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► **To cite this version:**

Charles Frances. Transformation groups in non-Riemannian geometry. Sophus Lie and Felix Klein: The Erlangen Program and Its Impact in Mathematics and Physics, European Mathematical Society Publishing House, pp.191-216, 2015, <10.4171/148-1/7>. <hal-03195050>

**HAL Id: hal-03195050**

**<https://hal.science/hal-03195050v1>**

Submitted on 10 Apr 2021

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# Transformation groups in non-Riemannian geometry

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2000 Mathematics Subject Classification: 22F50, 53C05, 53C10, 53C50.

Keywords: Transformation groups, rigid geometric structures, Cartan geometries.

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## 1 Introduction

The aim of this chapter is to survey some aspects of the links unifying a geometric structure  $\mathcal{S}$  on a smooth manifold  $M$ , and its group of automorphisms  $\text{Aut}(M, \mathcal{S})$ , namely the group of diffeomorphisms of  $M$  preserving  $\mathcal{S}$ . This domain of research, that could be called theory of geometric transformation groups, takes its roots in the pioneering ideas of S. Lie, and developed all along the past century, to blossom into a vast area where a wide range of mathematics meet: Lie theory of course, but also differential geometry, dynamical systems, geometry of foliations, ergodic theory, algebraic actions, etc.

One of the main motivations to study Lie group actions preserving a geometric structure comes from the fact that for a lot of interesting structures, the automorphism group itself is always a Lie transformation group<sup>1</sup>. What we call “fact” here is actually the result of a series of theorems. One of the earliest is due to S. Myers and N. Steenrod, who proved that the isometry group of a Riemannian manifold is always a Lie group.

**Theorem 1.1.** [31] *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Then the isometry group  $\text{Iso}(M, g)$ , endowed with the compact open topology, is a Lie transformation group of dimension at most  $\frac{n(n+1)}{2}$ . Moreover this group is compact as soon as  $M$  is compact.*

From the proof of the theorem, one gets also a local result, which is that on each open subset  $U \subset M$ , the Lie algebra of Killing fields, namely the vector fields defined on  $U$  and generating local flows of isometries, is finite dimensional. The dimension is again at most  $\frac{n(n+1)}{2}$ .

The geometric structures for which the two following facts always hold:

- (1) the automorphism group is a Lie transformation group;
- (2) the dimension of the Lie algebra of local Killing fields is finite

will be called *rigid* in this paper. Actually, there is a more precise, and very general definition of rigid geometric structures introduced by Gromov in [24], which turns out to imply points (1) and (2) above. To avoid too much technicalities, we will keep our “rough” definition here.

The Myers-Steenrod theorem was followed by many other works which increased the list of geometric structures which are known to be rigid. Thus, pseudo-Riemannian metrics, affine connections, projective structures, conformal structures in dimension  $\geq 3$  turned out to be rigid.

On the other hand, there are structures, like for instance symplectic structures, which are not rigid. Any Hamiltonian flow on a symplectic manifold

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<sup>1</sup>the precise definition of a Lie transformation group will be given at the beginning of Section 2

$(M, \omega)$  acts by symplectomorphisms, so that there are far too much symplectic automorphisms for Conditions (1) and (2) to be satisfied.

Observe also that some structures like complex ones display an intermediate behavior. By a theorem of Bochner and Montgomery, the automorphism group of a compact complex manifold is a Lie transformation group, so that for *compact* structures, Condition (1) is satisfied. Nevertheless, Condition (2) clearly always fails, and we won't retain complex structures as being rigid.

Assume now that we focus on a certain class of geometric structures, which are known to be rigid. For instance, we study Riemannian metrics, or Lorentzian ones, or affine connections. Given a manifold  $M$  endowed with a structure  $\mathcal{S}$  belonging to our given class, it is natural to ask what kind of Lie group  $\text{Aut}(M, \mathcal{S})$  can be. More precisely,

**Question 1.2.** *What are the possible Lie groups that can be the automorphism group  $\text{Aut}(M, \mathcal{S})$  of a structure  $(M, \mathcal{S})$  of the class we are considering, when  $M$  is a compact manifold?*

For instance, Theorem 1.1 says that the isometry group of a compact Riemannian manifold must be a compact Lie group. Conversely, it is shown in [37] that any compact Lie group may be realized as the full isometry group of a compact manifold. Hence, Question 1.2 is completely settled for Riemannian structures on compact manifolds.

Observe that Question 1.2 is meaningless if we don't put any global assumption on  $M$ , like compactness. To see this, let us consider any Lie group  $H$ , and put on its Lie algebra a Euclidean scalar product. Pushing this scalar product by left translations yields a left-invariant Riemannian metric on  $H$ . Thus, without any compactness assumption, we see that any Lie group may appear as a subgroup of isometries of a Riemannian manifold.

Answering Question 1.2 for other structures than Riemannian metrics is generally a hard problem, which is solved completely in very few cases. The subject got a renewed impulse thanks to the very influential works of Gromov and Zimmer in the eighties (see in particular the monumental [24], as well as [51]). We will present in Sections 3 and 4 definitive results on this problem, and partial ones in Section 5.

One of the difficulties to tackle Question 1.2 is that for most rigid geometric structures, there is no analogue of the Myers-Steenrod theorem. One can indeed exhibit instances of compact structures with a noncompact automorphism group. On the other hand, it is expected that such occurrences are rather unusual. A good illustration is the following result, proved by J. Ferrand, and independently (in a weaker form) by M. Obata in the early 1970s.

**Theorem 1.3.** *[32][17] A compact connected Riemannian manifold  $(M, g)$  of dimension  $n \geq 2$  having a noncompact group of conformal diffeomorphisms must be conformally diffeomorphic to the standard sphere  $\mathbf{S}^n$ .*

This result is extremely strong, since the mere assumption of noncompactness of the automorphism group is enough to single out only one space: the model space of compact conformal Riemannian structures.

In the survey [14], it is vaguely conjectured that compact rigid geometric structures with a large automorphism group should be peculiar enough to be classified. Rather than a conjecture, we should speak of a principle, which is indeed illustrated by a lot of beautiful results, as Theorem 1.3. The heuristics is as follows. Rigid geometric structures, *generically*, do not admit any symmetry at all (local or global). The reader could take the example of a “generic” Riemannian metric as an illustration. The presence of *a lot* of symmetries is thus extremely unlikely. At the end of the spectrum, one has the highly nongeneric case of structures which are homogeneous, or locally homogeneous, if we just consider local symmetries. Those structures are few, but generally beautiful and play a prominent role in the theory. We will be especially interested in results showing that rather mild assumptions on the automorphism group force local homogeneity.

Beside noncompactness, other notions of “largeness” for the automorphism group are of interest. For instance, the condition can be put on the action of the group itself. Noncompactness amounts to a nonproperness assumption, but stronger dynamical conditions can be relevant: ergodicity, existence of a dense orbit etc. In this regard, a very striking result was obtained by Gromov in [24]. It is often called *open dense orbit theorem*.

**Theorem 1.4.** [24] *Let  $(M, \mathcal{S})$  be a rigid geometric structure of algebraic type. Assume that the group  $\text{Aut}(M, \mathcal{S})$  acts on  $M$  with a dense orbit. Then there exists a dense open subset  $U \subset M$  which is locally homogeneous.*

We won’t define here what is the meaning of “algebraic type” in the statement above. The theorem applies to all the examples of rigid structures we have been considering so far: pseudo-Riemannian metrics, affine connections, etc. This is again a nice illustration of the principles presented above, even though the conclusion is slightly weaker than the one expected: the local homogeneity of the structure holds only on a dense open set. Passing from local homogeneity on this dense open set to the whole manifold is in general extremely difficult (actually it might be false in full generality), but is possible in certain cases [15], [4].

The organization of the text is as follows. We will first discuss several notions of geometric structures in Section 2, focusing on Cartan geometries and  $G$ -structures of finite type, for which the automorphism group is always a Lie transformation group. In Sections 3 and 4, we will present classification theorems for the automorphism group of conformal Riemannian structures, and Lorentzian metrics. Section 5 will be devoted to other geometric structures, for which the picture is less precise. We will emphasize results of Zimmer

yielding valuable information about the structure of the Lie algebra of automorphisms of  $G$ -structures, as well as some developments in the realm of Cartan geometries, focusing on pseudo-Riemannian conformal structures.

## 2 Rigid geometric structures

Until now, our use of expressions such as “geometric structure” or “rigid structure” was rather informal. In any case, it is always rather arbitrary to adopt a definition of what a geometric structure is. Nevertheless, it is nowadays broadly accepted that a decisive step toward the modern way of viewing geometry was achieved by Klein, who was the first to consider a geometric structure as a manifold  $M$  acted upon (transitively) by a group  $G$ , and the study of all properties which are invariant under the group action. The modern formulation is that of a *Klein geometry*, as a homogeneous space  $G/P$ , where  $G$  is a Lie group, and  $P$  a closed Lie subgroup of  $G$ . Whereas this point of view was extremely successful to unify the “classical” geometries such as Euclidean, hyperbolic, spherical, projective geometries, etc, it was later on considered as too restrictive for a general definition of geometric structure. As the mere example of Riemannian geometry shows, homogeneous structures are clearly the exception, not the rule. Thus, other attempts were made during the twentieth century to find broader definitions of geometric structures. We present briefly two of them below, which will be of interest for our purpose. Especially, those structures will have the remarkable property that *their automorphism group is always a Lie transformation group*.

Recall at this point that if  $M$  is a smooth manifold, a subgroup  $H \subset \text{Diff}^\infty(M)$  is a Lie transformation group if it can be endowed with a Lie group structure for which the action  $H \times M \rightarrow M$  is smooth. Moreover, one requires that any flow of diffeomorphisms included in  $H$  is a 1-parameter group for the Lie group structure of  $H$ .

### 2.1 Cartan geometries

A first broad generalization of Klein’s definition of geometry was introduced by E. Cartan under the name “*espaces généralisés*”. It is actually the most natural extension of Klein’s point of view, since it gives a precise meaning of what a “curved analogue” of a Klein geometry is. To understand the notion of a Cartan geometry, we start with a homogeneous space  $\mathbf{X} = G/P$ , and we reverse slightly Klein’s point of view, looking for a geometric data on  $\mathbf{X}$ , whose automorphism group is exactly  $G$ . By geometric data, we mean here something like a tensor, or a connection, on the space  $\mathbf{X}$ , or on some space naturally built from  $\mathbf{X}$ .

The nice thing is that there is a general answer to this question, whatever the homogeneous space  $\mathbf{X}$  is. Let us indeed consider the group  $G$  itself as a  $P$ -principal fiber bundle over  $G/P$ , and put on  $G$  the so called Maurer-Cartan form  $\omega^{MC}$ . This is the 1-form on  $G$ , with values in the Lie algebra  $\mathfrak{g}$ , such that for every left-invariant vector field  $X$ ,  $\omega^{MC}(X) = X(e)$ . It is not very hard to check that the automorphisms of the  $P$ -bundle  $G \rightarrow G/P$ , which moreover preserve  $\omega^{MC}$ , are exactly the left translations on the group  $G$ . So, on our homogeneous space  $\mathbf{X}$ , we get a nice differential geometric structure, namely the fiber bundle  $G \rightarrow \mathbf{X}$ , and the 1-form  $\omega^{MC}$ , which is natural in the sense that its automorphism group is exactly  $G$ .

The next step is to generalize this picture to an arbitrary manifold  $M$  of the same dimension as  $\mathbf{X}$ . This is easily done by considering the following data:

- A  $P$ -principal fiber bundle  $\hat{M} \rightarrow M$ .
- A 1-form  $\omega$  on  $\hat{M}$  with values in the Lie algebra  $\mathfrak{g}$ , which mimics the Maurer-Cartan form. In particular, one requires that at every point  $\hat{x}$  of  $\hat{M}$ , the map  $\omega : T_{\hat{x}}\hat{M} \rightarrow \mathfrak{g}$  is a linear isomorphism. One requires also equivariance properties of  $\omega$  with respect to the actions of  $P$  on  $\hat{M}$  and  $\mathfrak{g}$ , but we won't mention them here. The reader who wants to know more about Cartan geometries is referred to the very comprehensive [36] and [8].

The triple  $(\hat{M}, M, \omega)$  is called a *Cartan geometry modelled on  $\mathbf{X}$* , and the form  $\omega$  a *Cartan connection*. A Cartan geometry  $(\hat{M}, M, \omega)$  modelled on  $\mathbf{X} = G/P$  is often referred to as a *curved analogue* of the Klein geometry  $\mathbf{X}$ . The precise meaning of this sentence is made clear by the following remark. The Maurer-Cartan form on the Lie group  $G$  satisfies a property, known as the equation of Maurer-Cartan. If  $X$  and  $Y$  are two vector fields on  $G$ , then the equation reads:

$$d\omega^{MC}(X, Y) + [\omega^{MC}(X), \omega^{MC}(Y)] = 0. \quad (2.1)$$

Now, for a general Cartan geometry  $(\hat{M}, M, \omega)$  modelled on  $\mathbf{X} = G/P$ , one can introduce the *curvature form*  $K$ , which is defined, for any pair  $(X, Y)$  of vector fields on  $\hat{M}$  as

$$K(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

In general, the curvature form is not zero, and actually, one shows that the curvature form vanishes identically if and only if the Cartan geometry  $(\hat{M}, M, \omega)$  is *locally equivalent* to the model  $(G, \mathbf{X}, \omega^{MC})$ . Hence, spaces locally modelled on the homogeneous space  $\mathbf{X}$  (which are commonly called  $(G, \mathbf{X})$ -structures) are just *flat* Cartan geometries modelled on  $\mathbf{X}$ .

The notion of a Cartan geometry is an elegant way of formalizing what is a curved Klein geometry. But one drawback of the definition is that it

involves an abstract fiber bundle  $\hat{M}$  over our manifold  $M$ , whereas one would rather like to work with geometric data directly available on  $M$ . Thus, if one fixes once and for all the model homogeneous space  $\mathbf{X} = G/P$ , two natural questions arise:

- (1) Given a Cartan geometry, modelled on  $\mathbf{X}$ , on a manifold  $M$ , can we interpret the data  $(M, \hat{M}, \omega)$  on  $\mathbf{X}$  in terms of geometric data  $\mathcal{S}$  on  $M$  (such as tensors, connections, etc.).
- (2) Conversely, if such a set of geometric data  $\mathcal{S}$  is given on  $M$ , can we build a  $P$ -principal fiber bundle  $\pi : \hat{M} \rightarrow M$ , which is natural with respect to  $\mathcal{S}$ , as well as a Cartan connection  $\omega : T\hat{M} \rightarrow \mathfrak{g}$ , so that the procedure described in point (1), when applied to the triple  $(M, \hat{M}, \omega)$ , yields back  $\mathcal{S}$ . One would like moreover to find suitable normalization conditions which make the connection  $\omega$  unique.

A model homogeneous space  $\mathbf{X} = G/P$  being given, we say that *the equivalence problem* is solved for Cartan geometries modelled on  $\mathbf{X}$  if we can give a positive answer Problems (1) and (2). This is for instance the case for Cartan geometries modelled on the Euclidean space  $\mathbf{E}^n = O(n) \ltimes \mathbb{R}^n / O(n)$ . The data of such a geometry on a manifold yields a Riemannian metric  $g$  on  $M$ . Reciprocally, the existence of the Levi-Civita connection allows to build a Cartan connection  $\omega$  on  $\hat{M}$ , the  $O(n)$ -bundle of orthonormal frames associated to  $g$ . The fact that the Levi-Civita connection is the only torsion-free connection compatible with the metric ensures the uniqueness of this “normal” Cartan connection  $\omega$ .

**2.1.1 Classical examples of Cartan geometries** The most interesting Cartan geometries to consider are of course those for which the equivalence problem is solved. We give examples below, but the reader should keep in mind that except for very few cases, solving the equivalence problem, and proving the existence of a normal Cartan connection is not an easy matter at all. Since the pioneering works of E. Cartan ([9]), some deep advances on this problem were done in [12], [38], [7] among others.

- (1) Pseudo-Riemannian structures of type  $(p, q)$  are Cartan geometries modelled on type  $(p, q)$  Minkowski space  $\mathbf{E}^{p,q} = O(p, q) \ltimes \mathbb{R}^n / O(p, q)$ .
- (2) In dimension  $p+q \geq 3$ , conformal classes of type  $(p, q)$  metrics are Cartan geometries (this was proved by É. Cartan himself in the Riemannian case). The model space  $\mathbf{X}$  is the pseudo-Riemannian conformal analogue of the round sphere  $\mathbf{S}^n$ . It is called *Einstein’s universe of signature  $(p, q)$* , denoted  $\mathbf{Ein}^{p,q}$ , and is the product  $\mathbb{S}^p \times \mathbb{S}^q$  endowed with the conformal class of the product metric  $-g_{\mathbb{S}^p} \oplus g_{\mathbb{S}^q}$  (where  $g_{\mathbb{S}^m}$  stands for the round Riemannian metric on  $\mathbb{S}^m$ ). As a homogeneous space under its conformal group, it can be written as  $\mathbf{Ein}^{p,q} = O(p+1, q+1)/P$ , where  $P$  is the

stabilizer of a null direction. In Riemannian signature  $p = 0$ ,  $\mathbf{Ein}^{0,q}$  is just the round sphere.

- (3) Nondegenerate  $CR$ -structures (structures appearing on real hypersurfaces with nondegenerate Levi form in complex manifolds) are Cartan geometries modelled on the boundary of the complex hyperbolic space, namely the homogeneous space  $\mathbf{X} = \mathrm{PSU}(p+1, q+1)/P$ , where  $P$  is the stabilizer of a null direction.
- (4) Affine connections on an  $n$ -dimensional manifold  $M$  are Cartan geometries modelled on the affine space  $\mathbf{A}^n = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n / \mathrm{GL}(n, \mathbb{R})$ .
- (5) Projective classes of affine connections in dimension  $n$  are Cartan geometries modelled on the  $n$ -dimensional projective space  $\mathbf{RP}^n = \mathrm{PGL}(n+1, \mathbb{R})/P$ .
- (6) Conformal,  $CR$  and projective structures are instances of *parabolic geometries*, which are Cartan geometries modelled on a homogeneous space  $\mathbf{X} = G/P$ , for which  $G$  is a simple Lie group and  $P$  a parabolic subgroup. Thanks to the works [38], [7], the equivalence problem is solved for almost all parabolic geometries, meaning that there is a one-to-one correspondence between parabolic geometries with suitable normalizations made on the Cartan connection, and certain geometric data on the manifold.

**2.1.2 Rigidity of Cartan geometries** Let  $M$  be a manifold endowed with a Cartan geometry  $\mathcal{S}$  modelled on some homogeneous space  $\mathbf{X}$ . By  $\mathcal{S}$ , we mean the triple  $(\hat{M}, M, \omega)$ . One defines naturally an automorphism of this geometry as a bundle automorphism  $\hat{f} : \hat{M} \rightarrow \hat{M}$  satisfying  $\hat{f}^*\omega = \omega$ . When we deal with geometries for which the equivalence principle holds, and if  $\mathcal{S}$  stands as well for the geometric data on  $M$  equivalent to the triple  $(\hat{M}, M, \omega)$ , then the automorphisms are exactly the diffeomorphisms  $f : M \rightarrow M$  satisfying  $f^*\mathcal{S} = \mathcal{S}$  (in the sense that such  $f$  lift naturally to automorphisms  $\hat{f}$  of  $\hat{M}$  in the previous sense). There is also a notion of Killing field of  $\mathcal{S}$ , as a vector field generating local flows of automorphisms.

A very important and nice feature of the automorphism group of a Cartan geometry is the

**Theorem 2.1.** *Let  $M$  be a manifold endowed with a Cartan geometry  $\mathcal{S}$ . Then the automorphism group  $\mathrm{Aut}(M, \mathcal{S})$  is a Lie transformation group.*

*Moreover, on each open subset of  $M$ , the Lie algebra of Killing fields has dimension at most  $\dim \mathfrak{g}$ .*

Let us indicate why this theorem holds. The main point is that the Cartan connection  $\omega$  defines naturally a *parallelism* (one says also a *framing*) on  $\hat{M}$ , namely a family  $\hat{X}_1, \dots, \hat{X}_s$  of vector fields of  $\hat{M}$ , where  $s = \dim(\hat{M}) = \dim(\mathfrak{g})$ , such that at every  $\hat{x} \in \hat{M}$ ,  $(\hat{X}_1(\hat{x}), \dots, \hat{X}_s(\hat{x}))$  is a basis of  $T_{\hat{x}}\hat{M}$ . To build

this parallelism, it is enough to consider a basis  $(X_1, \dots, X_s)$  of  $\mathfrak{g}$ , and to define  $\hat{X}_i = \omega^{-1}(X_i)$ . Hence  $\text{Aut}(M, \mathcal{S})$  is identified with a closed subgroup of the group of automorphisms of our parallelism, the latter being itself a Lie transformation group, when endowed with the compact open topology. To see this, let us first observe that the group of diffeomorphisms preserving a parallelism  $\mathcal{P}$  on a manifold, must act freely. Indeed, if such an automorphism fixes a point, then it must fix pointwise any curve  $\gamma$  through this point, such that  $\gamma'$  has constant coordinates in the frame field defining the parallelism. But a straightforward application of the inverse mapping theorem shows that the set of such curves fills in an open neighborhood of  $x_0$ . Hence the set of fixed points of our automorphism is open, and since it is of course closed, the automorphism is trivial. We infer for instance that a Killing field of a parallelism having a zero must be identically zero. This yields the bound on the dimension of the Lie algebra of local Killing fields in Theorem 2.1.

Another consequence of the freeness of the action is that the automorphism group of a parallelism can always be identified with any of its orbits. One has then to show that this identification is a homeomorphism and the orbits are closed smooth manifolds, which is a little bit more involved (details can be found in [25, Theorem 3.2]).

The above argument shows that the topology on  $\text{Aut}(M, \mathcal{S})$ , making it a Lie transformation group, is the one inherited after identifying  $\text{Aut}(M, \mathcal{S})$  with one of its orbits on  $\hat{M}$ . In general,  $\hat{M}$  is a subbundle of the bundle of  $m$ -jets of frames over  $M$ . Thus a sequence  $(f_k)$  converges in  $\text{Aut}(M, \mathcal{S})$  when  $(f_k)$ , together with its  $m$  first derivatives, converges uniformly on compact subsets of  $M$ . In some cases, extra arguments show that this topology is actually the one induced by the compact-open topology on  $\text{Homeo}(M)$ .

## 2.2 $G$ -structures

Beside Cartan geometries, there is another family of geometric structures, called  $G$ -structures, which was studied a lot by differential geometers.

Let us consider a smooth  $n$ -dimensional manifold  $M$ , and the bundle  $R(M)$  of frames of  $M$  (this is a  $\text{GL}(n, \mathbb{R})$ -principal bundle). Let  $G$  be a closed subgroup of the linear group  $\text{GL}(n, \mathbb{R})$ . One defines a  $G$ -structure on the manifold  $M$  as a  $G$ -subbundle  $\hat{M}$  of the bundle  $R(M)$ . This means that at each point of  $M$ , we select a subclass of distinguished frames (or, equivalently, we select a distinguished class of charts at each point of  $M$ ).

There is a natural notion of automorphism for a  $G$ -structure: this is a diffeomorphism  $f : M \rightarrow M$  whose action on  $R(M)$  preserves the subbundle  $\hat{M}$  defining the  $G$ -structure. In the same way, one defines what is an isomorphism between two  $G$ -structures. Unlike for Cartan geometries, there is no direct notion of curvature for a  $G$ -structure, but there is still the notion of a *flat*

(one says also *integrable*)  $G$ -structure, as one which is locally isomorphic to  $\mathbb{R}^n \times G \subset \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$  (where the product  $\mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$  is identified with the frame bundle of  $\mathbb{R}^n$ ).

- (1) For  $G = \mathrm{O}(p, q)$ , a  $G$ -structure on a manifold  $M$  is equivalent to the data of a pseudo-Riemannian metric  $g$  of type  $(p, q)$ . The subbundle  $\hat{M}$  of the frame bundle defining the  $G$ -structure is then merely the bundle of orthonormal frames.
- (2) For  $G = \mathbb{R}^* \times \mathrm{O}(p, q)$ , where the factor  $\mathbb{R}^*$  denotes the homothetic maps in  $\mathrm{GL}(n, \mathbb{R})$ , a  $G$ -structure on  $M$  is the same as the data of a conformal class  $[g] = \{e^\sigma g \mid \sigma \in C^\infty(M)\}$  of type  $(p, q)$  pseudo-Riemannian metrics.
- (3) Let  $\omega = \Sigma dx_i \wedge dy_i$  be the standard symplectic form on  $\mathbb{R}^{2n}$  and  $\mathrm{Sp}(n, \mathbb{R})$  the group of linear transformations of  $\mathbb{R}^{2n}$  preserving  $\omega$ . A  $\mathrm{Sp}(n, \mathbb{R})$ -structure on a manifold  $M$  is called an *almost symplectic structure*. It is a genuine symplectic structure (*i.e.*  $d\omega = 0$ ) if and only if it is flat (this is Darboux's theorem).

Other examples of interesting  $G$ -structures are presented in [25, Chapter I].

Let us now make a trivial remark: a  $\mathrm{GL}(n, \mathbb{R})$ -structure on some  $n$ -dimensional manifold yields nothing more than the differentiable structure. Hence, any diffeomorphism of  $M$  is an automorphism of the structure. It is thus clear that we can not expect all  $G$ -structures to be rigid (another example is given by symplectic structures, see (3) above). So, a natural question is: *can we determine, among all  $G$ -structures, which ones are rigid?*

There is a reasonable answer to this question, and it depends only on data involving the Lie algebra  $\mathfrak{g}$ . Let us fix the Lie group  $G \subset \mathrm{GL}(n, \mathbb{R})$ . Let us identify the bundle of frames of  $\mathbb{R}^n$  as the product  $\mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$ , and let us consider the *flat  $G$ -structure* on  $\mathbb{R}^n$  as the subbundle  $\mathbb{R}^n \times G \subset \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$ . Let us try to exhibit a lot of Killing fields for this  $G$ -structure. Recall that by a Killing field, we mean a vector field  $X$ , the local flow of which acts by  $G$ -bundle automorphisms of  $\mathbb{R}^n \times G$ . Let  $k \geq 0$  be an integer, let  $L_i$ ,  $i = 1, \dots, n$ , be symmetric  $(k+1)$ -linear forms on  $\mathbb{R}^n$ , and let  $X$  be the polynomial vector field defined by

$$X(x) = \Sigma_{i=1}^n L_i(x, \dots, x) \frac{\partial}{\partial x_i}.$$

What is the condition for  $X$  to be a Killing field? Looking at the action of the local flow of  $X$  on the frame bundle, a necessary and sufficient condition is that for every  $x \in \mathbb{R}^n$ , the endomorphism of  $\mathbb{R}^n$  given by

$$u \mapsto \begin{pmatrix} L_1(x, \dots, x, u) \\ \vdots \\ L_n(x, \dots, x, u) \end{pmatrix}$$

is an element of  $\mathfrak{g}$ . Differentiating  $k$  times, we find another necessary (and sufficient) condition which is that for every vectors  $v_1, \dots, v_k$ , the endomorphism

$$u \mapsto \begin{pmatrix} L_1(v_1, \dots, v_k, u) \\ \vdots \\ L_n(v_1, \dots, v_k, u) \end{pmatrix}$$

is in  $\mathfrak{g}$ .

This motivates the definition of *the  $k$ -th prolongation  $\mathfrak{g}_k$  of  $\mathfrak{g}$*  as the set of symmetric  $(k+1)$ -linear maps

$$\varphi : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

such that the endomorphism  $u \mapsto \varphi(v_1, \dots, v_k, u)$  is in  $\mathfrak{g}$  for every choice of the  $k$  first entries  $v_1, \dots, v_k$ . One easily checks that  $\mathfrak{g}_0 = \mathfrak{g}$ , and whenever  $\mathfrak{g}_k = 0$ , then  $\mathfrak{g}_j = 0$  for every  $j > k$ .

The alternative is then as follows. Either  $\mathfrak{g}_k \neq 0$  for every  $k \in \mathbb{N}$ , and in this case the flat  $G$ -structure admits Killing fields of the form  $X = \sum_{i=1}^n L_i \frac{\partial}{\partial x_i}$ , with  $L_i$  a polynomial of arbitrary large degree. The algebra of Killing fields is thus infinite-dimensional for the flat structure, and we won't retain  $G$ -structures as rigid structures.

More interestingly, if for some  $k \in \mathbb{N}$ ,  $\mathfrak{g}_{k+1} = 0$ , one can define successive extensions of the frame bundle  $R(M)$  as well, such that the last one yields a  $P$ -principal bundle  $\hat{M}_k$  over  $M$ , *endowed with a natural parallelism* (the reader will find details in [25, Chapter I]). Here,  $P$  is a Lie group with Lie algebra  $\mathfrak{g} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ , and "natural" means that every automorphism  $f : M \rightarrow M$  of the  $G$ -structure lifts to a bundle automorphism of  $\hat{M}_k$  preserving the parallelism.

As for Cartan geometries, one can conclude:

**Theorem 2.2.** [25, Theorem 5.1] *Let  $M$  be a manifold endowed with a  $G$ -structure of finite type  $\mathcal{S}$ . Then the automorphism group  $\text{Aut}(M, \mathcal{S})$  is a Lie transformation group.*

One can also show, as for Cartan geometries, that the dimension of the Lie algebra of local Killing fields is finite. Examples of  $G$ -structures of finite type are, for instance:

- Pseudo-Riemannian metrics or type  $(p, q)$  ( $\mathfrak{g} = \mathfrak{o}(p, q)$  and  $\mathfrak{g}_1 = 0$ ).
- Conformal structures of type  $(p, q)$  when  $p + q \geq 3$  ( $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{o}(p, q)$ , and  $\mathfrak{g}_2 = 0$ ).
- One can also interpret the notion of affine connection, or projective class of such connections, but it is then necessary to introduce  $G$ -structures of higher order, what we won't do here.

### 3 The conformal group of a Riemannian manifold

In the previous section, we exhibited quite a large class of structures, whose the automorphism group is a Lie group. For these structures, we can consider Question 1.2 formulated in the introduction: *a given class of structures being fixed, what are the possible Lie groups  $\text{Aut}(M, \mathcal{S})$ , for  $M$  compact and  $\mathcal{S}$  belonging to the class we are considering.*

As we already mentioned, the first complete general result regarding this question was proved for Riemannian structures by S. Myers and N. Steenrod in [31]. Their result describes the isometry group of a compact Riemannian manifold  $(M, g)$ , namely the group of smooth diffeomorphisms  $\varphi : M \rightarrow M$  satisfying  $\varphi^*g = g$ . For the reader's convenience, we recall the statement:

**Theorem 3.1.** [31] *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Then the isometry group  $\text{Iso}(M, g)$ , endowed with the compact open topology is a Lie transformation group of dimension at most  $\frac{n(n+1)}{2}$ . Moreover this group is compact as soon as  $M$  is compact.*

This is now a direct consequence of Theorem 2.1 (or equivalently 2.2). Observe that the theorem contains an extra information about the topology making  $\text{Iso}(M, g)$  a Lie transformation group. It is here induced by the compact open topology, and we saw in Section 2.1.2 that this fact is generally not straightforward. This amounts to showing that  $\text{Iso}(M, g)$  is closed in the group of homeomorphisms of  $M$ , namely that a  $C^0$  limit of smooth isometries is still smooth. The reason why it is true is that a homeomorphism preserving the distance defined by  $g$  must send parametrized geodesics to parametrized geodesics. The smoothness of such a transformation follows.

#### 3.1 The theorem of Obata and Ferrand

Let us now start with a Riemannian manifold  $(M, g)$ , and let us take for geometric structure the conformal class  $[g] = \{e^\sigma g \mid \sigma \in C^\infty(M)\}$ . In dimension at least three, (pseudo)-Riemannian conformal structures are Cartan geometries (and  $G$ -structures of finite type as well), hence by Theorem 2.1, the group of conformal diffeomorphisms  $\text{Conf}(M, g)$  (namely the diffeomorphisms preserving the conformal class) is a Lie group. Observe that a conformal diffeomorphism is a transformation preserving angles between curves. Obviously, the group of conformal diffeomorphisms on  $M$  contains the isometry group of any metric in the conformal class  $[g]$ .

This inclusion can be strict, as shows the example of the round  $n$ -sphere  $\mathbf{S}^n = (\mathbb{S}^n, g_0)$ , where  $g_0$  is “the” metric with constant curvature  $+1$  on  $\mathbb{S}^n$ . The isometry group of  $\mathbf{S}^n$  is  $\text{O}(n+1)$ , whereas the conformal group is the Möbius group  $\text{PO}(1, n+1)$ . The latter is noncompact, showing that one can not

expect a generalization of Theorem 3.1 for Riemannian conformal structures. At first glance, it is not unreasonable to expect the noncompactness of the conformal group to be a rather general phenomenon. But if the reader tries to determine the conformal group of classical spaces in Riemannian geometry (real projective space  $\mathbf{RP}^n$ , flat tori, compact hyperbolic manifolds, etc.), he will always find a compact group. Actually, Lichnerowicz conjectured, in the middle of the sixties, that noncompactness of the conformal group for compact Riemannian manifold only occurs for the standard sphere. His guess became a theorem a few years later, thanks to independent works by Ferrand and Obata.

**Theorem 3.2.** [32][17] *A compact connected Riemannian manifold  $(M, g)$  of dimension  $n \geq 2$  having a noncompact conformal group must be conformally diffeomorphic to the standard sphere  $\mathbf{S}^n$ .*

Actually, the result obtained by Obata in [32] is weaker, since he made the stronger assumption that the identity component  $\text{Conf}^\circ(M)$  is noncompact. For instance, Obata's result did not cover the possibility (which *a posteriori* never occurs) of an infinite discrete conformal group.

Theorem 3.2 settles Question 1.2 for Riemannian conformal structures. Indeed, it says that for a compact Riemannian manifold  $(M, g)$ , the conformal group  $\text{Conf}(M, g)$  is either a compact Lie group, or the Möbius group  $\text{PO}(1, n+1)$ . The latter possibility only occurs for the standard sphere  $\mathbf{S}^n$ .

### 3.2 Ideas of the proof of the Ferrand-Obata theorem

We will work in dimension  $\geq 3$  (for  $n = 2$ , Theorem 3.2 is a consequence of the uniformization theorem for Riemann surfaces). The proof is made of two distinct steps. The first one is to use the noncompactness assumption on the conformal group to show that  $(M, g)$  is *conformally flat*, namely every sufficiently small open subset of  $(M, g)$  is conformally diffeomorphic to an open subset of Euclidean space  $\mathbf{E}^n$ . Then, the second step uses tools from the theory of  $(G, \mathbf{X})$ -structures to show that a compact conformally flat manifold which is not the standard sphere has a compact conformal group.

Let us recall that whereas any Riemannian metric on a surface is conformally flat (this was first showed by F. Gauss in the analytic case), the situation is completely different in dimension  $\geq 3$ . Then, there exists a tensor  $W$  on  $M$ , that we will call the *conformal curvature*, which vanishes if and only if  $(M, g)$  is conformally flat. This tensor  $W$  is the Weyl (resp. Cotton) tensor in dimension  $\geq 4$  (resp. in dimension 3).

On a Riemannian manifold  $(M, g)$ , the conformal curvature allows to build a conformally invariant “metric” putting  $h_g = \|W\|_g^\alpha g$ , where  $\|W\|_g$  denotes the norm of the tensor  $W$  with respect to  $g$ , and  $\alpha = 1$  (resp.  $\alpha = \frac{3}{2}$ ) when the dimension is  $\geq 4$  (resp. is 3). Of course,  $h_g$  is not really a metric because

$W$  may vanish at some points. Nevertheless, if we assume that  $(M, g)$  is not conformally flat,  $h_g$  is not identically zero. Hence we can define a nontrivial, conformally invariant, singular distance

$$d_h(x, y) = \inf_{\gamma} \int_{\gamma} h_g(\gamma', \gamma'),$$

the infimum being taken over all  $\gamma$ 's joining  $x$  to  $y$ .

Now, still assuming that  $(M, g)$  is not conformally flat, let us consider the closed subset  $K$  on which the conformal curvature vanishes. This is a proper subset of  $M$ , and if  $K_\epsilon = \{x \in M, d_h(x, K) \geq \epsilon\}$ , then  $(K_\epsilon, d_h)$  is, for  $\epsilon$  sufficiently small, a genuine (*i.e.* nonsingular) metric space. Moreover,  $K_\epsilon$  is invariant by the conformal group  $\text{Conf}(M, g)$ , and this group acts isometrically with respect to  $d_h$ . Using Ascoli's theorem, and elaborating a little bit, one infers that  $\text{Conf}(M, g)$  is compact.

This completes the first step: whenever  $\text{Conf}(M, g)$  is noncompact, then  $(M, g)$  must have *locally* the same conformal geometry as the standard sphere  $\mathbf{S}^n$ .

The second part of the proof aims at globalizing this local result. Once we know that  $(M, g)$  is conformally flat, then we also know since Kuiper (see [26]), that there is a conformal immersion  $\delta : \tilde{M} \rightarrow \mathbf{S}^n$  (where  $\tilde{M}$  stands for the universal cover of  $M$ ), called *developing map*, as well as a morphism  $\rho : \text{Conf}(\tilde{M}, \tilde{g}) \rightarrow \text{PO}(1, n+1)$  satisfying the equivariance relation

$$\delta \circ \gamma = \rho(\gamma) \circ \delta. \quad (3.1)$$

The noncompactness of  $\text{Conf}(\tilde{M}, \tilde{g})$  implies that of  $\rho(\text{Conf}(\tilde{M}, \tilde{g}))$ . Now, divergent sequences  $(g_k)$  in  $\text{PO}(1, n-1)$  have (up to extracting a subsequence) a "north-south" type dynamics, meaning that there is an attracting, and a repelling pole for the sequence  $(g_k)$ . The equivariance relation (3.1) allows to show that some noncompact sequence  $(f_k)$  in  $\text{Conf}(\tilde{M}, \tilde{g})$  has a repelling pole on  $\tilde{M}$ , and that there exists a small open set  $U \subset \tilde{M}$  on which  $\delta$  is one-to-one, such that  $f_k(U)$  is an increasing sequence, the union of which is  $\tilde{M}$ , or  $\tilde{M}$  minus a point. We infer that  $\delta$  is one-to-one on  $\tilde{M}$ , hence  $(M, g)$  is conformally equivalent to a quotient  $\Omega/\Gamma$ , where  $\Omega \subset \mathbf{S}^n$  is an open subset, and  $\Gamma \subset \text{PO}(1, n-1)$  is a discrete subgroup acting cocompactly on  $\Omega$ . The normalizer of  $\Gamma$  in  $\text{PO}(1, n+1)$  is then always compact (and so is the conformal group of  $\Omega/\Gamma$ ), except when  $\Gamma = \{\text{id}\}$ . This forces  $\Omega = \mathbf{S}^n$  by cocompactness and  $(M, g)$  is conformally diffeomorphic to the round sphere  $\mathbf{S}^n$ .

### 3.3 Generalizations to rank one parabolic geometries

Theorem 3.2 has been generalized to the noncompact case, independently by Ferrand in [19] and Schoen [35].

**Theorem 3.3.** [19][35] *Let  $(M, g)$  be a connected Riemannian manifold of dimension  $n \geq 2$ . If the group  $\text{Conf}(M, g)$  does not act properly on  $M$ , then  $M$  is conformally diffeomorphic to the standard sphere  $\mathbf{S}^n$ , or to Euclidean space  $\mathbf{E}^n$ .*

The proofs of this theorem are much more involved than for the compact case. The methods of Ferrand led to further generalizations of Theorem 3.3 for the group of  $K$ -quasi conformal mappings of a Riemannian manifold (see [20]).

The PDEs methods used by Schoen in [35] allowed him to obtain the same kind of statement for strictly pseudoconvex  $CR$  structures. This is not that surprising if we take the point of view of Cartan geometries, because the model spaces of conformal Riemannian structures and of strictly pseudoconvex  $CR$ -structures are the boundary at infinity of real and complex hyperbolic spaces respectively, hence have similar properties (for instance the north-south dynamics for divergent sequences of their automorphism group). These structures fit in the larger class of Cartan geometries for which the model space  $\mathbf{X}$  is a quotient  $G/P$ , where  $G$  is a noncompact simple Lie group of rank one, and  $P$  a parabolic subgroup. In other words,  $\mathbf{X}$  is the Hadamard boundary of a rank-one symmetric space of noncompact type. The Riemannian conformal case corresponds to  $G = \text{PO}(1, n+1)$ . There are three other types of geometries involved, respectively for  $G = \text{PU}(1, n+1)$  (strictly pseudoconvex  $CR$ -structures),  $G = \text{PSp}(1, n+1)$  and  $G = F_4^{-20}$ . Those Cartan geometries are called *rank one parabolic geometries*. It turns out that the results of Obata-Ferrand-Schoen quoted above generalize to all the geometries of this family. Indeed:

**Theorem 3.4** ([23]). *Let  $(M, \mathcal{S})$  be a rank one, regular, Cartan geometry modelled on  $\mathbf{X}$ . If the automorphism group  $\text{Aut}(M, \mathcal{S})$  does not act properly on  $M$ , then  $M$  is isomorphic, as a Cartan geometry, to the model  $\mathbf{X}$ , or to  $\mathbf{X}$  minus a point.*

Of course, one recovers Theorem 3.3 when the model space  $\mathbf{X}$  is the conformal sphere  $\mathbf{S}^n$ .

## 4 Lorentzian isometries

Like for conformal structures, there is no analogue of Myers-Steenrod's theorem in Lorentzian geometry. Recall that a Lorentzian metric  $g$  on a manifold  $M$  is a smooth field of indefinite nondegenerate bilinear forms of signature  $(1, n-1)$  (namely  $(-, +, \dots, +)$ ). Finding compact Lorentz manifolds having

a noncompact isometry group is not completely obvious, so that we begin by describing relevant examples.

### 4.1 Flat tori

We begin with the simplest example. Let us consider the matrix  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$  in  $\mathrm{SL}(2, \mathbb{R})$ . It has two distinct eigenvalues  $\lambda_1 < 1$  and  $\lambda_2 > 1$ , and we choose  $u$  and  $v$  two associated eigenvectors. Let  $g$  be the translation-invariant Lorentzian metric on  $\mathbb{R}^2$  given by  $g = dudv$ . Let  $\mathbb{T}^2$  be the quotient of  $\mathbb{R}^2$  by the lattice  $\mathbb{Z}^2$ , equipped with the Lorentzian metric  $\bar{g}$  induced by  $g$ . Because  $A$  normalizes  $\mathbb{Z}^2$ , it induces a Lorentzian isometry  $\bar{A}$  of  $(\mathbb{T}^2, \bar{g})$ , and the group generated by  $\bar{A}$  has noncompact closure in the homeomorphisms of  $\mathbb{T}^2$  (actually  $\bar{A}$  is an Anosov diffeomorphism).

One can elaborate on this example, by considering a quadratic form  $g$  on  $\mathbb{R}^n$ . We assume moreover that  $g$  has Lorentzian signature, and its coefficients are rational. We see  $g$  as a flat Lorentz metric on  $\mathbb{R}^n$ . Again, we consider the torus  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ , equipped with the metric  $\bar{g}$  induced by  $g$ . A theorem of Borel and Harish-Chandra ensures that the group  $O(g, \mathbb{Z}) \subset O(g)$ , comprising all linear transformations of  $O(g)$  with integral entries, is a lattice in  $O(g)$ . It is in particular noncompact. The isometry group of  $(\mathbb{T}^n, \bar{g})$  coincides, as a Lie group, with the (noncompact) semi-direct product  $O(g, \mathbb{Z}) \ltimes \mathbb{T}^n$ .

Observe that in those examples, the identity component of the isometry group is compact (a torus). The noncompactness comes from the discrete part of the group.

### 4.2 Anti-de sitter 3-manifolds

Here is a more general procedure to build compact pseudo-Riemannian manifolds with noncompact isometry group. Start with a Lie group  $G$ , such that there is on  $\mathfrak{g}$  an  $\mathrm{Ad}(G)$ -invariant, pseudo-Riemannian scalar product  $\lambda$ . If one pushes  $\lambda$  by left translations on  $G$ , one gets a pseudo-Riemannian metric  $g$  on  $G$ , which is left-invariant by construction, but also right-invariant because  $\lambda$  is  $\mathrm{Ad}(G)$ -invariant. Assume that  $G$  has a cocompact lattice  $\Gamma$ . Then, on  $G/\Gamma$ , there is an induced pseudo-Riemannian metric  $\bar{g}$ , which is invariant for the left action of  $G$  on  $G/\Gamma$ . We thus get a copy of  $G$  inside  $\mathrm{Iso}(G/\Gamma, \bar{g})$ . If  $G$  cannot be embedded in any compact Lie group, this forces  $\mathrm{Iso}(G/\Gamma, \bar{g})$  to be noncompact.

A nice example of this construction is obtained by choosing  $G$  to be a connected noncompact simple Lie group, and  $\lambda$  to be the Killing form on  $\mathfrak{g}$ . The Killing form is defined by  $\lambda(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ . It is always  $\mathrm{Ad}(G)$ -invariant, and for simple Lie groups, it is nondegenerate. Moreover, it

follows from the works of Borel and Harish-Chandra that noncompact simple Lie groups do admit cocompact lattices, and the above mentioned procedure works. Nevertheless, in general, the pseudo-Riemannian metric  $\bar{g}$  on  $G/\Gamma$  is not Lorentzian.

The only case leading to Lorentzian metrics is that of  $G = \mathrm{SL}(2, \mathbb{R})$  (or more generally when  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ ). The manifold  $\mathrm{SL}(2, \mathbb{R})$  endowed with the Killing form is a Lorentzian manifold of constant negative curvature, and is called *anti-de Sitter space*  $\mathbf{AdS}_3$ . For any cocompact lattice  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ , the quotient  $\mathrm{SL}(2, \mathbb{R})/\Gamma$ , endowed with the induced Lorentzian metric, is a compact anti-de Sitter manifold, the isometry group of which is noncompact.

Let us mention that when  $\Gamma$  is torsion-free, the manifold  $\mathrm{SL}(2, \mathbb{R})/\Gamma$  is naturally identified with (a double cover of) the unit tangent bundle  $T^1\Sigma$  of the hyperbolic surface  $\Sigma = \mathbb{H}^2/\Gamma$ . The flows  $\left\{ \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \right\}_{t \in \mathbb{R}}$  and  $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t \in \mathbb{R}}$  acting on the left on  $\mathrm{SL}(2, \mathbb{R})/\Gamma$  can be interpreted respectively as the geodesic and horocyclic flows on  $T^1\Sigma$ . Thus, geodesic and horocyclic flows on the unit tangent bundle of hyperbolic surfaces are instances of isometric flows for Lorentzian metrics. This gives an idea of the richness of Lorentzian dynamics.

### 4.3 Warped Heisenberg groups

Abelian and simple Lie groups are not the only instances of Lie groups admitting a bi-invariant pseudo-Riemannian metric. There is actually a classification of the possible Lie algebras for such groups (see [30]).

Here is a nice class of examples which will be important for our purpose. We start with the Heisenberg Lie algebra  $\mathfrak{heis}(2d+1)$ , generated as a vector space by  $2d+1$  elements  $X_1, Y_1, X_2, Y_2, \dots, X_d, Y_d$  and  $Z$ , with the bracket relations

$$[X_i, Y_i] = Z, [X_i, Z] = [Y_i, Z] = 0 \text{ for } i = 1, \dots, d.$$

In the sequel, we will denote by  $\mathrm{Heis}(2d+1)$  the connected simply connected Lie group having  $\mathfrak{heis}(2d+1)$  for Lie algebra (it can be seen as a subgroup of upper-triangular unipotent matrices in  $\mathrm{GL}(d+2, \mathbb{R})$ ).

Let  $\mu = (m_1, \dots, m_d) \in \mathbb{Z}_\pm$ , where  $\mathbb{Z}_\pm$  denotes the set of  $d$ -uples of integers all having the same sign. We introduce an extra element  $T$  satisfying the bracket relations

$$[T, X_i] = m_i Y_i, \text{ for } i = 1, \dots, d,$$

$$[T, Y_i] = -m_i X_i, \text{ for } i = 1, \dots, d,$$

and

$$[T, Z] = 0.$$

The Lie algebra  $\mathbb{R} \times_{\mu} \mathfrak{heis}(2k+1)$  spanned by  $T, X_1, \dots, X_d, Y_1, \dots, Y_d, Z$  is called a *warped Heisenberg algebra*.

The derivation  $\text{ad}(T)$  can be integrated into an action of  $\mathbf{S}^1$  by automorphisms of  $\mathfrak{heis}(2d+1)$ . Hence there exists a group  $G_{\mu}$ , called a *warped Heisenberg group*, isomorphic to a semi-direct product  $\mathbf{S}^1 \ltimes \text{Heis}(2d+1)$ , having  $\mathbb{R} \times_{\mu} \mathfrak{heis}(2k+1)$  for Lie algebra. Let us put on  $\mathbb{R} \times_{\mu} \mathfrak{heis}(2k+1)$  a Lorentzian product  $\lambda$  defined as follows:

- on  $\text{Span}(X_1, \dots, X_d, Y_1, \dots, Y_d)$ ,  $\lambda$  is a Riemannian scalar product, invariant by the action of  $\text{Ad}(\mathbf{S}^1)$ .
- the scalar product  $\lambda(T, Z)$  equals 1, and  $\lambda(T, T) = \lambda(Z, Z) = 0$ .
- the space  $\text{Span}(T, Z)$  is  $\lambda$ -orthogonal to  $\text{Span}(X_1, \dots, X_d, Y_1, \dots, Y_d)$ .

Then  $\lambda$  is an  $\text{Ad}(G_{\mu})$ -invariant Lorentzian product on  $\mathbb{R} \times_{\mu} \mathfrak{heis}(2k+1)$ , which can be pushed by left translations to get a bi-invariant Lorentz metric on  $G_{\mu}$ .

The last remark is that if  $\Gamma$  is a cocompact lattice in  $\text{Heis}(2d+1)$ , then  $\Gamma$  yields a cocompact lattice in  $G_{\mu}$ . It follows that the compact manifold  $G_{\mu}/\Gamma$  is endowed with a Lorentzian metric, for which the isometry group is noncompact (the connected component of this group is actually  $G_{\mu}$ ).

#### 4.4 The classification of isometry algebras

We now present the results classifying all the possible connected components for the isometry group of a compact Lorentz manifold. Actually, those results focus on the possible Lie algebras  $\mathfrak{Iso}(M, g)$ . They are due, independently and almost simultaneously, to Adams-Stuck and Zeghib, following some pioneering works of Zimmer and Gromov.

We already met examples of compact Lorentzian manifolds  $(M, g)$  where the Lie algebra  $\mathfrak{Iso}(M, g)$  is  $\mathfrak{sl}(2, \mathbb{R})$  or some warped Heisenberg algebra  $\mathbb{R} \times_{\mu} \mathfrak{heis}(2k+1)$ . These example can be enriched by the following warped-product procedure. Assume that  $(N, h)$  is a Riemannian manifold,  $(M, g)$  a Lorentzian one, and let  $\beta : M \rightarrow (0, \infty)$  be some smooth function. Then on the product manifold  $M \times N$ , the warped metric  $g' = h \oplus \beta g$  is Lorentzian, and it is clear that  $\text{Iso}(M, h) \times \text{Iso}(N, g)$  is included in  $\text{Iso}(M \times N, g')$ . Here is now the classification result we announced at the beginning of this section.

**Theorem 4.1.** [1], [45], [46]

*Let  $(M, g)$  be a compact Lorentzian manifold. Then the Lie algebra  $\mathfrak{Iso}(M, g)$  is isomorphic to a direct sum  $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{s}$ , where  $\mathfrak{k}$  is either trivial or the Lie algebra of a compact semisimple group,  $\mathfrak{a}$  is abelian, and  $\mathfrak{s}$  is either trivial, or of one of the following types:*

- (1) *The Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ .*
- (2) *The Heisenberg algebra  $\mathfrak{heis}(2k + 1)$ , for some integer  $k \geq 1$ .*
- (3) *A warped Heisenberg algebra  $\mathbb{R} \times_{\mu} \mathfrak{heis}(2k + 1)$ , for some integer  $k \geq 1$ , and  $\mu \in \mathbb{Z}_{\pm}$ .*

*Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentzian manifold.*

Beyond this algebraic result, the works [1], [45] give a quite precise picture of the geometry of the manifold  $(M, g)$ , when the factor  $\mathfrak{s}$  is nontrivial. For instance, if the group  $\mathrm{SL}(2, \mathbb{R})$  acts faithfully and isometrically on a compact Lorentzian manifold  $(M, g)$ , then the universal cover  $(\widetilde{M}, \widetilde{g})$  is a warped product of  $\mathrm{SL}(2, \mathbb{R})$  endowed with the Killing form, and some Riemannian manifold  $(N, h)$  (this case was actually first proved by Gromov in [24]). The situation for actions of warped Heisenberg groups is also well understood, but less easy to describe.

Those results illustrate once again the principle stated in the introduction: a “large” isometry group only occurs for very peculiar geometries.

To conclude this section about Lorentzian isometries, let us quote the following striking result by D’Ambra, the proof of which is a very nice application of Gromov’s ideas presented in [24]. It says that some kind of Myers-Steenrod theorem is available in Lorentzian geometry, under some analyticity assumption, and for simply connected manifolds.

**Theorem 4.2.** *[13] Let  $(M, g)$  be a compact, analytic, simply connected Lorentzian manifold. Then the group  $\mathrm{Iso}(M, g)$  is compact.*

## 5 Further structure results

In the previous sections, we presented very complete results answering Question 1.2 for peculiar geometric structures. Now, the question is: can we obtain more general statements for entire classes of structures, such as  $G$ -structures, or Cartan geometries?

The first significant theorems in this direction were obtained by Zimmer in [51], and also Gromov in [24]. They resulted from a wonderful mix of new ideas, involving ergodic theory and algebraic actions.

### 5.1 Zimmer’s embedding theorem

In [51], Zimmer investigated actions of connected, noncompact, simple Lie groups on compact manifolds, preserving a  $G$ -structure. The reader who does

not know the definition of a simple Lie group should have in mind  $\mathrm{SL}(m, \mathbb{R})$ ,  $\mathrm{O}(p, q)$ ,  $\mathrm{SU}(p, q)$ ,  $\mathrm{Sp}(m, \mathbb{R})$  etc. He obtained the following theorem, often called *Zimmer's embedding theorem*.

**Theorem 5.1** ([51], Theorem A). *Let  $H$  be a connected, noncompact simple Lie group. Assume that  $H$  acts faithfully on some  $n$ -dimensional manifold  $M$ , preserving a  $G$ -structure. Assume also that  $G$  is an algebraic subgroup of  $\mathrm{SL}'(n, \mathbb{R})$ , the group of linear transformations of  $\mathbb{R}^n$  with determinant  $\pm 1$ . Then :*

- (1) *There is a Lie algebra embedding  $\sigma : \mathfrak{h} \rightarrow \mathfrak{g}$ .*
- (2) *More precisely, there exists a linear subspace  $V \subset \mathbb{R}^n$ , a Lie subalgebra  $\mathfrak{h}_V \subset \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$  isomorphic to  $\mathfrak{h}$ , leaving  $V$  invariant, and such that the action of  $\mathfrak{h}_V$  on  $V$  is conjugate to the linear action of the algebra  $\mathrm{ad} \mathfrak{h}$  on  $\mathfrak{h}$ .*

Since  $H$  is a simple group, the map  $\mathrm{ad} : \mathfrak{h} \rightarrow \mathrm{End}(\mathfrak{h})$  is one-to-one. It follows that the second point of the theorem implies the first one but, as we will see it soon on some examples, it carries more information.

In the above statement, the  $G$ -structure we consider is not required to be of finite type. Hence, Zimmer's result applies for instance to symplectic structures, which are not rigid, in the sense we adopted in this text. Actually, the rigidity comes here from the algebraic assumption (simplicity) on the group  $H$ .

When the  $G$ -structure is of finite type, the automorphism group is a Lie group (see Section 2.2). The Levi decomposition allows to write the Lie algebra  $\mathrm{aut}(M)$  as a semidirect product  $\mathfrak{s} \ltimes \mathfrak{r}$ , where  $\mathfrak{r}$  is the solvable radical and  $\mathfrak{s}$  is a semisimple algebra. Zimmer's theorem puts some restrictions on the semisimple factor  $\mathfrak{s}$  : Noncompact factors in  $\mathfrak{s}$  must embed into  $\mathfrak{g}$ , hence cannot be "too big", with respect to  $\mathfrak{g}$ .

## 5.2 Illustration in the case of isometric actions on Lorentz manifolds

Zimmer's theorem 5.1 predates Theorem 4.1 of almost ten years. It allows to derive quickly results which are now particular cases of Theorem 4.1.

Let us consider a compact manifold  $M$  endowed with a Lorentz metric  $g$ . Assume that some noncompact, connected, simple Lie group  $H$  acts isometrically on  $(M, g)$ . We already saw that the data of a Lorentz metric  $g$  on  $M$  is the same as the data of an  $\mathrm{O}(1, n-1)$ -structure on  $M$ . Because  $\mathrm{O}(1, n-1)$  is an algebraic subgroup of  $\mathrm{SL}'(n, \mathbb{R})$ , Zimmer's embedding theorem applies: there exists a Lie algebra embedding  $\sigma : \mathfrak{h} \rightarrow \mathfrak{o}(1, n-1)$ .

This puts rather strong restrictions on  $\mathfrak{h}$ . For instance, we infer immediately from the theorem that  $\mathrm{SL}(3, \mathbb{R})$  cannot act isometrically on a compact Lorentz manifold. Indeed, such an action would provide a Lie algebra embedding  $\sigma : \mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{o}(1, n-1)$ . But such an embedding cannot exist, because the real rank of  $\mathfrak{sl}(3, \mathbb{R})$  is 2, whereas that of  $\mathfrak{o}(1, n-1)$  is 1, and the rank cannot decrease under an embedding of Lie algebras.

Actually, the second point of Theorem 5.1 allows to determine  $\mathfrak{h}$  completely. Indeed, it implies that the transformations of  $\mathrm{ad} \mathfrak{h}$  are skewsymmetric with respect to some bilinear form  $B$  on  $\mathfrak{h}$ . This form  $B$  is just obtained after identifying  $\mathfrak{h}$  with the subspace  $V$  (given by the statement of Theorem 5.1), and restricting to  $V$  a Lorentz scalar product which is  $O(1, n-1)$ -invariant. In particular, the totally isotropic subspaces of  $B$  have dimension at most 1. But on a simple real Lie algebra  $\mathfrak{h}$  of noncompact type, one checks easily that any bilinear form  $B$  for which the elements of  $\mathrm{ad} \mathfrak{h}$  are skewsymmetric must be zero on the root spaces  $\mathfrak{h}_\alpha$ , and two distinct root spaces  $\mathfrak{h}_\alpha, \mathfrak{h}_\beta$  with  $\alpha \neq -\beta$  must be  $B$ -orthogonal. As a consequence, if the isotropic subspaces of  $B$  have dimension at most 1,  $\mathfrak{h}$  can have only two roots  $\alpha, -\alpha$  with 1-dimensional root spaces. This only happens for the Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$ . We are thus led to the following

**Corollary 5.2.** *Let  $(M, g)$  be a compact Lorentz manifold, and  $H$  a connected, noncompact, simple Lie group acting isometrically on  $(M, g)$ . Then  $\mathfrak{h}$  is isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ .*

### 5.3 Ideas of the proof of Zimmer's embedding theorem

To illustrate the beautiful methods introduced by Zimmer to prove Theorem 5.1, we give an elementary exposition of the proof in the case of isometric actions on pseudo-Riemannian manifolds.

We are thus considering  $(M, g)$  a compact pseudo-Riemannian manifold of type  $(p, q)$  (without loss of generality, we will assume  $p \leq q$ ), and  $H$  a connected noncompact simple Lie group acting isometrically and faithfully on  $(M, g)$ .

The first important idea in Zimmer's proof is that an action of a connected Lie group on a manifold, when it preserves a geometric structure, often defines natural equivariant maps to algebraic varieties (called "Gauss maps" in [24]). Let us see this in the case of our isometric action. First, with each element  $X \in \mathfrak{h}$ , we associate the vector field  $X^*$  on  $M$ , defined as  $X^*(x) = \frac{d}{dt}|_{t=0}(e^{tX}.x)$ . We call  $\mathcal{S}(\mathfrak{h})$  the space of symmetric bilinear forms on  $\mathfrak{h}$ , and  $\mathrm{Gr}(\mathfrak{h})$  the Grassmanian of subspaces of  $\mathfrak{h}$ .

A first map we can consider is

$$\alpha : M \rightarrow \mathrm{Gr}(\mathfrak{h}),$$

which associates, with each point  $x \in M$ , the Lie algebra  $\mathfrak{h}_x$  of vectors  $X \in \mathfrak{h}$  satisfying  $X^*(x) = 0$ .

A second interesting map  $\beta : M \rightarrow \mathcal{S}(\mathfrak{h})$  is defined as follows:

$$\beta_x(X, Y) = g_x(X^*(x), Y^*(x)).$$

Those maps are natural in the sense that they are  $H$ -equivariant, where we make  $H$  act on  $\mathcal{S}(\mathfrak{h})$  and  $\text{Gr}(\mathfrak{h})$  through the representation  $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{h})$ .

Let us now consider the  $H$ -invariant open set  $\Omega$  where the  $H$ -orbits have maximal dimension  $m_0 \geq 1$ , and the map  $\alpha \times \beta : \Omega \rightarrow \text{Gr}_{n_0}(\mathfrak{h}) \times \mathcal{S}(\mathfrak{h})$ , where  $n_0 = \dim H - m_0$ . The next key idea in the proof of Zimmer is to notice that, roughly speaking,  $\text{Gr}_{n_0}(\mathfrak{h}) \times \mathcal{S}(\mathfrak{h})$  is an algebraic variety on which  $H$  acts algebraically (through the representation  $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{h})$ ). More precisely,  $\text{Gr}_{n_0}(\mathfrak{h}) \times \mathcal{S}(\mathfrak{h})$  is an open subset of a projective subvariety of  $\mathbf{RP}^m$ , for some integer  $m$ , and the  $H$ -action on it comes from that of a simple Lie subgroup of  $\text{GL}(m+1, \mathbb{R})$ . Thus, our Gauss map has transformed our initial dynamical system into an algebraic one. Moreover, our pseudo-Riemannian metric defines a volume on  $M$ , giving volume 1 to every direct orthonormal frame (if  $M$  is not orientable, this makes only sense locally, but this still defines a smooth measure). As a consequence,  $H$  preserves a Borel measure  $\mu$  on  $M$ , which is finite by compactness of  $M$ . Pushing forward the measure  $\mu$  by  $\alpha \times \beta$ , we get a (nonzero) finite Borel measure  $\nu$  on  $\text{Gr}_{n_0}(\mathfrak{h}) \times \mathcal{S}(\mathfrak{h})$ , and this measure  $\nu$  is  $H$ -invariant.

Now, algebraic actions preserving a finite measure are dynamically very poor, since from the measurable point of view, they factor through actions of compact groups. This is the content of the following statement, often called ‘‘Borel density theorem’’.

**Theorem 5.3** (Borel density theorem). *Let  $H \subset \text{GL}(m+1, \mathbb{R})$  be an algebraic subgroup. If the action of  $H$  on  $\mathbf{RP}^m$  preserves a finite Borel measure  $\nu$ , then there exists a cocompact, normal, algebraic subgroup  $H_0 \subset H$  which acts trivially on the support of  $\nu$ .*

In our situation, Borel’s density theorem 5.3 says that for  $\mu$ -almost every point  $x \in \Omega$ ,  $(\sigma(x), \beta(x))$  is  $\text{Ad } H$ -invariant. The  $\text{Ad } H$ -invariance of the subspace  $\sigma(x) = \mathfrak{h}_x$  means exactly that the Lie algebra  $\mathfrak{h}_x$  is an ideal of  $\mathfrak{h}$ . By simpleness of  $\mathfrak{h}$ , we get  $\mathfrak{h}_x = \{0\}$  or  $\mathfrak{h}_x = \mathfrak{h}$ . Points where  $\mathfrak{h}_x = \mathfrak{h}$  have orbits of dimension 0, so the definition of  $\Omega$  leads to  $\mathfrak{h}_x = \{0\}$   $\mu$ -almost everywhere on  $\Omega$ . Because  $\mu$  is of full support, this implies  $\mathfrak{h}_x = \{0\}$  on  $\Omega$ . It follows that the dimension of the orbits is that of  $H$ .

We now use the fact that  $\beta(x)$  is  $\text{Ad } H$ -invariant for  $\mu$ -almost every  $x$  of  $\Omega$ , which implies that the kernel of  $\beta(x)$  must be an ideal in  $\mathfrak{h}$ . We infer that  $\beta(x)$  is either zero, or non degenerate of type  $(p', q')$ ,  $p' \leq p$ ,  $q' \leq q$ .

(1) If  $\beta(x)$  is zero, the restriction of  $g$  to the orbit  $H.x$  is zero as well.

Because we already noticed the dimension of  $H.x$  is that of  $H$ , we infer

$\dim H \leq p$  (and thus  $p \geq 3$ ). The second point of Theorem 5.1 follows because through the representation  $\text{ad} : \mathfrak{h} \rightarrow \text{End}(\mathfrak{h})$ , the Lie algebra  $\mathfrak{h}$  embeds into  $\mathfrak{sl}(d, \mathbb{R})$ , where  $d = \dim H$ , and  $\mathfrak{o}(p, q)$  contains a subalgebra which is conjugate in  $\mathfrak{gl}(p+q, \mathbb{R})$  to:

$$\left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 0_{p+q-2d} & 0 \\ 0 & 0 & -{}^t A \end{pmatrix}, A \in \mathfrak{sl}(d, \mathbb{R}) \right\}.$$

- (2) If  $\beta(x)$  is nonzero, the restriction of  $g$  to the orbit  $H.x$  is non degenerate of type  $(p', q')$ . Thus  $p' \leq p$  and  $q' \leq q$ . The group  $\text{Ad}(H)$ , whose Lie algebra is  $\mathfrak{h}$  by simplicity, can be seen as a subgroup of  $O(p', q')$ , and Theorem 5.1 follows.

## 5.4 Extension of Zimmer's result to Cartan geometries

Zimmer's Theorem 5.1 does not apply to all  $G$ -structures. Indeed, the group  $G$  is required to be an algebraic subgroup of  $\text{SL}'(n, \mathbb{R})$  (the subgroup of linear transformations with determinant  $\pm 1$ ). This assumption is basically equivalent to the fact that the  $G$ -structure defines a natural (*i.e* invariant by automorphisms) smooth measure on the manifold  $M$ . That's why some natural geometric structures do not enter in the range of application of Theorem 5.1. Actually, the statement is even wrong for some of those structures. To check this, let us consider, for instance, the case of Riemannian conformal structures. The data of a conformal class of Riemannian metrics on some  $n$ -dimensional manifold amounts to the data of an  $\mathbb{R}_+^* \times O(n)$ -structure. Observe that  $\mathbb{R}_+^* \times O(n)$  is not included in  $\text{SL}'(n, \mathbb{R})$ . Now, the Möbius group  $\text{PO}(1, n+1)$  acts conformally on the round sphere  $\mathbf{S}^n$ . On the other hand, one can show that any Lie algebra morphism

$$\sigma : \mathfrak{o}(1, n+1) \rightarrow \mathbb{R} \oplus \mathfrak{o}(n)$$

must be trivial, hence can never be an embedding, and the conclusions of Theorem 5.1 do not hold in this case.

In light of this example, it would be desirable to obtain statements in the spirit of Zimmer's embedding theorem, for structures, like conformal ones, which do not define natural invariant measures.

Some results in this direction were proved in [5]. They yield significant informations about the automorphism groups of geometric structures which are not covered by Theorem 5.1, like conformal, or  $CR$ , or projective structures. The class of geometric structures covered by these results is no longer that of  $G$ -structures, but that of Cartan geometries. The Cartan geometries we will consider in the following will be modelled on homogeneous spaces  $\mathbf{X} = G/P$  satisfying the two properties:

- The action of  $G$  on  $\mathbf{X} = G/P$  has finite kernel.
- The image  $\text{Ad}_{\mathfrak{g}} P$  of  $P$  by the representation  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$  is almost algebraic, namely has finite index in its Zariski closure.

Those restrictions are actually harmless since they are satisfied for most relevant examples of Cartan geometries.

The main result of [5] is probably too technical to be stated here. It says roughly that if a connected Lie group  $H$  acts on a compact manifold  $M$ , preserving a Cartan geometry  $\mathcal{S}$  modelled on  $\mathbf{X} = G/P$ , then the adjoint representation of most solvable Lie subgroups  $S < H$  on  $\mathfrak{h}$ , is “contained” in the adjoint representation of  $P$  on  $\mathfrak{g}$ . Thus, the upshot is that for a Lie subgroup  $H < \text{Aut}(M, \mathcal{S})$ , relevant algebraic informations on  $H$  are controlled by the pair  $(G, P)$ .

Some of these algebraic informations are numerical invariants, whose definition we recall now. If  $L \subset \text{GL}(m, \mathbb{R})$  is a linear subgroup, one defines the *real rank of  $L$* , denoted  $\text{rk}(L)$ , as the maximal dimension of an abelian subgroup of  $L$  made of  $\mathbb{R}$ -split transformations. The *algebraic rank*  $\text{rk}^{\text{alg}}(L)$  is the maximal real rank of the Zariski closure of an abelian subgroup of  $L$  made of  $\mathbb{R}$ -split transformations. One always has  $\text{rk}(L) \leq \text{rk}^{\text{alg}}(L)$ , and the inequality can be strict (see examples in Section 5.5 below). The *nilpotency index* of  $L$ , denoted by  $\text{nilp}(L)$  is the maximal nilpotency index of a connected nilpotent Lie subgroup of  $L$ .

**Theorem 5.4.** [5, Theorems 1.3 and 1.5] *Let  $(M, \mathcal{S})$  be a Cartan geometry modelled on the homogeneous space  $\mathbf{X} = G/P$ . Let  $H$  be a connected Lie subgroup of  $\text{Aut}(M, \mathcal{S})$ . Assume that the manifold  $M$  is compact. Then*

(1)

$$\text{rk}^{\text{alg}}(\text{Ad } H) \leq \text{rk}(\text{Ad}_{\mathfrak{g}} P)$$

and

$$\text{nilp}(\text{Ad } H) \leq \text{nilp}(\text{Ad}_{\mathfrak{g}} P).$$

(2) *If moreover  $\mathbf{X} = G/P$  is a parabolic geometry, if  $\mathcal{S}$  is regular, and if the equality*

$$\text{rk}(\text{Ad } H) = \text{rk}(\text{Ad}_{\mathfrak{g}} P)$$

*holds, then  $(M, \mathcal{S})$  is isomorphic, as a Cartan geometry, to a quotient  $\Gamma \backslash \tilde{\mathbf{X}}$ , for some discrete group  $\Gamma \subset \tilde{G}$ .*

We refer to Section 2.1.1 for the definition of parabolic geometries. The regularity condition involves conditions on the curvature, and is harmless since it is part of the normalization made on the Cartan connection to ensure uniqueness when the equivalence problem is solved.

The second point of the theorem might be compared to Theorem 3.2. It is another nice illustration of the principle stated in the introduction, that rigid geometric structures with large automorphism group should be very peculiar. Here, the largeness of the automorphism group is expressed by the fact that  $\text{Aut}(M, \mathcal{S})$  has the maximal real rank allowed.

**Remark 5.5.** If  $H$  is not assumed to be connected, the inequality

$$\text{rk}^{\text{alg}}(\text{Ad } H) \leq \text{rk}(\text{Ad}_{\mathfrak{g}} P)$$

is still true, provided the kernel of the morphism  $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{h})$  is amenable

## 5.5 Illustrations

We are going to illustrate Theorem 5.4, by considering actions on several geometric structures. We will be interested in the following groups:

- For  $n \geq 2$ , we consider the subgroup of affine transformations of  $\mathbb{R}^n$  given by  $\Gamma_n = \text{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ . The real rank  $\text{rk}(\text{Ad } \Gamma_n)$  is zero, but its algebraic rank  $\text{rk}^{\text{alg}}$  is  $n - 1$ .
- For  $n \geq 2$ , we introduce  $R_n = L_n \ltimes \mathbb{R}^n$ , the subgroup of affine transformations of  $\mathbb{R}^n$ , for which

$$L_n = \left\{ \left( \begin{pmatrix} e^{t_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{t_n} \end{pmatrix}, (t_1, \dots, t_n) \in \mathbb{R}^n \right) \right\}.$$

The group  $R_n$  is a semi-direct product  $\mathbb{R}^n \ltimes \mathbb{R}^n$ . The real rank  $\text{rk}(\text{Ad}(R_n))$ , as well as the algebraic rank  $\text{rk}^{\text{alg}}(\text{Ad}(R_n))$  are equal to  $n$ .

- For every  $n \geq 2$ , the group  $U_n$  of unipotent, upper-triangular matrices in  $\text{GL}(n, \mathbb{R})$  is nilpotent, with index of nilpotency equal to  $n - 1$ . The index  $\text{nilp}(\text{Ad}(U_n))$  is equal to  $n - 2$ .

We can now state some consequences of the previous theorem.

- (1) Let  $(M, g)$  be a compact pseudo-Riemannian manifold of type  $(p, q)$ , where  $1 \leq p \leq q$ . Pseudo-Riemannian structures of type  $(p, q)$  are Cartan geometries whose model space is  $\mathbf{E}^{p,q} = G/P$  where  $G = \text{O}(p, q) \ltimes \mathbb{R}^{p+q}$  and  $P = \text{O}(p, q)$ . The real rank of  $\text{Ad}_{\mathfrak{g}} P$  is  $p$ , and its index of nilpotency is  $2p - 1$ . We knew thanks to Zimmer's theorem 5.1 that  $\text{SL}(m, \mathbb{R})$  cannot act isometrically on  $(M, g)$  if  $m \geq p + 2$ . Theorem 5.4 and Remark 5.5 yield the same conclusion for the group  $\Gamma_m$ . In the same way, there is no isometric action of  $R_m$  on  $(M, g)$  as soon as  $m \geq p + 1$ , and the same is true for  $U_m$  if  $m \geq 2p + 2$ .

- (2) When  $p+q \geq 3$ , the conformal class of the type  $(p, q)$  pseudo-Riemannian manifold  $(M, g)$  defines a unique normal Cartan geometry modelled on the space  $\mathbf{Ein}^{p,q} = O(p+1, q+1)/P$  (where  $P$  is the stabilizer of an isotropic line in  $O(p+1, q+1)$ , see Section 2.1.1). One computes  $\mathrm{rk}(\mathrm{Ad}_{\mathfrak{g}} P) = p+1$  and  $\mathrm{nilp}(\mathrm{Ad}_{\mathfrak{g}} P) = 2p+1$ .

Hence, for instance, a Lie group  $H$  acting conformally on a compact Lorentzian manifold must satisfy  $\mathrm{rk}(\mathrm{Ad} H) \leq 2$ . We thus infer that  $\mathrm{SL}(4, \mathbb{R})$ , which has real rank 3, does not admit such a conformal action. Actually, this is also true for  $\mathrm{SL}(3, \mathbb{R})$ , even though this group has rank 2. Indeed, by the second point of Theorem 5.4, a conformal action of  $\mathrm{SL}(3, \mathbb{R})$  on a compact Lorentzian manifold can only occur on a quotient  $\Gamma \backslash \widetilde{\mathbf{Ein}}^{1, n-1}$ , which is conformally flat. Hence, such an action would provide an embedding of Lie algebras  $\mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{o}(2, n)$ , and it is rather easy to check that it is impossible.

Using the bounds provided by Theorem 5.4 and Remark 5.5, we infer more generally that there does not exist any conformal action of  $\Gamma_m$  on a compact type  $(p, q)$  manifold  $(M, g)$  as soon as  $m \geq p+3$ . The same conclusion holds for  $R_m$  when  $m \geq p+2$ , and for  $U_m$  if  $m \geq 2p+3$ .

- (3) As a last example, let us consider a compact manifold  $M$  of dimension  $n \geq 2$ , endowed with a linear connection  $\nabla$ . This connection defines a unique Cartan geometry modelled on the affine space  $\mathbf{A}^n = G/P$ , where  $G = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$  and  $P = \mathrm{GL}(n, \mathbb{R})$ . One checks that  $\mathrm{rk}(\mathrm{Ad} P) = n$ , and  $\mathrm{nilp}(\mathrm{Ad} P) = n-1$ . We infer that whenever  $m \geq n+2$ , neither  $\Gamma_m$ , nor  $U_m$  can act on  $M$  preserving  $\nabla$ . The same conclusion holds for  $R_m$  if  $m \geq n+1$ .

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