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Fishery Management under Poorly Known Dynamics [☆]

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Abstract

Adapting fishing regulations in a highly uncertain environment remains a complex challenge for managers who have to deal with non-linear dynamics of fish population and harvest levels. In this research, a recent method of stochastic control is adapted to a general fishery management problem under multiples sources of uncertainty related to the dynamics of the fish population and the effect of fishing on its growth. The question is about adjusting permanently the management rule or to hold a fixed policy thus avoiding additional noise. The mathematical problem developed here, though oversimplified, represents an original approach to the fishery management issue inspired by the monetary policy challenge of a central bank (Brainard principle). It assumes that Control Variation Increases the level of Uncertainty (namely CVIU approach) under particular conditions, resulting in preferable inaction regions for managers. We specify these conditions to show that the management of a poorly known fishery is still possible by using a CVIU approach.

Keywords: Control; System Dynamics; Fishery Management; Uncertainty; CVIU

1. Introduction

Research in fisheries management has a long tradition of applying operational research methods to address optimal control issues through deterministic or stochastic dynamic programming approaches. Several reviews in the past such as Lane (1989); Rodrigues (1990); Bjørndal et al. (2004); Arnason (2009) mention methods that are developed in descriptive mathematical modelling, mathematical programming and optimisation, statistical analysis and estimation procedure, numerical simulation, decision (and game) theory (Lane, 1989). Operations research is deemed very relevant to find optimal pathways and levels for investment, search and vessel movements, target stocks, fishing effort in a very uncertain decisional context (Reed, 1979; Sethi et al., 2005). However, beyond the mere identification of optimal management objectives, a lack of operational research dealing with the design of the fisheries management, enforcement and judicial systems has been pointed out (Arnason, 2009).

The overexploitation of common renewable resources like fisheries may turn to a situation where it becomes optimal to overfish a natural stock to extinction (Clark, 1973). Such cases occur when the pace of biomass growth is too slow, property rights are not well defined, fishing costs are too small or hidden by subsidies, discount rates and fish prices are too high, and technical change make exploitation still profitable at low levels of biomass (Munro and Sumaila (2002), Grafton et al. (2010), Collette et al. (2011), Squires and Vestergaard (2013)).

As a political response to overexploitation, conservation measures encompassing the choice of a target stock size and a harvest time path to improve or stabilize it have been developed

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(Anderson and Seijo (2010)). For Arnason (2009) fishery management is technically demanding and such measures can be very complex to implement. As noticed by Bjørndal et al. (2004), one of the reasons is that the task of modelling stock population dynamics is at the heart of fisheries management, which implies dealing with complex and unpredictable dynamics. Given the degree of uncertainty surrounding models, the situation of a fishery may even worsen if the target of sustainable yield is overestimated by scientists or set above the advised level for political or short-sighted economic reasons (Borges (2018), Wiedenmann and Jensen (2018)).

Sources of uncertainty in the management of renewable resources are multiple. Several surveys and classifications of uncertainty are proposed by authors (Francis and Shotton (1997), Charles (1998), Regan et al. (2002), Hill et al. (2007)). Uncertainty in fisheries may come from random fluctuations influenced by environmental variability, wrong parameter estimates leading to stock assessment errors, and structural uncertainty stemming from a lack of knowledge about the nature of the fishery system and an inaccurate implementation of harvest quotas (Reed (1979), Clark and Kirkwood (1986), Roughgarden and Smith (1996), Charles (1998), Hill et al. (2007)). These distinct sources of uncertainty may have different implications for managers. If environmental variability is unpredictable because of a random multiplicative shock disturbing the growth but the stock is correctly assessed every period, then managers can choose a constant optimal escapement rule avoiding stock exhaustion (Reed, 1979). However, constant escapement is no longer possible when managers do not know accurately what the stock is in the current period (Clark and Kirkwood, 1986). As a result, the optimal management policy can be significantly different, justifying less cautious strategies and even extinction in some rare cases (Sethi et al., 2005). By introducing stochastic behaviors of fishers when choosing their catch levels on the basis of managers decision, the outcome becomes even more unpredictable and analytical solutions can hardly be discovered (Roughgarden and Smith, 1996).

Regarding specifically the fishing or natural resource management problems, there are few stochastic models, since most of the papers deal with deterministic simple models. We can nonetheless cite a number of works such as (Williams, 2009) that applies Stochastic Dynamic Programming (SDP) to solve sequential optimization problems. The model, built as Markov chain process, suggests several future uncertain states of the spawning stock biomass as a function of an initial state and decision model such as a TAC. It allows an optimal management strategy with respect to different possible states of the stock. Several other studies have relied on SDP to look at various sources of uncertainty, including the natural variability of stocks or market price changes (Sethi et al., 2005), and other authors applied SDP in more complex contexts such as straddling stocks by integrating stochastic processes (Kulmala et al., 2008; Kompas et al., 2010). A stochastic continuous time model that deals with the fishery problem is studied in (Yoshioka, 2019), driven by a Poisson jump process. Because of their complexities, most applications to fisheries are numerical and based on a deterministic framework (Marescot et al., 2013).

Uncertainty is likely to increase by considering the marine ecosystems and their dynamics. Beyond the errors of parameter estimates, the uncertainty about model structures has been poorly regarded by scientists. All ecosystem dynamics models simplify the structure of the food web, the nature of ecological interactions, and the demographic structure of populations (Hill et al., 2007), making predictions highly risky. Such a complexity has resulted in the development of new management policies called adaptive (or experimental) management, robust management, or balanced harvest, sometimes advocating opposite measures to what had been advised so far by scientists to managers, such as increasing selectivity (Levin and Lubchenco (2008), Garcia et al. (2012)). In this ecosystem approach of fisheries, what matters most is the conservation of ecosystem structure and functioning in order to maintain ecosystem services and marine biodiversity. By broadening the range of sizes and species being caught in fisheries, the balanced harvest policy would provide economic benefits derived from food security, higher productivity of the ecosystem and greater resilience to climate change (Charles et al., 2015). It has even been

reported for some African lake fisheries that no management at all could bring better outcomes than a strictly selective management, by conserving the structure of the ecosystem (Jul-Larsen et al., 2003). This option is challenged by other authors who consider that balanced harvesting should not mean a *laissez-faire* approach and the rejection of selective management, but rather upscale selectivity to the integrated ecosystem level (Garcia et al. (2012), Reid et al. (2016)).

Questions about the timely type of management is more than ever on top of the research agenda, because management decisions may directly influence the level of uncertainty surrounding the fishery. Taking this debate to the extreme, uncertainty can lead to two opposite options for managers: doing nothing (or keeping the rule unchanged whatever the circumstances), well known in the monetary economic literature as the Brainard principle (Brainard, 1967), or acting as vividly and often as possible to adapt decisions to the best existing knowledge of biomass, effort and harvest targets. The second strategy can be the optimal one in case of necessary learning process (Kling et al., 2017) but both solutions can also be deemed sub-optimal or unsatisfying solutions on both biological and economic grounds if the biomass dynamics is affected by inappropriate management decisions.

Indeed, social and natural systems may respond to management options in multiple unexpected ways. Despite its influence, the human behavior is much less studied than any other source of uncertainty (Fulton et al., 2011). However, investment decisions in fisheries, like in other economic activities, must rely on a certain amount of stability depending on the state of the resource stock, the total allowable catch (TAC) decided by managers, the fishers individual catch share, etc. Fishers behavior can become erratic if their regulated access to resources varies from period to period because of adaptive management. Within the framework of the Common Fisheries Policy, TACs and quotas for most stocks are set annually in December by the Council of fisheries ministers, creating an uncertain climate for European fishers whose short and mid-term business depends on such decisions. Most producers organizations claim for multi-annual quotas to secure their own activity, rather than operating in a context of incomplete information (Dwyer and Minnegal, 2006).

Natural systems are also profoundly affected by fishing (Glaser et al., 2014). Engwerda (Engwerda, 2017) analyzed the level of activities of fishers in an uncertain world where fishers may take care—or not—of a biomass stabilization constraint. As usual in robust control models, the growth of fish stocks (e.g. the state variable) depends on some aggregate uncertain factors (water quality, weather, etc.). This disturbance is somehow independent from any stock level or any amount of fishing effort. However, the way fish is caught may easily put stock growth onto non-linear dynamics (Anderson et al. (2008), Glaser et al. (2014), Gray (2016)). As reported by FAO (2003), fishing impacts natural stocks by reducing population abundance, its spawning capacity and, potentially, the population growth parameters (growth, fecundity, etc.) (Rochet, 1998). Fishing modifies the age and size structure, sex ratio, genetics and species composition of the targeted renewable resources, as well as their associated and dependent species (Jennings and Kaiser, 1998). For example, bottom trawling is a fishing method that drags a large net across the seafloor, destroying some essential habitats for marine fauna (Lindholm et al., 1999). Large amounts of bycatch are unintentionally caught and thrown overboard dead or dying, changing the community structure and trophic interactions (Jennings and Kaiser, 1998). All these negative externalities can obviously reduce the carrying capacity of the whole ecosystem.

Anderson et al. (2008) studied larval fish records over fifty years in California and showed that the intrinsic growth rate of the population had increased because of fishing, amplifying the non-linear dynamics of the biomass. This evidence may appear counter-intuitive because the truncation of the stock across the age structure cuts down the number of highly productive big old fat fecond female fish (BOFFFFs) (Hixon et al., 2014), and therefore should reduce population growth. However, this truncation produces other interesting dynamic effects. Higher survival rates of offspring individuals compensate their shorter life history traits. Fewer competitors for food and cannibalism enhance the somatic growth and the per-capita fecundity of

the population, therefore increasing the intrinsic growth rate which amplifies the stock fluctuations. It is well known that unstable dynamics may result from increased values of intrinsic growth rates, but fairly high values of this rate are required for a single species stock. However, a truncated age-size structure of a population interacting with other related species can create unpredictable variability with only a small increase of the growth rate parameter (Anderson et al., 2008). The resulting non-linear dynamics urges the need for a precautionary approach of fishery management.

In this article, we assume that the level of uncertainty is directly linked, and somehow proportional, to the evolution of the gap between the current fish stock and a desired one, which can be a biological optimized stock such as the maximum sustainable yield (MSY). Moving away from the desired level creates more uncertainty, thus affecting the growth function parameters and the biomass dynamics. We also assume that catch levels influence the amount of uncertainty, either because the ecosystem and the carrying capacity are modified (Jennings and Kaiser, 1998), or because it endogenously changes the intrinsic growth rate (Anderson et al., 2008). As recommended by FAO (2017), it becomes urgent to move away before being trapped in a situation where only extreme strategies are possible (stock exhaustion, area closure, fishing ban).

The central issue of the present research can be summarised as follows: what are the conditions under which changing the management rules creates more uncertainty and unpredictable outcomes than addressing the overexploitation problem in a context of poor knowledge about fish stocks, harvest levels and population dynamics? We expect situations where taking no other action than sticking to the long-term management rule (e.g. a fixed Total Allowable Catch) is more appropriate for both natural and social systems than adjusting permanently the amount of authorized catches on the basis of new but inconsistent knowledge. To approach these aspects, we apply a recent control theory framework where Control and State Variations Increase the level of Uncertainty (CVIU) due to a lack of a trustful model. Namely, the CVIU approach allows better understanding of resource management in a context of poorly known dynamics, cf. Calmon et al. (2009a), Calmon et al. (2009b), do Val and Souto (2017)). As stated by Powell (2018), the quality of a policy depends on the quality of the underlying model, and in this regards, we believe that the stochastic optimization model developed here might improve the relevance of management policies.

The paper is organized as follows. Section 2 presents the bioeconomic model and how uncertainty impacts the model. Section 3 presents the CVIU optimization problem. Section 4 provides the CVIU solutions and shows the existence of an Inaction Region. Section 5 discusses the characteristics of the inaction region in the bioeconomic model, and presents numerical simulations. It explores the impact of state uncertainty (e.g. surrounding the dynamics of biomass) and the discount factor as key elements of the rising of an inaction region. Control uncertainty (e.g. the impact of the TAC policy) sets the location of the inaction region around the desired state level. Section 6 provides some concluding remarks and policy implications.

2. Bio-Economics of the fisheries and the CVIU approach

2.1. Economic considerations

The fish population $\{z(t)\}_{t \geq 0}$, is harvested at any time t under a controlled rate $\{h(t)\}_{t \geq 0}$, and in this model the fishers will always harvest at their technological capacity and according to the level set by the Total Allowable Catch (TAC) instrument,

$$h(t) = \text{TAC}(t) = h_e + u(t) \quad (1)$$

meaning that fishers cannot catch more than the TAC at time t , but they will try to catch no less. The TAC is split into two components: a fixed element, h_e , and a flexible element, $u(t)$. Consequently, $u(t)$ acts as the incremental, $u(t) > 0$, or reduced, $u(t) < 0$, amount of allowed catches around the fixed TAC. Such a variable allows the policymaker to adjust

whenever necessary the TAC level at each period t , in order to align increasing profits of fishers and enhancing the biomass level, given the uncertainty involved.

Let us assume that fishers are identical and small (e.g. price takers). Let also assume that, at the industrial level, the individual total costs and revenues of fishers can be summed up to give the following overall profit function for the fishing sector:

$$\pi(h(t)) = ph(t) - ch(t)^2 \quad (2)$$

under a constant market price p , the quadratic cost with $c > 0$ expresses the decreasing returns to scale. With no dynamic considerations, maximization of the profit function in (2) leads to the static optimal solution $h^*(t) = p/2c$. Using the decomposition in (1), the profit as a function of the managers flexible instrument, $u(t)$ is:

$$\pi(h_e, u(t)) = (ph_e - ch_e^2) + ((p - 2ch_e)u(t) - cu(t)^2) \equiv \bar{\pi} + \pi(u(t)) \quad (3)$$

This instantaneous profit is made up with two corresponding components: a maximum fixed value, $\bar{\pi} = ph_e - ch_e^2$ depending on the fixed component of the TAC value, and a maximum variable value, $\pi(u(t))$, relying on the amount by which the policymaker restricts ($u(t) < 0$) or increases ($u(t) > 0$) catches below or over, the fixed component of the TAC at time t . Note from (3) that $p - 2ch_e > 0$ necessarily to yield profit.

Let us assume that the government wishes to minimize the following expected cost function (in the case of constant price) by setting an optimal sequence $\{u(t)\}, t \geq 0$,

$$E \left[\int_0^T e^{-\alpha t} (-\pi(h_e, u(t)) + x(t)^2 + qx(t)) dt \right] \quad (4)$$

where α a discount rate and $x(t)$ is the gap value between a desired biomass level, as aimed in the choice of the nominal TAC value h_e having considered a corresponding population equilibrium, see (6) below for more details.

Given (3), one can rewrite this minimization problem in a standard linear quadratic setting such as:

$$J(s, x, u(\cdot)) = E^x \left[\int_s^\infty e^{-\alpha t} (Qx(t)^2 + qx(t) + Ru(t)^2 + ru(t) - K) dt \right], \quad (5)$$

with $Q = 1$, $r = -(p - 2ch_e)$, $R = c > 0$, $K = \bar{\pi} > 0$. The term $qx(t)$ added to (4)–(5) produces one of the two impacts. If $q < 0$, it means that when a biomass is over the desired level, it can provide a small reward by itself. On the contrary, if $q > 0$, a biomass over the desired level is seen as a cost.

By defining $\{u(t)\}_{t \geq 0}$ it is possible to offset (if $u(t) \neq 0$) on the agreed nominal harvesting rate h_e . In other words, the manager has to trade off between maximizing a profit level and keeping $x(t)$ closed to zero, i.e. keeping the biomass level at its desired level.

2.2. Biological considerations

The fishery manager wishes to control, via authorized harvesting, the dynamics of stocks for some species of economic interest, striving to avoid depletion or even exhaustion of the population. Assume that the population follows a natural growth behavior subject to small fluctuations on its growth rate, that could be well represented by additive noise with variable intensity. However, due of the complexity of ecosystems, the actual parameters defining the growth curve are not known, and only approximated estimates of the parameters can be used. This assumption is quite realistic since these values are highly influenced by the fluctuations of the environment as discussed in Section 1. Such a large degree of uncertainty should be taken into account, specially the impact on the intensity of the control and on the evolution of the state, both having to be properly accounted in the model.

Typically, the growth of the biomass is expressed as a function of the biomass itself, $z(t)$, and as a function of the aggregated catch level, $h(t)$, to model the population dynamics. Among the potential growth function candidates, the logistic function is frequently used by fisheries economists. This logistic function will lead to an inverted U-shaped curve at a given level of biomass or effort. It is well known that the maximum sustainable yield (MSY) is reached when the growth (e.g. the productivity of the biomass) is at its maximum. It is also well known that the biological equilibrium may not correspond to an optimal economic solution when taking into consideration other parameters such as prices, costs, technological change, discount rate, and the access regime to resources (Grafton et al., 2010), (Collette et al., 2011), (Squires and Vestergaard, 2013).

Interestingly, even in the simplest scenario, both parameters of the logistic curve, the *carrying capacity* and the *intrinsic growth rate* r , may not be perfectly known.

At least four sources of uncertainty can be defined as a function of some noise-error ϵ . Firstly, the policymaker may not know what is the exact current biomass, $z(t) + \epsilon_x$. Secondly, the intrinsic growth rate can be either underestimated or overestimated, $r + \epsilon_r$. Thirdly, one may ignore the accurate carrying capacity, $K + \epsilon_K$. Finally, the specified TAC policy may not be strict in force, $\text{TAC}(t) = h_e + u(t) + \epsilon_u$. Traditionally, these noises are set exogenously, but here we consider that uncertainties on intrinsic growth rate and on carrying capacity are functions of the gap between the observed biomass and the desired one, and of the value of the control offset $u(t)$ in use.

To discuss these assumptions, along with a nonlinear growth curve, first consider that at a given point of interest, one can locally approximate the nonlinear dynamics $\dot{z}(t) = f(z(t)) - h(t)$, by a linear one. The dynamics may locally evolve according to:

$$\dot{z}(t) \approx \dot{x}(t) = Ax(t) + f_0 - u(t) \quad (6)$$

where $x(t) = z(t) - z_e$, $u(t) = h(t) - h_e$, and $f_0 = f(z_e) - h_e$ is set to zero by the obvious choice $h_e = f(z_e)$ in the linearization. For the logistic model,

$$A = r \left(1 - \frac{2z_e}{K} \right), \quad f_0 = r \left(1 - \frac{2z_e}{K} \right) z_e \quad (7)$$

Because of uncertainty in the logistic curve, the linear equation may exhibit the following parameter uncertainties

$$\dot{x}(t) = (A + \epsilon_z)x(t) + (f_0 + \epsilon_0) - u(t) \quad (8)$$

In Eq. (8), a direct link between the value of the state and the degree of uncertainty is considered.

Another source of uncertainty may come from the harvesting impact. By fishing “big fish” rather than small ones, and because fishing gears can somehow degrade the ecosystem, either the carrying capacity (K) or the growth rate can be negatively impacted (Lindholm et al. 1999, Anderson et al. 2008). Including this harmful effect in equation (6) will give:

$$\dot{x}(t) = Ax(t) + f_0 - (1 + \epsilon_h)u(t) \quad (9)$$

Which means that the net impact of fishing maybe distinct to an one-to-one to an “average fish”. Taking into account all uncertainties by aggregating (8) and (9), and adding another source of uncertainty due to external and unpredictable events (such as climate oscillations), the general bio-economic dynamics equation becomes:

$$\dot{z} = Az(t) + f_0 - u(t) + (A\epsilon_z x(t) + \epsilon_0 - \epsilon_h u(t)) + \epsilon \quad (10)$$

Such a dynamic equation means that we poorly understand the population dynamics, even in a locally chosen equilibrium point (z_e, h_e) . It means also that, *ceteris paribus*, there exists a link between the state (biomass) level, $z(t)$, the harvest action value, $h(t)$, and the degree of

uncertainty. Finally, and from the managers perspective, managing uncertainty is likely to be more important when the biomass value is lower than the targeted value z_e , than on the other side, the surplus side. The system is actually more dangerously uncertain on the shortage side because, for example, the whole ecosystem may be affected by some irreversible damages. In the next section, we show that the management of a poorly known fishery system, with the described features, is still made possible by using a CVIU approach (do Val and Souto, 2017) based on the control of an associate stochastic differential equations (SDE).

3. The CVIU for Growth Modeling

3.1. CVIU state dynamics

Let us denote for some scalar variable y , $y^+ = \max(y, 0)$ and $y^- = \min(y, 0)$. We apply this notation to the process $t \rightarrow x(t)$ and $t \rightarrow u(t)$ to write, $t \rightarrow x(t)^+$ and $t \rightarrow u(t)^+$ and $t \rightarrow x(t)^-$ and $t \rightarrow u(t)^-$, to elaborate on sideways values.

The uncertain model sketched in (10) can be made precise by the following SDE model. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions, cf. (Yong and Zhou, 1999, Ch 1, def 2.6), on which a three-dimensional Brownian motion (BM) $W = \{W(t), W^x(t), W^u(t)\}_{t \geq 0}$ is defined and \mathcal{F}_t is the P -augmentation of the natural filtration of W . Consider a real-valued controlled Markov diffusion process $x = \{x(t)\}_{t \geq 0}$ described by the Itô's SDE,

$$dx(t) = (Ax(t) + Bu(t)) dt + \sigma dW(t) + (\bar{\sigma}_x + (\sigma_x^+ x(t)^+ - \sigma_x^- x(t)^-)dW^x(t) + (\bar{\sigma}_u + (\sigma_u^+ u(t)^+ - \sigma_u^- u(t)^-)dW^u(t), \quad (11)$$

$x(0) = x \in \mathbb{R}, t \geq 0$, where,

A and B are scalars and W^x, W^u are added one-dimensional standard BM each. The two groups of scalars $\bar{\sigma}_x, \sigma_x^+, \sigma_x^-$ and $\bar{\sigma}_u, \sigma_u^+, \sigma_u^-$ possesses the same signal and without loss we assume that they are all nonnegative. The control strategy $u = \{u(t)\}_{t \geq 0}$ is a $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted stochastic process taking its values in a compact set U of the real line and admissible controls, denoted by $\mathcal{U}[0, T]$, are U -valued Markov functions $t \rightarrow u(t, \omega) = u(t, x(t))$ and $u(\cdot) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R}) := \{\phi(\cdot) = \{\phi(t, \omega) : 0 \leq t \leq T\}$ such that $\phi(\cdot)$ is an $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted, real-valued measurable process on $[0, T]$, and $E[\int_0^T \|\phi(t, \omega)\|_X^2 dt] < \infty\}$. For the infinite horizon α -discounted problem to be studied, we consider $\mathcal{U}[0, \infty]$ as admissible class, similar as above, but with elements satisfying $E[\int_0^\infty e^{-\alpha t} \|\phi(t, \omega)\|_X^2 dt] < \infty$ for $\alpha > 0$.

With these elements, the control problem of minimizing (5) in the class $\mathcal{U}[0, \infty]$ can be framed under the weak formulation of Itô diffusion stochastic control problems; cf., (Yong and Zhou, 1999, Ch 2, sec 4).

In the CVIU class of models, the modeling errors are associated to extra independent BM driven terms that are modulated by constant terms and offsets terms, according with (11). The uncertainties introduced on the state and control vectors at the equilibrium (z_e, h_e) are represented by the products $\bar{\sigma}_x dW^x(t)$ and $\bar{\sigma}_u dW^u(t)$ respectively. These are possibly due to the choice of inaccurate parameters A and B in the linearization procedure. In addition, when the state of the system deviates from the equilibrium, the multiplicative portions of the noise, namely, $(\sigma_x^+ x(t)^+ - \sigma_x^- x(t)^-)dW^x(t)$ and $(\sigma_u^+ u(t)^+ - \sigma_u^- u(t)^-)dW^u(t)$ stand for the uncertainty increase due to the loss of accuracy of the adopted linear model.

If $\sigma_x^+ = \sigma_x^-$ and $\sigma_u^+ = \sigma_u^-$ one can write $\sigma_x^+ x^+ - \sigma_x^- x^- = \sigma_x |x|$, $\sigma_u^+ u^+ - \sigma_u^- u^- = \sigma_u |u|$ and one retrieves the symmetric CVIU model studied in do Val and Souto (2017) for the scalar case. It is appropriate for engineering purposes but here the model takes the unsymmetrical and more general form than that in do Val and Souto (2017) because in the fishery management problem it is desirable that the uncertainty on the nature could be stressed asymmetrically in the model. If the biomass is above of what is the assigned equilibria for one reason or another,

it is not so critical as if it falls below the equilibria, in the sense that the depletion due to harvesting may be getting too harsh on the population number. No matter if its due to a coarse model parametrization or other effects that are not accounted for, the uncertainty here is not symmetrically taken into the model, by setting $\sigma_x^- > \sigma_x^+$ in (11). Similarly, the uncertainty should be amplified if the decision is to harvest more than the agreed rate h_e , compared to the decision of harvesting less than what had been previously consented, and $\sigma_u^- < \sigma_u^+$ in the model.

The management problem for fisheries is seen as the choice of a harvesting nominal point h_e , a rule that can be reviewed to a value $h(t) = u(t) + h_e$ at any time $t \geq 0$, subject to unknown dynamics, which is “roughly modeled” by (11). As indicated before, even for the simplest logistic model, possibly $r \neq r_0$ and $K \neq K_0$, and the model is equipped for this realistic framework.

The robust approach is a possible way to deal with this scenario, e.g. see Engwerda (2017). However since it provides policies that need to be adjusted to the worst case scenario, it is bound to be conservative from its inception. In a situation when the probability of the worst case scenario to happen is quite low, the robust policies do not account for this fact. Moreover, the policies that optimize a CVIU model are richer than the ones obtained by the robust approach based on LMIs. For a deeper discussion on that matter, see Silva et al. (2017).

3.2. Asymmetric Scalar CVIU

With the SDE model (11), the objective in the control problem is to solve the family of problems, which amounts to determine J^* , the value function,

$$J^*(s, x) := \inf_{u(\cdot) \in \mathcal{U}[s, \infty]} J(s, x, u(\cdot)), \quad (12a)$$

of J such as (5), associated to the scalar asymmetric CVIU diffusion model,

$$dx(t) = (Ax(t) + Bu(t)) dt + \hat{\sigma}(x(t), u(t)) d\hat{W}(t), \quad (12b)$$

$t \in [s, \infty], x(s) = x \in O$, with

$$\hat{\sigma}(x, u) = [\sigma \quad \bar{\sigma}_x + \sigma_x^+ x^+ - \sigma_x^- x^- \quad \bar{\sigma}_u + \sigma_u^+ u^+ - \sigma_u^- u^-] \quad (12c)$$

and

$$d\hat{W}(t) = [dW(t) \quad dW^x(t) \quad dW^u(t)]^\top,$$

with the assumption that

$\sigma_x^+, \sigma_x^-, \bar{\sigma}_x$ and $\sigma_u^+, \sigma_u^-, \bar{\sigma}_u$ are all nonnegative.

The goal is to find an admissible pair $u^*(\cdot) \in \mathcal{U}[s, \infty]$ and the corresponding $x^*(\cdot)$, such that $(x^*(\cdot), u^*(\cdot))$ achieves the minimum of $J(s, x, u(\cdot))$ over $\mathcal{U}[s, \infty]$. The above problem can be framed under the weak formulation of Itô diffusion stochastic control problems, cf., (Yong and Zhou, 1999, Ch 2, sec 4).

The elements above place the control problems (12) for a general asymmetric CVIU and the idea here is to explore, within the general framework, important features of the control problem, and whenever possible, we quote from the results in do Val and Souto (2017).

The first important aspect is stated in the next proposition. The proof of it in the case of symmetric multivariable CVIU model is presented in do Val and Souto (2017), and the proof for the asymmetric case is derived without much effort, thus omitted.

Proposition 1. *Consider the problem in (12) with the quadratic running cost in (5). It follows that the cost $J(\cdot, \cdot, \cdot)$ and the value $J^*(\cdot, \cdot)$ are continuous and $J(t, \cdot, \cdot), J^*(t, \cdot)$ are (strictly) convex for each $t \geq 0$.*

For each $(x, u, p, P) \in \mathbb{R}^4$ define the Hamiltonian function as,

$$H(x, u, p, P) := \frac{1}{2} P \hat{\sigma}(x, u)^2 + p(Ax + Bu) + f(x, u) \quad (13)$$

with $f(x, u) = Qx^2 + qx + Ru^2 + ru$. Provided that J^* is a continuous function, then it is also the unique solution of the Hamilton-Jacobi-Bellman (HJB) equation, in the viscosity sense, cf. Crandall et al. (1992). The following definition indicates the adequate HJB for the problem.

Definition 1. *An upper semicontinuous (respectively, lower semicontinuous) function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a viscosity subsolution (respectively, supersolution) of the HJB equation if*

$$-\alpha\varphi(x) + \inf_{u \in U} H(t_0, x_0, u, \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \leq (\geq) 0, \quad (14)$$

whenever $v - \varphi$ attains a local maximum (respectively, minimum) in a neighborhood of (t_0, x_0) for $\varphi \in C^2(\mathbb{R}^n)$, where φ_x and φ_{xx} indicate respectively, the gradient and the Hessian matrix of φ at (t, x) . A function v is called a viscosity solution of the HJB equation if it is both a viscosity subsolution and a viscosity supersolution of (14).

It follows that the viscosity solution for the problem studied here is a convex and continuous function with $n = 1$. We employ the notions of super and subdifferentials, $(p, P) \in D^{1,2,sp}J^*(x)$ and $(p', P') \in D^{1,2,sb}J^*(x)$, and since J^* is convex, $P, P' \geq 0$. These are the tools to deal with the control problem in the next section.

4. The Design of a CVIU Controller

Before going, we define some auxiliary elements, notations and equivalences that will be necessary to the next sections.

For the use of representing subdifferentials, we consider the following notations. For some $v \in \mathbb{R}$, $\tilde{\mathcal{S}}^+(v)$ and $\tilde{\mathcal{S}}^-(v)$ are sets defined as

$$\tilde{\mathcal{S}}^+(v) := \begin{cases} +1, & \text{if } v > 0, \\ [0, +1], & \text{if } v = 0, \\ 0 & \text{otherwise.} \end{cases} \quad \tilde{\mathcal{S}}^-(v) := \begin{cases} -1, & \text{if } v < 0, \\ [-1, 0], & \text{if } v = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

They correspond to the sets $D_v^{1,sb}v^+ = D_v^{1,sb} \max(v, 0)$ and $D_v^{1,sb}v^- = D_v^{1,sb} \min(v, 0)$, respectively.

Analogously, set for $v \in \mathbb{R}$, $\mathcal{S}^+(v) : \mathbb{R} \rightarrow \{0, +1\}$ and $\mathcal{S}^-(v) : \mathbb{R} \rightarrow \{0, -1\}$ the sign functions, defined as

$$\mathcal{S}^+(v) = \begin{cases} +1, & \text{if } v > 0 \\ 0, & \text{otherwise} \end{cases} \quad \mathcal{S}^-(v) = \begin{cases} -1, & \text{if } v < 0 \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

respectively. We also employ the notation $\mathcal{S}(v)$ to express $\mathcal{S}^+(v) + \mathcal{S}^-(v)$, and also add the subscript \mathcal{S}_x and \mathcal{S}_u whenever necessary.

Proposition 2. *For $A \in \mathbb{R}$ and $v \in \mathbb{R}$, the following subdifferentials with respect to v can be calculated:*

$$D_v^{1,sb}(A(v^+)) = A\tilde{\mathcal{S}}^+(v), \quad D_v^{1,sb}(A(v^+)^2) = 2Av^+ \quad (17)$$

$$D_v^{1,sb}(A(v^-)) = -A\tilde{\mathcal{S}}^-(v), \quad D_v^{1,sb}(A(v^-)^2) = 2Av^- \quad (18)$$

We define the following operators. Let $\mathcal{H} : \mathbb{R} \times \{-1, 0, +1\} \rightarrow \mathbb{R}$, $\mathcal{G} : \mathbb{R} \times \{-1, 0, +1\} \rightarrow \mathbb{R}$,

$\mathcal{Q} : \mathbb{R} \times \{-1, 0, +1\} \rightarrow \mathbb{R}$, $\mathcal{R} : \mathbb{R} \times \{-1, 0, +1\} \rightarrow \mathbb{R}$ and $\mathcal{T} : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathcal{H}(U, s) = 2U\bar{\sigma}_x (\sigma_x^+ s^+ + \sigma_x^- s^-) \quad (19)$$

$$\mathcal{G}(U, s) = 2U\bar{\sigma}_u (\sigma_u^+ s^+ + \sigma_u^- s^-) \quad (20)$$

$$\mathcal{Q}(U, s) = U \left((\sigma_x^+ s^+)^2 + (\sigma_x^- s^-)^2 \right) \quad (21)$$

$$\mathcal{R}(U, s) = U \left((\sigma_u^+ s^+)^2 + (\sigma_u^- s^-)^2 \right) \quad (22)$$

$$\mathcal{T}(U) = U (\sigma^2 + \bar{\sigma}_x^2 + \bar{\sigma}_u^2) \quad (23)$$

Note that if $U \geq 0$ then $\mathcal{Q}(U, \cdot)$, $\mathcal{R}(U, \cdot)$ and $\mathcal{T}(U) \geq 0$.

Proposition 3. *The Hamiltonian in (13) can be written equivalently as*

$$\begin{aligned} H(x, u, p, P) = & f(x, u) + \frac{1}{2}\mathcal{Q}(P, \mathcal{S}(x))x^2 + \frac{1}{2}\mathcal{R}(P, \mathcal{S}(u))u^2 \\ & + \frac{1}{2}\mathcal{H}(P, \mathcal{S}(x))x + \frac{1}{2}\mathcal{G}(P, \mathcal{S}(u))u + p(Ax + Bu) + \frac{1}{2}\mathcal{T}(P) = 0 \end{aligned} \quad (24)$$

Inspecting the pointwise minimization in the HJB equation (14) that involves the Hamiltonian function H in (24), one notices that the values u^+ and u^- makes it nondifferentiable w.r.t. u at the origin.

Lemma 1. *Consider $v : \mathbb{R} \rightarrow \mathbb{R}$, a convex function and $p \in \mathbb{R}$ and $P \in \mathbb{R}$, with $(p, P) \in D_x^{1,2,\text{sp}}v(x)$, for some fixed x . If such a v is the solution of the HJB equation in (14), the minimum $u = u^*$ of the HJB satisfies*

$$\begin{aligned} 0 \in D_u^{1,\text{sb}}H(x, u, p, P)|_{u=u^*} = & \frac{\partial_u f(x, u)}{\partial u}|_{u=u^*} + Bp \\ & + P (\bar{\sigma}_u \sigma_u^+) \tilde{\mathcal{S}}^+(u)|_{u=u^*} + P (\bar{\sigma}_u \sigma_u^-) \tilde{\mathcal{S}}^-(u)|_{u=u^*} \\ & + P ((\sigma_u^+)^2) u^+|_{u=u^*} + P ((\sigma_u^-)^2) u^-|_{u=u^*}. \end{aligned} \quad (25)$$

Proof. First order condition for the optimality of u in the HJB equation in (14) provides (25) (see Proposition 2). Since $P \geq 0$, the condition in the provides the global minimum of H on the set U , namely, the function $H(t, x, \cdot, p, P)$ is indeed convex and the minimum is attained for u^* that satisfies (25). \square

Regarding (15) and (25), note that it is necessary to determine the sets of signals $\mathcal{S}^+(u^*)$ and $\mathcal{S}^-(u^*)$ to best determine u^* itself. In fact, it is possible to divide the state space line \mathbb{R} into three segments for which the optimal control u^* has known sign. The next result is drawn mutatis mutandis from do Val and Souto (2017), and thus, the proof is omitted.

Proposition 4. *Consider that $f(\cdot, \cdot)$ and $g(\cdot)$ are convex and $f(x, \cdot)$ is continuously differentiable for each x . Then the optimal control u^* for the problem in (12) satisfies the following sign conditions,*

$$\begin{cases} u^* > 0, \text{ if } x \in \mathcal{R}^+, \text{ with} \\ \mathcal{R}^+ := \{x \in \mathbb{R} : \lim_{u \downarrow 0} \partial H(x, u, p, P)/\partial u < 0\}, \\ u^* < 0, \text{ if } x \in \mathcal{R}^-, \text{ with} \\ \mathcal{R}^- := \{x \in \mathbb{R} : \lim_{u \uparrow 0} \partial H(x, u, p, P)/\partial u > 0\}, \\ u^* = 0, \text{ if } x \in \mathcal{R}^0 := \{x \in \mathbb{R} : (\mathcal{R}^+ \cup \mathcal{R}^-(t))^c\}, \end{cases} \quad (26)$$

for some $(p, P) \in D_x^{1,2,\text{sp}}J^*(t, x)$, a.e. $t \geq 0$, \mathbf{P} -a.s.

4.1. The Inaction Region

Note from Proposition 4 that region \mathcal{R}^0 can be equivalently represented by

$$\mathcal{R}^0 = \left\{ x \in \mathbb{R} : \lim_{u \uparrow 0} \frac{\partial H(x, u, p, P)}{\partial u} \leq 0 \leq \lim_{u \downarrow 0} \frac{\partial H(x, u, p, P)}{\partial u} \right\}. \quad (27)$$

Note also that $u^*(x) \equiv 0, \forall x \in \mathcal{R}^0$ is optimal and for this reason \mathcal{R}^0 is called *Inaction Region* for the control input. Note however that “inaction” does mean that a *zero-control variation* should be optimal inside region \mathcal{R}^0 , since, $u^*(t) = 0$ implies that $h^*(t) = h_e$ and the nominal effort is optimal.

The next lemma gives conditions for the inaction region \mathcal{R}^0 to be a nondegenerate interval of the line, from the fact that $J^*(t, \cdot)$ is a convex function.

Lemma 2. *Assume that $\sigma_x^+, \sigma_x^-, \bar{\sigma}_x$ and $\sigma_u^+, \sigma_u^-, \bar{\sigma}_u$ are all nonnegative scalars and that $f(x, \cdot)$ is a continuously differentiable function for each x . Let $(p, P) \in D_x^{1,2\text{sp}} J^*(x)$, for each $x \in \mathbb{R}$, and define.*

$$\delta^+ := P\bar{\sigma}_u\sigma_u^+, \quad \delta^- := P\bar{\sigma}_u\sigma_u^-. \quad (28)$$

The region $\mathcal{R}^0 \subset \mathbb{R}$ is an interval defined by

$$\mathcal{R}^0 = \left\{ x \in \mathbb{R} : -\delta^+ \leq \frac{\partial f(x, u)}{\partial u} \Big|_{u^*=0} + Bp \leq \delta^- \right\}. \quad (29)$$

Note that $P\bar{\sigma}_u\sigma_u^+ \geq 0$ and $P\bar{\sigma}_u\sigma_u^- \geq 0$ hold from the assumption on the lemma and the fact that $P \geq 0$. It is enough that one of the products indicated in (28) is positive to conclude that the region $\mathcal{R}^0(t) \subset \mathbb{R}$ is of nonzero length in the state space line.

Proof. Consider the definition of the regions \mathcal{R}^0 in (27). By applying the limits to the Hamiltonian function derivatives in (25), (see (24)), we obtain

$$\lim_{u \uparrow 0} \frac{\partial H(x, u, p, P)}{\partial u} = \lim_{u \uparrow 0} \left[\frac{\partial f(x, u)}{\partial u} + P(\sigma_u^-)^2 u \right] + Bp - P\bar{\sigma}_u\sigma_u^- \leq 0, \quad (30)$$

$$\lim_{u \downarrow 0} \frac{\partial H(x, u, p, P)}{\partial u} = \lim_{u \downarrow 0} \left[\frac{\partial f(x, u)}{\partial u} + P(\sigma_u^+)^2 u \right] + Bp + P\bar{\sigma}_u\sigma_u^+ \geq 0. \quad (31)$$

Thus, the condition for the inaction gap to appear is that at least δ^+ or δ^- be positive. \square

4.2. Discounted Quadratic-linear Running Cost

Let us consider the control problem in (12) with the quadratic costs in (5). Since the data of the problem is time-homogenous and from the structure of the cost, one can introduce the following representation,

$$J^*(s, x) = e^{-\alpha s} V^*(x), \quad (32)$$

where V^* is a time-invariant solution, see do Val and Souto (2017) for details. The announced control problem has its solution connected to the HJB equation in (14) and V^* would be its unique solution. The Hamiltonian function as detailed in (24), with $f(x, u) = Qx^2 + qx + Ru^2 + ru$, allow us to write the HJB equation as

$$\begin{aligned} \alpha\varphi(x) - \inf_{u \in U} \left\{ (Q + \frac{1}{2}Q(P, \mathcal{S}(x)))x^2 + (R + \frac{1}{2}\mathcal{R}(P, \mathcal{S}(u)))u^2 \right. \\ \left. + \frac{1}{2}\mathcal{H}(P, \mathcal{S}(x))x + \frac{1}{2}\mathcal{G}(P, \mathcal{S}(u))u \right. \\ \left. + p(Ax + Bu) + qx + ru + \frac{1}{2}\mathcal{T}(P) \right\} = 0 \end{aligned} \quad (33)$$

with $p = \varphi_x$ and $P = \varphi_{xx}$ if $\varphi \in C^2(\mathbb{R})$. As previously mentioned, one cannot presume that $V^* \in C^2(\mathbb{R})$ and the solution for the HJB equation (33) should be coined in terms of viscosity solution. In that case, for each $(p, P) \in D_x^{1,2\text{sp}}\varphi(x)$ (33) holds with ' ≤ 0 ' (φ is a viscosity subsolution), and for each $(p', P') \in D_x^{1,2\text{sb}}\varphi(x)$ (33) holds with ' ≥ 0 ' (φ is a viscosity supersolution).

Remark 1. Since $\varphi(\cdot)$ that satisfies (24) in the classic or in the viscosity sense is such that $\varphi = V^*$, it is a convex function according with Theorem 1, and therefore, either $\varphi_{xx} \geq 0$, or for each P with $(p, P) \in D_x^{1,2\text{sp}}\varphi(x)$ one has $P \geq 0$.

Section 4.1 indicated that the optimal control is related to three distinct regions on the state space \mathcal{R}^+ , \mathcal{R}^- and \mathcal{R}^0 , related to the corresponding signs of the optimal u^* . In the next section we show that the value V^* coincides piecewisely with some quadratic solutions and the inaction region is completely determined from this analysis.

4.3. Characterization of the inaction region

In Section 4.1 a region of inaction \mathcal{R}^0 for control input u was identified in the state space, possibly of nonzero length interval of the real line.

Recall that the time-invariant solution $V^*(\cdot)$ is a convex function, and one can conclude that its minimum is attained next to the origin and near region \mathcal{R}^0 . The idea is to solve the control problem inside \mathcal{R}^0 , relying on the quadratic expansion of V^* sufficiently close to an arbitrary point x_0 inside \mathcal{R}^0 , and realize that the choice of x_0 does not affect the representation. Let us consider the following type of equation,

$$(2A - \alpha)X + Q + \mathcal{Q}(X, s) = 0, \quad (34)$$

with $s = +1$ or $s = -1$.

Lemma 3. *Consider the control problem in (12). Suppose that*

$$X^+ = -\frac{Q}{2A - \alpha + (\sigma_x^+)^2}, \quad \text{or,} \quad X^- = -\frac{Q}{2A - \alpha + (\sigma_x^-)^2}, \quad (35)$$

are such that $X^+, X^- > 0$. Then set

$$v_0^+ = -\frac{q + 2X^+\bar{\sigma}_x\sigma_x^+}{A - \alpha} \quad \text{or,} \quad v_0^- = -\frac{q - 2X^-\bar{\sigma}_x\sigma_x^-}{A - \alpha} \quad (36)$$

The inaction region \mathcal{R}^0 is the connected interval,

$$\begin{aligned} \mathcal{R}_0 = & \{x \leq 0 : -\bar{\sigma}_u\sigma_u^+ \leq Bx + (Bv_0^- + r)/2X^- \leq \bar{\sigma}_u\sigma_u^-\} \\ & \cup \{x \geq 0 : -\bar{\sigma}_u\sigma_u^+ \leq Bx + (Bv_0^+ + r)/2X^+ \leq \bar{\sigma}_u\sigma_u^-\}. \end{aligned} \quad (37)$$

If X^+ or X^- is not positive, the corresponding subinterval is empty.

The proof of Lemma 3 is presented in Appendix A.

4.4. Asymptotic solutions in regions \mathcal{R}^+ and \mathcal{R}^-

In the previous section the optimal solution was obtained inside region \mathcal{R}^0 of the state space line, which contains the minimum of V^* . The region \mathcal{R}^0 separates two distinct regions for which either $u^*(x) > 0$ or $u^*(x) < 0$, denoted respectively by \mathcal{R}^+ and \mathcal{R}^- .

Here the situation in which $u^* \neq 0$ is considered, whether the interval \mathcal{R}^0 is of zero length or not. If one takes points $x \in \mathbb{R}$ such that $|x|$ is sufficiently large, the control sign is either positive or negative, and it remains the same for a large neighborhood of such a point x .

In parallel to Section 4.3, we seek steady state solutions for the HJB equation inside regions \mathcal{R}^+ and \mathcal{R}^- for $|x| > L$ for large enough L . The following equations, in X^{s_x} , $v^{s_x, u}$ and $\ell^{s_x, u}$ are considered here.

$$(2A - \alpha)X^{s_x} - \frac{(BX^{s_x})^2}{R + \mathcal{R}(X^{s_x}, s_u)} + Q + \mathcal{Q}(X^{s_x}, s_x) = 0, \quad (38a)$$

$$\left(A - \alpha - \frac{B^2 X^{s_x}}{R + \mathcal{R}(X^{s_x}, s_u)} \right) v^{s_x, u} - \frac{BX^{s_x}}{R + \mathcal{R}(X^{s_x}, s_u)} (\mathcal{G}(X^{s_x}, s_u) + r) + \mathcal{H}(X^{s_x}, s_x) + q = 0 \quad (38b)$$

$$- \alpha \ell^{s_x, u} + \mathcal{T}(X) - \frac{1}{4(R + \mathcal{R}(X, s_u))} (Bv^{s_x, u} + \mathcal{G}(X^{s_x}, s_u) + r)^2 = 0 \quad (38c)$$

for certain combinations of $s_x = \pm 1$ and $s_u = \pm 1$. Consider also the feedback control

$$u(x) = -\frac{1}{R + \mathcal{R}(X^{s_x}, s_u)} \left(BX^{s_x} x + \frac{1}{2} (Bv^{s_x, u} + \mathcal{G}(X^{s_x}, s_u) + r) \right) \quad (39)$$

The next lemma presents some simple asymptotic solution for the HJB equation, valid when $L \rightarrow \infty$.

Lemma 4. *Consider the control problem in (12). If there exists a solution for (38a) with $X^{s_x} > 0$, then the value V^* tends asymptotically to $V_a^{s_x, u}$ for $|x| > L \rightarrow \infty$, where,*

$$V_a^{s_x, u}(x) := X^{s_x} x^2 + v^{s_x, u} x + \ell^{s_x, u}, \quad (40)$$

with $X_{s_x}, v^{s_x, u}$ and $\ell^{s_x, u}$ as in (38) for suitable combination of signals $s_x = \pm 1, s_u = \pm 1$. Moreover, for $|x| > L$ with $L \rightarrow \infty$, the optimal control policy $u^*(x)$ tends asymptotically to $u(x)$ in (39) as $|x| \rightarrow \infty$.

The proof of Lemma 4 is presented in Appendix B.

Remark 2. To set the right combination of signals of x and the optimal u one needs a matched signals reasoning. For the fishery management problem $B < 0$ thus, if $x \ll 0$ then $u^* < 0$ and if $x \gg 0$ then $u^* > 0$ and in this case the signals are coupled, i.e., $s_x = s_u$. Writing (38) more explicitly for this case,

$$s_x = s_u = +1 \begin{cases} (2A - \alpha + (\sigma_x^+)^2)X^+ + Q - \frac{(X^+ B)^2}{R + X^+(\sigma_u^+)^2} = 0 \\ \text{Then } X^+ = (-b + \sqrt{b^2 - 4ac})/2a, \text{ with,} \\ a = (\sigma_u^+)^2(2A - \alpha + (\sigma_x^+)^2) - B^2 \\ b = R(2A - \alpha + (\sigma_x^+)^2) + Q(\sigma_u^+)^2, \quad c = RQ \\ v^+ = -\left(A - \alpha - \frac{B^2 X^+}{R + X^+(\sigma_u^+)^2} \right)^{-1} \left(q + 2X^+ \bar{\sigma}_x \sigma_x^+ (+1) - \frac{BX^+(r + 2X^+ \bar{\sigma}_u \sigma_u^+ (+1))}{R + X^+(\sigma_u^+)^2} \right) \\ \ell^+ = \frac{1}{\alpha} \left(\mathcal{T}(X^+) - \frac{(Bv^+ + 2X^+ \bar{\sigma}_u \sigma_u^+ (+1) + r)^2}{4(R + X^+(\sigma_u^+)^2)} \right) \\ \text{Asymptotic optimal solution:} \\ u^*(x) = -\frac{1}{2(R + X^+(\sigma_u^+)^2)} (2X^+ Bx + v^+ B + X^+ \bar{\sigma}_u \sigma_u^+ (+1) + r) \\ \text{valid when } x \rightarrow +\infty. \end{cases}$$

For $s_x = s_u = -1$ replace ‘+1’ above by ‘-1’ and superscripts ‘+’ by ‘-’. The first equation is a modified version of the Riccati equation that appears in the discounted quadratic cost problem for deterministic controls and it resembles the ones that apply to stochastic optimal control with multiplicative BM noise. In classic problems $\sigma_x^+ = \sigma_x^- = 0$ and $\sigma_u^+ = \sigma_u^- = 0$, and the Riccati theory reveals in the scalar case that if $B \neq 0$, there exist unique solutions $X^+ > 0$ and $X^- > 0$.

Remark 3. So far the optimal solution was established inside the inaction region, and such a region was determined in Lemma 3. In addition, Lemma 4 outlines the asymptotic solution for any $|x|$ sufficiently large. The exact solution outside the region of inaction but for small values of state x is hard to find and it has to rely on the HJB equation in Proposition 3. However a good approximation is obtained at each half line by a linear interpolation between the optimal control $u^* = 0$ at the inaction region and the asymptotic control (39) obtained in Lemma 4. For more details, including the stochastic stability notion for the solutions see (do Val and Souto, 2017, sec. V).

4.5. About the Inaction Region

Remark 4. The characterization in Lemma 3 above allows the following considerations:

- (i) From (35) note that it is necessary that $2A - \alpha + (\sigma_x^+)^2 < 0$ to exist an inaction region for $x > 0$ and correspondently, $2A - \alpha + (\sigma_x^-)^2 < 0$ to exist an inaction region for $x < 0$. Hence, by increasing α or decreasing the uncertainty represented by σ_x^+ and σ_x^- respectively, one eventually gives rise to the inaction region in the solution.
- (ii) From (37) one can understand how the length of the inaction interval varies. For the positive line write equivalently the expression in (37).

$$-\bar{\sigma}_u \sigma_u^+ \leq Bx + p_0 \leq \bar{\sigma}_u \sigma_u^-, \quad p_0 = \frac{Bv_0^+ + r}{2X^+}$$

For the case depicted in Fig. 1, $B > 0$, $\sigma_u^+ > \sigma_u^-$ and $e^- > 0$. In this situation, $\{x \leq 0 : -\bar{\sigma}_u \sigma_u^+ \leq Bx + (Bv_0^- + r)/2X^- \leq \bar{\sigma}_u \sigma_u^-\} = \emptyset$.

Conversely, in the negative line, if $B > 0$, and the upper end of the interval $e^+ = (\bar{\sigma}_u \sigma_u^- - p_1)/B$ with $p_1 = (Bv_0^- + r)/2X^-$ is such that $e^+ < 0$ then the set $\{x \geq 0 : -\bar{\sigma}_u \sigma_u^+ \leq Bx + (Bv_0^+ + r)/2X^+ \leq \bar{\sigma}_u \sigma_u^-\}$ must be empty.

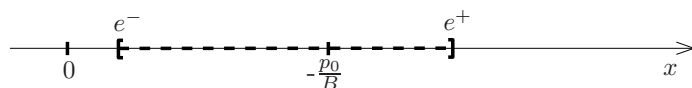


Figure 1: The inaction interval with extremes values $e^- = -(\bar{\sigma}_u \sigma_u^+ + p_0)/B$ and $e^+ = (\bar{\sigma}_u \sigma_u^- - p_0)/B$ for $B > 0$.

- (iii) From (ii) one can conclude that the larger $\bar{\sigma}_u$, σ_u^+ and σ_u^- are, the wider the inaction region will be; in particular, $\bar{\sigma}_u$ influences both e^- and e^+ , the lower and upper ends. The way that σ_x^+ , σ_x^- and $\bar{\sigma}_x$ influence the inaction region is more complex. First recall (i) above for the role of σ_x^+ and σ_x^- . Then note that points $-p_0/B$ and $-p_1/B$ depend linearly on $\bar{\sigma}_x$, which brings them closer as $\bar{\sigma}_x$ increases.

5. Application to the bio-economic fishery model

Recall that the objective of the policymaker is to minimize

$$J(s, x, u(\cdot)) = E^x \left[\int_s^\infty e^{-\alpha t} (x(t)^\top Q x(t) + q^\top x(t) + u(t)^\top R u(t) + r^\top u(t) - K) dt \right], \quad (41)$$

with $Q = 1$, $r = -(p - 2che)$, $R = c > 0$, $K = \bar{\pi} > 0$. When $q < 0$ a reward for having a biomass surplus is introduced, otherwise, if $q > 0$ a penalty holds for a biomass increase.

About sign of r . If h_e is set to the value that maximizes the static profit, then $h_e = h_{\text{Max}} = p/2c$ and $r = 0$. If $h_e < h_{\text{Max}}$ then $r < 0$, otherwise, $h_e > h_{\text{Max}}$ and then $r > 0$. We assume that the TAC value is chosen to be smaller than the one that maximizes the static profit, thus $h_e < h_{\text{Max}}$ and $r < 0$.

The dynamic is as in (12b)–(12c),

$$dx(t) = (Ax(t) + Bu(t)) dt + \hat{\sigma}(x(t), u(t))d\hat{W}(t), \quad (42a)$$

with

$$\hat{\sigma}(x, u) = [\sigma \quad \bar{\sigma}_x + \sigma_x^+ x^+ - \sigma_x^- x^- \quad \bar{\sigma}_u + \sigma_u^+ u^+ - \sigma_u^- u^-] \quad (42b)$$

and

$$d\hat{W}(t) = [dW(t) \quad dW^x(t) \quad dW^u(t)]^\top, \quad (42c)$$

with the assumption that $\sigma_x^+, \sigma_x^-, \bar{\sigma}_x \geq 0$, and $\sigma_u^+, \sigma_u^-, \bar{\sigma}_u \geq 0$.

Since $x(t) = z(t) - z_e$, when $x(t) > 0$, the biomass is larger than the targeted level, whereas $x(t) < 0$ means that the biomass is smaller than its desired level. It is reasonable to assume that $\sigma_x^+ \leq \sigma_x^-$, meaning that above the targeted level is somehow less “risky” than falling below. We set $B = -1$ which means that fishing removes the biomass at the same unit rate.

Finally, it is reasonable to assume that $A \leq 0$, which means that, when there is an excess of biomass ($x(t) > 0$), any increase of the living biomass will decrease the flow of new biomass due to carrying capacity (for the logistic model it implies that $z_e \geq K/2 = z_{\text{MSY}}$). It also means that when the biomass is low, $x(t) < 0$, but not too far from z_e , the biomass may grow back to its desired level. As consequence of setting $A > 0$ ($z_e < K/2$ in the logistic model), the biomass dynamics will never settle at z_e without continuous intervention, moving away from this desired level, even without fishing. In this situation, an inaction region would not be optimal.

5.1. Existence of the Inaction Region

Recall from remark 4 that the condition of existence of the Inaction Region is

$$X^+ = -\frac{Q}{2A - \alpha + (\sigma_x^+)^2} > 0, \quad \text{and} \quad X^- = -\frac{Q}{2A - \alpha + (\sigma_x^-)^2} > 0, \quad (43)$$

Since $Q = 1 > 0$, the condition of existence is that

$$2A - \alpha + \sigma_x^{+2} < 0 \text{ or } 2A - \alpha + \sigma_x^{-2} < 0 \quad (44)$$

should be satisfied. From (44), the following propositions are straightforward.

Proposition 5 (Biomass (State) uncertainty). *Ceteris Paribus, when the biomass drift coefficient $A < 0$, an increase of the state uncertainty (represented by $(\sigma_x^\pm)^2$), will reduce the chance of an inaction region to exist. In the case of $A > 0$, the dynamics implies that $x(t)$ will move away from z_e and hence inaction is no longer an optimal policy, or equivalently, the size of inaction region will be empty.*

Proposition 6 (Discount rate). *Ceteris Paribus, an increase of the discount rate α will increase the chance of an inaction region to exist, even if $A > 0$.*

Proposition 7 (Stability around z_e). *The more (less) negative is the biomass drift coefficient A is around z_e , the more (less) likely the inaction region will exist.*

5.2. Size and position of the Inaction Region

By definition, $\bar{\sigma}_u, \sigma_u^-, \sigma_u^+, \bar{\sigma}_x, \sigma_x^+, \sigma_x^- > 0$, $B = -1 < 0$, $r < 0$, and we assume $A < 0$, meaning that z^e is stable equilibrium point. Assume also that $X^+ > 0$ and/or $X^- > 0$, i.e., the inaction

region is nonempty. From Lemma 3, we know that the inaction region \mathcal{R}^0 is the connected interval,

$$\begin{aligned} \mathcal{R}_0 = \mathcal{R}_0^- \cup \mathcal{R}_0^+ = \{x \leq 0 : -\bar{\sigma}_u \sigma_u^- \leq x + (v_0^- - r)/2X^- \leq \bar{\sigma}_u \sigma_u^+\} \\ \cup \{x \geq 0 : -\bar{\sigma}_u \sigma_u^- \leq x + (v_0^+ - r)/2X^+ \leq \bar{\sigma}_u \sigma_u^+\}. \end{aligned} \quad (45)$$

with

$$v_0^+ = -\frac{q + 2X^+ \bar{\sigma}_x \sigma_x^+}{A - \alpha} \quad \text{or,} \quad v_0^- = -\frac{q - 2X^- \bar{\sigma}_x \sigma_x^-}{A - \alpha} \quad (46)$$

One can easily check that the signs of v_0^+ and v_0^- will depend on the sign of q . If $q < 0$, which means a reward for the state being above the desired state level, then $v_0^- < 0$ and $v_0^+ \geq 0$. If $q > 0$, which means a penalty for the biomass state being above the desired state, then $v_0^+ > 0$ and $v_0^- \leq 0$. Finally of $q = 0$, then $v_0^+ > 0$ and $v_0^- < 0$. Taking into account these remarks, and given (45)-(46), the following propositions hold.

Proposition 8 (Control uncertainty). *Ceteris Paribus, an increase of the control uncertainties, $\bar{\sigma}_u, \sigma_u^-, \sigma_u^+$, will increase the size of the inaction region.*

Thus, the more uncertain is the impact of the control on the state evolution, the wider the Inaction Region should be. In other words, when the control action affects the state evolution with high uncertainty, the best policy is to do nothing up to a considerable drifting of the state away from its desired level. The proof of this proposition is straightforward and comes from the fact that an increase of σ_u^+ and/or σ_u^- will increase the interval for the conditions in (45) to hold. This eases the existence and enlarge the size of the connected region, i.e. the inaction region.

Corollary 1 (No Control Uncertainty). *Without uncertainty on the control effect, the size of inaction region is empty.*

The corollary means that if there is no doubt about the consequences of a change in the TAC policy, then the management rule should always be flexible, accompanying the evolution of $t \rightarrow x(t)$.

From (46) it is possible to check that v_0^+ increases with σ_x^+ and $\bar{\sigma}_x$, whereas v_0^- decreases with σ_x^- and $\bar{\sigma}_x$. As a consequence, the conditions $-\bar{\sigma}_u \sigma_u^- \leq x + (v_0^- - r)/2X^-$, for $x < 0$, and the condition $x + (v_0^+ - r)/2X^+ \leq \bar{\sigma}_u \sigma_u^+$, for $x > 0$, in (45) will be less easily verified. As a result, the size of the inaction region will reduce and the following proposition holds.

Proposition 9 (State uncertainty). *An increase of the state uncertainties, σ_x^+, σ_x^- , will decrease the size of the inaction region.*

In summary, the more uncertain is the impact of the state level on the state evolution, the less the Inaction Region may exist. In other words, if doubt exists about the impact of the state evolution, one should intervene as much as possible into the state evolution via the control. Notice that an increase of Q , the cost parameter in the cost function related to the state gap, will have a similar effect to reduce the size of the inaction region (by increasing the value of X^+ and X^-). The consequences of A and α on the size of the inaction region are less obvious since they will impact both X^+ and X^- , and, in an opposite way, v_0^+ or v_0^- .

Finally, there is an obvious proposition which depends on the biomass level being drifted away from the desired level.

Proposition 10 (State deviation). *If the current level of biomass is far away from its desired level ($|x| \gg 0$), distant it will be from a possible Inaction Region.*

5.3. Numerical Simulations

The purpose of this section is to explore some numerical simulations to highlight the previous propositions. In all cases, we set $B = -1$, $R = 0.5$ and $r = -0.1$.

5.3.1. Impact of q and α

The next figures show the range of the inaction region as a function of the q values. In all cases: $A = -0.2$, $\bar{\sigma}_x = \bar{\sigma}_u = 0.2$, $\sigma_x^+ = \sigma_u^+ = 0.3$, $\sigma_x^- = \sigma_u^- = 0.5$. Below the solid line the optimal action u^* is to reduce the TAC ($u^* < 0$, points C and D in in Fig. 2), while above the dashed line, the optimal action is to increase the TAC ($u^* > 0$, points B and E in Fig. 2). Between the two lines lies the inaction region (no change of TAC, $u^* = 0$, point A, Fig. 2).

In Fig. 2, at point A, the state is close to its desired value ($x \simeq 0$) and no reward or penalty is given ($q = 0$). The most appropriate strategy is to do nothing, given the level of uncertainty associated with the use of the control (e.g. uncertainty created by a change of the fixed TAC value). One can see that the position of the inaction region at $q = 0$ is asymmetric around $x = 0$, leading to more room for inaction when $x < 0$ than when $x > 0$. This is explained by the uncertainty values employed in the model for the control, which are asymmetric and it weights more a decrease of u rather than an increase of u , e.g. $\sigma_u^+ = 0.3 < \sigma_u^- = 0.5$.

As a result, when the fishery manager is expected to change the fixed TAC, the optimal choice will be not to act so as to avoid uncertainty created by the change of the fixed TAC. Of course, when the deviation from the desired state becomes too important, e.g. $|x| \gg 0$, then the manager should take action whatever the level of uncertainty. Compared to A, points B and C require to act and change the rule, either to increase the fixed TAC (point B), or to reduce it (point C). Points D and E have the same state level than A but lead to different interpretations of the situation with respect to the desired state value. In point D, since $q < 0$, there is a reward for the state being above the desired value, e.g. a reward for $x > 0$. In other words, $q < 0$ creates an incentive for the policymaker to set up a reduction of the TAC in order to increase the value of x . On the opposite side, point E with $q > 0$ prevails a penalty for letting the state be above the desired level, e.g. a penalty if $x > 0$. At point E, the policymaker tends to increase the TAC despite of the uncertainty.

More generally, the following remarks apply to Figs. 2–10. Firstly, the position of the inaction region changes with the value of q : a reward for being above the desired state level (e.g. $q < 0$) is more compatible with the problem since the inaction interval arises more for $x > 0$ than for $x < 0$. Secondly, when x drifts away from its desired level, it will leave at some point the inaction region, and the fishery manager should take action by changing the TAC rule. Thirdly, an increase of the discount rate $\alpha = 0.9$ rather than $\alpha = 0.2$ will increase the likelihood of existence of an inaction region but not necessarily its size.

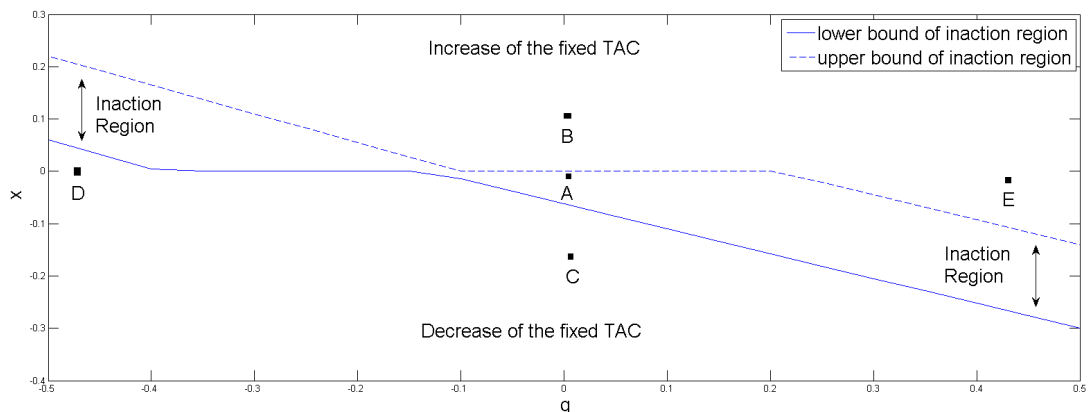


Figure 2: Position of the inaction region in the state x with respect to q , with $\alpha = 0.9$

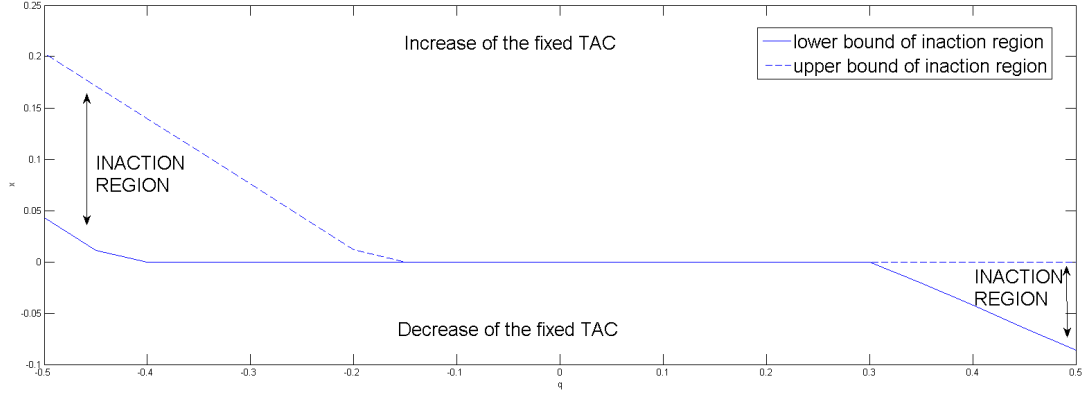


Figure 3: Position of the inaction region in the state x with respect to q , with $\alpha = 0.2$

5.3.2. Impact of control uncertainties

We set in all simulations: $A = -0.2$, $\alpha = 0.9$, $\bar{\sigma}_x = \bar{\sigma}_u = 0.2$, and $\sigma_x^+ = \sigma_x^- = 0$. In this case, we do not assume any endogenous uncertainty generated by the evolution of the state value but only consider the role of control uncertainties. Figs. 4–5 highlight the impact of having asymmetric control uncertainties. Firstly, note that without any state uncertainty, the inaction region has a constant size. Secondly, a comparison of Figs. 4–5 indicates the fact that, *ceteris paribus*, the existence of control uncertainties will shift down (up) the inaction region when uncertainty is mainly due to negative (positive) value of the control, e.g. $\sigma_u^- > 0$ in Fig. 4 (e.g. $\sigma_u^+ > 0$ in Fig. 5). When the inaction region is moving downward (upward), the likelihood of cases where the manager should decrease (increase) the TAC is decreasing.

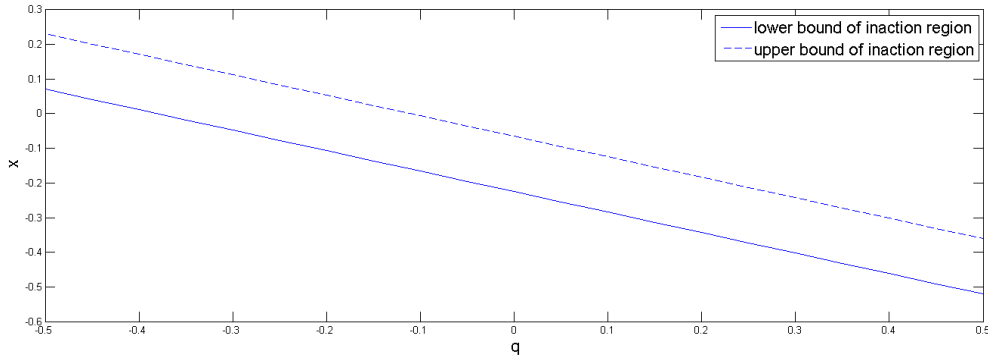


Figure 4: Position of the inaction region in the state x with respect to q , with $\sigma_u^- = 0.8$ and $\sigma_u^+ = 0$

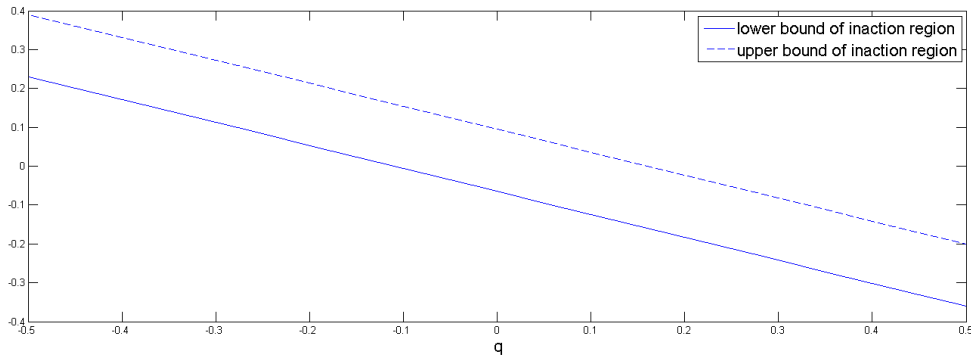


Figure 5: Position of the inaction region in the state x with respect to q , with $\sigma_u^- = 0$ and $\sigma_u^+ = 0.8$

5.3.3. Impact of asymmetric state uncertainties

Figures 6–7 highlight the impact of increasing the state uncertainties given the other uncertainty values. For all simulations, we set: $A = -0.2$, $\alpha = 0.9$, $\bar{\sigma}_x = \bar{\sigma}_u = 0.2$, $\sigma_x^- = \sigma_u^- = 0$, and $\sigma_u^+ = 0.8$. Under this set of parameters, control uncertainty is only created when the policymaker increases the TAC value, that is, when $u^* > 0$. In Fig. 6 one assumes a small positive state uncertainty ($\sigma_x^+ = 0.3$), whereas in Fig. 7 a higher positive state uncertainty value ($\sigma_x^+ = 0.8$) is set. It is seen from the figures that when the level (positive value) of state uncertainty increases, for a given control uncertainty, the inaction region is reduced (Fig. 7). In addition, even when a reward for the state being above the desired level exists, e.g. when $q < 0$, the inaction region moves down (compare with Fig. 6). Since positive values of x create a high level of uncertainty, the best strategy is to reduce x by increasing the TAC.

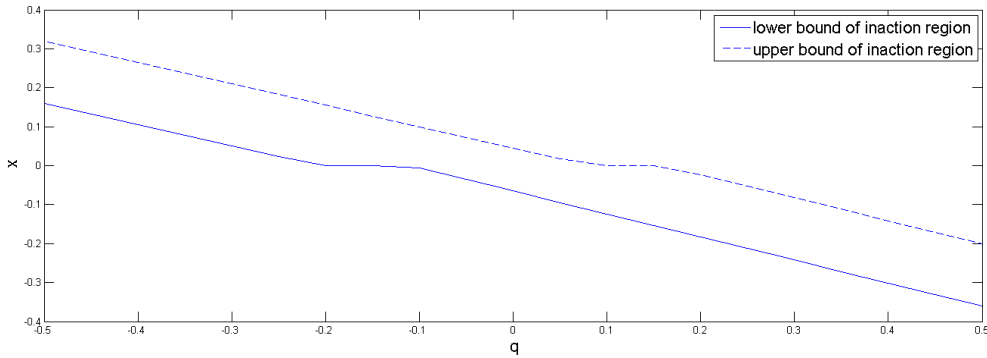


Figure 6: Position of the inaction region in the state x with respect to q , with $\sigma_x^- = \sigma_u^- = 0$ and $\sigma_x^+ = 0.3$ and $\sigma_u^+ = 0.8$

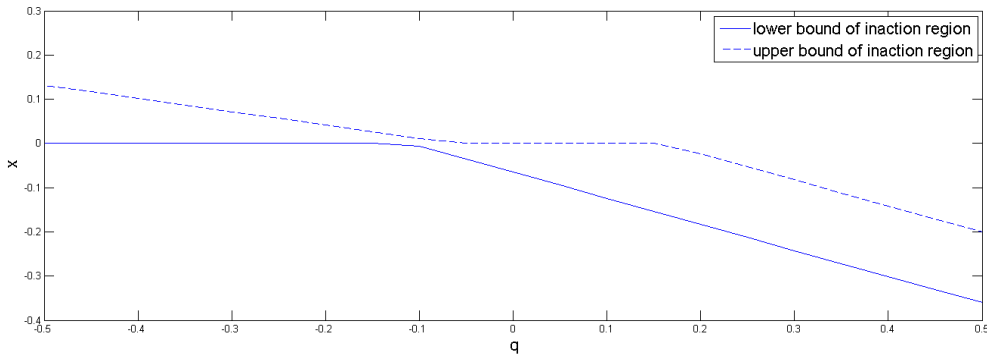


Figure 7: Position of the inaction region in the state x with respect to q , with $\sigma_x^- = \sigma_u^- = 0$ and $\sigma_x^+ = 0.8$ and $\sigma_u^+ = 0.8$

5.3.4. Effect of strictly biological objectives

Here, we consider the case where the policymaker does not pay attention to the profit of fishers, but only to the stock evolution. In that respect, we set $R = r = 0$ in the cost function (41). Figure 8 highlights the impact of setting a policy which is only resource-oriented. The main effect is to move upward the inaction region. For example, when the stock is over its desired level (point A), without any economic (profit) consideration, the stock is still seen high enough to allow for an increase of TAC. On the other hand, when including an economic objective, the TAC will be augmented.

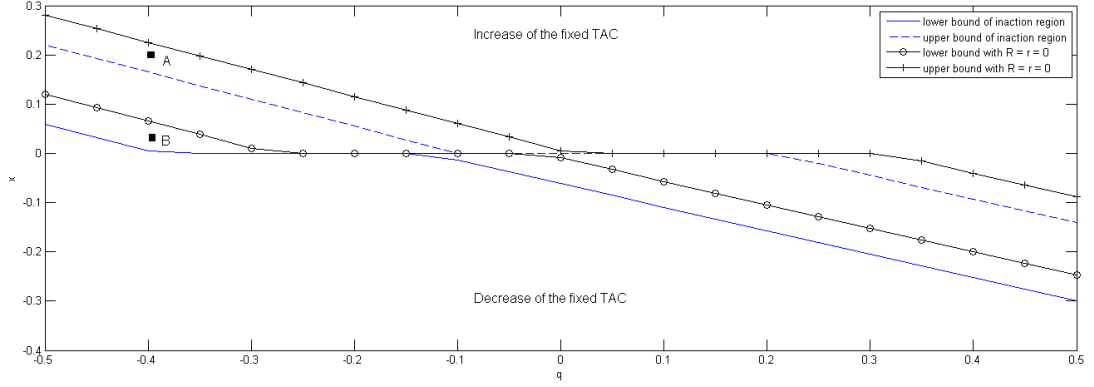


Figure 8: Comparison of the position of the inaction region with respect to q , with $\alpha = 0.9$, and with $(R, r \neq 0)$ or without $(R = r = 0)$ profit consideration in the cost function.

Point B shows that, although the stock is only slightly above the desired level, the policy maker will not reduce the TAC if profit is taken into consideration, although the TAC will be decreased with a mere biological attention.

5.3.5. Two biological opposite cases

We conclude the numerical illustrations with a discussion on the desired state value, z_e , and possible consequences of this choice.

K as the desired level

Let assume that $z^e = K$, the carrying capacity. From an ecological perspective, K is a stable equilibrium, which means that we would have $A < 0$. On that basis, there may exist a general agreement between conservationists and fishers that would target an acceptable level of biomass below the K value. Consequently, the fishery manager may include a term $q \geq 0$ in the cost function, which gives a reward for $x < 0$, i.e. $z < K$, and a penalty if $x > 0$, i.e. $z > K$. In Fig. 9, we set $A = -0.1$, with $\alpha = 0.9$, $B = -1$, $\sigma_x^- = \sigma_u^- = 0.2$, $\sigma_x^+ = 0.5$ and $\sigma_u^+ = 0.8$. In other words, we assume here that there is more uncertainty when the TAC is decreasing ($u < 0$) and when the biomass is greater than K ($x > 0$), than when the TAC is increasing ($u > 0$), and when the biomass level is less than K ($x < 0$).

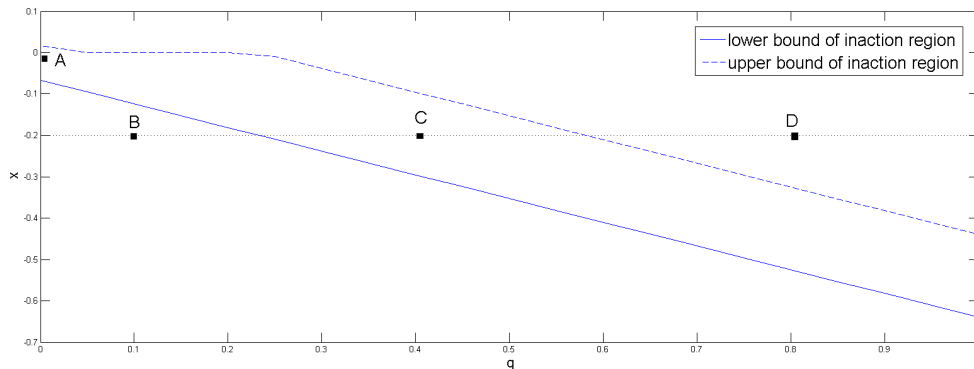


Figure 9: Position of the inaction region in the state x with respect to q , with $\alpha = 0.9$ and with $\sigma_x^- = \sigma_u^- = 0.2$ and $\sigma_x^+ = 0.5$ and $\sigma_u^+ = 0.8$

The following observations are put forward. At point A, $q = 0$, which means that no reward or penalty of being away from K is implemented, the best option for the manager is to do nothing as long as the biomass is sufficiently close to its desired value K . In points B, C and D, we have $x_B = x_C = x_D = -0.2 < 0$, meaning that the biomass is below the K values. The three points differ because of their q -values: $q_B = 0.1 < q_C = 0.4 < q_D = 0.8$. As q increases, the reward of being below the K value increases. As a consequence, with the same state level, the fishery manager with q_B value chooses to reduce the TAC since the biomass is below K , while with another ruler with q_D value would trigger an opposite choice by increasing the TAC. Between these two strategies, a manager using the q_C value would not change the fixed TAC rule.

MSY as the desired level

Let us assume that $z_e = z_{\text{MSY}}$. At the MSY biomass level occurs the maximum growth rate of the population and the growth derivative will always be zero. Thus, $A = 0$ in the model and it is not, per se, an asymptotic equilibrium. Figure 10 shows the inaction region when we set $A = 0$, together with $\alpha = 0.9$ and $B = -1$.

Let us compare situations A, B and C where $x_B = x_C = x_A < 0$. In A, we have $q = 0$, meaning no reward or penalty for being away from the MSY value. At x_A , the biomass level is below the MSY value but belongs to an inaction region, commanding the manager not to act here. Compared to A, in B, we have $q = -0.1$ rewarding any biomass level that would overcome the MSY value, which is desirable from a conservationist viewpoint. In such circumstances, the optimal decision will lead to reduce the TAC. Conversely, in C, the $q = 0.4$ penalty makes it profitable to accept a biomass level below the MSY value, in line with an economic target. The optimal solution will then switch to an increase of the TAC.

Similarly, if the two cases of $q = -0.4$ and $q = 0.4$ were to be compared at $x = 0.025$, an economic-driven strategy will claim for an increase of the TAC, while the conservationist will stick to the fixed TAC rule until the biomass far exceed the MSY level, preferring a sort of precautionary approach.

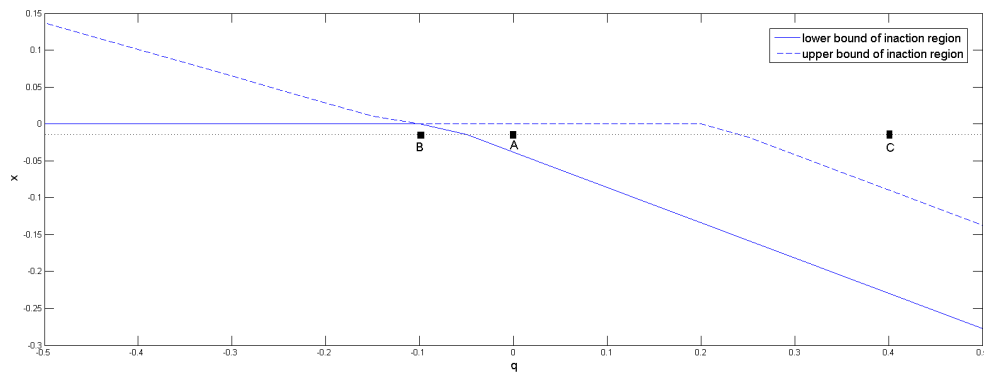


Figure 10: Position of the inaction region in the state x with respect to q , with $\alpha = 0.9$ and with $\sigma_x^- = \sigma_u^- = 0.2$ and $\sigma_x^+ = \sigma_u^+ = 0.5$

6. Discussion and conclusion

The paper develops an asymmetric model to deal with the control problem of continuous stochastic systems with a poorly known dynamics, when the deviations of the desired value should be taken with unbalanced risk. It is based on the CVIU approach that inbuilt in the model the fact that control variations increase the overall state uncertainty. This method is

proposed as an alternative to the robust approach in the context of a poor knowledge of the system dynamics.

The use of dynamic programming to design the CVIU model controller requires tools from viscosity analysis, and simple assumptions are made to show that the value function is convex. The optimal feedback control policy is derived and reveals a region in the state space in which the policy is to sustain the action while inside it. Making use of a nomenclature already used in economics, this region is called inaction region, and it creates a precautionary control policy, not seen in the robust worst-case analysis. It is an expected behavior, since the decision maker should exercise care to intervene in a not well known system.

The problem is formulated in an infinite time horizon control framework, as a discounted quadratic cost problem, to study time-invariant optimal solutions. Closed-forms are derived for the solution inside the inaction region, and asymptotically, far away from it. These solutions are quadratic functions obtained as a type of Lyapunov (inside the inaction interval) or Riccati equations (asymptotically, as $|x| \rightarrow \infty$). Motivated by the fishery management problem, the case of asymmetric uncertainties is studied here for the first time.

We found that the CVIU framework is particularly appropriate in the case of fishery management. It has been recognised that fishery systems dynamics is rather poorly understood, both on the natural and social grounds. The most conventional management measures are based on setting catch (TAC) or effort limits, as implemented by the European Common Fisheries Policy for a number of species (with TACs and quotas fixed every December). However, these management measures rely on accurate stock assessment which is altered by many sources of uncertainties: random shocks caused by environmental variability, data based on catches, errors in parameter estimates, structural uncertainty in ecosystemic models and trophic interactions, inefficient enforcement of harvest quotas resulting in illegal, unregulated and unreported fishing, etc.

According to the source of uncertainty (natural resources facing random oscillations of the environment, stochastic behaviors of fishers responding to harvest control rules), choosing a constant optimal escapement rule is not always possible, as shown in previous studies (Clark and Kirkwood (1986), Roughgarden and Smith (1996), Sethi et al. (2005)). Optimal control becomes even more erratic when the system-to-be-governed includes uncertainty within the model structure itself (Hill et al. (2007), Bavinck et al. (2013)). Increasingly, fishery scientists advocate new forms of management policy towards an ecosystemic approach of fisheries, including robust, experimental or adaptive management, harvest control rules, balanced harvest management striving to keep the ecosystem structure and functioning intact, in spite of fishing mortality (Garcia et al. (2012), Charles et al. (2015)). Fishing across ages and sizes and giving up the golden rule of selectivity would enhance the fishery system better than usual recommendations setting an age limit and targeting the older individuals of the fish population. To the extreme, in some cases, the exploited system could be better off with no management at all, avoiding costly and impacting changes of management rules for users (Jul-Larsen et al. (2003), Boettiger et al. (206)), a position argued by other scientists (Garcia et al. (2012), Reid et al. (2016)).

With a poorly known dynamics of the fishery system, the CVIU approach points out the limit cases within which fishery managers should rather stick to a fixed management rule (e.g. TAC) instead of adapting it permanently to the latest state of knowledge surrounding stock assessment and harvest levels. Following the well-known Brainard principle (Brainard, 1967) applied to the implementation of a monetary policy by a central bank, we demonstrate and explain the existence of inaction regions for fishery managers. In this particular case, inaction would not mean for a fishery manager to do nothing at all, but instead to keep a time fixed harvest control rule like a TAC, rather than adjusting it permanently, in view of the inherent uncertainties surrounding the model.

Such inaction regions are more likely to be met when the discount rate is high, the biomass dynamics is stable around the desired level of biomass, and the (state) uncertainty of the biomass

decreases. When it comes to uncertainty created by the harvest control rules (e.g. changing the TAC level), we found an asymmetric response of the system dynamics and distinct trends of the inaction region. When the uncertainty coming from harvest control is caused by a decrease of the TAC, the fishery manager ought to adjust the control rule on a regular basis. Conversely, if uncertainty is endogenously created by modifying positively the control rule (i.e. increasing the TAC), then the inaction region is likely to expand and the laissez-faire policy (no new action) should be the rule. Whatever the response of the system to new control rules, whenever the current state of biomass drifts away from the desired (e.g. MSY) level, the inaction region tends to fade away and managers should change the output control (TAC) level.

The implications for fishery management are important. As long as the biomass state level is likely to drift away from the desired level, the risk of critical thresholds and tipping points (Scheffer et al. (2003), Schill et al. (2015), Boettiger et al. (206)) would increase and jeopardize the survival of the population. As long as changing the TAC is not costly, in terms of creating additional uncertainty, the precautionary approach should be imposed in such circumstances and the fishery manager ought to monitor closely the fishery by adjusting downward or upward more frequently the authorized harvest level. On the other hand, if the biomass fluctuates around the desired level of conservation, even in the case of a high state uncertainty, the fishery manager should hold the same control rule and avoid a too frequent intervention. The choice of the desired state level, together with the willingness to reward or penalize more any deviation from this desired state level, will impact on existence and magnitude of the inaction region.

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Appendix A. Proof of Lemma 3

Let us assume that point $x \in \mathcal{R}_0$ and knowing that $u^* = 0$, (24) yields,

$$\alpha V^*(x) - \left\{ (Q + \frac{1}{2} \mathcal{Q}(P, \text{sign}(x))x^2 + \frac{1}{2} \mathcal{H}(P, \text{sign}(x))x + (pA + q)x + \frac{1}{2} \mathcal{T}(P)) \right\} = 0 \quad (\text{A.1})$$

with $p = V_x^*$ and $P = V_{xx}^*$ if V^* is twice differentiable at x ; otherwise, $(p', P') \in D_x^{1,2\text{sp}}V^*(x)$ and $(p'', P'') \in D_x^{1,2\text{sb}}V^*(x)$ with (A.1) changed to corresponding inequalities.

It was already hinted that the value function $V^*(\cdot)$ inherits convexity as result of Theorem 1, and a quadratic function provides a good approximation to the optimal cost in a neighborhood of any point x_0 . Set here $x_0 \leq 0$ or $x_0 \geq 0$ and a sequence $\{\zeta_k\}_{k \geq 0}$ with each ζ_k an Alexandrov point of function V^* and $\zeta_k \rightarrow x_0$ having homogenous signals, in such a way that $\text{sign}(\zeta_k) = -1, \forall k$ for the former, or $\text{sign}(\zeta_k) = +1, \forall k$, for the latter choice. Let us set a quadratic approximation for V^* near x_0 and for some $\epsilon > 0$ and k sufficiently large, we write,

$$\left| V^*(x) - \left(V^*(\zeta_k) + V_x^*(\zeta_k)(x - x_0) + \frac{1}{2} V_{xx}^*(\zeta_k)(x - x_0)^2 \right) \right| < \epsilon,$$

provided that $x \in \mathcal{R}_0$, and $\|x - x_0\| < \delta = \delta(\epsilon)$. It follows from Alexandrov theorem for convex functions (Niculescu and Persson, 2006, Th 3.11.2) that $0 \leq V_{xx}^*(\zeta_k) \leq \rho I$, for some $\rho > 0$.

Consider a point x in the neighborhood of x_0 as above. Substituting the approximation of V^* at such a x near x_0 into (A.1), one can write that

$$\begin{aligned} & \alpha \left(V^*(\zeta_k) + V_x^*(\zeta_k)(x - x_0) + \frac{1}{2} V_{xx}^*(\zeta_k)(x - x_0)^2 \right) \\ & - \left\{ (Q + \frac{1}{2} \mathcal{Q}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k))x^2 + \frac{1}{2} \mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k))x + qx \right. \\ & \quad \left. + (V_{xx}^*(\zeta_k)(x - x_0) + V_x^*(\zeta_k))Ax + \frac{1}{2} \mathcal{T}(V_{xx}^*(\zeta_k)) \right\} \quad (\text{A.2}) \end{aligned}$$

is dominated from above and from below by some $\pm\gamma(\epsilon)$ with $\gamma(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Rearranging (A.2),

$$\begin{aligned} & \frac{1}{2} \left((2A - \alpha) V_{xx}^*(\zeta_k) + 2Q + \mathcal{Q}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) \right) x^2 \\ & + \left(q + \frac{1}{2} \mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) + (\alpha - A) V_{xx}^*(\zeta_k) x_0 + (A - \alpha) V_x^*(\zeta_k) \right) x \\ & - \alpha V^*(\zeta_k) + \frac{1}{2} \mathcal{T}(V_{xx}^*(\zeta_k)) + \alpha V_x^*(\zeta_k) x_0 - \frac{1}{2} \alpha V_{xx}^*(\zeta_k) x_0^2. \quad (\text{A.3}) \end{aligned}$$

Taking the limit of $\{\zeta_k\}_{k \geq 0}$, the choices $V_{xx}^*(x) = 2X^s$ and $V_x^*(x) = v^s$, with $s = -1$ or $s = +1$ as appropriate, the above expression reads as:

$$\begin{aligned} & \left((2A - \alpha) X^s + Q + \mathcal{Q}(X^s, s) \right) x^2 \\ & + \left(q + \mathcal{H}(X^s, s) + (\alpha - A)(2X^s x_0 - v^s) \right) x \\ & - \alpha V^*(x_0) + \mathcal{T}(X^s) - \alpha(X^s x_0^2 - v^s x_0). \quad (\text{A.4}) \end{aligned}$$

Suppose that $(2A - \alpha)X^s + Q + \mathcal{Q}(X^s, s) = 0$ has a positive solution for X^s . Then one can set the Hamiltonian to zero by a qualified choice of the parameter v^s and obtain the corresponding value of $V^*(x_0)$ from (A.4). That provides the quadratic representation for the value function,

$$V^*(x) = X^s(x - x_0)^2 + v^s(x - x_0) + \frac{\mathcal{T}(X^s)}{\alpha} - (X^s x_0^2 + v^s x_0) \quad (\text{A.5})$$

valid for each $x \in \mathcal{R}_0$ in a neighborhood of x_0 of same signal. Note that one may choose x_0 such that $\text{sign}(x_0) = s$, but otherwise arbitrarily, to set (A.3) to zero.

Recall the definition of region \mathcal{R}^0 in (29), Lemma 2. In the present situation, with quadratic running costs, the HJB equation in (24) specialized for region \mathcal{R}^0 in (A.1) for the time-invariant solution, V^* in (A.5), one obtains $P = 2X^s$, $p = 2X^s(x - x_0) + v^s$ with $v^s = -\frac{q + \mathcal{H}(X^s, s)}{A - \alpha} + 2X^s x_0$, valid for $x, x_0 \in \mathcal{R}^0$ of same signals. Now let $x_0 \uparrow 0$ if $s = -1$, and $x_0 \downarrow 0$ if $s = +1$, to obtain from Lemma 2 the characterization,

$$\begin{aligned} \mathcal{R}_0 = \{ & x \leq 0 : -2X^- \bar{\sigma}_u \sigma_u^+ \leq B(2X^- x + v_0^-) + r \leq 2X^- \bar{\sigma}_u \sigma_u^- \} \cup \\ & \{ x \geq 0 : -2X^+ \bar{\sigma}_u \sigma_u^+ \leq B(2X^+ x + v_0^+) + r \leq 2X^+ \bar{\sigma}_u \sigma_u^- \} \end{aligned} \quad (\text{A.6})$$

with $v_0^+ = -\frac{q + \mathcal{H}(X^+, +)}{A - \alpha}$ and $v_0^- = -\frac{q + \mathcal{H}(X^-, -)}{A - \alpha}$ as in (36). Rearranging (A.6), we obtain (37).

It remains to show that \mathcal{R}^0 expressed by (A.6) is a connected region. Note for this purpose that $x \rightarrow V_x^*(x)$ is defined for $x < 0$ or $x > 0$ inside \mathcal{R}^0 , and it should be monotone increasing function from the fact that V^* is convex (piecewise quadratic) function¹. If $B > 0$, consider the upper limit of the first interval and the lower limit of the second one. Otherwise, consider the lower limit of the first interval and the upper limit of the second one. Region \mathcal{R}_0 will be connect provided that

$$B > 0 : \begin{cases} BV_x^-(z - \frac{\bar{\sigma}_u \sigma_u^-}{B}) + r \leq 0, \\ BV_x^+(y + \frac{\bar{\sigma}_u \sigma_u^+}{B}) + r \geq 0. \end{cases} \quad B < 0 : \begin{cases} BV_x^-(z + \frac{\bar{\sigma}_u \sigma_u^+}{B}) + r \geq 0, \\ BV_x^+(y - \frac{\bar{\sigma}_u \sigma_u^-}{B}) + r \leq 0. \end{cases}$$

with $z \geq y$.

Let us assume that the interval \mathcal{R}_0 is not connected for the case $B > 0$. This is equivalent to assume that there exist a z^+ with $BV_x^-(z^+ - \bar{\sigma}_u \sigma_u^-/B) + r = 0$ and a y^- with $BV_x^+(y^- + \bar{\sigma}_u \sigma_u^+/B) + r = 0$ with $z^+ < y^-$. It then follow that

$$V_x^-(z^+ - \frac{\bar{\sigma}_u \sigma_u^-}{B}) = V_x^+(y^- + \frac{\bar{\sigma}_u \sigma_u^+}{B}) = -\frac{r}{B}$$

Since $x \rightarrow V_x^*(x)$ is monotone increasing function, the above would imply that $V_x^*(x) = -r/B$ in the interval $z^+ - \bar{\sigma}_u \sigma_u^-/B \leq x \leq y^- + \bar{\sigma}_u \sigma_u^+/B$ and hence $V_{xx}^*(x) = 0$ in the same interval. However, from the HJB equation (24) comes the fact that the optimal control would be $u^*(x) = 0$ in the same interval, which denies the assumption and implies the fact that region \mathcal{R}_0 is a connected interval.

When $B < 0$ the proof follows exactly the same arguments considering the interval $z^+ + \bar{\sigma}_u \sigma_u^+/B \leq x \leq y^- - \bar{\sigma}_u \sigma_u^-/B$, and the details are omitted. \square

Appendix B. Proof of Lemma 4

We take into account the HJB equation (24) in Proposition 3. As in Section 4.3, let us employ the quadratic approximation here in a neighborhood of a point x_0 with $|x_0| > L$ for some large value of L . For this, consider the HJB equation in (24) with a quadratic approximation valid

¹A generalized version of the monotocity pointed out can be developed when V^* is not differentiable but convex. We choose to write these arguments when V^* is continuously differentiable and V_x^* exists as a standard function, to not impair the clarity with too many technical details. However, they can be shown in the general case.

for x near x_0 ,

$$\begin{aligned} & \alpha \left(V^*(\zeta_k) + V_x^*(\zeta_k)(x - x_0) + \frac{1}{2}V_{xx}^*(\zeta_k)(x - x_0)^2 \right) \\ & - \inf_{u \in U} \left\{ \left(Q + \frac{1}{2}\mathcal{Q}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) \right) x^2 + \frac{1}{2}\mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k))x + qx \right. \\ & \quad + \left(R + \frac{1}{2}\mathcal{R}(V_{xx}^*(\zeta_k), \text{sign}(u)) \right) u^2 + \frac{1}{2}\mathcal{G}(V_{xx}^*(\zeta_k), \text{sign}(u))u + ru \\ & \quad \left. + (V_{xx}^*(\zeta_k)(x - x_0) + V_x^*(\zeta_k))(Ax + Bu) + \frac{1}{2}\mathcal{T}(V_{xx}^*(\zeta_k)) \right\} \quad (\text{B.1}) \end{aligned}$$

Let us denote,

$$\tilde{R}_k = R + \frac{1}{2}\mathcal{R}(V_{xx}^*(\zeta_k), \text{sign}(u)) \quad (\text{B.2a})$$

$$\tilde{Q}_k = Q + \frac{1}{2}\mathcal{Q}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) \quad (\text{B.2b})$$

$$\tilde{\Upsilon}_k = \Upsilon_k^1 x + \Upsilon_k^0 \quad (\text{B.2c})$$

with $\Upsilon_k^1 = V_{xx}^*(\zeta_k)B$, and

$$\Upsilon_k^0 = (V_x^*(\zeta_k) - x_0 V_{xx}^*(\zeta_k))B + \frac{1}{2}\mathcal{G}(V_{xx}^*(\zeta_k), \text{sign}(u)) + r$$

Rearranging (B.1) it gives,

$$\begin{aligned} & \frac{1}{2} \left((2A - \alpha)V_{xx}^*(\zeta_k) + 2\tilde{Q}_k \right) x^2 \\ & + \left(q + \frac{1}{2}\mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) + (\alpha - A)V_{xx}^*(\zeta_k)x_0 + (A - \alpha)V_x^*(\zeta_k) \right) x \\ & - \alpha V^*(\zeta_k) + \frac{1}{2}\mathcal{T}(V_{xx}^*(\zeta_k)) + \alpha V_x^*(\zeta_k)x_0 - \frac{1}{2}\alpha V_{xx}^*(\zeta_k)x_0^2 \\ & \quad + \inf_{u \in U} \{ \tilde{R}_k u^2 + \Upsilon_k^1 x u + \Upsilon_k^0 u \} \quad (\text{B.3}) \end{aligned}$$

and since

$$\begin{aligned} & \tilde{R}_k u^2 + \Upsilon_k^1 x u + \Upsilon_k^0 u = \\ & \left(u + \frac{1}{2}\tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0) \right) \tilde{R}_k \left(u + \frac{1}{2}\tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0) \right) - \frac{1}{4}(\Upsilon_k^1 x + \Upsilon_k^0) \tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0) = \\ & \quad \tilde{R}_k \left(u + \frac{1}{2}\tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0) \right)^2 - \frac{1}{4}\tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0)^2 \quad (\text{B.4}) \end{aligned}$$

and $\tilde{R}_k > 0$, one has that

$$\inf_{u \in U} \{ \tilde{R}_k u^2 + \Upsilon_k^1 x u + \Upsilon_k^0 u \} = -\frac{1}{4}\tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0)^2$$

Hence, the optimal control at x is

$$u^* = -\frac{1}{2}\tilde{R}_k^{-1}(\Upsilon_k^1 x + \Upsilon_k^0)$$

and (B.3) can be written as

$$\begin{aligned} & \frac{1}{2} \left((2A - \alpha)V_{xx}^*(\zeta_k) + 2\tilde{Q}_k - \frac{1}{2}\tilde{R}_k^{-1}(\Upsilon_k^1)^2 \right) x^2 \\ & + \left(q + \frac{1}{2}\mathcal{H}(V_{xx}^*(\zeta_k), \text{sign}(\zeta_k)) + (\alpha - A)V_{xx}^*(\zeta_k)x_0 + (A - \alpha)V_x^*(\zeta_k) - \frac{1}{2}\tilde{R}_k^{-1}\Upsilon_k^1\Upsilon_k^0 \right) x \\ & - \alpha V^*(\zeta_k) + \frac{1}{2}\mathcal{T}(V_{xx}^*(\zeta_k)) + \alpha V_x^*(\zeta_k)x_0 - \frac{1}{2}\alpha V_{xx}^*(\zeta_k)x_0^2 - \frac{1}{4}\tilde{R}_k^{-1}(\Upsilon_k^0)^2 \end{aligned} \quad (\text{B.5})$$

Denote $\text{sign}(x_0) = s_x$ and $\text{sign}(u) = s_u$. Taking the limit of any sequence $\{\zeta_k\}_{k \geq 0}$ with each ζ_k an Alexandrov point, $\text{sign}(\zeta_k) = s_x$ and $\zeta_k \rightarrow x_0$, together with the choice $V_{xx}^*(x) = 2X$ to get from (B.5) that

$$\begin{aligned} & \left((2A - \alpha)X + Q + \mathcal{Q}(X, s_x) - (R + \mathcal{R}(X, s_u))^{-1}(XB)^2 \right) x^2 \\ & + \left(q + \mathcal{H}(X, s_x) + 2(\alpha - A)Xx_0 + (A - \alpha)V_x^*(x_0) \right. \\ & \left. - (R + \mathcal{R}(X, s_u))^{-1}XB\{(V_x^*(x_0) - 2x_0X)B + \mathcal{G}(X, s_u) + r\} \right) x \\ & - \alpha V^*(\zeta_k) + \mathcal{T}(X) + \alpha V_x^*(\zeta_k)x_0 - \alpha Xx_0^2 \\ & - \frac{1}{4}(R + \mathcal{R}(X, s_u))^{-1}((V_x^*(x_0) - 2x_0X)B + \mathcal{G}(X, s_u) + r)^2 \end{aligned} \quad (\text{B.6})$$

holds. Now, assume that the quadratic equation,

$$(2A - \alpha)X + Q + \mathcal{Q}(X, s_x) - (R + \mathcal{R}(X, s_u))^{-1}(XB)^2 = 0$$

has a solution $X > 0$. Since x_0 is arbitrary, except for its signal, one can set $x_0 \uparrow 0$ or $\downarrow 0$ accordingly. For that, the quadratic of form $V(x) = Xx^2 + vx + \ell$ with an appropriate choice of v and ℓ can be set $V^*(x) = V(x)$ and $V_x^*(x) = V_x(x) = 2Xx + v$ in such a way that (B.6) is set to zero by simply choosing v and ℓ such that,

$$q + \mathcal{H}(X, s_x) + (A - \alpha)v - (R + \mathcal{R}(X, s_u))^{-1}XB\{Bv + \mathcal{G}(X, s_u) + r\} = 0 \quad (\text{B.7})$$

$$- \alpha\ell + \mathcal{T}(X) - \frac{1}{4}(R + \mathcal{R}(X, s_u))^{-1}(Bv + \mathcal{G}(X, s_u) + r)^2 = 0 \quad (\text{B.8})$$

Such a quadratic functions V defines an upper bound for the optimal V^* at the positive and negative line, since they satisfy the HJB equation at a point x with $|x| \rightarrow \infty$. \square