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Data Simulations for Voting by Evaluation

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Abstract

Considering voting rules based on evaluation inputs rather than preference rankings modifies the paradigm of probabilistic studies of voting procedures. We propose in this paper several simulation models for generating evaluation-based voting inputs. These models can cope with dependent and non identical marginal distributions of the evaluations received by the candidates. A last part is devoted to fitting these models to real data sets.

Keywords: Voting rules, Evaluation-based voting rules, Simulation, IC model.

1 Introduction

Voting rules can be seen as functions which aim at determining a winner from a set of candidates considering preferences of a set of voters. Both social and mathematical approaches consider positive or negative properties satisfied by a given voting rule as a matter of interest. Studying the properties of voting rules can be done either in an axiomatic approach or in a probabilistic approach. The axiomatic approach supposes to determine which properties characterize a specific voting process, *i.e.* which properties are to be observed, and which are not, via formal theorems. The probabilistic approach aims at determining

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whether these properties are likely to be observed, *i.e.* determining the frequency of occurrence of such properties using a given voting rule. This pragmatic approach is often based on simulations of voting situations. Tideman and Plassmann [44, 46], Plassmann and Tideman [39], Green-Armytage et al. [22] contain examples of simulation-based studies of voting rules. The latter considers only voting rules based on rankings on candidates. Our objective here is to propose simulation frameworks for a probabilistic approach of evaluation-based voting rules. A review of both ranking-based and evaluation-based voting processes can be found in Felsenthal and Machover [19], Section 3.3.

Several simulation models exist, which have been well studied for years. One can refer to Diss and Kamwa [14] for a recent state-of-the-art of simulation techniques for a probabilistic approach of voting theory. Chapter 8 of Alvo and Philip [1] also provide an overview of probabilistic models on rankings. Two main approaches can be distinguished: modeling directly the preferences rankings or introducing latent variables corresponding to the preferences of the voters, that is, the intensity of accordance between voters and candidates. These preferences are often called utilities. Rankings are deduced from the orders of these latent variables. In the latter approach, the latent variables are continuous and the modeling model is defined through their probabilistic distribution. The setting is, at first sight, very similar to a continuous evaluation framework. Yet, to our knowledge, latent preferences are defined with an infinite support (going from $-\infty$ to ∞ , while the evaluation of a candidate by a voter is defined on a finite support. The context is, hence, different, and a new paradigm has to be set up.

We propose in this paper to investigate several models and methods to simulate data for studying evaluation-based voting rules. Our objective is to generate new voting situations. This should be useful to study the occurrence of statistical properties or paradoxes for different voting rules in a general context. To this aim, we need modeling voting situations through the rationalization of probabilistic distributions of evaluations upon candidates. We propose some general models with a large scope, which could be used for such a modeling. The package `VoteSim` in R [41], available on the CRAN repository, provides the corresponding code.

We first introduce evaluation-based voting methods in Section 2. Section 3 proposes some distributions for univariate evaluations. We discuss next about the relevance of Impartial Culture (IC) framework in evaluations. Independent and identically distributed (i.i.d.) voters' evaluations on each candidate yield IC models on preference rankings. The relaxations of the i.i.d. model enlarge the scope of generating models, introducing the possibility of modeling given situations such as the polarizing character of a candidate, or the positive or negative correlations between the evaluations for two candidates. Section 4 is devoted to the different characteristics of the generating models, where voter's evaluations on each candidate can be i.i.d., can be differently distributed (but still independent) or can be dependent. We introduce among others copula-based evaluations models and spatial models. Section 5 is devoted to an illustration on fitting on observed situations.

2 Evaluation-based voting methods

Many classical voting methods are based on preference rankings : first-past-the-post voting, runoff voting, Borda count, etc. These methods are vulnerable to numerous known “paradoxes” [19] and impossibility theorems. The most famous is Arrow’s impossibility theorem [3]. Alternative methods based on evaluations exist. Even if these methods are also vulnerable to some paradoxes (e.g. Gibbard-Satterthwaite theorem, [40]), obtained results are promising and stretch some limits of the rankings-based methods. We refer to [4] for a global vision of these evaluations-based methods from a statistical point of view. The three most famous methods based on evaluations are approval voting, range voting and majority judgment.

The approval voting (see [10] for a complete study) is maybe the most famous of these methods: each voter evaluates candidates on a scale of 2 grades, which is the simplest possible scale. The voter grades 1 if the candidate is acceptable, else 0. The voter can then approve for several candidates, even all of them or none of them, accordingly to his/her convictions. Note that this method is very simple to apply in practice. The two other methods are based on more nuanced classes of grading, which can be continuous or discrete. Approval voting can be seen either as a range voting or as a majority judgment, with a grading scale reduced to a binary scale 0 or 1. With the range voting, described by Smith [42], the winner is the candidate with the highest average grade. With the majority judgment, introduced by Balinski and Laraki [5, 6], the voters give mentions to the candidates and the winner is the candidate with the highest median mention. Tie-break situations are, here, a matter of importance and are taken into account for example in [7] and [18].

The recent interest in evaluation-based votes yields a need of simulation methods to study their properties. Hence, our objective is to extend preference simulation procedures to evaluations.

Notations

In the following, we will consider situations with n voters and m candidates. Each voter associates a grade from a set \mathcal{E} to each candidate. Evaluation of voter v for candidate c will be denoted e_{vc} , for $v = 1, \dots, n$, $c = 1, \dots, m$. Observations $\{(e_{vc})_{v=1, \dots, n, c=1, \dots, m}\}$ are n independent realizations of a random variable $E = (E_1, \dots, E_m)$, which takes values in \mathcal{E}^m . Defining a simulation setting can be seen as defining a multivariate probability distribution on \mathcal{E}^m . We will consider two cases with respect to the amount of information contained in the set \mathcal{E} :

- continuous grades: without loss of generality, $\mathcal{E} = [0, 1]$;
- discrete grades: without loss of generality, $\mathcal{E} = \{0, \dots, K\}$ with $K \in \mathbb{N} \setminus \{0\}$.

As observed by Critchlow et al. [11], one can generate any distribution on rankings from continuous

random variables. The *Thurstone order statistics models* (see e.g. [1, chapter 8] and references therein) aim at building probability distributions on rankings by ordering continuous random variables. Considering continuous grades, a strict preference relation on the candidates can be obtained from the evaluations for each voter. Following Thurstone models, we can deduce a distribution on rankings. In case of discrete grades, *ex-aequos* may occur, and then a pre-order can be obtained from evaluations, but not a complete order.

Note that approval voting needs specific preference information as input: the discrete grades scale have only two levels 0 and 1. Direct approval scores generators can be proposed, based on Bernoulli's distribution for example. However, it is also very simple to obtain approval preferences on the candidates from discrete or continuous grades. For example, [31] proposes several different approval voting rules based on evaluations / preference orders on candidates. Therefore we do not further explore the specific case of approval voting in the following.

Evaluations versus preference rankings

In voting framework based on preference rankings, the Impartial Culture model (IC model) seems to be the most widely used in simulation. Introduced by Guilbaud [23] in 1952, IC model supposes that each preference order on the candidates is equally likely to be selected by each voter.

IC models correspond to a uniform distribution on rankings. Other distributions were introduced. The two main approaches are I. Thurstone order statistics modeling and II. modeling directly on rankings. Approach I. considers a modeling of the ranking process, using latent variables. Dating back to [43], the idea is to describe mathematically the psychological processes that a voter undergoes to produce a ranking of the candidates. It leads to latent continuous random variables measuring how close a voter is to a candidate. Rankings are then obtained by the ordering of these latent variables. Approach II. aims to describe parametrically the distribution of rankings. Following [1], it can be decomposed in three main families: paired comparison models, distance-based models, and multistage models.

As observed by [30], approach I helps in developing models for approach II, and vice versa, and the dichotomy is not an issue. We also refer to [11] and [1, Chapter 8] for an overview of the above models. Observe that with evaluation-based voting processes, approach I appears much more adequate, since it leads to both evaluations and rankings modeling. In our context, the main difference with this approach is that we are interested in the distribution of the evaluations itself, but not in the rankings. Moreover, in Thurstone order statistics models, the latent variables measuring the preferences of voters are informal, and defined on infinite supports. The evaluations in our context are defined on a fixed bounded support, possibly discrete.

The easiest distributions one can consider for the m random variables E_1, \dots, E_m are independent and identically distributed (i.i.d.) distributions. In such a case, the preferences resulting from the evaluations

will satisfy an IC model, that is, a uniform distribution on rankings. This model can be extended to more complex ones following two ways. The first way is to remove the identical distribution assumption: two candidates may not have the same evaluation distribution in the population of voters. The second way is to introduce dependence between the evaluations of the candidates. Indeed, two candidates with a similar profiles may lead to positively correlated evaluations, while two candidates with opposite profiles may lead to negatively correlated evaluations.

Reasoning on the resulting model on rankings seems too reducing for evaluation-based processes. It appears more appropriate to discuss about the independence hypothesis and the identity of marginal distributions, which provides finer information on the evaluations. Introducing these complexities in generation yields more general models. It is interesting to analyze the behavior of voting processes in different contexts of distribution or/and dependencies.

3 Univariate distributions on evaluations

We propose in this section several distributions to model the evaluation distribution on a single candidate by several voters.

Continuous case

When the evaluations are continuous, that is, $\mathcal{E} = [0, 1]$, we propose to use Uniform, truncated Normal or Beta distributions. Uniform distribution is the easiest model. Truncated Normal distribution is useful to model uni-modal distributions when the proportions of evaluations close to 0 or 1 are significant (see Figure 1). Beta distribution is a very versatile model that can model several situations, depending on the chosen parameters; it is particularly adapted in the case of bi-modal distributions, when most of the evaluations are around 0 or 1, but also in the case of uni-modal distribution with a quick decreasing curve.

Examples of probability distribution functions for Uniform, truncated Normal and Beta distributions are presented in Figure 1. It can be seen that these three families of distributions cover a large scope of distributions.

The initial distribution proposed by [43] is the standard Gaussian. Our truncated Normal distribution is, hence, a generalization to bounded-support evaluations. Other distributions were introduced in Thurstone’s like approach. The most famous is the Gumbel distribution proposed by [29], which has the advantage of providing a closed form distribution of rankings. Considering a Beta distribution is a natural extension to bounded-support distributions, even if the main interest is its wide range of shapes.

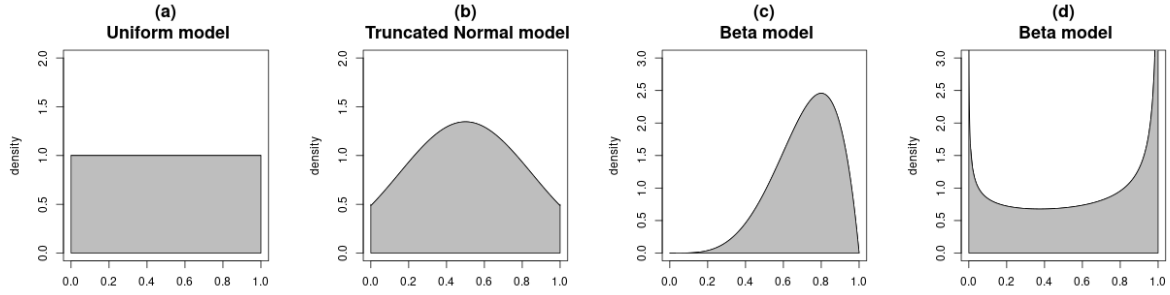


Figure 1: Probability distribution functions of evaluations on one candidate. (a) Uniform distribution, (b) truncated Normal distribution with $\mu = 0.5$ and $\sigma = 0.35$, (c) Beta distribution with $\alpha = 5$ and $\beta = 2$ and (d) Beta distribution with $\alpha = 0.7$ and $\beta = 0.5$.

Discrete case

When the evaluations are discrete, i.e. $\mathcal{E} = \{0, \dots, K\}$, we propose to use discrete Uniform, Binomial or Beta-Binomial distributions, which can be seen as the discrete counterpart of the Uniform, truncated Normal and Beta continuous distributions. Examples are presented in Figure 2.

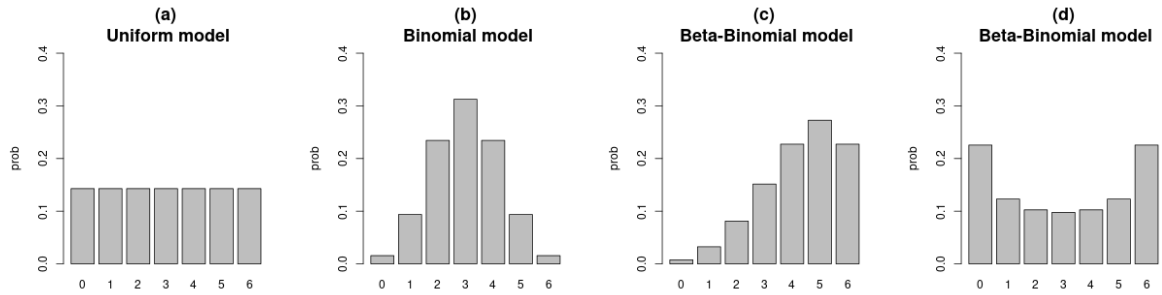


Figure 2: Examples of probability distributions on 7 levels. (a) Uniform distribution, (b) Binomial distribution with $p = 0.5$, (c) Beta-Binomial distribution with $\alpha = 5$ and $\beta = 2$, and (d) Beta-Binomial distribution with $\alpha = \beta = 0.5$.

Parameters for each model are summed up in Table 1.

Marginal continuous distributions	Parameters	Marginal discrete distributions	Parameters
Continuous Uniform $\mathcal{U}[0, 1]$.	Discrete Uniform $\mathcal{U}\{0, \dots, K\}$.
Truncated Normal $\mathcal{N}_T(\mu, \sigma^2)$	(μ, σ)	Binomial $\mathcal{B}(K, p)$	p
Beta $\mathcal{B}(\alpha, \beta)$	(α, β)	Beta-Binomial $\mathcal{B}(K, \alpha, \beta)$	(α, β)

Table 1: Marginal distributions proposed for the distribution of the evaluations. K is not a flexible parameter since it corresponds to the known scale of the evaluations.

General remarks

A high dispersion of evaluations corresponds to candidates that divide the voters population between strongly pro and strongly cons voters, like for example for the distribution displayed in Figure 1(d). Whereas some candidates are less polarizing, like for example for the distribution displayed in Figure 1(c).

Of course, these distributions are not exhaustive and many choices are tractable. We focus here on the most natural, which already seem to cover various shapes. An extension can be proposed by the use of a mixture model of distributions. For example, [34] proposed mixture models for rankings. This is especially interesting when fitting models on real voting elections [16, 21]. As our main objective is to propose generating models, we do not explore this direction here. But note that none of the distributions presented here is multimodal without modes in the extremities. This is a limit for the fitting of real data, that mixtures can overcome.

The earlier methods require certain assumptions about the distribution of the evaluations, which must be validated. These assumptions can be assessed using goodness-of-fit tests such as the Chi-Square test, Kolmogorov-Smirnov test, Anderson test, and others. In instances where the theoretical distribution does not align with the observed data, more versatile techniques become pertinent. Specifically, non-parametric approaches like kernel density estimation or projection-based estimation of evaluation densities can be employed. This spectrum of methods ensures both adaptability and verifiability of the approach.

4 Multivariate evaluation models

To generate a voting framework, it is necessary to define a multivariate distribution. This section introduces distributions for the random variables E_1, \dots, E_m corresponding to voters' evaluations for each of the m candidates. First, i.i.d distributions can be considered, resulting in an IC structure on rankings. Subsequently, the assumption of identical distributions across candidates and the assumption of independence can be progressively relaxed.

4.1 Independent and identically distributed evaluation models

The simplest case of evaluation-based voting process is to consider that voters' evaluations of each candidate are i.i.d., which we will call an Independent and Identically Distributed evaluations (Ev-IID) model.

Definition 1. Ev-IID model

The Independent Identically Distributed evaluations (Ev-IID) model based on a distribution \mathcal{D} on \mathcal{E} is such that random variables E_1, \dots, E_m are independent and identically distributed with distribution \mathcal{D} .

As far as the evaluations are i.i.d., Ev-IID models yield IC models when considering only rankings. We propose hereafter alternatives to Ev-IID models. The first possibility is to deal with independent and not identically distributed distributions. Such models will be called Independent and Differently Distributed evaluations (Ev-IDD) Models.

4.2 Independent and differently distributed evaluation models

In social choice, there occur situations where some candidates divide the voters population between strongly pro and strongly cons voters, whereas some other candidates are less polarizing, as discussed previously. Therefore, the evaluation distribution for each candidate can be independent from the others, but the distributions are not identical. Such models, even if independence holds, allow different evaluation distributions for each candidates, and they will be denoted as Ev-IDD models.

Definition 2. Ev-IDD model

The Evaluation Independent and Differently Distributed (Ev-IDD) model is such that the random variables E_1, \dots, E_m are independent.

The marginal distributions of (E_1, \dots, E_m) can be different by changing parameters of a given distribution family, or by changing the distribution family. As noticed previously, Beta (continuous) distributions and Beta-Binomial (discrete) distributions enable to consider various shapes of distributions.

4.3 Dependent evaluation simulation models

A possibility to extend Ev-IID models (and Ev-IDD models) is to remove the independence hypothesis between the evaluations of each candidate. In such a case, the marginal distributions can be either identical or different. Hence, these models can be considered as Dependent and Identically Distributed evaluations models (Ev-DID). But considering non identical distributions provide Dependent and Differently Distributed evaluations models (Ev-DDD). Note that contrary to Ev-IID models, Ev-DID models do not imply IC models on preference rankings.

Definition 3. Ev-DID model

The *Dependent and Identically Distributed evaluations (Ev-DID)* model based on a distribution \mathcal{D} on \mathcal{E} is such that random variables E_1, \dots, E_m are dependent and identically distributed with distribution \mathcal{D} .

Definition 4. Ev-DDD model

The *Dependent and Differently Distributed evaluations (Ev-DDD)* model is such that the random variables E_1, \dots, E_m are dependent and non identically distributed.

We propose in the following three Ev-DDD models based on three different approaches : the multinomial or Dirichlet distribution, the use of copulas, and the spatial model.

Multinomial and Dirichlet models

Ev-IID and Ev-IDD models suppose that each voter evaluates all the candidates independently, without any constraint on the evaluations vector. However, an alternative evaluation process consists in dividing a total score on the candidate evaluations, such that the evaluations' sum on the set of candidates is the same for each voter. Therefore the evaluations are not independent as there is a link between the evaluations for all the candidates given by a single voter.

Both discrete and continuous models are available, using multinomial distribution (a multivariate version of the Binomial distribution) in the discrete case and Dirichlet distribution (a multivariate generalization of the Beta distribution) for the continuous case.

When the evaluations are discrete, i.e. $\mathcal{E} = \{0, \dots, K\}$, the Ev-DDD multinomial model is the following.

Definition 5. Ev-DDD multinomial model

The *Ev-DDD multinomial model* is defined by $(E_1, \dots, E_m) \sim \mathcal{M}\{K, p_1, \dots, p_m\}$, where $\mathcal{M}\{K, p_1, \dots, p_m\}$ is the multinomial distribution of parameters K and p_1, \dots, p_m , with for all $c = 1, \dots, m$, $p_c \geq 0$ and $\sum_{c=1}^m p_c = 1$.

The marginal distributions of a multinomial distribution are Binomial distributions. The multinomial introduces dependence between the evaluations of the candidates. More precisely, if parameters $(p_c)_{c=1, \dots, m}$ do not have the same value, then the evaluations are not identically distributed and the associated multinomial model is then an Ev-DDD model. If for all $c = 1, \dots, m$, $p_c = 1/m$, then the associated multinomial model is an Ev-DID model. Note that K is the maximal authorized grade.

Figure 3 shows an example of a Ev-DDD multinomial model for 3 candidates and $K = 6$, with the probability vector $(0.5, 0.3, 0.2)$.

The continuous counterpart of the multinomial distribution is obtained through the use of a Dirichlet distribution on $[0, 1]^m$ as follows (see [37] for details about Dirichlet distribution).

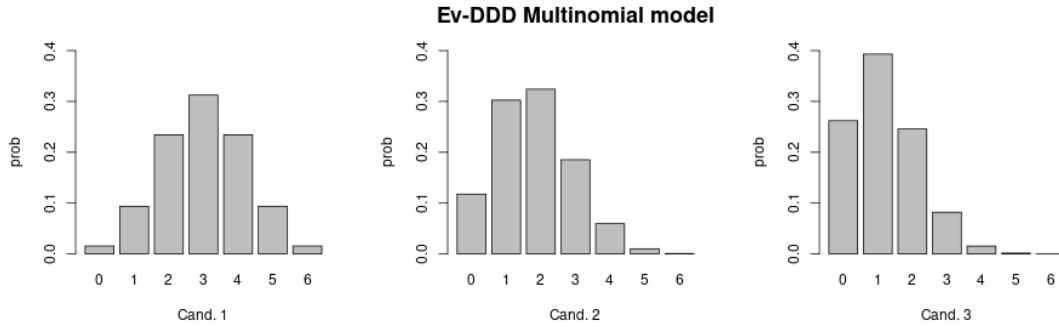


Figure 3: Barplots of evaluations for each of the $m = 3$ candidates in a Ev-DDD multinomial model with $K = 6$ and a probability vector $(0.5, 0.3, 0.2)$.

Definition 6. Ev-DDD Dirichlet model

The Ev-DDD Dirichlet model is defined by $(E_1, \dots, E_m) \sim \text{Dir}\{\alpha_1, \dots, \alpha_m\}$, where $\text{Dir}\{\alpha_1, \dots, \alpha_m\}$ is the Dirichlet distribution of parameters $\alpha_1, \dots, \alpha_m$, with for all $c = 1, \dots, m$, $\alpha_c > 0$.

The Dirichlet distribution is an extension of the Beta distribution to the multivariate case. The marginal distributions are Beta distributions and Dirichlet modeling introduces dependence between the evaluations of the candidates. If parameters $(\alpha_c)_{c=1, \dots, m}$ do not have all the same value, then the Dirichlet model is an Ev-DDD model. If for all $c = 1, \dots, m$, $\alpha_c = 1$, then the Dirichlet model is an Ev-DID model. We refer to Ng et al. [37, Chapter 1 and Chapter 2] and Lin [27] for an overview on Dirichlet distribution.

An example of simulation with 3 candidates using different probabilities for the candidates is shown in Figure 4. The links between the evaluations of the three candidates is shown in Figure 5.

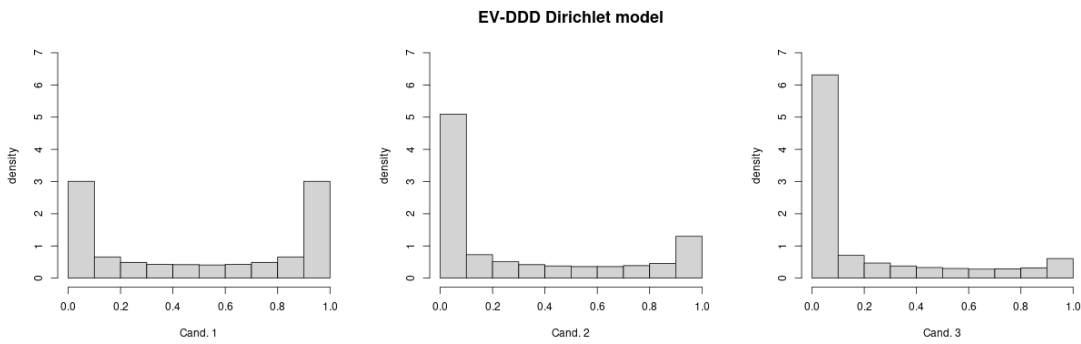


Figure 4: Histograms of evaluations for each of the $m = 3$ candidates in a Ev-DDD Dirichlet model with parameters vector $(5, 3, 2)$.

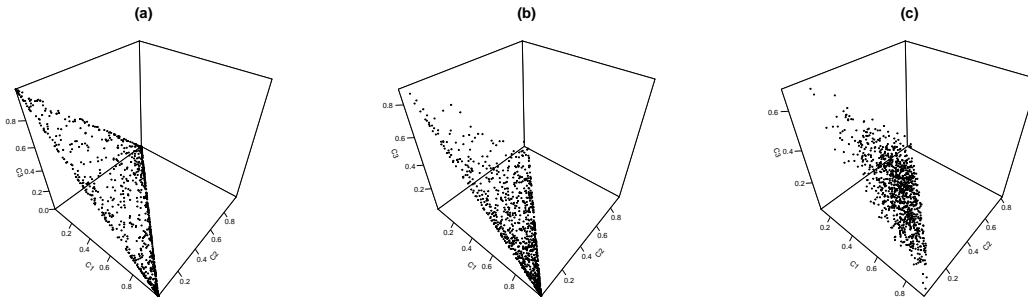


Figure 5: Simulations of evaluations for each of the $m = 3$ candidates ($n = 1000$ voters) in a Ev-DDD Dirichlet model with (a) $(\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.3, 0.2)$, (b) $(\alpha_1, \alpha_2, \alpha_3) = (2, 0.5, 0.5)$, (c) $(\alpha_1, \alpha_2, \alpha_3) = (5, 3, 2)$.

Copula-based models

Copulas are useful tools to represent dependencies between variables. If the distributions of the evaluations of each candidate are not independent, a multivariate copula can be used to take into account their dependencies. In a nutshell, a copula is a multivariate cumulative distribution function which has all its margins uniformly distributed on the unit interval. It can also be applied on transform of random variables to generate dependence with non Uniform marginals. See [20, 36] for a formal presentation of the subject.

Definition 7. Copula Ev-DDD models

The Copula Ev-DDD models are defined by

$$E = (E_1, \dots, E_m) \sim C(\delta_1, \dots, \delta_m),$$

where C is a multivariate copula and $\delta_1, \dots, \delta_m$ are distributions on \mathcal{E} .

A strength of copulas is that they allow any marginal distributions. Therefore, the model should specify both the marginal distribution for each candidate (*e.g.* with continuous or discrete distributions of Section 2), and the copula used to model the dependencies between variables. Note that in Thurstone order statistics modeling, with continuous variables, copulas were introduced by McFadden [32] and studied *e.g.* by [25]. In the following, we distinguish the cases of a continuous set \mathcal{E} and a discrete set \mathcal{E} .

Copulas are a wide family which offers several possibilities to model dependencies. Among others, in the continuous case Gaussian copulas offer a simple way to model dependencies between each pairs of candidates, through the correlation coefficients. The dependencies of the copula between two variables

are exactly characterized by the correlation coefficients. This model has therefore the advantage to be easy to simulate and to enable to define in a very comprehensive way the dependence between the evaluations. Another interesting copula class is the checkerboard copula class [12], which represents a good compromise between the richness of the expression and the complexity of the model. See *e.g.* [17] for a discussion on the choice of a copula.

Evaluations on discrete scales need the use of specific discrete copulas for simulation, as copulas are different for discrete and continuous cases. Among others, pair-copulas [38] and Gaussian copulas [8] have been proposed to simulate dependent discrete data, and therefore can be used also to model discrete evaluations in a social choice framework.

Next, marginal distributions must be chosen. One can consider the same distribution for each candidate and obtain Ev-DID models, or different distributions following Ev-DDD models. Except for the Copula Ev-DID Uniform discrete and continuous models, which belong to the Impartial Culture setting since all candidates have the same distributions, each of these models allows for different marginal distributions for the evaluations of each candidate. Additionally to this non identical setting, the dependence modeling, through the copula, yields a large scope of models. These copula-based models appear very rich and adapted for covering much framework of simulations of evaluations.

Figure 6 presents a simulation based on the use of two different marginal distributions for two candidates (Beta distribution with parameters (0.7,0.5) for the first candidate and (0.5,0.7) for the second one), using a Gaussian copula of parameter 0.8.

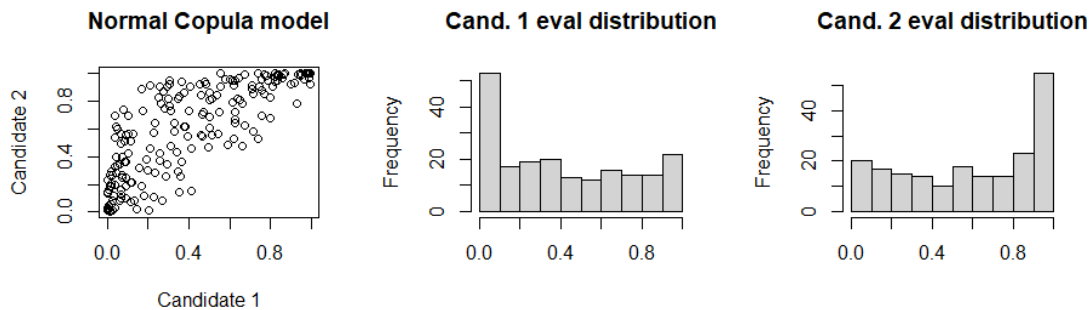


Figure 6: Simulation examples of Ev-DDD Copula evaluations for $m = 2$ candidates and $n = 200$ voters.

Spatial models

Spatial voting simulations have been developed following the early work of [15] for votes based on rankings. The model is based on the use of an euclidean distance between the candidates and the voters, living

in the same uni- or multi-dimensional space: the smaller the distance, the better the rank. Tideman and Plassmann [45] conclude that a spatial model “describes the observations in data sets much more accurately” than other models.

We propose an adaptation of the spatial model in the framework of evaluation-based voting. Let d be a given dimension parameter. Parameter d should be seen as the number of latent characteristics which are used to build an opinion on the candidates. Typically, $d = 2$ or 3 , see [2] for a discussion on the choice of d . Voters and candidates are then randomly generated as points inside the hypercube $[0, 1]^d$. Spatial voting is next based on the distances between the generated points. The closer a voter to a candidate, the higher their evaluation of this candidate.

Definition 8. Ev-DDD Spatial models

Let $x_v, v = 1, \dots, n$, and $y_c, c = 1, \dots, m$ be independent realizations from a distribution on $[0, 1]^d$, $d \in \mathbb{N} \setminus \{0\}$. The Ev-DDD spatial model for the evaluation e_{vc} of candidate c by voter v is defined as

$$\forall v = 1, \dots, n, \forall c = 1, \dots, m, \quad e_{vc} = f(\delta(x_v, y_c)),$$

where δ is a distance between x_v and y_c and f a non-increasing function mapping \mathbb{R}^+ to $[0, 1]$.

Typically, an intuitive spatial simulation model is given by the choices of

- a spatial distribution for the voters and the candidates, $x_v, v = 1, \dots, n$ and $y_c, c = 1, \dots, m$. A classical choice is the Uniform distribution without any additive information about the voters, see Figure 7. More specific distributions like a Gaussian distribution (resp. a mixing of Gaussian distribution) allow to obtain a bigger concentration of voters in a specific area of the unit cube (resp. different areas). See e.g. [35].
- a distance. A usual distance is the Euclidean distance δ_e . Numerous distances exist, each with specific properties.
- a link function, f . For instance, for $v = 1, \dots, n$ and $c = 1, \dots, m$, $e_{vc} = \max\{0, (1 - \ell \times \delta_e(x_v, y_c))\}$ with δ_e the Euclidean distance and $\ell > 0$. The parameter ℓ defines the decreasing rate of the evaluations with respect to the distance. For example, ℓ greater than 2 ensures that a voter being anywhere on the frontier of the unit cube will give a null score to a candidate who is on the center of the unit cube or, equivalently, that a voter who is on the center of the cube will give a null score to an extreme candidate on the frontier of the cube.

Other link functions f are also possible, as for example the sigmoid which is defined as follows: for $v = 1, \dots, n, c = 1, \dots, m, e_{vc} = (1 + e^{\lambda(\beta\delta_e(x_v, y_c) - 1)})^{-1}$, with $\lambda > 0$ and $\beta > 0$. Figure 8 presents an example of such a function for $\lambda = 5$ and $\beta = 2$.

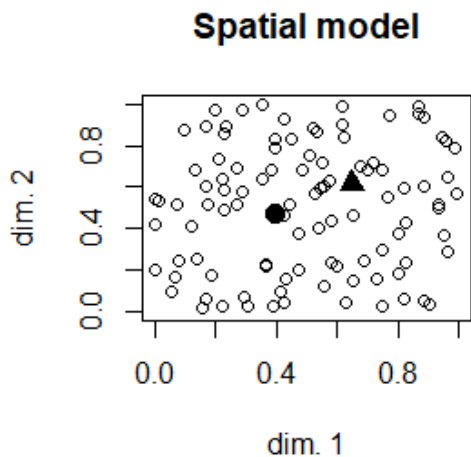


Figure 7: Simulation example of spatial model, with $m = 2$ candidates (\blacktriangle is candidate 1 and \bullet is candidate 2) and $n = 100$ voters in a 2-dimensional space ($d = 2$), obtained through an Uniform distribution on $[0, 1]^2$.

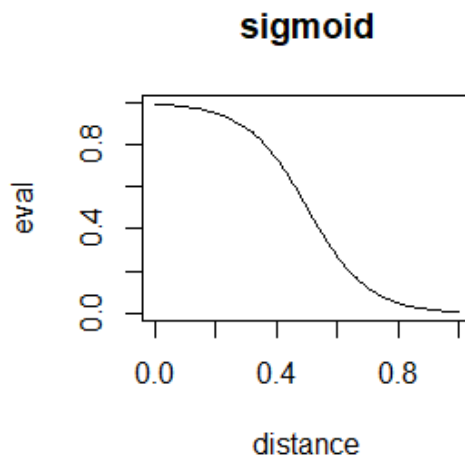


Figure 8: Example of sigmoid transformation, $\delta \mapsto (1 + e^{\lambda(\beta\delta-1)})^{-1}$, with $\lambda = 5$ and $\beta = 2$.

For a given position of candidates in $[0, 1]^d$, the spatial model is clearly an Ev-DDD model, since the marginal distributions are different (see for example Figure 9) and potentially correlated (see for example Figure 10).

An example of evaluations obtained through a spatial model is presented in Figure 7. Two candidates and $n = 100$ voters have been randomly generated in $[0, 1]^2$, that is, with $d = 2$. Evaluations are then obtained from the euclidean distance of each voter to each candidate. We propose in Figure 9 histograms of three different ways to obtain evaluations, based on the same spatial situation (the one described in Figure 7), for each candidate:

- evaluations in model 1 are obtained using $e_{vc} = \max\{0, (1 - 2 \times \delta_e(x_v, y_c))\}$,
- evaluations in model 2 are obtained using $e_{vc} = (1 + e^{\lambda(\beta\delta_e(x_v, y_c)-1)})^{-1}$ with $\lambda = 5$ and $\beta = 2$,
- evaluations in model 3 are obtained using $e_{vc} = (1 + e^{\lambda(\beta\delta_e(x_v, y_c)-1)})^{-1}$ with $\lambda = 2$ and $\beta = 2$.

Spatial models with discrete evaluations on $\{0, \dots, K\}$ can easily be obtained from continuous models by dividing the $[0, 1]$ interval onto the $K + 1$ intervals. Then if the continuous grade obtained with the

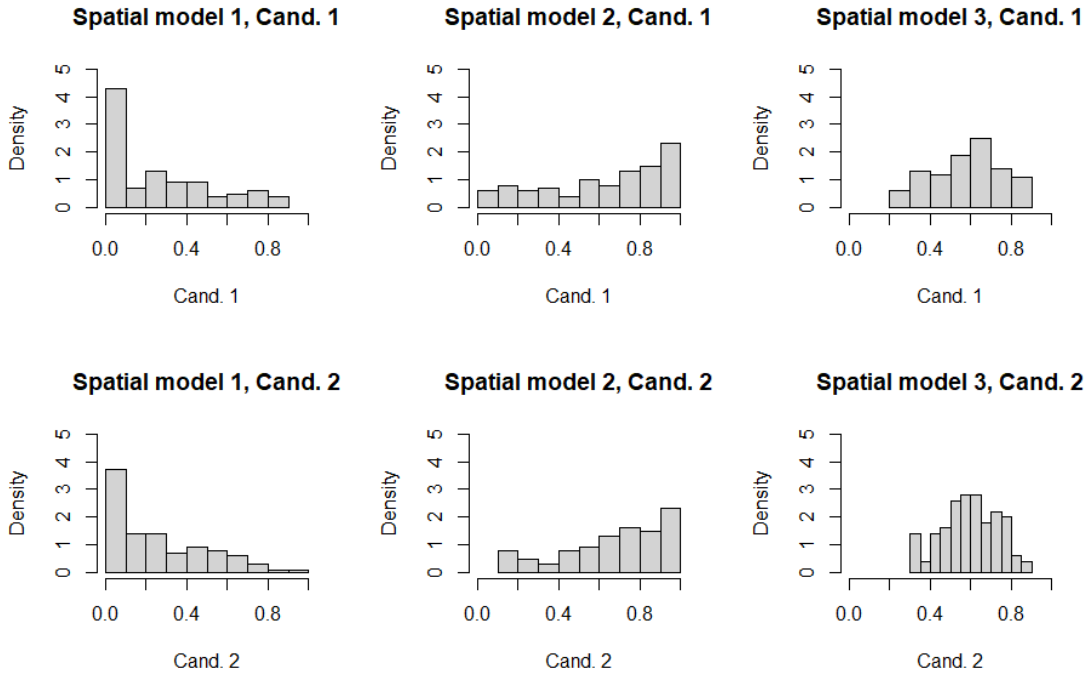


Figure 9: Histograms of evaluations obtained through the spatial model of Figure 7 and calculation models 1, 2 and 3 defined above.

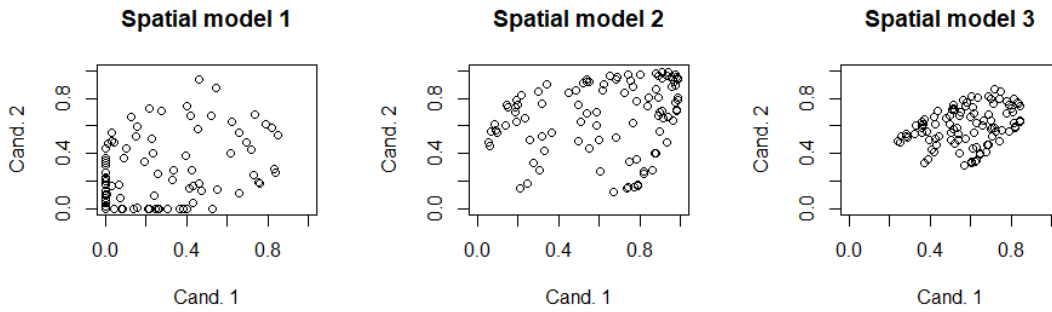


Figure 10: Plot of evaluations of candidate 2 vs candidate 1 given by each voter, with calculation models 1, 2 and 3 defined above.

spatial model belongs to the l^{th} interval, the discrete evaluation is set to $l - 1$. That is, for a continuous

evaluation e_{vc} on $[0, 1]$, we consider $\lfloor (K + 1)e_{vc} \rfloor$ as the resulting discrete evaluation, where $\lfloor u \rfloor$ denotes the greatest integer lower than u . The spatial interpretation of such a process is to determine K spheres centered on the candidate and to give the evaluation of K if the voter is into the smallest sphere, $K - 1$ for the second smallest sphere and so on, until 0.

5 Fitting real data

Simulation models proposed above are based on theoretical considerations. Real voting situations do not follow pure theoretical models. Voters evaluations in real life depend on many latent factors that cannot be easily modeled. Moreover, the voters do not always have homogeneous behaviors. There may exist several groups of voters, with different distributions of evaluations of the candidates. Therefore, it is an illusion to think that a single model can capture a real vote situation.

Remark 1. Mixture models can be a solution to deal with these heterogeneous situations and capture the complexity of real life situations. But a danger is then to overfit the data with too much parameters. That's why we do not propose here mixture models as our goal is just to illustrate how the proposed models are convenient to represent real-world situations.

Remark 2. Note that a Bayesian procedure could also be used to fit the models (see e.g. [48, 33, 24] for Bayesian estimation of Thurstone order statistics models). We rather use a frequentist approach here.

A guideline for the choice of a specific model could be the following:

1. Choose a general parametric model for each distribution.
2. Fit the marginal distributions of the evaluations of each candidate, E_1, \dots, E_m to the chosen model.
3. Test if these marginal distributions can be considered as identical.
4. Test the independence of E_1, \dots, E_m .

The results of the distributions equality (id. distrib.) and independence tests lead to the cases summarized in Table 2.

	Identical Distributions = True	Identical Distributions = False
Independence = True	Ev-IID	Ev-IDD
Independence = False	Ev-DID	Ev-DDD

Table 2: Correspondence between results of statistical tests and models on evaluations.

This guideline is relevant only in case of small amount of data, as it is well-known that statistical tests are inclined to systematically reject the null hypothesis when the sample size is too large [26, 28].

As a matter of illustration, we propose in the following two examples of fitting real voting situations through the proposed models. The first one deals with continuous evaluations, whereas the second one focuses on discrete evaluations. As the high number of voters prevents from the pertinence of goodness-of-fit tests, any χ^2 test or Kolmogorov test should conduct to reject any regular hypothesis on the distributions. Hence, we only provide the corresponding statistics, which can be interpreted as distances to the fitted model, rather than the p-values. Note also that we only provide here an example of how to fit an evaluation-based voting situation, and not a general study of which model is the best one.

5.1 Continuous case

The first example, in a continuous framework, is based on the use of a survey concerning the 2017 presidential election in France. Data are available in [9], and deal with 13 candidates evaluations on $\{0, \dots, 100\}$ by $n = 20210$ voters. We transform this 0-100 scale into a continuous scale, adding to any value given by a voter to a candidate a random value uniformly distributed between 0 and 1. These values included in the $[0, 101]$ interval are then scaled to the $[0, 1]$ interval.

We focus on $m = 3$ candidates: François Fillon (FF), Benoit Hamon (BH) and Emmanuel Macron (EM) (who finally won the election). Illustrations of the observed distributions are presented in Figure 11, with the distributions of Uniform, truncated Normal, Beta and checkerboard copula models.

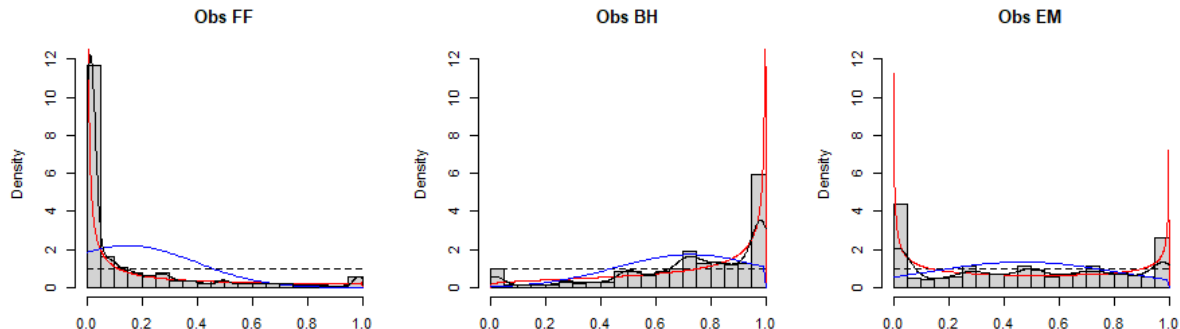


Figure 11: Histograms of observed evaluations of $m = 3$ candidates given by $n = 20210$ voters at the French 2017 election. Uniform model distribution is in dashed black, Beta model in red, truncated Normal in blue and checkerboard copula in solid black.

Marginal distributions. We first propose to model the marginal distributions using either Uniform distributions, Beta distributions or Truncated Normal distribution as a matter of example. Obviously,

truncated Normal distribution does not seem adequate with these data, since, as illustrated in Figure 11, the shapes of the distributions are far from a Gaussian curve.

The Uniform distribution does not need any parameter estimation. The Beta distribution depends on two parameters α and β . These parameters can be estimated using the method of moments. Let $\hat{\mu}$ denote the sample mean and s the sample standard deviation. Then parameters α and β can be estimated respectively by $\hat{\alpha} = \hat{\mu} \left(\frac{\hat{\mu}(1-\hat{\mu})}{s^2} - 1 \right)$ and $\hat{\beta} = (1 - \hat{\mu}) \left(\frac{\hat{\mu}(1-\hat{\mu})}{s^2} - 1 \right)$. The truncated Normal distribution depends on two parameters, the mean and standard deviation, which are easily estimated.

The first three lines of Table 3 gives the Kolmogorov-Smirnov statistic, corresponding to the distance between the distribution of observed evaluation and the distribution of simulated evaluations. The smaller the distance, the more the model fits the sample distribution.

One can see that the Beta model is the most adequate in any case. Note that the critical value for a significance level $\alpha = 0.05$ is equal to 0.009 and the critical value for a significance level $\alpha = 0.01$ is equal to 0.011.

	FF	BH	EM
Uniform marginals, Ev-IID or Ev-DID models	0.572	0.355	0.182
Beta marginals, Ev-IDD or Ev-DDD models	0.268	0.100	0.117
Trunc. normal marginals, Ev-IDD or Ev-DDD models	0.492	0.244	0.191
Marginals from nonparametric Copula Ev-DDD model (40)	0.267	0.141	0.103
Marginals from nonparametric Copula Ev-DDD model (101)	0.005	0.007	0.005

Table 3: Kolmogorv-Smirnov statistic. The number of intervals is provided in parenthesis for nonparametric Copula Ev-DDD models.

Dependence Let us now focus on the independence between the evaluations. Evaluations given to FF and BH are negatively correlated, whereas evaluations given to FF and EM are slightly positively correlated, and evaluations given to BH and EM are not correlated, as one can see in Table 4. Hence, the independence assumption is not realistic and we consider Copula-based Ev-DDD models in order to capture the coupling between the three distributions. We consider two Copula Ev-DDD models:

Parametric Copula Ev-DDD model. We consider first a Normal copula, with marginal distributions fitted by Beta distributions as established above, and correlation coefficients equal to the observed correlation coefficients shown in Table 4. It has the advantage to be parametric and to provide a reproducible modeling.

Non parametric Copula Ev-DDD model. We also consider a checkerboard Copula model, which does not require any assumption on the marginal distributions. The marginal distributions are

simply the sample distributions, divided into G classes. We tried $G = 101$ to recover the initial discrete values on the 0-100 scale, and, as suggested in [12], we also tried $G = 40$. In practice it is often efficient to fit a dataset but not appropriate to generate a predictive modeling.

Since the parametric Copula model is based on Beta distributions, the Kolmogorov-Smirnov statistics for the fitting of the marginals has already been calculated. Concerning the non parametric approach, the statistics are presented in the last two lines of Table 3.

One can see that the non parametric Copula model better fits the data than the parametric Copula model, at least when the number of class is big enough. This is not surprising due to the parametric/non parametric nature of the models.

Table 5 displays the correlation coefficients obtained with the two modelings.

	FF	BH	EM
FF	1	-0.41	0.30
BH	-0.41	1	0.0008
EM	0.30	0.0008	1

Table 4: Empirical correlations between three candidates at the French 2017 presidential election.

	FF	BH	EM		FF	BH	EM
FF	1	-0.33	0.23	FF	1	-0.41	0.30
BH	-0.33	1	0.002	BH	-0.41	1	0.01
EM	0.23	0.002	1	EM	0.30	0.01	1

Table 5: Correlations between candidates obtained with copula Ev-DDD models at the French 2017 presidential election. On the left, correlations obtained with a Normal copula and Beta marginals, on the right correlations obtained through the use of a checkerboard copula with empirical marginal distributions and 40 classes.

As a conclusion, Copula DDD-models are more appropriate. A non-parametric approach of course better fits the data distributions, but a parametric approach seems more appropriate to model the situation without overfitting.

5.2 Discrete case

We study data from the Comparative Studies of Electoral Systems project (<https://cses.org/>), and especially from the “module 5” which consists in surveys about 38 elections between 2016 and 2020

worldwide, for a total of 258 candidates and numbers of voters between 385 and 3548. Future voters are invited to evaluate several candidates competing at each election. The CSES - module 5 dataset includes evaluations of candidates by voters on a discrete 0-10 scale. A basic treatment has been necessary to remove missing answers.

Marginal distributions. We first propose to model the marginal distributions on the candidates using Uniform distributions, Binomial distributions and Beta-Binomial distributions.

The Binomial distribution needs the estimation of a parameter p , which can be estimated by $\hat{p} = \frac{\hat{\mu}}{K+1}$, where $\hat{\mu}$ is the sample mean and $K + 1$ is the number of scales in the evaluations. The Beta-Binomial distribution needs the estimation of parameters α and β , obtained as follows: $\hat{\alpha} = \frac{K\hat{\mu} - \hat{\mu}^2 - s^2}{K(s^2/\hat{\mu} - 1) + \hat{\mu}}$ and $\hat{\beta} = \frac{(K - \hat{\mu})(K - \hat{\mu} - s^2/\hat{\mu})}{K(s^2/\hat{\mu} - 1) + \hat{\mu}}$, with $\hat{\mu}$ the sample mean and s the sample standard deviation [47].

For each candidate of each election, we compute the χ^2 statistics of the distances between the distribution of observed evaluations and the theoretical Uniform, Binomial and Beta-Binomial distributions. The smallest the distance, the more the observed distribution is fitted by the model. On the 258 candidates of the dataset, the Beta-Binomial distribution is the best one 228 times, the Uniform distribution 29 times and the Binomial distribution only once.

Table 6 shows the summary statistics of the χ^2 distances between the observed distributions of the evaluations and the simulated distributions for the 258 candidates from the CSES dataset. As one can see, the Binomial model is very far to fit the real candidate evaluation distributions and therefore Beta-Binomial or Uniform distribution seem more convenient.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Uniform	35.3	319.2	567.1	835.2	977.0	8609.7
Binomial	153	8592	26270	2.837e+07	2.09e+05	6.47e+09
Beta-binomial	11.03	104.57	176.48	245.44	305.70	1625.51

Table 6: Summary of χ^2 distances between the observed distributions of the evaluations and the simulated distributions for the 258 candidates from the CSES dataset.

Table 7 gives the χ^2 statistics of distances for 3 candidates of the Danish 2019 election as an example. One can see that the Beta-Binomial distribution better fits the data for candidates F and G, whereas the Uniform distribution is the best for candidate B. Figure 12 shows bar plots for observed distributions and models.

Dependence. As in the continuous case, the evaluations in the discrete case are not independent. The 38 elections of the CSES corpus lead to 798 pairs of candidates. On these 798 pairs, 32% have a negative

	B	F	G
Uniform	43.8	236.3	204.8
Binomial	14476.3	53236.7	4607.2
Beta-Binomial	120.3	78.5	126.8

Table 7: χ^2 distance between the observed distributions of the evaluations and the simulated distributions of $m = 3$ candidates and 3 simulation models at the Danish 2019 election.

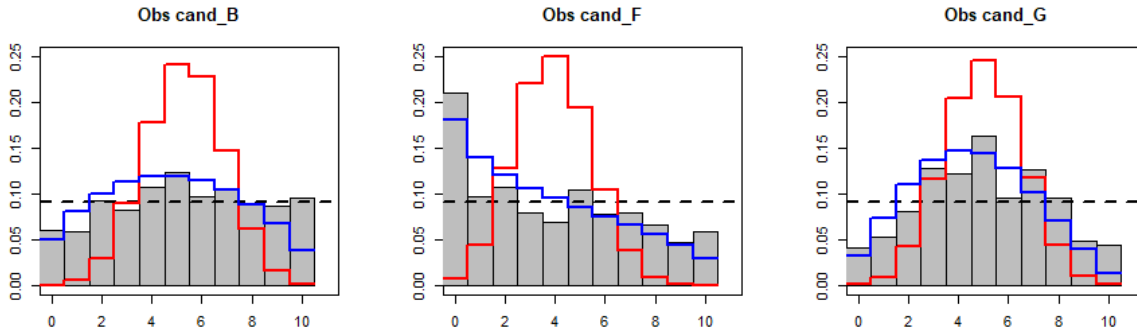


Figure 12: Bar plots of observed values for $m = 3$ candidates at the Danish 2019 election. The dashed black line represents the uniform distribution. The distribution in red corresponds to the binomial model, and the distribution in blue corresponds to the beta-binomial model.

correlation, and 68% have a positive correlation. The correlation coefficient is significantly different to zero for 90% of the pairs of candidates. This shows the importance of taking into account dependence in modeling.

As a matter of example, Table 8 shows the correlations between the three candidates at the Danish 2019 elections already introduced above. First, the marginal distributions are fitted as described above by a Beta-Binomial distribution. Then, a Gaussian discrete copula is used to model the correlation between candidates. The obtained correlations are presented in Table 8, and bar plots are presented in Figure 13. The discrete copula captures the dependence structure, but may introduce overfitting of the dependence, when generating simulated observations. No parametric modeling, among the ones proposed, are well adapted in such context.

	B	F	G
B	1	-0.58	0.63
F	-0.58	1	-0.40
G	0.63	-0.40	1

	B	F	G
B	1	-0.57	0.65
F	-0.57	1	-0.41
G	0.65	-0.41	1

Table 8: Correlations between the evaluations of each candidates at the Danish 2019 election. On the left the observed correlations, on the right the correlations obtained with a Copula Ev-DDD Beta-Binomial model.

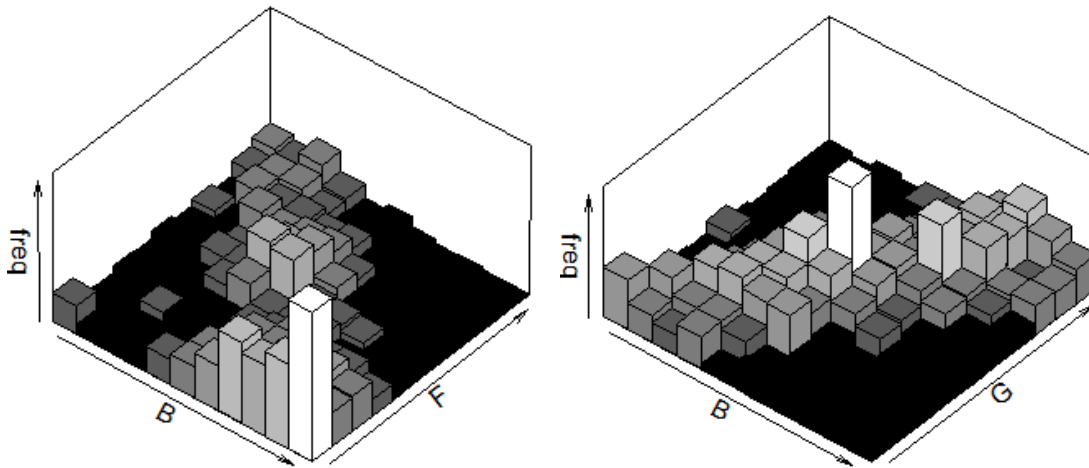


Figure 13: Barplots of observed evaluations between candidates B and F (left) and candidates B and G (right) at the Danish 2019 election.

5.3 Spatial representation

The spatial model introduced in Section 4.3 can also be used as a representation of candidates and voters in the same space. However, the experiments show that it is difficult to simulate new data from a spatial representation: even if the spatial representation of the voting situation is accurate, it is not an easy task to identify the latent space (its dimension d and its metric) nor the distribution of the voters in the voting space. Suppose the dimension d known, that the metric is given, and let us focus on the distributions of voters in the latent space.

In order to simulate new data, we propose a two steps process:

1. estimate both candidates' and voters' positions into a d -dimensions space, for example by the use of the SMACOF method [13] .

2. estimate the distribution of voters into the d -dimensions space, in order to generate new voters with the same distribution.

The second step needs to fit a multidimensional distribution. The fitting is not done directly on the data but on the latent positions obtained with the first step. This can be done for example with a copula model, as previously. We choose hereafter to consider $d = 2$, and the Euclidean distance. The spatial representations obtained by the SMACOF algorithm respectively on the continuous and discrete data are shown in Figure 14. Hence, from real data, we have calibrated the parameters of a spatial model, which allows generating new data. Note that the intrinsic dimension and the used distance are to be set by the user. Some post-hoc measures of the quality of the adjustment are available for comparing different choices [13, Chapter 4].

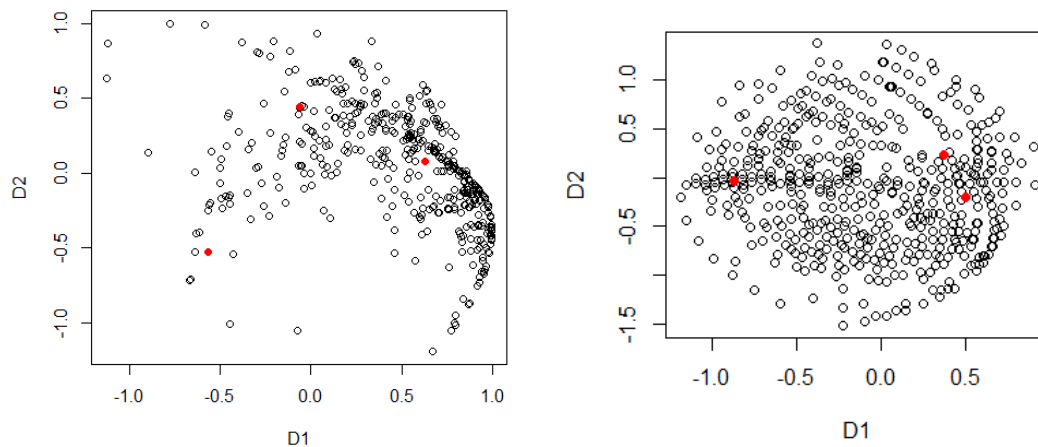


Figure 14: Spatial representation of candidates (filled red bullets) and a sample of voters of the French 2017 election (left) and the Danish 2019 election (right) with by the SMACOF algorithm.

6 Conclusion

As explained in the preamble, simulations can be done in two different settings.

- On the one hand, simulations can be done without any specific context, and the tuning of the distribution of the evaluations is let free or determined by external considerations. One has therefore to choose a model and set the parameters to arbitrary values. Examples of such simulation settings have been proposed with the description of each model above (Sections 3 and 4.3).

- On the other hand, one can wish to simulate observations in harmony with real data. In that case, an adjustment of the model to the observed data is necessary. The aim is therefore 1) to choose the appropriate model 2) to infer the model parameters from the available data. This situation is detailed in Section 5 for both discrete and continuous cases.

We introduced in this paper several models to simulate evaluation-based voting data in a probabilistic-based analysis perspective of evaluation-based voting rules. Three main families of distributions were proposed for the marginal distributions of the evaluations, in a continuous setting and in a discrete setting. On the contrary to preference rankings models, where the key notion is the impartiality, a more refined discussion is needed for evaluation-based processes. Independent and identically distributed modeling (Ev-IID models, Section 4.1) yields Impartial Culture on preferences, but there are two possibilities for relaxing this assumption. We propose first to distinguish either the marginal distributions are identical or not (Ev-IDD models, Section 4.2). Such models do not imply Impartial Culture on preferences. Next, introducing dependence (Ev-DDD models, Section 4.3) creates more complex models. We give examples of dependent distributions with identical marginals (Ev-DID models) which provide Impartial Culture on preferences. In particular, we introduce Copula Ev-DID and Ev-DDD models which allow to model the dependence between the evaluations. The variety of modeling described here offers the possibility of studying the properties of evaluation-based voting processes with an extensive probabilistic approach. It also provides new IC and non IC simulation approaches for preferences, since preferences rankings can be deduced from evaluations. Finally, as some proposed settings are parametric, the studied models can be fitted to real dataset to deduce more realistic frameworks. We present examples of such an approach on real data for continuous and discrete evaluations (Section 5). Finally, a R package named *voteSim* containing generation functions described in this paper has been developed and is available on the CRAN repository.

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