\} : u \in U\} \rangle : \mathbb{p} \in \mathbb{P}\}$$

$$(\text{ind.}) \text{truc}(\mathcal{K}) = \{\langle \mathbb{p}, \{\{u', u'' : \mathcal{R}_1(\{\mathbb{p}\})(\dots, u', \dots, u, \dots, u'', \dots)\} : u \in U\} \rangle : \mathbb{p} \in \mathbb{P}\}$$

With such nice pair functions (the proofs are omitted here), the respective extended classes of frames are  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$  with the corresponding specification  $\mathbf{S}_0$ . For instance, for the extended indiscernibility relation,

$$\mathbf{S}_0 = \langle \forall f (f(x_1)=f(x_2)) \wedge \dots \wedge (f(x_{n-1})=f(x_n)) \rangle$$

### 5.3 $\Sigma$ -frames with multiple families of binary relations

The informational representability of classes of  $\Sigma$ -frames with multiple families of binary relations can be found in [37, 39, 40, 42]. Other classes of  $\Sigma$ -frames with multiple families of relations can be shown to be informationally representable by using the results of the previous sections and [37, 39, 40, 42]. In order to avoid the boredom of repetitive arguments, we shall present herein a single and original example. Let  $\Sigma = \langle \mathbf{P}, \langle 2, 2 \rangle \rangle$  be the signature and  $\mathbf{S}_0 = \langle \forall f f(x_1) = -f(x_2), \forall f f(x_1) = f(x_2) \rangle$  be a  $\Sigma$ -specification in  $\text{spec}_{\Sigma}^1$ .

We write  $\mathcal{F}_{\mathbf{S}_0}$  to denote the set of  $\Sigma$ -frames  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, 2\}\})$  such that ,

- for all  $l \in \{1, 2\}$   $\mathcal{R}_l(\emptyset) = U \times U$  and for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_l(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})$ ;
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_2(\{\mathbf{p}\})$  is an equivalence relation;
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  is a complementarity relation;
- for all  $u, v \in U$ , for all  $\mathbf{p} \in \mathbf{P}$ ,  $(u, v) \in \mathcal{R}_2(\{\mathbf{p}\})$  iff there exists a  $\mathcal{R}_1(\{\mathbf{p}\})$ -path of even length between  $u$  and  $v$ .

Let  $\text{truc}$  be the mapping defined in Section 5.1.2, that is for all  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, 2\}\}) \in \mathcal{F}_{\mathbf{S}_0}$ ,

$$\text{truc}(\mathcal{K}) = \{ \langle \mathbf{p}, \{ \bigcup_{u \in Y} C_{u, \mathcal{R}_1(\{\mathbf{p}\})}^{U'} : \exists U' \subseteq U, \exists Y \in W_{\mathbf{p}} \} : \mathbf{p} \in \mathbf{P} \}$$

$$\text{with } W_{\mathbf{p}} = \{ Y \subseteq U : \forall u, v \in Y, C_{u, \mathcal{R}_1(\{\mathbf{p}\})} \neq C_{v, \mathcal{R}_1(\{\mathbf{p}\})}, \bigcup_{u \in Y} C_{u, \mathcal{R}_1(\{\mathbf{p}\})} = U \}$$

LEMMA 5.8. Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, 2\}\}) \in \mathcal{F}_{\mathbf{S}_0}$ . For all  $\langle \mathbf{p}, X \rangle \in \text{truc}(\mathcal{K})$ ,

- (1)  $\bigcup_{Y \in X} Y = U$ ,
- (2) for all  $u, v \in U$ ,  $(u, v) \in m_{(U, \{at(\mathbf{p}, X)\})}(\forall f f(x_1) = f(x_2))$  iff (for all  $Y \in X$ ,  $u \in Y$  iff  $v \in Y$ ) iff  $(u, v) \in \mathcal{R}_2(\{\mathbf{p}\})$ ,
- (3) for all  $u, v \in U$ ,  $(u, v) \in m_{(U, \{at(\mathbf{p}, X)\})}(\forall f f(x_1) = -f(x_2))$  iff (for all  $Y \in X$ ,  $u \in Y$  iff  $v \notin Y$ ) iff  $(u, v) \in \mathcal{R}_1(\{\mathbf{p}\})$ .

PROOF: (1) and (3) follow from Section 5.1.2. (2) The proof is omitted here and is inspired by the different cases analyzed in the proof of Lemma 5.5. Q.E.D.

COROLLARY 5.9.  $\mathcal{F}_{\mathbf{S}_0}$  is  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

## 6 Examples of non-representability

Until now, only representability results have been shown. However, non representability results are also very informative for understanding the relevance of the notion of representability we introduced.

## 6.1 Class of $\Sigma$ -frames closed under subframes

The example below provides some insight into classes of information systems closed under subsystems and classes of  $\Sigma$ -frames closed under subframes. Although Proposition 6.1 might appear quite natural, it has some unexpected consequences (see for instance Corollary 6.2).

PROPOSITION 6.1. Let  $\Sigma$  be a signature,  $\mathbf{S}$  be a  $\Sigma$ -specification,  $X \subseteq \mathcal{F}_\Sigma$  be a set of  $\Sigma$ -frames and  $Y \subseteq \mathcal{IS}$  be a set of information systems closed under subsystems. If  $X$  is  $(=, \mathbf{S})$ -inf-representable in  $Y$ , then  $X$  is closed under subframes.

PROOF: The proof is by an easy verification. For any  $S = (OB, AT) \in Y$ , for any subsystem  $S' = (OB', AT')$  of  $S$ , for any contribution function  $I$  for  $S$ , let  $I'$  be the contribution function for  $S'$  such that for all  $a \in AT$ ,

$$I'(a_{OB'}) = \bigcup \{I(a) : \exists a' \in AT, a_{OB'} = a'_{OB'}\}$$

Let  $\mathcal{K} = (U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\}) \in X$ . There exist an information system  $S \in Y$  and a contribution function  $I$  for  $S$  such that  $D_{\Sigma, \mathbf{S}}(S, I) = \mathcal{K}$ . Let  $\mathcal{K}' = (U', \{\mathcal{R}'_l(P) : P \subseteq \mathbf{P}, l \in \{1, \dots, k\}\})$  be a subframe of  $\mathcal{K}$ . It can be shown that  $D_{\Sigma, \mathbf{S}}(S', I') = \mathcal{K}'$  where  $S'$  (resp.  $I'$ ) is the restriction of  $S$  (resp.  $I$ ) to  $U'$ . Since the information system  $S'$  belongs to  $Y$  (by hypothesis) and by the soundness condition in the definition of informational representability, we have  $\mathcal{K}' \in X$ . Q.E.D.

COROLLARY 6.2. Let  $\Sigma$  be a signature,  $\mathbf{S}$  be a strong  $\Sigma$ -specification and  $X$  be a set of  $\Sigma$ -frames such that for all  $l \in \{1, \dots, k\}$ ,

- $\mathcal{R}_l(\emptyset) = U \times U$ , for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_l(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_l(\{\mathbf{p}\})$
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_l(\{\mathbf{p}\})$  is serial (resp. atomic, weakly dense, discrete).

$X$  is not  $(=, \mathbf{S})$ -inf-representable in  $\mathcal{IS}$ .

Some relationships might exist between *subframe logics* (see [45]) and the logics characterized by classes of frames informationally representable by frames derived from a class of information systems closed under subsystems. However, it is not in the scope of this work.

## 6.2 Non representability of local agreement frames in $\mathcal{IS}$

In [15], a class of models for logics of indiscernibility is defined. The relations in those models are equivalence relations that additionally satisfy a condition referred to as local agreement. Two equivalence relations  $R$  and  $R'$  (on  $U$ ) are said to be in *local agreement* [15] iff for all  $u \in U$ , either  $R(u) \subseteq R'(u)$  or  $R'(u) \subseteq R(u)$ .

We write  $\mathcal{F}_{LA}$  to denote the set of  $\Sigma$ -frames  $(U, \{\mathcal{R}_l(P) : P \subseteq \mathbf{P}, l \in \{1\}\})$  such that

- $\mathcal{R}_1(\emptyset) = U \times U$ , for all  $\emptyset \neq P \subseteq \mathbf{P}$ ,  $\mathcal{R}_1(P) = \bigcap_{\mathbf{p} \in P} \mathcal{R}_1(\{\mathbf{p}\})$ ;
- for all  $\mathbf{p} \in \mathbf{P}$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  is an equivalence relation;
- for all  $P, Q \subseteq \mathbf{P}$ ,  $\mathcal{R}_1(P)$  and  $\mathcal{R}_1(Q)$  are in local agreement.

Observe that  $\mathcal{F}_{LA}$  is closed under subframes and  $\mathcal{IS}$  is closed under subsystems (see Section 6.1). The rest of the section is devoted to showing that  $\mathcal{K}_{LA}$  is not  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$  where  $\mathbf{S}_0 = \langle \forall f f(x_1)=f(x_2) \rangle$ .

PROPOSITION 6.3. For all  $\Sigma$ -frames  $\mathcal{K}$  in  $\mathcal{F}_{LA}$ , there is an information system  $S \in \mathcal{IS}$  and a contribution function  $I$  for  $S$  such that  $D_{\Sigma, I}(S, I) = \mathcal{K}$ .

PROOF: See the construction `truc` in Section 5.1.1. Q.E.D.

But,

PROPOSITION 6.4. If  $\mathbf{P}$  has at least 2 elements, then  $\mathcal{K}_{LA}$  is not  $(=, \mathbf{S}_0)$ -inf-representable in  $\mathcal{IS}$ .

PROOF: Let  $S = (\{o_1, o_2, o_3, \}, \{a, a'\})$  be the information system such that  $a(o_1) = a(o_2) = \{1\}$ ,  $a'(o_1) = a'(o_3) = \{1\}$ ,  $a(o_3) = a'(o_2) = \{2\}$  ( $Val_a = Val_{a'} = \{1, 2\}$ ). For some  $\mathbf{p}, \mathbf{p}' \in \mathbf{P}$  define the contribution function  $I$  such that  $I(a) = \{\mathbf{p}\}$  and  $I(a') = \mathbf{P} \setminus \{\mathbf{p}\}$ . It is easy to show that in the  $\Sigma$ -frame  $\mathcal{K} = D_{\Sigma, I}(S, I)$ ,  $\mathcal{R}_1(\{\mathbf{p}\})$  and  $\mathcal{R}_1(\{\mathbf{p}'\})$  are not in local agreement. Q.E.D.

A generalization of the local agreement is introduced in [4] and a broad class of information logics whose frames satisfy this condition is investigated in [13, 6].

## 7 Conclusion

In this work we have defined a general framework for proving informational representability of classes of frames. The nice pair proof technique has been illustrated with a number of examples. In particular, we have characterized the class of frames informationally representable by the so-called complementarity relations. Considerations about non representability have shown the limits of the notion of informational representability. Some possible continuations of the present work might take into account the following open problems:

- (1) For which classes of specifications a construction analogous to the nice pair construction defined in the paper can be developed and the informational representability theorem can be proved?
- (2) What is the exact relationships between first-order definability and informational representability?
- (3) How to extend or restrict the language  $\mathbf{LS}$  in order to characterize the classes of informationally representable frames in a systematic way?

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