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Overview on the bending effects distorting axisymmetric capillary bridges.

G rard Gagneux¹ and Olivier Millet²

LaSIE, UMR-CNRS 7356, Universit  de La Rochelle, avenue Michel Cr peau,
17042 La Rochelle cedex 1, France.

Abstract It is well known that the constant mean curvature surfaces, highly studied, are obtained by minimizing the only surface tension energy at fixed volume, the constant corresponding to the Lagrange multiplier. Implicitly, this means that the Gaussian curvature and the total curvature are not taken into account and that therefore, the bending energy is disregarded or *a priori* considered as having negligible effects compared to the effects of surface tension. Admittedly, the spherical or distorted water drops, freely evolving in the air, exactly agree with this simplifying method. Indeed, under boundaryless manifold condition, the most common formulation of the Gauss-Bonnet integration theorem indicates that for a spherical drop or a soap bubble without contact, with or without bump, the integral of the Gaussian curvature over the surface, proportional to the bending energy, is invariant if one bends and deforms the surface (this value is a topological invariant). However, this result is not valid when the drop is subjected to contact boundary conditions. Therefore, because of the constrained boundaries, the study of capillary bridges requires to take into account the Gaussian curvature of the surface and the total geodesic curvature of the boundaries. We study here, theoretically, the real effects of bending forces and therefore of the Gaussian curvature in order to identify a discriminating criterion according to the sign of the Gaussian curvature. Concerning the bending effects, the true shape of the static bridge surfaces is here described by parametric equations, generalizing Delaunay formulas. The related generalized Young-Laplace boundary value system is then solved as an inverse problem from experimental data for the unknown parameters identification. The subject of this study presents strong correspondences with the Gullstrand equation of geometrical optics, with also the gravitational bending angle of light for finite distance and in geometrical approaches to gravitational lensing theory in the astrophysical context.

Keywords: Distortion of capillary bridges · Mean and Gaussian curvatures impact · Generalized Young-Laplace equation · Bending effects · Gauss-Bonnet Theorem.

MSC: 49N45 · 53A10 · 58E12 · 74F10 · 74G05 · 74G15

¹Corresponding author. Email address : gerard.maryse.gagneux@gmail.com

²Email address : olivier.millet@univ-lr.fr

1 Terminology, scope and focus of the topic

1.1 Some liminary considerations on the subject and the various formulations of the Gauss-Bonnet theorem.

This study is underpinned by the following question: why studies on capillary bridges often seem to ignore the role of Gaussian curvature and therefore of bending effects? In addition, the local shape of a surface cannot generally be described by a single scalar. It is well known that the principal curvatures are intrinsically the two eigenvalues of the shape operator, the Gauss curvature being its determinant and the mean curvature is its trace [5].

Ignoring the Gaussian curvature amounts to misregarding the role of the local geometry of the free lateral surface in the distribution of pressures, that is a very rough method in various geometrical situations that we will characterize and which are very common in experimental practice.

We assume that the shape of the capillary surfaces remains axisymmetric in the deformations.

Let's start with some terminology elements.

Synclastic surfaces are those in which the centres of curvature are on the same side of the surface (dome-shape or elliptic surface). The Gaussian curvature is everywhere strictly positive; for examples among the Delaunay constant mean curvature surfaces of revolution (see *in* [8] a synoptic table for identifying the capillary bridges of revolution) : a portion of unduloid, catenoid or nodoid with concave upper meridian, the axis of the bridge being horizontal.

This is opposed to **antyclastic surfaces**, which are those in which the centres of curvature are located on opposing sides of the surface (saddle shape or hyperbolic surface for the confined liquid). The Gaussian curvature is then everywhere strictly negative; for example: a portion of unduloid, catenoid, nodoid, or sphere with convex upper meridian, the axis of the bridge being horizontal. It is then not mathematically correct to say without further information that a nodoid is an anticyclastic surface.

The Gaussian curvature of a right circular cylinder is everywhere equal to zero and therefore accordingly, this case is outside the scope of this study.

In the case of minimal surfaces, necessarily anticyclastic surfaces, such as catenoids with upper convex meridian, where the mean curvature is zero, the strictly negative Gaussian curvature has the specificity of being the **only contribution** of the curvature tensor to the energy formulation; therefore, the Gaussian curvature plays a critical role in the determination of the equilibrium state of this capillary system.

According to the Gauss-Bonnet integration theorem³, for a closed free surface, the energy contribution of the Gaussian curvature during deformation is constant (as long as the topol-

³For any compact, boundaryless two-dimensional Riemannian manifold Σ , the integral of the Gaussian curvature K over the entire manifold with respect to area measure is 2π times the Euler characteristic of Σ , also called the Euler number of the manifold, *i.e.*

$$\int_{\Sigma} K d\Sigma = 2\pi\chi(\Sigma).$$

For example, for a sphere Σ of radius R in \mathcal{R}^3 , it comes:

$$\int_{\Sigma} K d\Sigma = \frac{1}{R^2} 4\pi R^2 = 4\pi \quad \text{and here } \chi(\Sigma) = 2.$$

ogy of the surface, described by the Euler characteristic of the surface, does not change [35]) and can be ignored when determining the shape of such a membrane. This has probably favored the in-depth study of constant mean curvature surfaces, excluding gravity effects [7][15][28][29]. In the case of capillary bridges, the presence of contact surfaces does not allow this simplification (the total geodesic curvature of the boundary is to be taken into account to apply the Gauss-Bonnet integration theorem⁴) and makes it *a priori* necessary to take into account the Gaussian curvature, to establish a hierarchy of the various configurations with regard to the bending effects and to introduce the generalized Young-Laplace equation.

By way of illustration, we indicate that the geodesic curvature of a circle of radius r on the sphere of radius R is given by the formula

$$k_g = \frac{\sqrt{R^2 - r^2}}{Rr} = \frac{\cot \theta}{R}$$

where θ is the colatitude.

This study proposes a resolutely theoretical contribution to the problem of the various distortions affecting axisymmetric capillary bridges in order to establish a structured and practical framework for experimentation and numerical approach [16]. We deduce a clear hierarchization of effects between various reference configurations. This requires introducing some notions of differential geometry.

1.1.1 The original aspects of this theoretical approach.

A recognition problem.

The second aspect, in practice covering a wide area, concerns the strong distortions for which the bending effects, bending the interface⁵, are modeled by an additional curvature-related term: the introduction of C_K , a multiplier coefficient of the Gaussian curvature K , at the dimension of a force and standing for the bending stress. Under appropriate boundary conditions, the shape of an interface between two non-solid substances is then described by the so-called generalized Young-Laplace equation, thus involving both mean and Gaussian curvatures. Indeed structurally analogous to the Gullstrand equation of geometrical optics, the resulting equation, at the downward vertical measurement x linked to the value Δp_0 at $x = 0$, comes in the following form [3], [12], [27] that will be detailed more specifically:

$$\gamma \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + C_K \frac{1}{\rho_c N} = \Delta p_0 - \Delta \rho g x ,$$

where the force C_K divided by the area $\rho_c N$ stands for the bending stress, ρ_c and N for the principal radii of curvature (evaluated algebraically, positively when the curvature is

Suppose M is a compact two-dimensional Riemannian manifold with boundary δM and let k_g the signed geodesic curvature of δM . Then

$$\int_M K dM + \int_{\delta M} k_g ds = 2\pi\chi(M).$$

The geodesic curvature k_g of an arbitrary curve at a point P on a surface is defined as the curvature of the projection of the curve onto a plane tangent to the surface at P .

⁴We should write in all fairness Gauss-Binet-Bonnet theorem (1848).

⁵Bending the interface, *i.e.* changing its curvature, in a first approach.

turned into the interior of the capillary bridge) and the pressure deficiency is Δp_0 at $x = 0$. Thenceforth, a major difficulty is to estimate the influence of C_K on two determinant data: the modified contact angles and the spontaneous curvature $\frac{\Delta p_0}{\gamma}$ at $x = 0$ after distortion. It is also reported [31] that in electro-capillarity, at the nanoscale, the presence of electric fields leads to an extra stress term to be added in the Young-Laplace equation.

As expected, the length $\frac{|C_K|}{\gamma}$ that occurs in the exact formulas that we will establish allows in a certain way to assess the relative importance of bending effects (**a typical scale of tension versus bending**). In particular, for this purpose, the smallness or not of the dimensionless number $\frac{|C_K|}{2\gamma Y^*}$ appears significant, Y^* being the gorge radius of the distorted bridge. In the form $\frac{\pi |C_K|}{2\pi\gamma Y^*}$, this number appears as the quotient of the contributions of the bending and liquid surface tension forces at the distorted bridge neck. Strictly speaking, the formulas obtained retain the value $\frac{\pi |C_K|}{2\pi\gamma Y^* - \Delta p_0 \pi Y^{*2}}$ as the most accurate criterion, taking then into account the contribution of the hydrostatic pressure.

It will also be noted that, in respect of certain theoretical issues, a capillary bridge may be considered as an optic system because it is composed of two interfaces [27].

The main tool is to highlight, in both cases, an exact first integral for the Young-Laplace equations, classical or generalized. These relationships, which are actually total energy conservation laws, are taken advantage of to obtain the theoretical expression of the varying inter-particle force, quantified effects of flexural strength. When considering the only bending effects, the method allows to easily obtain **a parameterization of the profile** by generalizing together a Delaunay formula related to constant mean curvature surfaces [6], p.313 and the resolution method of the Young-Laplace equation as an inverse problem developed in [8].

1.1.2 Synclastic versus anticlastic surfaces

As expected *a priori*, under the condition: $\frac{1}{\rho_c} + \frac{1}{N} \neq 0$, excluding temporarily the minimal surfaces, the quotients

$$\left| \frac{\frac{1}{\rho_c N}}{\frac{1}{\rho_c} + \frac{1}{N}} \right| \text{ i.e. } \frac{1}{|\rho_c + N|} \text{ and the scalar } \frac{|C_K|}{\gamma} \frac{1}{|\rho_c + N|}$$

that occur in the generalized Young-Laplace equation allow also in a certain way to assess the relative importance of bending effects. It is then a question of studying the variations of the function $(\rho_c, N) \rightarrow |\rho_c + N|$.

By placing oneself out of gravity for a simple illustration, it appears that, as expected by experience, the bending is to be considered in the case of the **anticlastic⁶ surfaces** and are of little importance for **synclastic surfaces**. Hence, the common horizontal axis nodoid **with convex upper meridian** is certainly sensitive to bending effects.

⁶etymologically: opposed to being broken into pieces

The Gaussian curvature of a right circular cylinder is everywhere equal to zero and therefore accordingly, this case is outside the scope of this study.

In the case of minimal surfaces such as catenoids⁷ where the mean curvature is zero, the Gaussian curvature has the specificity of being determined by the direct relationship at established equilibrium:

$$\frac{1}{\rho_c N} = -\frac{1}{\rho_c^2} = \frac{1}{C_K} \Delta p_0,$$

Δp_0 being here an unknown spontaneous value to be identified by the data of an additional boundary condition. This implicit unknown value, *a priori* non-zero, would highlight the significant interest in considering the bending effects after experimental verifications.

1.2 Analytical evaluation of capillary distortions by weak gravity effects.

As a benchmark to be used for comparative purposes, consider, first in the classical theory, an axisymmetric vertical liquid bridge (*i.e.* the x -axis is vertical and Δp_0 is the pressure difference through the interface at the neck level $x = 0$). I is an open interval on which we can define by Cartesian representation, say $x \rightarrow y(x)$, a portion of the Delaunay roulette strictly containing the convex profile of the bridge considered without taking into account the gravity (a zero or low gravity environment) [8]. So the shear stress is zero in the y direction and at first, we place ourselves in the relevant cases in which $y''(0) > 0$.

Taking then, if necessary [10], [13], [21] into account the effects of gravity, *via* an over-pressure, results conventionally in the modified nonlinear differential equation for the distorted profile $x \rightarrow Y(x)$, according to the volumic mass densities difference between the liquid and the surrounding fluid

$$\Delta\rho = \rho_{int} - \rho_{ext}$$

a quantitated balance between the surface tension and gravity forces:

$$\frac{Y''(x)}{(1 + Y'^2(x))^{3/2}} - \frac{1}{Y(x) \sqrt{1 + Y'^2(x)}} = -\frac{\Delta p_0}{\gamma} + \frac{g \Delta\rho}{\gamma} x \quad (1)$$

$$=: H + Bx, \quad x \in I.$$

As the only parameter of the disturbance, such an apparent density formula $\Delta\rho = \rho_{int} - \rho_{ext}$ can be discussed but, as a first approximation, remains conventional although the bridge fluid is not completely embedded in the surrounding fluid as for a wall-bound pendant drop without frictional contact constraints on the low boundary, possibly strongly distorting⁸. In continuum mechanics, this equation is obtained in the absence of motion when gravity is the only body force present. It is counterintuitive that the sign and the order of magnitude of the Gaussian curvature do not come into consideration for defining the distorted shape of the free capillary surface. This implicitly assumes that bending effects are neglected and that we are *de facto* limited here to studying rather moderate distortions. It is presumably a question of finding a balance between what is too simple and therefore necessarily false and

⁷surface with strictly negative Gaussian curvature and therefore, *a priori*, really subject to bending effects.

⁸See in [7], p.780, figure 15, a stable pendent water drop in a bath of castor oil exhibiting inflection on the profile, also neck and bulge (artificial low gravity: $\Delta\rho = 39 \text{ kg/m}^3$).

the too complicated, unusable in the context of an experimental validation. By repeating the terms of [12], p. 33, "it is apparent that the free energy representation appropriate for nonmoderately curved capillary systems produces a modified form of the Laplace equation, containing the bending moment and the Gaussian curvature". The question of bending and its impact on the deformation will be thoroughly discussed below.

1.3 An exact energy invariant related to distorted bridges with neck, influenced by Earth gravity

1.3.1 A first integral of the (H, B) –Young-Laplace equation

We still find ourselves in the framework and notations of the previous subsection, concerning essentially any bridge with strictly negative Gaussian curvature K (the product of the two principal curvatures). The free surface is then saddle shaped, precisely the case mainly concerned by the bending effects.

It is particularly proposed to provide a theoretical justification for an extension of the conventional gorge method in order to evaluate the interparticle capillary force under gravitational perturbation at the neck level as a special case of an energy conservation principle. Unlike the situation of axisymmetric bridges with constant mean curvature, the capillary force is no longer constant at all points of the distorted profile. The analytic expression of the interparticle force $\mathcal{F}_{cap}(x)$ is given with exactness at the generic level x ; it can be used by direct calculation from observed data and takes into account the gravitational forces versus the upward buoyancy forces.

For other approaches, we can consult the Russian authors about low-gravity fluid mechanics [22].

First, we introduce (X^*, Y^*) the coordinates of the moved neck (*i.e.* the point such that $Y(X^*) = Y^*$, $Y'(X^*) = 0$) and the two branches x^+ and x^- of Y^{-1} in the set-theoretical sense, respectively defined on $\{x \geq X^*\}$ and $\{x \leq X^*\}$, subsets of the vertical x - axis, with the convention that $x^+(Y) \geq 0$ if $Y \geq Y^*$, $x^+(Y^*) = 0$ in the Cartesian coordinate system linked to the neck level of the distorted bridge.

Keep in mind that the capillary bridge profile loses its symmetry: the gravitational perturbation modifies the localization of the contact points and hence, also the domain of definition for the modified nonlinear differential Young-Laplace equation; the associated boundary value problem does not admit locally symmetric solutions that are physically relevant.

According to [9], the mass of water is displaced toward the lower solid and in consequence, the real gorge radius is lower, *i.e.* more precisely,

$$X^* < 0, Y^* < y^* \quad \text{since } Y'(0) \approx B y^{*2} \text{ is strictly positive.}$$

Moreover, the upper boundary of the liquid thus slides over a wetted part of the solid, while the lower part spreads over a dry part, which should substantially affect the resulting values of the wetting angles; as is well known, the observed contact angle hysteresis depends on whether the liquid is advancing or receding on the surface. Let us add that the capillary phenomena are known to be highly sensitive to all types of microscopic non-uniformity (canthotaxis effects).

The **main result** is stated as follows:

Result 1

Whatever the shape taken by the distorted axisymmetric bridge, we have in relation to the case where the effects of gravity are neglected, the following relationship which is a generalization of an energy conservation principle: along each concerned branch of the profile, the two following functional expressions are **constant and equal**, at the dimension of a force.

For any $x \geq X^*$, H being rigorously evaluated as the mean curvature at the neck of the distorted bridge, we have:

$$\mathcal{F}^+ = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right)$$

and, if $x \leq X^*$,

$$\mathcal{F}^- = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B \int_{Y^*}^{Y(x)} x^-(y) y dy \right).$$

Moreover, the common value is

$$\begin{aligned} \mathcal{F}^+ = \mathcal{F}^- &= 2\pi\gamma Y^* + \pi\gamma H Y^{*2} \\ &= -\pi\Delta p_0 Y^{*2} + 2\pi\gamma Y^*, \end{aligned}$$

Δp_0 being evaluated, strictly speaking, at the neck of the distorted bridge.

The demonstration will be established at the end of the next subsection, after highlighting the important practical consequences of this abstract result.

These analytical expressions are generalizations of formulas obtained in [8], equation (14).

1.3.2 The general expression of the interparticle capillary force

The previous relationship opens the way for an **extension of the conventional gorge method** in order to evaluate the interparticle capillary force at the neck level. This result accurately quantifies the effect of gravitational forces only by the impact of the reduced neck diameter. These formulas are independent of the experimental device (horizontal plates, spheres, *etc.*) insofar as the distorted bridge keeps a saddle-shaped free capillary surface.

Indeed, the general analytical expression of the interparticle capillary force at the level x on $\{x \geq X^*\}$ for illustration, H being rigorously evaluated as the mean curvature at the neck of the distorted bridge, is

$$\begin{aligned} \mathcal{F}_{cap}(x) &= 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) \right) \\ &\quad + 2\pi g \Delta\rho \left(\int_{Y^*}^{Y(x)} x^+(y) y dy - \frac{x - X^*}{2} Y^2(x) \right) \end{aligned}$$

and therefore, conjugate with the expression of the first integral, becomes in practice immediately usable, by the various mathematical relationships:

$$\mathcal{F}_{cap}(x) = 2\pi\gamma Y^* + \pi\gamma H Y^{*2} - \pi g \Delta\rho (x - X^*) Y^2(x).$$

In other words, on $\{x \geq X^*\}$, at the generic abscissa x , the influence of gravity is thus explicitly reflected by the weight of the right cylindrical column of radius $Y(x)$, height $(x - X^*)$ and apparent density $\Delta\rho$, but also implicitly by the modified profile representation $x \rightarrow Y(x)$. Consequently,

$$\begin{aligned}\mathcal{F}_{cap}(X^*) &= 2\pi\gamma Y^* + \pi\gamma H Y^{*2}, \\ \mathcal{F}_{cap}(x) &= \mathcal{F}_{cap}(X^*) - \pi g \Delta\rho (x - X^*) Y^2(x), \quad x \geq X^*, \\ \mathcal{F}_{cap}(0) &= \mathcal{F}_{cap}(X^*) + \pi\gamma B X^* Y^2(0), \quad X^* < 0, \\ &\approx \mathcal{F}_{cap}(X^*) + \pi\gamma B X^* y^{*2}.\end{aligned}$$

In addition to affecting the lateral surface area of the bridge (denoted by $|\Sigma(d)|$, d as distorted by gravity) and the corresponding capillary energy, distortion causes a surface energy interchange by the wetting modifications of the solids boundaries. The contact lines are subject to a frictional force generating a dissipation mechanism. Hence, quantities $|\Lambda_i|(d)_{i=1,2}$, the area of the wetted region on solid S_i , are among the unknown of the problem and the capillary surface energy functional at prescribed volume, taking into account the free boundaries contribution, is to be transcribed in the following revised formulation

$$\begin{aligned}\mathcal{E}_{frictional}(d) &= -\gamma(\cos \delta_1(d) |\Lambda_1(d)| + \cos \delta_2(d) |\Lambda_2(d)|), \quad \delta_i(d) \in [0, \pi], \\ \mathcal{E}_{cap}(d) &= \gamma(|\Sigma(d)| - \cos \delta_1(d) |\Lambda_1(d)| - \cos \delta_2(d) |\Lambda_2(d)|),\end{aligned}$$

where $\delta_i(d)$ are the contact angles values, after distortion effects.

In the case of the consideration of bending effects, one must also introduce the expression of the variation of the bending stress *via* the Gaussian curvature K in the integral form

$$\mathcal{E}_{bending\ stress} = C_k \int_{\Sigma} K d\Sigma,$$

expression that can be easily reevaluated by the Gauss-Bonnet theorem and boundary data.

The particular situation of axisymmetric bridges considerably simplifies the calculation of the integral of the geodesic curvature along the boundaries.

By way of example, let us consider the rather academic situation of a bridge M whose free surface is a portion of the sphere of radius R between two parallel plates of distinct physical characteristics. It is *a priori* a toroid with a concave meridian, little concerned by the effects of bending. The boundary consists of two circles C_1 and C_2 of radius R_1 and R_2 . The calculation of the contribution of the boundary geodesic curvature is then presented in the following form

$$\int_{\Sigma} K d\Sigma + \int_{C_1} \frac{\sqrt{R^2 - R_1^2}}{R R_1} ds + \int_{C_2} \frac{\sqrt{R^2 - R_2^2}}{R R_2} ds = 2\pi\chi(M)$$

i.e.

$$\int_{\Sigma} K d\Sigma = 2\pi\chi(M) - \frac{2\pi}{R} \left(\sqrt{R^2 - R_1^2} + \sqrt{R^2 - R_2^2} \right).$$

The more general case of the geodesic curvature of a boundary circle of an axisymmetric free surface with a Delaunay roulette profile comes from classical exercises of differential geometry [4], [5].

Certainly, the marked distortion of a capillary bridge significantly alters its shape, its geometry and stability properties (varying main and Gaussian curvatures) and its resulting capillary and bending forces.

Limiting us to the interesting case of the distortion of a nodoid with a convex meridian in $\{y \geq 0\}$ (*i.e.* intrinsically, with negative Gaussian curvature) and assuming that H varies very little, we observe that the positive capillary force decreases weakly at the moved neck under the effect of gravity, due to the decrease of the gorge radius. At the level x , on $\{x \geq X^*\}$, the force decreases continuously under the effect of the apparent weight of the cylindrical liquid column with radius $Y(x)$ and height $(x - X^*)$.

The integral involved in the expression of \mathcal{F}^+ and \mathcal{F}^- is linked to the external volume of the figure of revolution generated by the rotation about the x - axis of a profile arc from the moved neck, preparatory term to transcribe the Archimedes' principle. Note that this would be the internal volume, counted negatively, for the observation of a hypothetical convex bridge.

Despite appearances, this last expression of \mathcal{F}^+ and $\mathcal{F}^- = 2\pi\gamma Y^* + \pi\gamma H Y^{*2}$ depends on B since (X^*, Y^*) , the coordinates of the neck, depend on B .

The key to understanding how to get this first integral is to rewrite locally the (H, B) -modified nonlinear differential Young-Laplace equation in the following local form, separately in the two branches related to $\{x \geq X^*\}$ and $\{x \leq X^*\}$, **H being rigorously evaluated as the mean curvature at the neck of the distorted bridge:**

$$-\frac{1}{Y} \frac{d}{dY} \frac{Y}{\sqrt{1+Y'^2}} = H + B x^\pm(Y).$$

Hence, by quadrature, quantities \mathcal{F}^+ and \mathcal{F}^- are constant respectively on $\{x \geq X^*\}$ and $\{x \leq X^*\}$; write $\mathcal{F}^+ = C^+$ and $\mathcal{F}^- = C^-$.

A continuity argument at (X^*, Y^*) implies

$$C^+ = C^- = 2\pi\gamma Y^* + \pi\gamma H Y^{*2}.$$

The demonstration lends itself to various easy generalizations, especially when a surface inflection exists and the Gaussian curvature changes sign.

1.3.3 The mean curvature recognition *in situ* at the distorted neck

The visual observation of a distorted capillary bridge allows by some geometric measurements to calculate analytically, at any point, the expression of the mean curvature (here varying, *a priori* unknown according to the formula $H + Bx$ in the Young-Laplace equation, H being a spontaneous value and not a known data).

For this purpose, we will take advantage of the exact first integral.

It is assumed that by a photographic process on the meridian profile, we could measure the coordinates of the neck (X^*, Y^*) , of the lower point (X_l, Y_l) and of the midpoint whose ordinate is $\frac{1}{2}(Y^* + Y_l)$, more precisely the point $(x^\pm(\frac{1}{2}(Y^* + Y_l)), \frac{1}{2}(Y^* + Y_l))$ and ϕ_l ,

the angle made by the tangent vector to the meniscus with the x -axis at the lower point. According to the exact first integral applied on the interval $[Y^*, Y_l]$, we note that

$$Y_l \cos \phi_l + \frac{H}{2} Y_l^2 + B \int_{Y^*}^{Y_l} x^+(y) y dy = Y^* + \frac{H}{2} Y^{*2}.$$

Therefore

$$H = 2 \frac{Y^* - Y_l \cos \phi_l - B \int_{Y^*}^{Y_l} x^+(y) y dy}{Y_l^2 - Y^{*2}}.$$

Simpson's rule will provide an adequate approximation to the exact integral. This is valid for very small "discretizations"; namely, when the distance between the neck and the "lower point" $|Y_l - Y^*| \ll 1$. In this case, it comes more precisely, so immediately usable:

$$\int_{Y^*}^{Y_l} x^+(y) y dy \approx \frac{Y_l - Y^*}{6} \left(X^* Y^* + 2(Y^* + Y_l) x^+ \left(\frac{Y^* + Y_l}{2} \right) + X_l Y_l \right).$$

Note here that the method is easily generalized when subsequently, bending effects will also be considered. The corresponding relationships are then written:

$$Y_l \cos \phi_l + \frac{H}{2} Y_l^2 - \frac{C_K}{2\gamma} \cos^2 \phi_l + B \int_{Y^*}^{Y_l} x^+(y) y dy = Y^* + \frac{H}{2} Y^{*2} - \frac{C_K}{2\gamma}.$$

$$H = 2 \frac{Y^* - Y_l \cos \phi_l - \frac{C_K}{2\gamma} \sin^2 \phi_l - B \int_{Y^*}^{Y_l} x^+(y) y dy}{Y_l^2 - Y^{*2}}.$$

1.4 On the generalized Young-Laplace equation. Effects of the Gaussian curvature at strong distortions

We set out here, for the convenience of the reader and in order to point out various directions of research, a generalization of the Young-Laplace equation presented in 1996 and which may be of particular interest in the case of strong capillary distortions.

The chapter 1 of the book "Applied Surface Thermodynamics" presents a generalized theory of capillarity [12]. The approach is entirely Gibbsian [3], based on the concept of the dividing surface, according to the statement of the authors. However, it extends and differs from the classical theory in that it is not restricted to moderately curved liquid-fluid interfaces by taking into account the bending stress via mechanical curvature potentials along the interface (see also [2], p. 63-64, [25], [30]).

Various independent approaches lead to the generalized Young-Laplace equation involving both the mean curvature and the Gaussian curvature K , two intrinsic curvature measures of the free surface related to the Weingarten map (in particular, bulk thermodynamics, minimization of an appropriate free energy function).

The shape of an interface between two non-solid substances is then described in the zero-gravity case by the **generalized Young–Laplace** equation:

$$\gamma \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + C_K \frac{1}{\rho_c N} = \Delta p$$

where C_K has the dimension of a force and stands for the account taken for the bending stress, ρ_c and N for the principal radii of curvature (evaluated algebraically). The new bending term discriminates the saddle-shaped interfaces (Gaussian curvature $K = \frac{1}{\rho_c N}$ is everywhere negative) and the synclastic interfaces on which Gaussian curvature is everywhere positive (locally convex axisymmetric bridges). A meridional profile of revolution with an inflection point juxtaposes the opposing effects of bending. It must be borne in mind that for very distorted profiles, the surface tension γ may be surface temperature and curvature-dependent, which severely complicates the mathematical treatment. It follows the specified formulation, with suitably dimensioned coefficients c_T and c_J , c_K ([12], p.9, eq. 27; here, interfacial tension equals local Gibbs free energy per non-planar surface area for chemically pure fluids):

$$\gamma = \gamma_0 + c_T T + c_J \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + c_K \frac{1}{\rho_c N} .$$

Taking then into account simultaneously the combined effects of gravity and flexure results in the generalized highly nonlinear differential equation for the distorted profile $x \rightarrow Y(x)$ of axisymmetric capillary systems:

$$\begin{aligned} & \frac{Y''(x)}{(1 + Y'^2(x))^{3/2}} - \frac{1}{Y(x) \sqrt{1 + Y'^2(x)}} - \frac{C_K}{\gamma} \frac{Y''(x)}{Y(x) (1 + Y'^2(x))^2} \\ & = -\frac{\Delta p_0}{\gamma} + \frac{g \Delta \rho}{\gamma} x \quad =: H + Bx \quad , \quad x \in I. \end{aligned}$$

Mathematically isomorphic but with different variables and physical units, this differential equation has the same structure as the **Gullstrand equation** of geometrical optics, which relates the optic power P'_{op} of a thick lens (in dioptres, the reciprocal of the equivalent focal length) to its geometry and the properties of the media. For example, the superficial tension γ is equivalent to the refractivity $\frac{n_1}{n_2} - 1$, where n_i is a refractive index, C_K is analogous to

the expression $-\left(\frac{n_1}{n_2} - 1\right)^2 \frac{n_2}{n_1} d$, d the lens thickness and Δp corresponds to P'_{op} .

Such an analogy is the subject of a thorough analysis in the publication [27] whose title is enlightening: The Young–Laplace equation links capillarity with geometrical optics.

See also [18] about the gravitational bending angle of light for finite distance or [14] in the astrophysical context. In a geometrical approach to gravitational lensing theory, the Gauss–Bonnet theorem is applied to the optical metric of a lens. Among the many related themes, we can also mention the principles of thin plate and shell theories, the depressions at the surface of an elastic spherical shell submitted to external pressure [26], the wrapping of an adhesive sphere with an elastic sheet [17] or the light deflection and the Gauss–Bonnet theorem for a definition of total deflection angle [1].

1.5 A generalization of an exact energy invariant related to strongly distorted bridges with neck

The qualitative results elaborated in the framework of the constant mean curvature theory are essentially based on the existence of an exact invariant (in fact, a first integral for the second order nonlinear differential equation which reveals the conservation of the total energy of the free surface). With minor adaptations, they are immediately applicable to the situation where the Gaussian curvature and bending effects are taken into account. Indeed, as we will see, we still highlight in this case a first integral for the generalized Young-Laplace equation by limiting ourselves to a presentation concerning essentially any bridge with strictly negative Gaussian curvature.

For the spontaneous but a priori unknown value of H , $H = H_B$, the generalized equation can be rewritten, with the previous notations, in the differential form:

$$-\frac{1}{Y} \frac{d}{dY} \left(\frac{Y}{\sqrt{1+Y'^2}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2} \right) = H + Bx^\pm(Y).$$

Hence, along each concerned branch of the strongly distorted profile, the two following functional expressions are constant and equal, at the dimension of a force. Moreover, we have for any $x \geq X^*$, (X^*, Y^*) being the coordinates of the moved neck,

$$\mathcal{F}_{C_K}^+ = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right)$$

and, if $x < X^*$,

$$\mathcal{F}_{C_K}^- = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^-(y) y dy \right).$$

By highlighting a continuous connection at the neck, the common value is

$$\begin{aligned} \mathcal{F}_{C_K}^+ &= \mathcal{F}_{C_K}^- = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2} \\ &= -\pi\Delta p_0 Y^{*2} - \pi C_K + 2\pi\gamma Y^*, \end{aligned}$$

Δp_0 being evaluated, strictly speaking, at the neck of the distorted bridge.

2 The special case of only bending effects

It is interesting to note that when considering the only bending effects (*i.e.* $C_K \neq 0$, $B = 0$), then for any axisymmetric capillary bridge, the interparticle capillary force

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma \left(\frac{Y}{\sqrt{1+Y'^2}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2} + \frac{H}{2} Y^2 \right) \quad (2)$$

is constant at all points of the profile (generalization of the proposition 1 *in* [8] on the conservation of the total energy of the liquid bridge free surface).

The evaluation of $\mathcal{F}_{C_K}^{cap}$ at the gorge radius Y^* leads to, rather than the "gorge method" :

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2}.$$

As explicated at the beginning of the paper, this exact formula allows to assess the relative importance of bending effects.

Of course equivalently, this expression may be evaluated at one or the other triple line. More generally, in fact

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma Y(x) \cos \Theta(x) - \pi C_K \cos^2 \Theta(x) + \pi\gamma H Y^2(x)$$

where $\Theta(x)$ is the easily calculable angle made by the tangent vector to the meniscus with the x -axis at the generic point $(x, Y(x))$.

2.1 Explicit parameterization of the profile

This latter relationship is interesting because it allows to easily obtain a parameterization of the profile *via* Θ by generalizing formulas established in an other way (by pure geometry) by C.H. Delaunay in 1841 [6], p.313. By a direct calculation and solving a quadratic equation if $H \neq 0$, we get at $H > 0$, for example

$$\begin{aligned} Y(\Theta) &= Y\left(\Theta, \frac{C_K}{\gamma}\right), \text{ the positive root such that } Y\left(0, \frac{C_K}{\gamma}\right) = Y^*, \\ &\text{i.e.} \\ Y(\Theta) &= \frac{1}{H} \left(-\cos \Theta + \sqrt{(HY^* + 1)^2 - \left(1 + H \frac{C_K}{\gamma}\right) \sin^2 \Theta} \right), \\ x(\Theta) &= X^* + \int_0^\Theta dY(\theta) \cot(\theta) d\theta, \text{ i.e., } x(\Theta) = X^* + \\ &\frac{1}{H} \left(\sin \Theta + \left(1 + H \frac{C_K}{\gamma}\right) \int_0^\Theta \frac{\cos^2 \theta d\theta}{\sqrt{(HY^* + 1)^2 - \left(1 + H \frac{C_K}{\gamma}\right) \sin^2 \theta}} \right). \end{aligned}$$

It should be kept in mind that the value of the gorge radius Y^* measured by observation *in situ*, and the same goes for H , depends on $\frac{C_K}{\gamma}$. As a practical observation, for the experimenter, the value of the capillary pressure \hat{H} is *a priori* an implicit unknown of the problem (a spontaneous value) whose mode of identification will be detailed below.

Note in particular that for a catenoid

$$Y\left(\Theta, \frac{C_K}{\gamma}\right) = \frac{2Y^* - \frac{C_K}{\gamma} \sin^2 \Theta}{2 \cos \Theta} \quad \text{if } H = 0.$$

If, moreover, $C_K = 0$, then the quantity $Y(\Theta) \cos \Theta$ is proved to be constant, a well-known geometrical property for catenaries).

2.2 Missing data recovery using the first integral

When the three dimensional constant terms $\mathcal{F}_{C_K}^{cap}$, C_K , H are considered as missing values to be restored (*a priori* unknowns of the problem as spontaneous values resulting *a posteriori* from a static equilibrium), the relation (2) makes it possible to obtain the result by solving a linear system. To do this *via* an inverse problem, it suffices to apply the specific invariance relation at three points of the profile (experimental data), taking care to verify that the corresponding determinant of the 3×3 matrix is not close to zero (numerical stability condition). For this purpose, considering, for example, the observation of three distinct points $Y(0) = Y^*$, $Y(\Theta_1) = Y_1$, $Y(\Theta_2) = Y_2$, we note that the explicit determinant of the involved system is expressed, in absolute value, by

$$|\sin^2 \Theta_1 (Y^{*2} - Y_2^2) - \sin^2 \Theta_2 (Y^{*2} - Y_1^2)|.$$

So, in that case, are generalized the analytical calculation methods of distorted capillary bridges properties deduced as an inverse problem from experimental data, developed in [8].

2.3 Gravity and bending effects on the interparticle capillary force

Renewing with the same principles the previous procedures by minor adaptations leads to the general analytical expression of the interparticle capillary force at any level.

Indeed, the interparticle capillary force at the level x , on $\{x \geq X^*\}$ for illustration, can be evaluated as follows:

$$\begin{aligned} \mathcal{F}_{cap, C_K, B}(x) = & 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2} Y^2(x) \right) \\ & + 2\pi\gamma B \left(\int_{Y^*}^{Y(x)} x^+(y) y dy - \frac{x-X^*}{2} Y^2(x) \right) \end{aligned}$$

and therefore, conjugate with the expression of the first integral of the generalized Young-Laplace equation, becomes in practice immediately usable, by the mathematical relationships:

$$\mathcal{F}_{cap, C_K, B}(x) = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2} - \pi\gamma B(x - X^*) Y^2(x),$$

and, in other words, rather than the "gorge method",

$$\mathcal{F}_{cap, C_K, B}(X^*) = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2}.$$

It follows then that concisely, by referring to the moved gorge point,

$$\mathcal{F}_{cap, C_K, B}(x) = \mathcal{F}_{cap, C_K, B}(X^*) - \pi g \Delta \rho (x - X^*) Y^2(x), \quad x \geq X^*.$$

These formulas remain valid on $\{x \leq X^*\}$ insofar as the distorted bridge keeps a saddle-shaped free capillary surface and thus, $x \rightarrow \mathcal{F}_{cap, C_K, B}(x)$ is everywhere a decreasing function. They are independent of the experimental device (horizontal plates, spheres, *etc.*).

It must be kept in mind that the values Y^* , $Y(x)$ on $\{x \geq X^*\}$ and H depend henceforth on the value of the force C_K , besides B and the contact parameters.

This also opens the way to build different numerical schemes of the predictor-corrector type with great flexibility of use for any strongly distorted bridge with strictly negative Gaussian curvature.

2.4 Gravity and bending effects on the strongly distorted profiles

For principal information, it should be noted that the numerical predictor-corrector scheme, previously presented out of the bending context (in the $H - B$ Young-Laplace framework) and corresponding to this more general situation, reveals a new term involving C_K as [9]:

$$\begin{aligned} & \left(\frac{H}{2} + Bh \left(k - \frac{1}{2} \right) \right) Y_k^2 + Y_k \cos \Psi (x_k) - hB \left(\frac{1}{2} Y^{*2} + \sum_{j=1, \dots, k-1} Y_j^2 \right) \\ & = Y^* + \frac{1}{2} H Y^{*2} - \frac{C_K}{2\gamma} \sin^2 \Psi (x_k) \quad , \quad x_k \geq X^* , \end{aligned}$$

(X^*, Y^*) are the coordinates of the moved neck having been observed and measured *in situ* in the simplified case of a direct experience.

In situation of a behavior prediction from the zero-gravity case, the evaluation of the coordinates (X^*, Y^*) as initialization value is more complicated on the basis of the previously developed considerations, reconstructing the shape of the local curvature. We can again approximate (X^*, Y^*) by the coordinates of the point with smallest ordinate on the osculating circle at $(0, Y(0))$ whose radius $R_S(0)$ is given by the following relationship [9], according to the generalized Young-Laplace equation:

$$R_S(0) = \frac{Y(0) \sqrt{1 + Y'^2(0)} - \frac{C_K}{\gamma}}{HY(0) \sqrt{1 + Y'^2(0)} + 1}.$$

The point $(0, y(0))$ can be in first approximation considered as an invariant point of the distortion and therefore $(0, y(0)) = (0, y^*) = (0, Y(0))$. At this point, only vary the local profile shape and the angle formed by the tangent to the distorted meniscus and the x -axis, from value 0 to α_0 , $\alpha_0 > 0$. We can use a successive approximations method (each new approximation is calculated on the basis of the preceding approximation) and the choice of the initial approximation for α_0 is, to some contextual extent, arbitrary but rather to underestimate: it is interesting to build ordered profiles approximations for the sake of convergence of the method (the value α_0 such as $\tan \alpha_0 = B y^{*2}$ if $B^2 y^{*4} \ll 1$ corresponds to a really weak distorsion).

Therefore, taking here as a first approximation the value of H at zero gravity [9],

$$R_S(0) \approx \frac{y^* \cos \alpha_0 - \frac{C_K}{\gamma}}{Hy^* \cos \alpha_0 + 1}$$

$$X^* \approx -R_S(0) \sin \alpha_0$$

$$Y^* \approx y^* - R_S(0) (1 - \cos \alpha_0).$$

By performing then a simple iteration from the generalized Young-Laplace equation, we obtain an estimate of the spontaneous value of H under the formulation:

$$H = H \left(B, \frac{C_K}{\gamma} \right) \approx \frac{1}{R_S(0)} \left(1 - \frac{C_K}{\gamma Y^*} \right) - \frac{1}{Y^*} - B X^*.$$

So, for the profile of the lower part of the distorted bridge, we have just to successively solve quadratic equations in the unknown Y_k , for $k = 1, 2, \dots$, the sequence of positive ordinates $\{Y_k\}_{k \geq 1}$ to be realistically increasing. The sequence $\{Y_k\}$ obtained at the first iteration of the process can serve as a new predictor and so on, which then defines an iterative numerical method by successive approximations and readjustment of the neck position and the H -value at each iteration.

Failing to know a first approximation of the distortion, the configuration of the bridge by neglecting both gravity and flexion effects, when it is known (by a microgravity experiment in parabolic flight for example) can be used to initialize the iterative algorithm, with readjustment of the neck position and the H -value at each iteration.

2.5 Adaptations to the case of strictly positive Gaussian curvature bridges

This rather academic case concerns situations of weak bending effects.

Taking again the previous argument by adapting to the case of strictly positive Gaussian curvatures, it shows that formally, the expression of the interparticle capillary force, *a priori* repulsive, formally keeps the same expression at any level x , on $\{x \geq X^*\}$:

$$\mathcal{F}_{cap, C_K, B}(x) = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2} - \pi g \Delta\rho (x - X^*) Y^2(x).$$

However, it must be kept in mind that the spontaneous and a priori unknown values X^* , Y^* , $Y(x)$ on $\{x \geq X^*\}$ and H (negative) depend on the value of the force C_K , on B and on the Gaussian curvature sign, besides the contact parameters. The physical interpretation of this relationship reminds us of the Pascal's experiment, known as "crève-tonneau" (barrel-buster).

The sign of variations of the interparticle capillary force depends on the expression:

$$\text{sign} \frac{d\mathcal{F}_{cap, C_K, B}}{dx}(x) = -\text{sign} (Y(x) + 2(x - X^*) Y'(x)).$$

In addition, one can still take advantage of the first integral on $\{x \geq X^*\}$, to fix ideas, still valid if $B = 0$,

$$\begin{aligned} 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1 + Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1 + Y'^2(x)} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right) \\ = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2}. \end{aligned}$$

This again opens the way to implement different numerical schemes of the predictor-corrector type with great flexibility of use for any strongly distorted bridge with strictly positive Gaussian curvature.

3 Conclusions

In summary of the key aspects, recall *via* the Gauss-Bonnet theorem, a situation where the issue of the importance of bending effects is irrelevant: for a closed free surface, the

energy contribution of the Gaussian curvature during deformation is constant, as long as the topology of the surface (*i.e.* the Euler characteristic) does not change, and therefore can be ignored when determining the shape of the profile.

The situation is quite different for capillary bridges. The total geodesic curvature of the contact boundaries must be taken into account, mainly for bridges with strictly negative Gaussian curvature.

Concerning the only bending effects in the zero-gravity case, the true shape of the static bridge surfaces is here described by parametric equations, generalizing Delaunay formulas; we have presented a forecasting method for evaluating the axisymmetric capillary distortions, from the zero-gravity case assumed to be known precisely. We took advantage of the knowledge of an exact first integral for the generalized Young-Laplace equation with missing data values and a double estimate of the position of the moved neck and the spontaneous value H after gravity and bending effects. It must of course be kept in mind that conduct such a predictive modeling for the motion of the contact lines by gravity and flexure effects is a problem considerably more difficult than to model the static distorted case, observed *in situ*. The isomorphic structure between the Gullstrand and generalized Young-Laplace equations may be thought to allow experimenters to consider a capillary bridge as an optical system. Although the two physical phenomena seem *a priori* disjoined but intellectually close, it could be deduced new practices for curvature measurements and fast, effective parameters identification. Nevertheless, the combined effect of volume, bending and axial gravity on the axisymmetric liquid bridge stability is a broad research subject to explore. The considerations on the numerical treatment of the distortion problem are given here as an indication of a research direction necessary to the advancement of the topic; this rather new subject could be of interest to specialists in finite elements or spline functions techniques. It would be interesting to reconsider, in taking into account the bending effects and the Gauss-Bonnet theorem, an analytical framework for evaluating the cohesion effects of coalescence between saddle shaped capillary bridges [11].

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