Time-Frequency varying beta estimation - a continuous wavelets approach

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Abstract

The Beta coefficient theorized by the CAPM is estimated by the Market Line. By hypothesis, the Beta is stable over time but empirical studies on it volatility don't confirm this fact. One of them is related to with agent heterogeneity hypothesis. In this paper; we study this hypothesis by continuous wavelets decomposition of the market line components. We use the wavelet Coherence to calculate a time-frequency Beta. We apply this methodology on three French listed stocks (AXA-LVMH-ORANGE) with different OLS beta for the daily period from 2005 to 2015. We show that the coherence and the time-frequency Betas improve our understanding of the equity characteristics and nature according to their time and frequency dynamics. AXA and LVMH have globally an high coherence with the market whereas ORANGE coherence is low (whatever frequencies). These results can affect the time-frequency betas values. By analyzing the betas we see different evolutions and dynamics which can be considered by portfolio managers to optimize their investment horizon. The continuous wavelets is a powerful tool for emphasize the time-frequency instabilities of betas. The hypothesis of heterogeneity of agents have an impact on systematic risk estimations and need to be considered in financial calculations.

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1 Introduction

The Capital Asset Pricing Model (or CAPM) developed by Treynor (1961, 1962) and Sharpe (1964) establishes a relation between the risk premium of asset and the market premium called the Securities Market Line.

$$r_{i,t} = \alpha + \beta.r_{m,t} + \epsilon_t \tag{1}$$

With $r_{i,t}$ the risk premium of the asset i (defined as the difference between returns and risk free rate) and $r_{m,t}$ the market premium; ϵ_t is an $i.i.d(0, \sigma_\epsilon)$ process.

The Beta parameter is a measure of systematic risk. For portfolios managers, the Beta value is useful to appreciate the asset sensitivity to market movements. It is estimated by OLS whose one of the hypothesis is the stability of the coefficients over time. Black, Scholes and Jensen (1972), Fama and McBeth (1973) Fabozzi and Francis (1978) Bos and Newbold (1984) discuss this hypothesis highlighting the instability of Beta. Followings these results, many methods are developed in order to estimate time-varying parameter as the rolling forward regression (or recursive) and the GARCH processes (see Bollerslev *et al.* 1988 , Groenwold and Fraser 1997 and Brooks *et al.* 1998).

Moreover, the CAPM implicitly assumes that agents have homogeneous behaviours. So, two investors with different investment horizons make choice base on the same systematic risk measure. But they do not adjust the Beta to their appetence. This claim is contestable because of the risk could be differentiated according to the investment frequency. On this basis, Gençay *et al.* (2005) and Mestre and Terraza (2017*a*) indicate the existence of frequency systematic risk by using a wavelets methodology. The wavelets has been developed by Haar in 1909 and used in Signal Processing by Morlet and Grossmann (1984). Thereafter, Meyer $(1986-1987)$[1] , Mallat $(1989-2001-2009)$ and Daubechies (2002) study their properties and create different wavelets filters. Thanks to more computing power, Auth (2013) and Bekiros *et al.* (2016) use continuous wavelets transforms (CWT) and the time-frequency coherence and phase to analyze the causality and interactions between variables.

In this paper, we propose a formula (and a methodology) based on Continuous Wavelets in order to estimate time-frequency betas at each time t and each frequency s. We illustrate the applicability of our formula, in the CAPM framework, by estimating the Beta of three stocks listed on the french market (CAC40 index) for the daily period $2005-2015$: AXA, LVMH, ORANGE. We select these stocks according to their different risk-profiles established with OLS Beta values (AXA with a Beta greater than 1, LVMH with a Beta equal to 1 and ORANGE with a beta lesser than 1)[2].

We indicate that it is possible to estimate a Time-Frequency varying Beta. An analysis of its time-frequency characteristics improves our understanding of systematic risk dynamics and its changing. Consequently, equities can be distinguished by the time-frequency dynamics of Betas offering managers more risk-profiles and assets classifications.

In a first part, we present a synthesis of the multivariate wavelets methodology, and then we describe results for the selected stocks.

[1] Abel price in 2017

[2] OAT 10 years rate is assimilated as risk free rate

2 Multivariate Time-Frequency Analysis

We use CWT to provide the time-frequency characteristics of the market's line as the Coherence (correlation coefficient). The complex Morlet wavelet, $\varphi^M(t)$, is used as filter because it represents a good balance between time and frequency localization of events. During the decomposition process, the function $\varphi^M(t)$ is shifted by τ and dilated by s in order to extract information on several frequency scales at a particular time t. All shifted-dilated versions of $\varphi^M(t)$ constitute the wavelets family providing the filtering properties.

The mathematical expression of the Morlet wavelet is written as follow:

$$\varphi^M(t) = \pi^{-1/4} e^{(if_0 t)} e^{(-t^2/2)}. \tag{2}$$

With f_0 the non-dimensional frequency equal to 6 in order to satisfy the admissibility condition.[3]

The wavelets coherence formula between two functions $x(t)$ and $y(t)$ (with same length N) or *Time-Varying Coherence*, is similar as the Fourier Coherence (Cf. bibliographies $14, 19, 22, 23$) . We define a spectral covariance between the wavelets of $x(t)$ and $y(t)$ called the cross-wavelets spectrum $SW_{xy}(s,\tau)$. This spectrum is associated with the (auto) power spectra SW_x^2 and SW_y^2 and we can establish the Wavelets Coherence $WQ_{\tau,s}$:

$$WQ_{\tau,s} = \frac{\left|G(s^{-1}.SW_{xy})\right|^2}{G(s^{-1}.\left|SW_x\right|^2).G(s^{-1}.\left|SW_y\right|^2)} \tag{4}$$

Because of the Complex Morlet wavelet, the coherence coefficients are complex. Consequently, the coherence is equal to 1 in its real representation (whatever τ and s). We need to smooth the outputs by a time-frequencey smoothing noted G. G is composed by a Time-smoothing, G_{time} for a given frequency scale, and by a Frequency-smoothing G_{scale} for a given time t. The G operator is build as follow:

$$G(W) = G_{scale}(G_{time}(W)) \tag{5}$$

G_{scale} and G_{time} expressions is developed by Torrence et Webster (1998):

$$G_{time}(W_N) = W_N.c1^{-t^2/2s^2} \tag{6}$$

$$G_{scale}(.) = W_N.c2\Pi(0.6s) \tag{7}$$

$c1$ and $c2$ are normalization constants, W_N is the wavelets coefficients and Π represent the rectangle function[4].

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[3] A more precise presentation of the continuous wavelets methodology in univariate case is available in reference 19.

The admissibility condition guarantees the nullity of wavelet mean and the energy preservation during the decomposition process.

$$C_\varphi = \int_{-\infty}^{+\infty} \frac{\left|\widehat{\varphi(s)}\right|^2}{|s|} \,\mathrm{d}f < +\infty \tag{3}$$

with $\widehat{\varphi(s)}$ the Fourier transform of $\varphi(s)$

[4] The rectangle function is a function equal to a in the interval $[-1/2, 1/2]$ and equal to 0 outside

The $WQ_{\tau,s}$ formula is equivalent to the determination coefficient. For each frequency scale s and at each time t, we have a coefficient between 0 and 1 representing the squared correlation between the two series (the explanatory power of $x(t)$ on $y(t)$ at time t). It is theoretically possible to link this formula with the Beta.

$$\sqrt(WQ_{\tau,s}) = \frac{|G(s^{-1}.SW_{xy})|}{G(s^{-1}.|SW_x|^2)^{1/2}.G(s^{-1}.|SW_y|^2)^{1/2}} \tag{8}$$

$$\sqrt(WQ_{\tau,s}) = \frac{|G(s^{-1}.SW_{xy})|}{G(s^{-1}.|SW_x|^2)^{1/2}.G(s^{-1}.|SW_y|^2)^{1/2}} * \frac{G(s^{-1}.|SW_x|^2)^{1/2}}{G(s^{-1}.|SW_x|^2)^{1/2}} \tag{9}$$

Showing the term $\frac{|G(s^{-1}.SW_{xy})|}{G(s^{-1}.|SW_x|^2)}$ we have:

$$\sqrt(WQ_{\tau,s}) = \frac{|G(s^{-1}.SW_{xy})|}{G(s^{-1}.|SW_x|^2)} * \frac{G(s^{-1}.|SW_x|^2)^{1/2}}{G(s^{-1}.|SW_y|^2)^{1/2}} \tag{10}$$

$$\sqrt(WQ_{\tau,s}) * \frac{G(s^{-1}.|SW_y|^2)^{1/2}}{G(s^{-1}.|SW_x|^2)^{1/2}} = \frac{|G(s^{-1}.SW_{xy})|}{G(s^{-1}.|SW_x|^2)} \tag{11}$$

The right term of this equation is the absolute value of Time-Frequency Beta $|\beta_{\tau,s}|$:

$$|\beta_{\tau,s}| = (WQ_{\tau,s})^{1/2} * \frac{G(s^{-1}.|SW_y|^2)^{1/2}}{G(s^{-1}.|SW_x|^2)^{1/2}} \tag{12}$$

This equation represents the relationship between the Beta and the Coherence like in the case of a simple regression model. The Coherence is weighted by the ratio of the variables standard-deviations.

The formula (12) doesn't provide the Beta sign because it is based on the squared root of the coherence. In order to overcome this problem, we use the wavelets phase function $\theta_{\tau,s}$. This function is the ratio of the imaginary $\Im$ and real $\Re$ part of the cross-spectrum:

$$\theta_{\tau,s} = \arctan(\Im(SW_{xy}(\tau,s))/\Re(SW_{xy}(\tau,s))) \tag{13}$$

The phase function values are between $-\pi$ et π, so we establish a sign parameter $\vartheta_{\tau,s}$ defined as follow:

- If $|\theta_{\tau,s}| \in (0\ ,\ \frac{\pi}{2}) \rightarrow \vartheta_{\tau,s} = 1$ The two chronics are *in phase* and so positively correlated .
- If $|\theta_{\tau,s}| \in (\frac{\pi}{2},\pi) \rightarrow \vartheta_{\tau,s} = -1$ The two chronics are *out of phase* and so negatively correlated.

By including the parameter $\vartheta_{\tau,s}$ in equation 12 we have the definitive formula of $\beta_{\tau,s}$:

$$\beta_{\tau,s} = \vartheta_{\tau,s} * (WQ_{\tau,s})^{1/2} * \frac{G(s^{-1}.|SW_y|^2)^{1/2}}{G(s^{-1}.|SW_x|^2)^{1/2}} \tag{14}$$

3 Results and Discussions

The OLS estimates of Equity-Market relationship are considered as Benchmark by investors. They assume the homogeneous Behavioral hypothesis of agents. In this section, we illustrate the applicability of this formula discussing this hypothesis, and we analyze the wavelet coherence and Time-Frequency Betas of the selected stocks.

3.1 OLS Estimations of Equity-Market Relationship

OLS Estimations of the three market's line need to verify the stationary of the variables (see Appendixe 1). The intercepts estimators are not significant contrary to the Beta estimators. The Beta value indicates the equity risk-profil : AXA is an offensive stock amplifying the market fluctuations because its beta is greater than 1. LVMH has a beta equal to 1 so it follows the market movements in the same proportions. Orange is a defensive stock because it attenuates the market effects (beta lesser than 1). The coefficients of determination indicate that Market Variance explain a relatively high percentage of returns variance. However,residuals of these models are autocorelated, heterescadastic and non-normally distributed. From a statistical point of view, these results are insufficient, the Beta is not BLUE because of it doesn't respect the mimimal variance property.

Moreover, the effects of crises (shocks) on Beta value are ignored. The Equity-Market links (as causality) are supposed stable over time. It is possible to illustrate the high volatility of beta across time by using rolling regression (see bibliography 18), but there is no distinction between short or long-run sensitivities. This method supposes that the short and long run shocks have similar effects, but the frequency aspects of the investment are not considered. These OLS betas are noted BMCO throughout the paper.

To improve this static approach, we use a multidimensional time-frequency analysis of the wavelet coherence and betas to determine the time and frequency dynamics of equity risk-profile.

3.2 Coherence between Equities and Market

Wavelet Coherence (Figures 1) is illustrated by a color system and represents the evolutions of equity-market links in time-frequency space. [5]

- Red represents a strong (high) correlation between the CAC40 and the equity. Correlation level decreases with the color intensity (red,orange, yellow).
- Blue represents a weak (low) correlation between the two variables.

The bold lines delimit the areas for which the correlation (the R^2 in this case) is significant at 5% risk level (calculated with Monte-Carlo Simulations). The white shaded area is the Cone of Influence and it represents edge effects. Because of the finite samples sizes (non-infinite length), edge effects can appear and bias the calculations of cross-power spectra.

[5] The phase is represented by arrows. We use the Package-R Biwavelets of Gouthier, Grinsted and Simko based on Torrence et Compo paper, see references 22

On the following figures 1, the frequency scales (the y-axis) represents the Period (in days) and the Time (in days) starting from 0 (the first observation) until 2869 (the last) is on the x-axis.

We highlight a Red dominance for AXA and LVMH illustrating a high correlation with the CAC40 but blue areas indicate a lesser correlation. Then, the systematic risk represents 80%-100% of total risk but not necessarily at each time t and frequency s. ORANGE coherence is globally dominated by Blue but we note an huge red areas between $1200 - 2000$ days for 0 to 256 periods (in y-axis), and an other area since 2500 days. In this case, the intensity of the Equity-Market relationship is not homogenous over time and frequency. Reb and Blue alternate at high-frequencies illustrating the instability of Equity-Market links for short-run horizon (2 days till 1 week). The Red areas become larger at Low-Frequencies, so the coherence is globally Homogenous for long-run horizon.

Considering these observations, we conclude that the Equity-Market links are time and frequency dependent.Globally, equities with strong Beta (AXA) or unit Beta (LVMH) are highly correlated with the market as opposed to equities with low beta (ORANGE). But these results provide a day-by-day and frequency-by-frequency framework, so the breaks and changes are highlighted as we can see for ORANGE (red area). In this case, the Time-Frequency Betas estimators are required to analyze a varying systematic risk according to the instability of Equity-Market relationship.

Figure 1.1 Coherence between AXA and CAC40

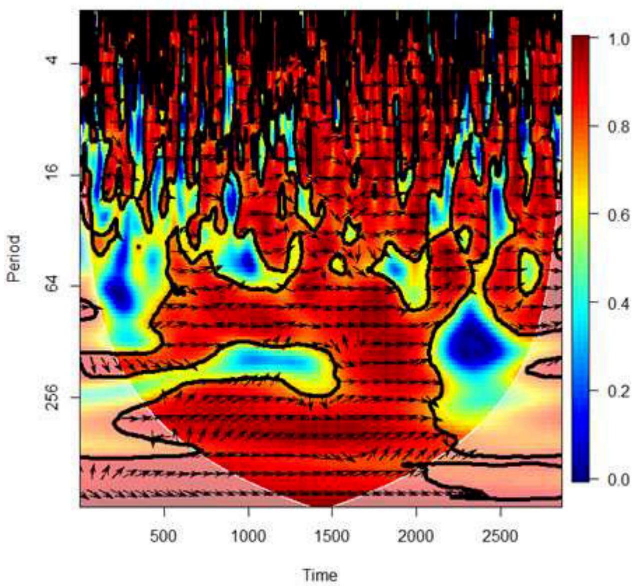


Figure 1.2 Coherence between LVMH and CAC40

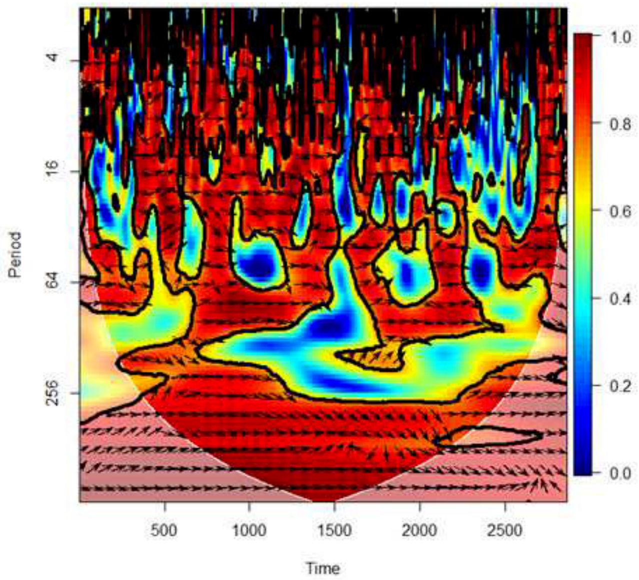
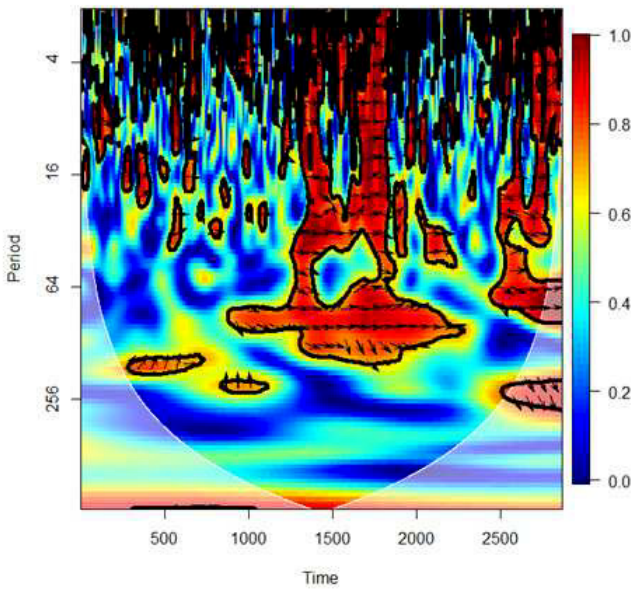


Figure 1.3 Coherence between Orange and CAC40



3.3 Time-Frequency estimates of Beta

The Betas are estimated with a 1/2 frequency-step in order to reduce the computational time but with a relatively fine scale. We have $300,000$ Betas per stocks corresponding to the Time-Frequency evolution of systematic risk. The scales s are linked to a Period of investment representing the time horizon of investors. For each scale s, we have $2,868$ Betas (1 at each time t) describing the time dynamics of systematic risk conditionally to the investors horizons. With this bi-dimensional information, we can analyze both the time and the frequency stability of equity-risk-profile.

To consider heterogeneous behaviours, we select the following frequency scales to represent a particular trading horizon:

- The $s2$ scale illustrates a sine function with a 2-days period in Fourier space so it represents a short-run investment horizon.
- The $s5$ scale illustrates a sine function with a 5-days period in Fourier space so it represents a 1-week investment horizon.
- The $s128$ scale illustrates a sine function with a 128-days period in Fourier space so it represents a 6-months investment horizon.

In order to appreciate the frequency variations of beta, we estimate the Beta by OLS on the three previous scales. We compare their values with the static OLS betas calculated without wavelets (as reminder BMCO). The OLS Beta estimators on scale i, noted Bsi ($Bs2$, $Bs5$ and $Bs128$) are not time varying because their are based on OLS hypothesis.

The Table I summarizes the differences between these two types of Betas.

Table I: (BMCO-Bsi), ($i = 2, 5, 128$).

	s2	s5	s128
AXA	0.07	-0.08	0.34
LVMH	0.03	-0.07	0.22
ORANGE	-0.02	0.06	0.13

The OLS global Beta estimators and the wavelets OLS betas are different confirming the frequency variations of systematic risk. The sign of the difference indicates if the OLS beta is lesser or greater than the wavelets estimators. In absolute value, the differences are minor for short-run investment (scale 2) whereas it is more important for low-frequencies scales (long-run investment) as on $s128$. This simple observation justifies the wavelets using and supporting the heterogeneous behaviors hypothesis during the investment choices. So, we have to complete our study by analyzing the time-frequency Betas dynamics which are useful for investors to appreciate their risks over time and according to their investment horizon.

Figures 2 illustrates the Time-Frequency Betas values of AXA stock on the y-axis and the time on x-axis. The results for the others stocks are recorded in appendixes 2. To improve our analysis, we represent on these figures the different OLS beta estimates. Moreover, we divide the overall period from 2005 to 2015 into sub-periods in order to considerate the crisis effects.

- The ante-subprimes crisis period from 2005 to 2006.
- The Subprimes Crisis Period 2007 – 2009.
- The Debt Crisis in Europe 2010 – 2012.
- The post-crisis starting in 2013.

These periods are represented on Figures 2 by vertical lines.

Figure 2.1 Time-Frequency Betas of AXA 2 days

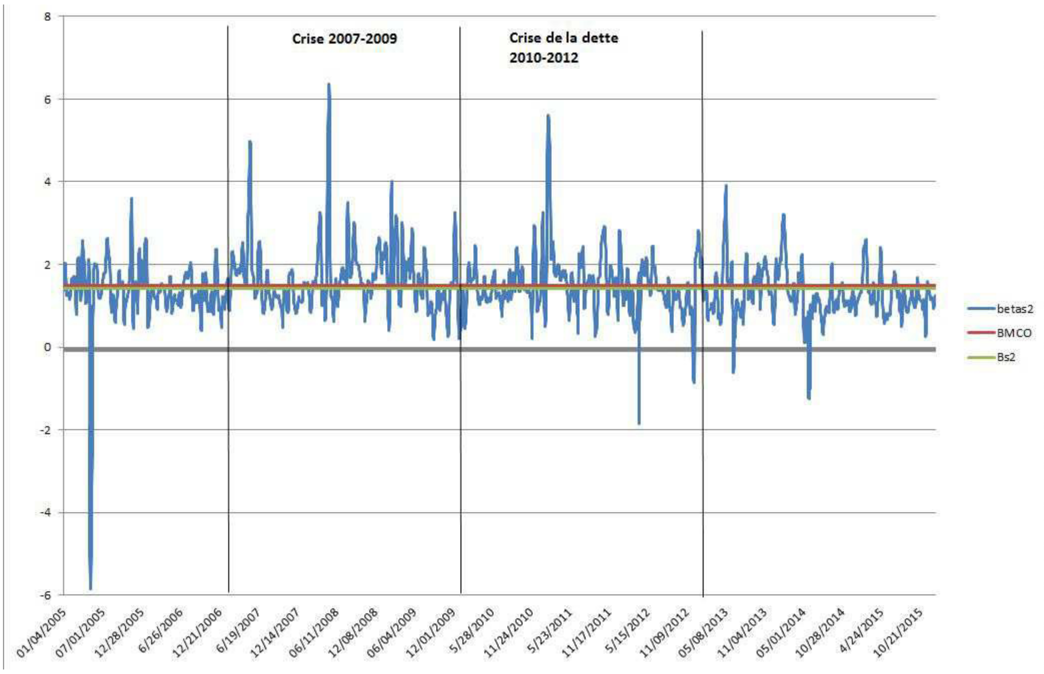


Figure 2.2 Time-Frequency Betas of AXA 1 week

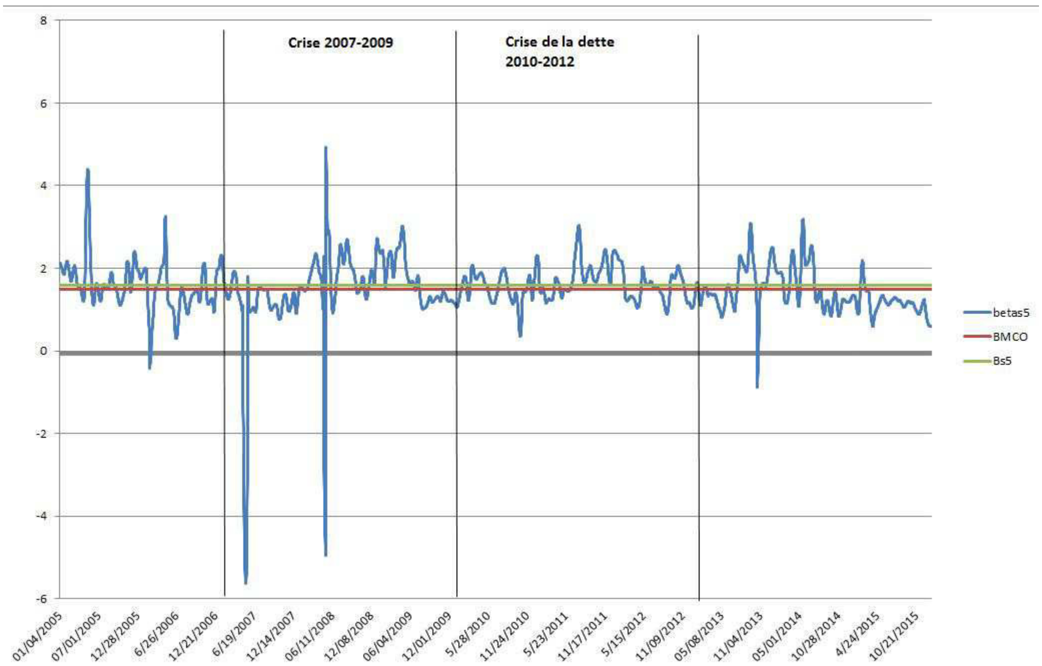
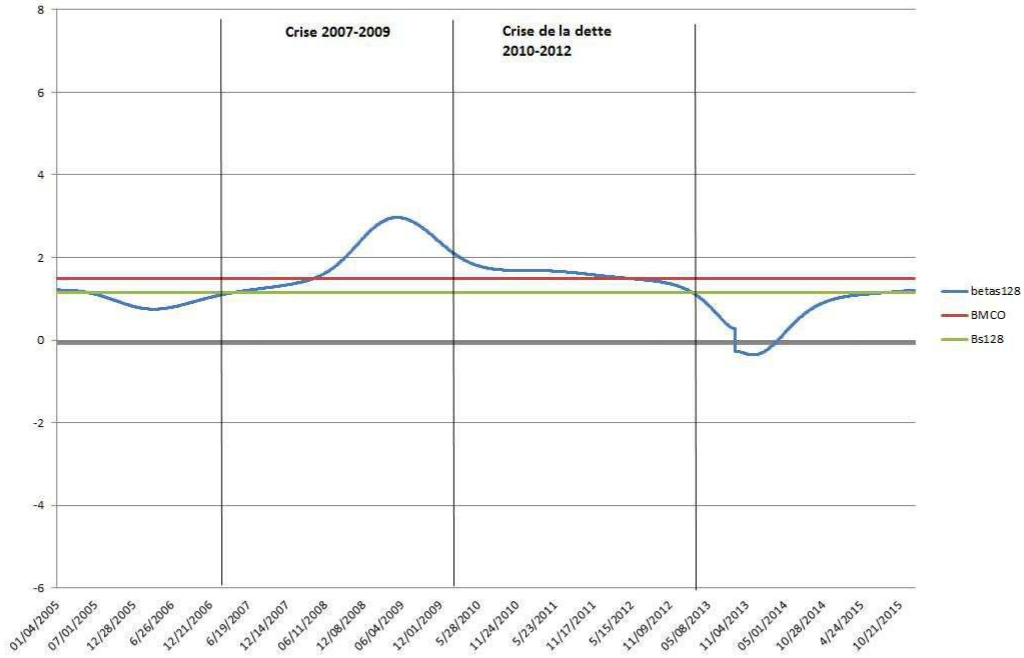


Figure 2.3 Time-Frequency Betas of AXA 6 months



The time stability of betas (the robustness) is ensured if betas values are not significantly different over time. But graphically, we observe the lesser or greater volatility of betas across time and across frequencies. We calculate the mean and the standard-deviation of Time-Frequency Betas on the four sub-periods in order to synthesize these graphs and simplify interpretation. Results are in Table II.

Table II : Means and Standard-Deviation of Time-Frequency Betas by period

		2005-2006 Mean-Sdev	2007-2009 Mean-Sdev	2010-2012 Mean-Sdev	2013-2015 Mean-Sdev	Overall Period 2005-2015 Mean-Sdev
Axa	s2	1.32 0.95	1.66 0.8	1.5 0.71	1.3 0.61	1.46 0.78
	s5	1.59 0.62	1.48 1.12	1.62 0.41	1.44 0.52	1.52 0.73
	s128	0.96 0.16	1.92 0.68	1.64 0.18	0.67 0.53	1.33 0.69
LVMH	s2	1.11 0.9	1.04 0.53	1.18 0.57	1.14 0.66	1.12 0.61
	s5	1.5 0.51	1.12 .39	1.1 0.38	1.02 0.66	1.15 0.52
	s128	1 0.08	0.88 0.13	0.78 0.12	1.15 0.14	0.95 0.19
Orange	s2	1.11 0.91	0.75 0.54	0.85 0.49	1.4 1.01	1.02 0.8
	s5	0.97 0.66	0.78 0.61	0.75 0.3	1.23 0.83	0.93 0.66
	s128	0.79 0.069	0.67 0.22	1.15 0.17	1.04 1.34	0.92 0.74

The Time-Frequency Betas of the three equities have a relatively high and erratic volatility at the high-frequencies as opposed to the low-frequencies where the evolutions seem smoother. This fact is related to the coherence values: strong for AXA and LVMH and weak for ORANGE. The Equity-Market relationship seems more stable at low-frequencies for AXA and LVMH because the strong correlation areas are wider. Consequently, the Betas are more robust and less volatile compare to High-Frequencies.

The crisis effects on Time-Frequency Betas evolutions are especially noticeable for the AXA stock: the Beta mean increases significantly at high-frequencies during the two crisis periods. Then, the aggressive risk-profile of AXA is challenged outside crisis periods. At the opposite, ORANGE and LVMH profiles have contrary evolutions. For LVMH, crisis effects on its short-run Betas are tenuous. For ORANGE, we notice similar evolutions at short-run but the long-run volatility is explosive, so the risk-profile is non-robust.

4 Conclusion

In this paper, we establish a new method to estimate time-frequency varying parameters based on CWT. We derive a Beta formula from the wavelet-coherence providing the value of the coefficient at each time t and each frequency s. We use the wavelet-phase to determine its sign. The CAPM model is used as an example to emphasize the usefulness and the interest of the method for portfolio managers.

The 3 selected stocks illustrate the results obtained with our methodology according to the initial equities risk-profiles.

On the one hand, the stocks have globally high (AXA and LVMH) or low coherence (ORANGE) but not totally homogeneous at each time or frequencies. This result indicates that the intensity of the relationship between variables is time-frequency varying. So, we can suppose that the systematic risk intensity vary over time and frequencies.

On the second hand, we estimate the Time-Frequency varying Betas. By analyzing the time evolutions of betas,we show that initial risk-profiles are not robust over time especially during crisis period. Moreover, we note different frequency dynamics: at short-run ($s2$), the betas are erratic whereas it seems smoother at long-run ($s128$). We can so classify differently the stocks according to their time-frequency characteristics.

With this methodology, portfolio managers can appreciate time changes in systematic risk according to its investments horizon. And also, they can measure the crisis impact on the risk, the volatility intensity or find the generating process of beta in forecasting view. Finally, they can optimize and adjust their decisions (choices).

This methodology can be applied for any equities or for any regression model to appreciate dynamically the parameter value with frequency aspects.

5 Appendixes

A1-Tests on variables

Phillips-Perron Stationary test on equities returns

Stokcs	Test Value	Critical Value at 1%
CAC	-56.11	-3.96
AXA	-51.22	-3.96
LVMH	-55.7	-3.96
Orange	-54.42	-3.96

Tests on residuals

Stocks	Beta	T-Stat	Constant	T-Stat	R^2	LB	ARCH	JB
AXA	1,5	31.74	4.25E-04	1.54	0.68	21.07	62.13	41993.2
LVMH	1	36.81	3.48E-04	1.68	0.62	13.24	38.34	10867.6
Orange	0.73	18.83	3.5E-05	0.157	0.43	17.7	37.81	4480.43

At 5% risk level, LB is Ljung-Bpx Test: $\chi^2(5) = 11.1$, ARCH is ARCH-LM Test: $\chi^2(2) = 5.99$, JB is Jarque-Bera Test): $\chi^2(2) = 5.99$.

Figure $A2.1$ Time-Frequency Betas of LVMH 2 days

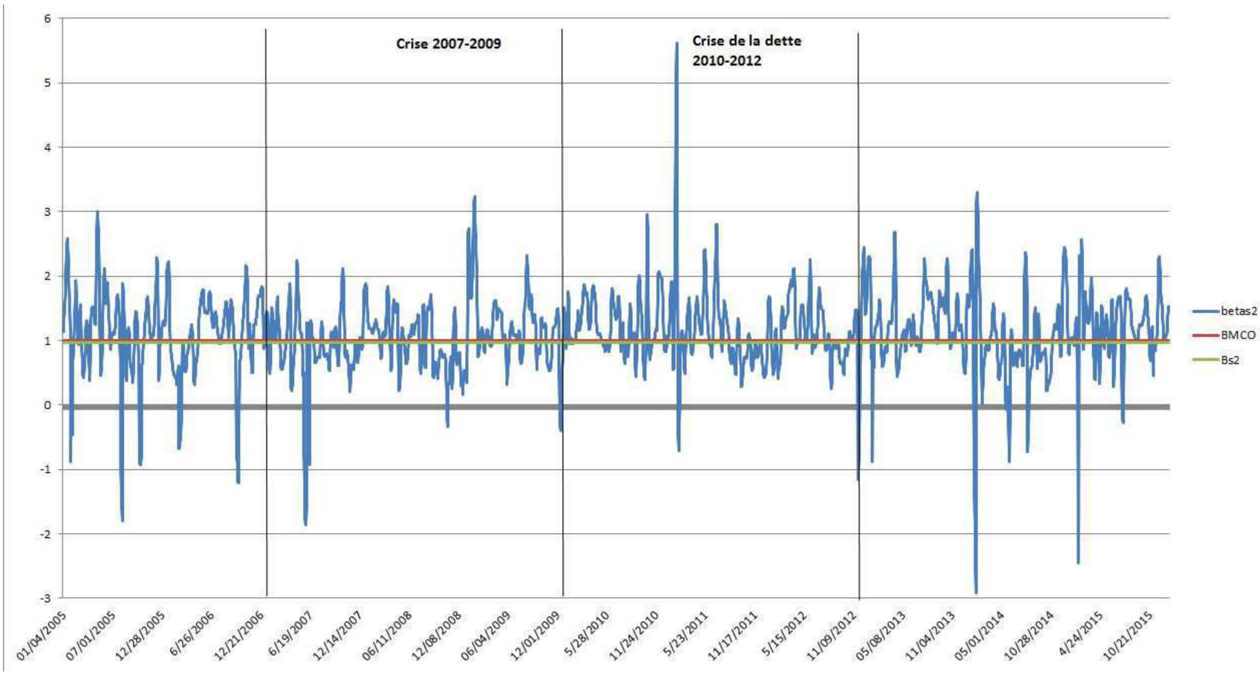


Figure $A2.2$ Time-Frequency Betas of LVMH 1 week

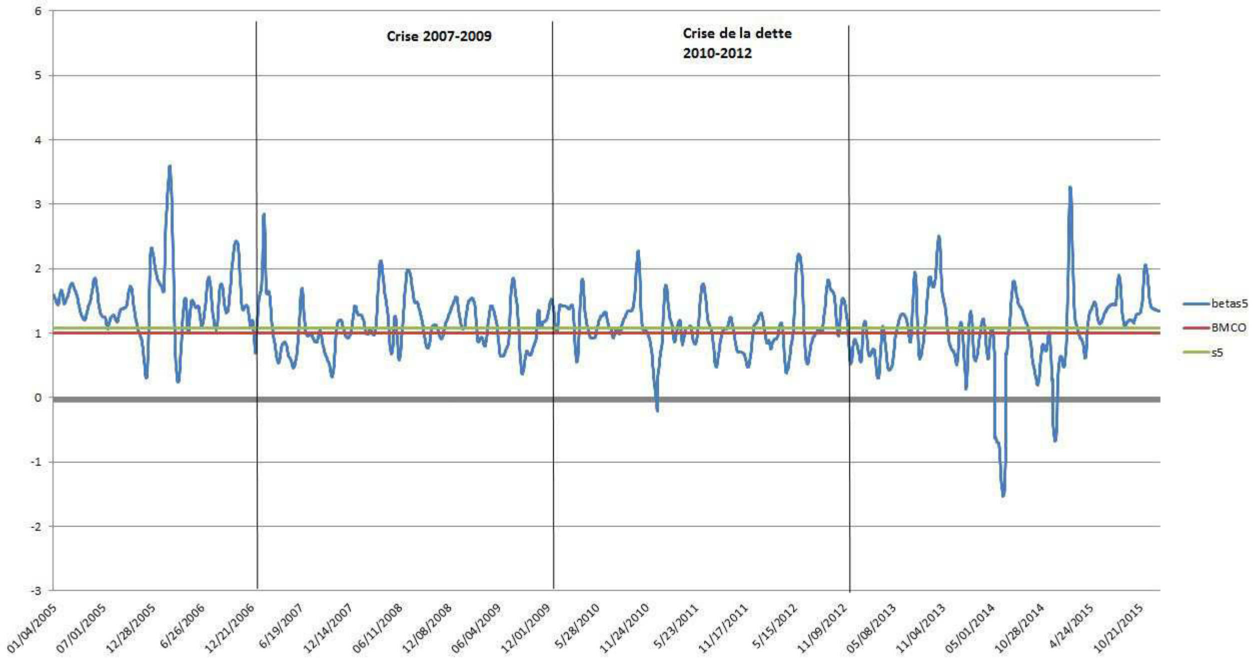


Figure $A2.3$ Time-Frequency Betas of LVMH 6 months

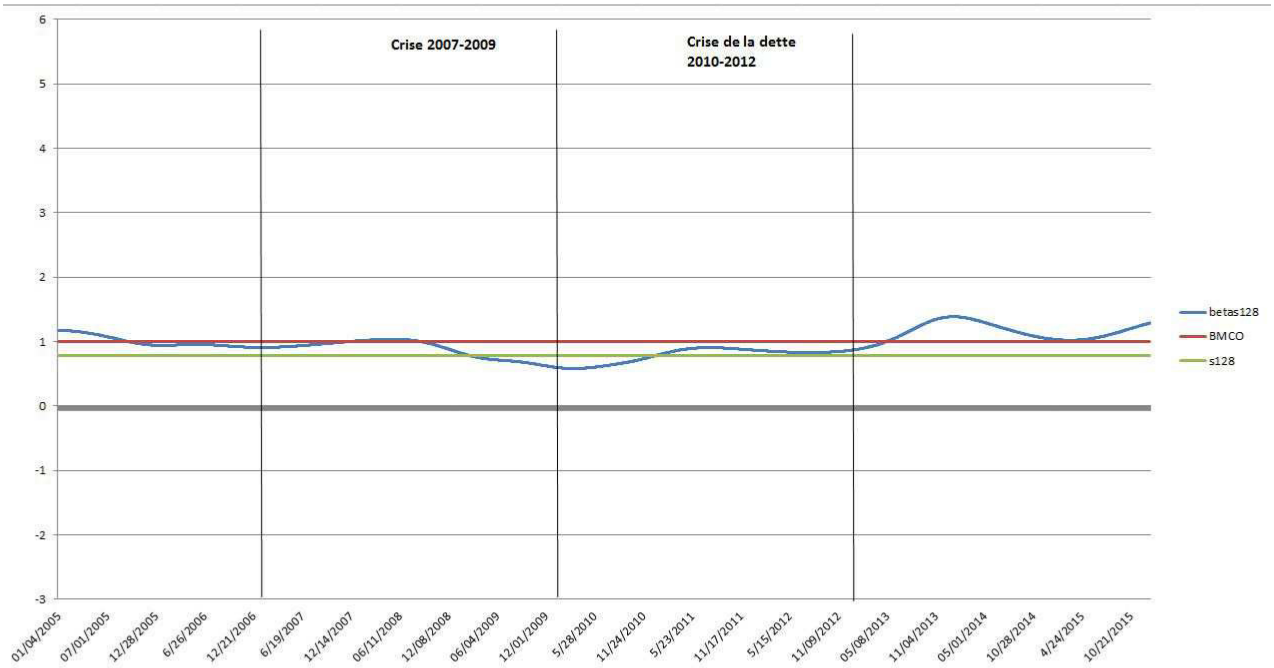


Figure $A2.4$ Time-Frequency Betas of Orange 2 days

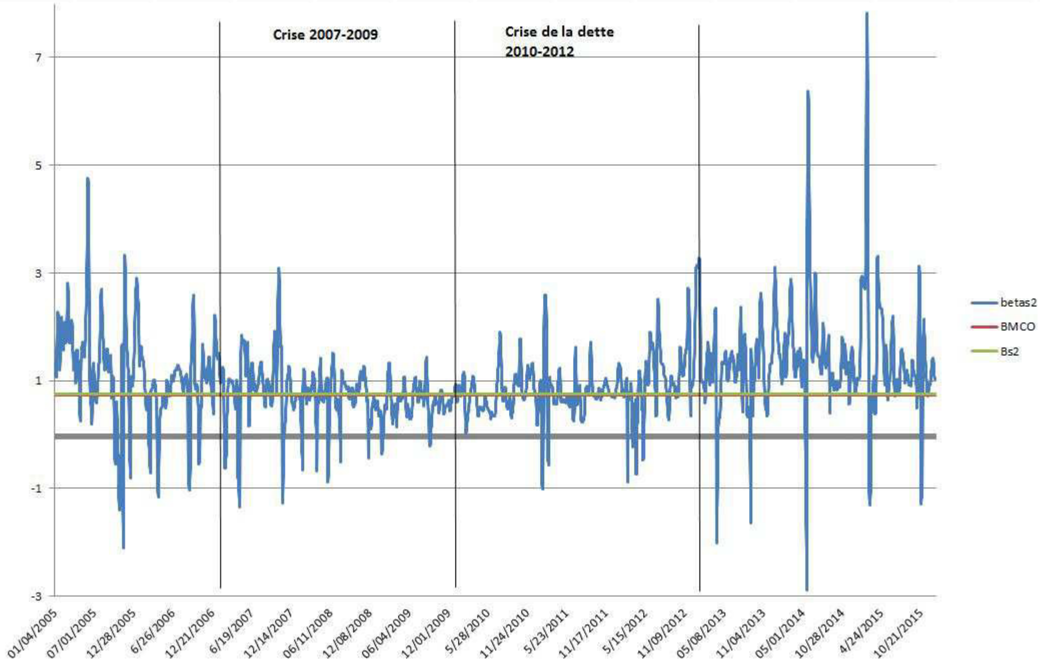


Figure 1: Figure $A2.5$ Time-Frequency Betas of Orange 1 week

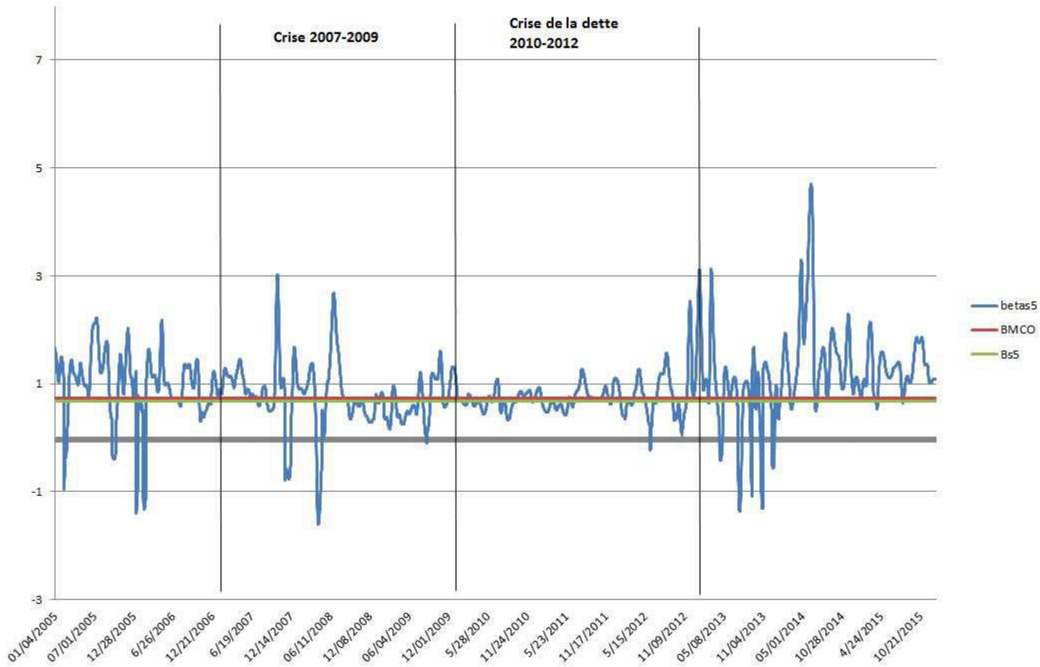
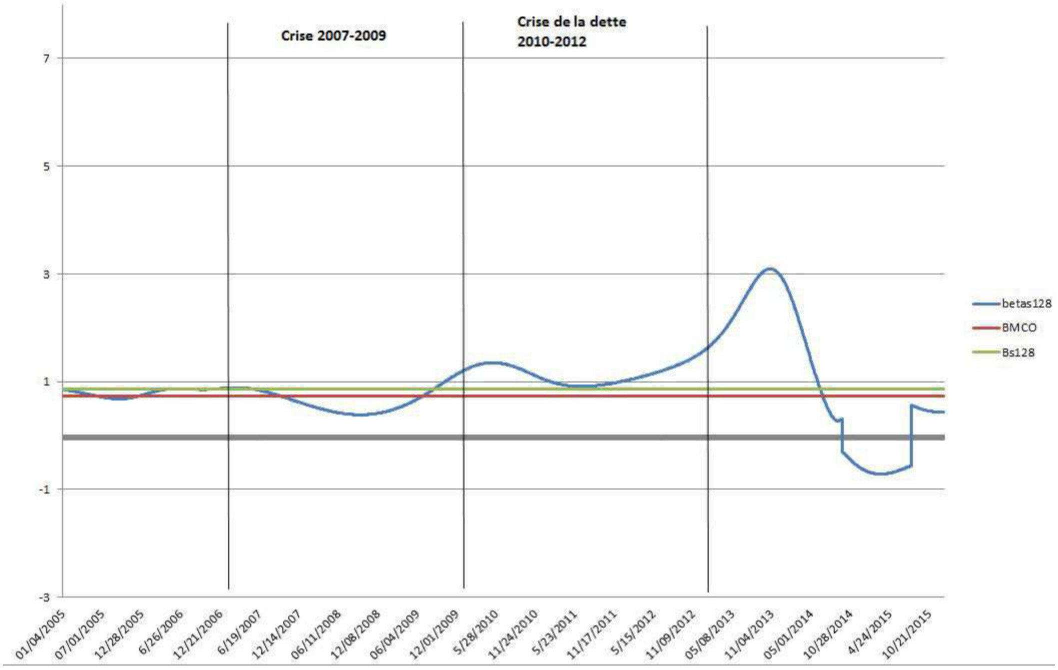


Figure $A2.6$ Time-Frequency Betas of Orange 6 months



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