# Extensions of modal logic S5 preserving NP-completeness 

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# EXTENSIONS OF MODAL LOGIC S5 PRESERVING NP-COMPLETENESS 


#### Abstract

We present a family of multi-modal logics having NP-complete satisfiability problems and admitting in the language S5-like modal operators, common knowledge and distributed knowledge operators. Our motivation is to find out interaction conditions between the modal operators that affect the computational complexity of the logics.


Key-words: multi-modal logics, interdependent modal operators, computational complexity

## 1. Introduction

In the fields of Artificial Intelligence and Computer Science, the modal logic S5 (see e.g. [11]) has been used in many ways to define logics modelling agents' knowledge (see e.g. [9], [15]). Since such logics often admit in their language a finite family of modal operators (in that sense they are multimodal logics) and since the modal operators can be rather expressive (for instance the common knowledge operator), it is not surprising that, their satisfiability problems can be PSPACE-complete, or even EXPTIMEcomplete (see e.g. [9]). For instance the satisfiability problem for S 5 is NP-complete [13] and the satisfiability problem for multi-modal logic $\mathrm{S} 5{ }_{n}^{\mathrm{C}}$ ( $n \geq 2$ ) with the common knowledge operator is EXPTIME-complete [9].

Although in [17], Vardi has defined knowledge logics having NPcomplete satisfiability problems (using Montague and Scott's semantics), the problem of finding out conditions between the agents to collapse the satisfiability problem to NP has been seldom considered. That is why in this work, we are mainly interested in finding out assumptions about interactions between the agents that affect the complexity of reasoning about the agents' knowledge.

Although the logics introduced in the paper can be viewed as knowledge logics (it is nowadays highly controversial whether logics admitting the logical onmiscience principle are good candidates to model agents' knowledge), our work is rather a contribution to multi-modal logic theory (see e.g. [16]). Hence, in the paper we shall consider the interpretations in terms of knowledge representation as secondary. In the realm of transfer theorems for multi-modal logics (see e.g. [5], [18]), join (or independent) union between uni-modal logics has been the most common way of combining logics (see also [10], [6]). The difficulty of extending results for uni-modal logics to multi-modal logics where new interactions appear between the different modal operators has thoroughly been described in [2]. This is precisely an instance of this general problem that we tackle in the present work.

The present paper shows the NP-completeness of non-join unions of the logic S5 admitting common knowledge and distributed knowledge operators in their language. Indeed, the paper presents a class of multimodal logics whose language contains a finite family of modal operators $\left\{\square_{i}: i \in\{1, \ldots, n\}\right\}$, the common knowledge operators $C_{X}$ and the distributed knowledge operators $\mathrm{D}_{X}$ for $\emptyset \neq X \subseteq\{1, \ldots, n\}$. $\mathrm{C}_{X} \mathrm{~A}$ can be read as "it is common knowledge among the agents in $X$ that A holds" whereas $\mathrm{D}_{X} \mathrm{~A}$ can be read as "the combined knowledge of the members of $X$ implies A". The main result of the paper is the NP-completeness of the satisfiability problem for such logics although similar existing knowledge logics have a PSPACE-complete or EXPTIME-complete satisfiability problem (see e.g. [9]). In order to state such results, we first prove that the logics considered have the finite model property by using a construction of models extending the construction in [4].

The rest of the paper is organized as follows. In Section 2, a class of multi-modal logics is defined. In Section 3, an original model construction is presented in order to show that every defined logic has the finite model property. Moreover, sufficient conditions are introduced in order to obtain decidable validity problem and NP-complete satisfiability problem.

## 2. The class of $* S 5_{\mathrm{C}}^{\mathrm{D}}(n)$-logics

The (propositional) modal language L ( $n \geq 1$ is fixed) is determined by three sets which are supposed to be pairwise disjoint: a countable set $\mathrm{F}_{0}$ of propositional variables, a set of propositional operators $\{\wedge, \neg\}$, and a set of modal
operators $O P=\left\{\square_{i}: i \in\{1, \ldots, n\}\right\} \cup\left\{\mathrm{C}_{X}, \mathrm{D}_{X}: \emptyset \neq X \subseteq\{1, \ldots, n\}\right\}$. The propositional operators $\vee, \Rightarrow, \Leftrightarrow$ are used as abbreviations in the standard way. The set F of L-formulae is the smallest set that satisfies the following conditions: $\mathrm{F}_{0} \subseteq \mathrm{~F}$, if $\oplus$ is any $k$-ary propositional operator and $\mathrm{A}_{1}, \ldots, \mathrm{~A}_{k} \in \mathrm{~F}$ then $\oplus\left(\mathrm{A}_{1}, \ldots, \mathrm{~A}_{k}\right) \in \mathrm{F}$ and if $\mathrm{A} \in \mathrm{F}$ and $\ominus \in O P$ then $\ominus \mathrm{A} \in \mathrm{F}$. We write $\operatorname{sub}(\mathrm{A})$ (resp. $m w(\mathrm{~A})$ ) to denote the set of subformulae of the formula A (resp. the modal weight of A, i.e. the number of occurrences of modal operators in A).

As usual, by an L-model we understand a structure $\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$ such that $U$ is a non-empty set of states, for all $i \in\{1, \ldots, n\}, \mathcal{R}_{i}$ is a binary relation on $U$ and $V$ is a mapping $\mathrm{F}_{0} \rightarrow \mathcal{P}(U)$; as usual $\mathcal{P}(U)$ denotes the power set of $U$. Let $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$ be an L-model and $\emptyset \neq U^{\prime} \subseteq U$. The model $\mathcal{M}_{\mid U^{\prime}}=\left(U^{\prime},\left(\mathcal{R}_{i}^{\prime}\right)_{i \in\{1, \ldots, n\}}, V^{\prime}\right)$ is said to be a submodel of $\mathcal{M}$ iff for all $i \in\{1, \ldots, n\}, \mathcal{R}_{i}^{\prime}=\mathcal{R}_{i} \cap U^{\prime} \times U^{\prime}$ and for all $\mathrm{P} \in \mathrm{F}_{0}, V^{\prime}(\mathrm{P})=V(\mathrm{P}) \cap U^{\prime}$. Let $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$ be an L-model. As usual, we define "the formula A is satisfied by the state $u \in U$ in $\mathcal{M}$ " (written $\mathcal{M}, u \models \mathrm{~A}$ ) by induction:

- $\mathcal{M}, u \models \mathrm{P}$ iff $u \in V(\mathrm{P})$, for all $\mathrm{P} \in \mathrm{F}_{0}$,
- $\mathcal{M}, u \models \neg \mathrm{~A}$ iff not $\mathcal{M}, u \models \mathrm{~A}$,
- $\mathcal{M}, u \models \mathrm{~A} \wedge \mathrm{~B}$ iff $\mathcal{M}, u \models \mathrm{~A}$ and $\mathcal{M}, u \models \mathrm{~B}$,
- $\mathcal{M}, u \models \square_{i} \mathrm{~A}$ iff for $\operatorname{all}^{1} v \in \mathcal{R}_{i}(u), \mathcal{M}, v \models \mathrm{~A}$,
- $\mathcal{M}, u \models \mathrm{C}_{X} \mathrm{~A}$ iff for all $v \in\left(\bigcup_{i \in X} \mathcal{R}_{i}\right)^{*}(u) \mathcal{M}, v \models \mathrm{~A}$,
- $\mathcal{M}, u \models \mathrm{D}_{X} \mathrm{~A}$ iff for all $v \in\left(\bigcap_{i \in X} \mathcal{R}_{i}\right)(u) \mathcal{M}, v \models \mathrm{~A}$.

For any binary relation $R$ on $U$, we write $R^{*}$ to denote the reflexive and transitive closure of $R$. A formula A is true in an L-model $\mathcal{M}$ (written $\mathcal{M} \models \mathrm{A})$ iff for all $u \in U, \mathcal{M}, u \models \mathrm{~A}$. In the sequel, by an ${ }^{*} S 5_{\mathrm{C}}^{\mathrm{D}}(n)$-logic ( $n \geq 1$ ), we understand a triple $\mathcal{L}=\left\langle\mathrm{L}, \mathcal{S}, \models_{\mathcal{L}}\right\rangle$ such that

1. $\mathcal{S}$ is a non-empty set of L-models closed under submodels
2. for all L-models $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$, if $\mathcal{M} \in \mathcal{S}$ then
(a) for all $i \in\{1, \ldots, n\}, \mathcal{R}_{i}$ is an equivalence relation and
(b) for all $u \in U$, there exists a permutation $s_{u}$ on $\{1, \ldots, n\}$ such that $R_{s_{u}(1)}(u) \subseteq \ldots \subseteq R_{s_{u}(n)}(u)$ ( $s_{u}$ may not be unique).
3. $=_{\mathcal{L}}$ is the restriction of $\models$ to the sets $\mathcal{S}$ and L (satisfiability relation).
[^0]The condition 1. is a feature shared by numerous non-classical logics determined by classes of possible-world structures (see e.g. [12], [18]). The condition 2(b) expresses that locally the relations of the models can be linearly ordered. For all models $\mathcal{M} \in \mathcal{S}, \mathcal{M}$ is said to be a model for $\mathcal{L}$. An L-formula A is said to be $\mathcal{L}$-valid iff A is true in all L-models of $\mathcal{S}$. An Lformula A is said to be $\mathcal{L}$-satisfiable iff there is $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in$ $\mathcal{S}, u \in U$ such that $\mathcal{M}, u=_{\mathcal{L}}$ A. Observe that for any ${ }^{*} 5_{\mathrm{C}}^{\mathrm{D}}(n)$-logic $\mathcal{L}$, for all $i, j \in\{1, \ldots, n\}$, for all $\emptyset \neq X \subseteq\{1, \ldots, n\}$,
(1) $\square_{i} \mathrm{P} \wedge \square_{j} \mathrm{P} \Leftrightarrow \square_{i} \square_{j} \mathrm{P} \wedge \square_{j} \square_{i} \mathrm{P}$,
(2) $\mathrm{C}_{X} \mathrm{P} \Leftrightarrow \bigwedge_{i \in X} \square_{i} \mathrm{P}$ and $\mathrm{D}_{X} \mathrm{P} \Leftrightarrow \bigvee_{i \in X} \square_{i} \mathrm{P}$
are $\mathcal{L}$-valid. By the way, none of the interaction axioms presented in [1], [2] captures the axiom schema (1) above. Moreover, consider the mapping $t: \mathrm{F} \rightarrow \mathrm{F}$ such that:

- $t(\mathrm{P})=\mathrm{P}$ for all $\mathrm{P} \in \mathrm{F}_{0}$,
- $t(\oplus \mathrm{~A})=\oplus t(\mathrm{~A})$ for all $\oplus \in\left\{\neg, \square_{1}, \ldots, \square_{n}\right\}$,
- $t(\mathrm{~A} \wedge \mathrm{~B})=t(\mathrm{~A}) \wedge t(\mathrm{~B})$,
- $t\left(\mathrm{C}_{X} \mathrm{~A}\right)=\bigwedge_{i \in X} \square_{i} t(\mathrm{~A})$,
- $t\left(\mathrm{D}_{X} \mathrm{~A}\right)=\bigvee_{i \in X} \square_{i} t(\mathrm{~A})$.

It is easy to show that $\mathrm{A} \Leftrightarrow t(\mathrm{~A})$ is $\mathcal{L}$-valid. However $t$ is not polynomial. That is why the NP-completeness of the $\mathcal{L}$-satisfiability problem might not straightforwardly follow from the possible NP-completeness of the satisfiability problem restricted to formulae without the operators $\mathrm{C}_{X}$ and $\mathrm{D}_{X}$.

Example 2.1. The set of ${ }^{*} \mathrm{~S}_{\mathrm{C}}^{\mathrm{D}}(n)$-logics contains the logic $\mathcal{L}=$ $\langle\mathrm{L}, \mathcal{S}, \models\rangle$ (resp. $\left.\mathcal{L}^{\prime}=\left\langle\mathrm{L}, \mathcal{S}^{\prime}, \models^{\prime}\right\rangle\right)$ where $\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}$ (resp. $\left.\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}^{\prime}\right)$ iff $\mathcal{R}_{1} \subseteq \ldots \subseteq \mathcal{R}_{n}$ (resp. ${ }^{2}$ for all $u \in U$ and for all $i, j \in\{1, \ldots, n\}$, either $R_{i}(u) \subseteq R_{j}(u)$ or $\left.R_{j}(u) \subseteq R_{i}(u)\right)$ and the $\mathcal{R}_{i}$ 's are equivalence relations.

For any finite sequence of natural numbers $\sigma$, $\operatorname{set}(\sigma)$ (resp. $|\sigma|$ ) denotes the set of elements occurring in $\sigma$ (resp. the length of $\sigma$ ). For example $\operatorname{set}((1,2,3,3,4))=\{1,3,2,4\}$. As usual $\sigma_{1} \cdot \sigma_{2}$ denotes the concatenation of two sequences.

[^1]
## 3. NP-complete satisfiability problems

In this section we shall show that every $* \mathrm{~S} 5 \mathrm{C}(n)$-logic $\mathcal{L}$ has the finite model property. Moreover we shall provide sufficient conditions to establish decidability and NP-completeness of the $\mathcal{L}$-satisfiability problem.

In the rest of this section $\mathcal{L}$ denotes an ${ }^{*} S_{5}^{\mathrm{D}}(n)$-logic $\left\langle\mathrm{L}, \mathcal{S}, \models_{\mathcal{L}}\right\rangle$ with $n \geq 2$ unless otherwise stated. The case $n=1$ is omitted here since it roughly corresponds to the modal logic S5.

Proposition 3.1. Let $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}$ and $\operatorname{let} \emptyset \neq U^{\prime} \subseteq U$. For all $u \in U^{\prime}$, for all $i, j \in\{1, \ldots, n\}$, if $\mathcal{R}_{i}(u) \subseteq \mathcal{R}_{j}(u)$ then $\mathcal{R}_{i}^{\prime}(u) \subseteq$ $\mathcal{R}_{j}^{\prime}(u)$ with $\mathcal{M}_{\mid U^{\prime}}=\left(U^{\prime},\left(\mathcal{R}_{i}^{\prime}\right)_{i \in\{1, \ldots, n\}}, V^{\prime}\right)$.
Proof. By an easy verification. Q.E.D.
Proposition 3.2 below states that when the chain of inclusions for the successors of $x$ is known then the chain for $y$ is partially known when $x$ and $y$ are $R_{i_{k}}$-related for some $i_{k} \in\{1, \ldots, n\}$.

Proposition 3.2. Let $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}, x \in U$ and $\mathcal{R}_{i_{1}}(x) \subseteq \ldots \subseteq \mathcal{R}_{i_{n}}(x)$ with $\{1, \ldots, n\}=\left\{i_{1}, \ldots, i_{n}\right\}$. Assume $(x, y) \in R_{i_{k}}$ for some $k \in\{1, \ldots, n\}$. Then,

1. for all $k^{\prime} \in\{k, \ldots, n\}, R_{i_{k^{\prime}}}(x)=R_{i_{k^{\prime}}}(y)$
2. for all $k^{\prime} \in\{1, \ldots, k-1\}$, $R_{i_{k^{\prime}}}(y) \subseteq R_{i_{k}}(y)$.

Proof. The proof is by an easy verification. Q.E.D.
Figure 1 illustrates Proposition 3.2. In order to prove the finite model property for $\mathcal{L}$, the standard filtration construction for modal logics (see e.g. [8]) does not work for the logic $\mathcal{L}$ unless additional assumptions are made. Instead of using the filtration construction, we propose a submodel construction. Let A be an L-formula, $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$ be a model for $\mathcal{L}, w \in U$ such that $\mathcal{M}, w \models \mathrm{~A}$. Assume that $\mathcal{R}_{i_{1}}(w) \subseteq \ldots \subseteq \mathcal{R}_{i_{n}}(w)$ with $\{1, \ldots, n\}=\left\{i_{1}, \ldots, i_{n}\right\}$.

We shall construct a set $U^{\prime} \subseteq U$ such that

- $w \in U^{\prime}$,
- $U^{\prime}$ is finite and,
- $\mathcal{M}_{\mid U^{\prime}}, w \mid=\mathrm{A}$.


Fig. 1. Illustration of Proposition 3.2

The construction informally described below generalizes the construction in [3] in order to include the operators $\mathrm{C}_{X}$ and $\mathrm{D}_{X}$. To build such a set $U^{\prime}$, we first consider the set Nec of necessity formulae $\square_{i} \mathrm{~A}^{\prime}$ such that $\ominus \mathrm{A}^{\prime} \in \operatorname{sub}(\mathrm{A})$ for some $\ominus \in O P$. Then the construction of $U^{\prime}$ is done recursively, that is $U^{\prime}=\bigcup_{0 \leq i \leq N} U_{i}$ for some $0 \leq N \leq n$ where each $U_{i}$ is finite. $U_{0}$ is initialized to $\{\bar{w}\}$ and then $U_{i+1}$ is defined from $U_{i}$. Everytime a formula in Nec is not satisfied at some element of $U_{i}$, we add a witness of this fact in $U_{i+1}\left(\mathcal{M}, u \not \models \ominus \mathrm{~A}^{\prime}\right.$ iff there exists $u^{\prime}$ such that $\left(u, u^{\prime}\right) \in \mathcal{R}_{\ominus}$ and $\mathcal{M}, u^{\prime} \not \vDash \mathrm{A}^{\prime}$ where $\mathcal{R}_{\ominus}$ is the binary relation attached to the modal operator $\ominus$ in $\mathcal{M})$. Moreover, assume that for some $u \in U_{I}, \mathcal{M}, u \not \vDash \square_{i} \mathrm{~A}^{\prime}$ requires a witness and $\mathcal{R}_{i}(u) \subseteq \mathcal{R}_{j}(u)$. There exists $u^{\prime} \in U_{I+1}$ such that $\mathcal{M}, u^{\prime} \neq \mathrm{A}^{\prime}$ and $\left(u, u^{\prime}\right) \in \mathcal{R}_{i}$. We can show that if $\mathcal{M}, u^{\prime} \not \vDash \square_{j} \mathrm{~A}^{\prime \prime}$ for some $\square_{j} \mathrm{~A}^{\prime \prime} \in \mathrm{Nec}$, there is no need to consider a new witness (the set $\mathrm{Nec}_{x}^{\sigma}$ defined below contains the necessity formulae that require the introduction of a new witness generated from the world $x$ and $\sigma$ encodes some information about $x$ and about some accessibility relations of the model). There exists $u^{\prime \prime}$ such that $\left(u^{\prime}, u^{\prime \prime}\right) \in \mathcal{R}_{j}$ and $\mathcal{M}, u^{\prime \prime} \notin \mathrm{A}^{\prime \prime}$. Since $\mathcal{R}_{j}$ is an equivalence relation, $\left(u, u^{\prime \prime}\right) \in \mathcal{R}_{j}$ and therefore $\mathcal{M}, u \not \vDash \square_{j} \mathrm{~A}^{\prime \prime}$. If the set $U_{I+1}$ has been properly built (this point should become clear in the formal definition), there exists
$v \in U_{I+1}$ such that $(u, v) \in \mathcal{R}_{j}$ and $\mathcal{M}, v \not \vDash \mathrm{~A}^{\prime \prime}$. Since $\left(u^{\prime}, v\right) \in \mathcal{R}_{j}, v$ is already a witness for $\mathcal{M}, u^{\prime} \not \vDash \square_{j} \mathrm{~A}^{\prime \prime}$. This observation allows us to find $N \leq n$ such that for all $u^{\prime} \in U_{N}$, no witness is needed.

We shall give in the sequel the formal definitions. The set Nec is defined as follows,

$$
\operatorname{Nec}=\left\{\square_{i} \mathrm{~A}^{\prime}: \ominus \mathrm{A}^{\prime} \in \operatorname{sub}(\mathrm{A}) \text { for some } \ominus \in O P, i \in\{1, \ldots, n\}\right\} .
$$

For all $x \in U$ and all sequences $\sigma=\left(j_{1}, \ldots, j_{k}\right)$ such that $\operatorname{set}(\sigma) \subseteq$ $\{1, \ldots, n\}$ and $\mathcal{R}_{j_{1}}(x) \subseteq \ldots \subseteq \mathcal{R}_{j_{k}}(x)$, the set $\operatorname{Nec}_{x}^{\sigma}$ is defined as follows:

$$
\begin{gathered}
\operatorname{Nec}_{x}^{\sigma}=\left\{\square_{j_{k^{\prime}}} \mathrm{A}^{\prime} \in \operatorname{Nec}: k^{\prime} \in\{1, \ldots,|\sigma|\}, \mathcal{M}, x \models \neg \square_{j_{k^{\prime}}} \mathrm{A}^{\prime},\right. \text { and } \\
\text { if } \left.k^{\prime} \geq 2 \text { then } \mathcal{M}, x \mid=\square_{j_{k^{\prime}-1}} \mathrm{~A}^{\prime}\right\}
\end{gathered}
$$

Remember the set $\mathrm{Nec}_{x}^{\sigma}$ allows to reduce the set of witnesses generated from $x$. For each natural number $i \leq n$, we are defining a set $W_{i}$ of 3 -tuples ( $u^{\prime}, \sigma, ?$ ) where,

- $u^{\prime} \in U$,
- $\sigma$ is a sequence of elements of $\{1, \ldots, n\}$ without repetition,
- ? is either the symbol ' $\Lambda$ ' or some $\square_{p} \mathrm{~A}^{\prime} \in \operatorname{Nec}$ with $p \notin \operatorname{set}(\sigma)$.

The set $U_{i}$ shall be later defined as the set $\left\{u^{\prime}:\left(u^{\prime},\left(i_{1}, \ldots, i_{n}\right), ?\right) \in\right.$ $\left.W_{i}\right\} . W_{i}$ is an intermediate set that contains some information about the elements of $U_{i}$. Let $W_{0}=\left\{\left(w,\left(i_{1}, \ldots, i_{n}\right), \Lambda\right)\right\}$. Assume $W_{i}$ is defined. We will now define $W_{i+1}$. Initialize $W_{i+1}$ to the empty set $\emptyset$.

For each $\left(w^{\prime}, \sigma, ?\right) \in W_{i}$, for each $\square_{j} \mathrm{~A}^{\prime} \in \operatorname{Nec}_{w^{\prime}}^{\sigma}$,
choose $u \in U$ such that $\left(w^{\prime}, u\right) \in \mathcal{R}_{j}$ and $\mathcal{M}, u \models \neg \mathrm{~A}^{\prime}$.
If $\sigma=\left(j_{1}, \ldots, j_{k}\right)$ then we write $k^{\prime}$ to denote the element of $\{1, \ldots, k\}$ such that $j_{k^{\prime}}=j$. The existence of $k^{\prime}$ is guaranteed by the definition of $\mathrm{Nec}_{w^{\prime}}^{\sigma}$. Add the triple $\left(u,\left(j_{1}^{\prime}, \ldots, j_{k^{\prime}-1}^{\prime}\right), \square_{j} \mathrm{~A}^{\prime}\right)$ to $W_{i+1}$ such that $\operatorname{set}\left(\left(j_{1}^{\prime}, \ldots, j_{k^{\prime}-1}^{\prime}\right)\right)$ $=\operatorname{set}\left(\left(j_{1}, \ldots, j_{k^{\prime}-1}\right)\right)$ and $\mathcal{R}_{j_{1}^{\prime}}(u) \subseteq \ldots \subseteq \mathcal{R}_{j_{k^{\prime}-1}^{\prime}}(u)-$ whenever $k^{\prime}=1$ the sequence $\left(j_{1}^{\prime}, \ldots, j_{k^{\prime}-1}^{\prime}\right)$ is the empty sequence $\Lambda$. Let $W_{i+1}$ be the set augmented this way. There exists $N \in\{0, \ldots, n\}$ such that $W_{N} \neq \emptyset$ and $W_{N+1}=\emptyset$ since the length of the sequences of natural numbers strictly decreases. Moreover if $\left(w^{\prime}, \sigma, ?\right) \in W_{i}$ then the length of $\sigma$ is at most $n-i$. Let $W$ and $U^{\prime}$ be the sets defined in the following way:

$$
W=\bigcup_{i=0}^{N} W_{i} \quad U^{\prime}=\bigcup_{i=0}^{N} U_{i} \text { with } U_{i}=\left\{w^{\prime}:\left(w^{\prime}, \sigma, ?\right) \in W_{i}\right\}
$$

Proposition 3.3. For all $i \in\{1, \ldots, N\}$ and all $\left(u_{i},\left(j_{1}, \ldots, j_{k}\right), \square_{p} \mathrm{~A}^{\prime}\right) \in$ $W_{i}$,

1. $\mathcal{R}_{j_{1}}\left(u_{i}\right) \subseteq \ldots \subseteq \mathcal{R}_{j_{k}}\left(u_{i}\right), \mathcal{R}_{j_{k}}\left(u_{i}\right) \subseteq \mathcal{R}_{p}\left(u_{i}\right), \mathcal{M}, u_{i} \models \neg \mathrm{~A}^{\prime}$, and
2. for all $j \in\{1, \ldots, n\} \backslash \operatorname{set}\left(\left(j_{1}, \ldots, j_{k}\right)\right), \mathcal{R}_{j_{k}}\left(u_{i}\right) \subseteq \mathcal{R}_{j}\left(u_{i}\right)$.

Proof. (1) Obvious from the construction of $\left(u_{i},\left(j_{1}, \ldots, j_{k}\right), \square_{p} \mathrm{~A}^{\prime}\right)$.
(2) By induction on $i$ using Proposition 3.2. Q.E.D.

Observe that ${ }^{3} \operatorname{card}(\mathrm{Nec}) \leq n \times m w(\mathrm{~A}), \operatorname{card}\left(\mathrm{Nec}_{x}^{\sigma}\right) \leq m w(\mathrm{~A})$ and $\mathrm{Nec}_{x}^{\sigma}=\emptyset$ when $\sigma$ is the empty sequence $\Lambda$. For all $i \in\{0, \ldots, N-1\}$, $\operatorname{card}\left(W_{i+1}\right) \leq \operatorname{card}\left(W_{i}\right) \times m w(\mathrm{~A})$ and therefore $\operatorname{card}\left(U^{\prime}\right) \leq 1+n \times m w(\mathrm{~A})^{n}$. Proposition 3.4 below states that the set $U^{\prime}$ contains enough states.

Proposition 3.4. For all $u \in U^{\prime}$ and for all $\square_{j} \mathrm{~A}^{\prime} \in \operatorname{Nec}$, if $\mathcal{M}, u \not \vDash \square_{j} \mathrm{~A}^{\prime}$ then there is $u^{\prime} \in U^{\prime}$ such that $\left(u, u^{\prime}\right) \in \mathcal{R}_{j}$ and $\mathcal{M}, u^{\prime} \not \vDash \mathrm{A}^{\prime}$.
Proof. By induction on $i$ when $(u, \sigma, ?) \in W_{i}$. Similar to the proof of Proposition 6 in [3]. Q.E.D.

A polynomial bound can be obtained for the size of some model satisfying A.

Proposition 3.5. An L-formula A is $\mathcal{L}$-satisfiable iff A is satisfiable in a model for $\mathcal{L}$ with at most $1+n \times m w(\mathrm{~A})^{n}$ states.
Proof. Assume there is a model for $\mathcal{L}, \mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right), w \in U$ such that $\mathcal{M}, w \models \mathrm{~A}$. Consider the model for $\mathcal{L}, \mathcal{M}^{\prime}=\mathcal{M}_{\mid U^{\prime}}$. We then prove that for all states $u^{\prime} \in U^{\prime}$ and for all $\mathrm{B} \in \operatorname{sub}(\mathrm{A}), \mathcal{M}, u^{\prime} \vDash \mathrm{B}$ iff $\mathcal{M}^{\prime}, u^{\prime} \models \mathrm{B}$ (including A ). We proceed by induction on the structure of B . The only nontrivial cases are when B has the form $\mathrm{C}_{X} \mathrm{~B}^{\prime}, \mathrm{D}_{X} \mathrm{~B}^{\prime}$ and $\square_{i} \mathrm{~B}^{\prime}$. Only the first case is presented herein. Take $u^{\prime} \in U^{\prime}$. If $\mathcal{M}, u^{\prime} \models \mathrm{C}_{X} \mathrm{~B}^{\prime}$ then for all $v \in U$ such that $\left(u^{\prime}, v\right) \in\left(\bigcup_{i \in X} \mathcal{R}_{i}\right)^{*}$ we have $\mathcal{M}, v \vDash \mathrm{~B}^{\prime}$. There is $i_{0} \in X$ such that $\mathcal{R}_{i_{0}}\left(u^{\prime}\right)=\left(\bigcup_{i \in X} \mathcal{R}_{i}\right)^{*}\left(u^{\prime}\right)$. So for all $v \in U^{\prime}$ such that $\left(u^{\prime}, v\right) \in \mathcal{R}_{i_{0}}$, we have $\mathcal{M}, v \models \mathrm{~B}^{\prime}$. Hence for all $v \in U^{\prime}$ such that $\left(u^{\prime}, v\right) \in \mathcal{R}_{i_{0}}^{\prime}$ we have $\mathcal{M}, v \models \mathrm{~B}^{\prime}$. By the induction hypothesis, for all $v \in U^{\prime}$ such that $\left(u^{\prime}, v\right) \in \mathcal{R}_{i_{0}}^{\prime}, \mathcal{M}^{\prime}, v \models \mathrm{~B}^{\prime}$. So $\mathcal{M}^{\prime}, u^{\prime} \models \square_{i_{0}} \mathrm{~B}^{\prime}$. By Proposition

[^2]3.1, $\mathcal{M}^{\prime}, u^{\prime} \models \mathrm{C}_{X} \mathrm{~B}^{\prime}$. Now assume $\mathcal{M}, u^{\prime} \not \vDash \mathrm{C}_{X} \mathrm{~B}^{\prime}$. There is $i_{0} \in X$ such that $\mathcal{M}, u^{\prime} \models \mathrm{C}_{X} \mathrm{~B}^{\prime} \Leftrightarrow \square_{i_{0}} \mathrm{~B}^{\prime}$. From Proposition 3.4, there exists $v \in U^{\prime}$ such that $\left(u^{\prime}, v\right) \in \mathcal{R}_{i_{0}}^{\prime}$ and $\mathcal{M}, v \not \vDash \mathrm{~B}^{\prime}$. It follows that $\left(u^{\prime}, v\right) \in \mathcal{R}_{i_{0}}^{\prime}$ and by the induction hypothesis $\mathcal{M}^{\prime}, v \not \vDash \mathrm{~B}^{\prime}$. Hence $\mathcal{M}^{\prime}, u^{\prime} \notin \square_{i_{0}} \mathrm{~B}^{\prime}$. By Proposition 3.1, $\mathcal{M}^{\prime}, u^{\prime} \models \mathrm{C}_{X} \mathrm{~B}^{\prime} \Leftrightarrow \square_{i_{0}} \mathrm{~B}^{\prime}$. Hence $\mathcal{M}^{\prime}, u^{\prime} \notin \mathrm{C}_{X} \mathrm{~B}^{\prime}$. Q.E.D.

Proposition 3.6. Let $\mathcal{L}=\left\langle\mathrm{L}, \mathcal{M}, \models_{\mathcal{L}}\right\rangle$ be an ${ }^{*} S 5_{\mathrm{C}}^{\mathrm{D}}(n)$-logic such that the problem: $<$ Is the finite L-model $\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S} ? \gg$ is decidable. Then the $\mathcal{L}$-satisfiability problem is decidable.
Proof. Take any formula A for which one wishes to know whether A is $\mathcal{L}$-satisfiable. By Proposition 3.5, A is $\mathcal{L}$-satisfiable iff there exist an $\mathcal{L}$-model $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$ and $u \in U$ such that $\mathcal{M}, u \models \mathrm{~A}$ and $\operatorname{card}(U) \leq 1+n \times m w(\mathrm{~A})^{n}$. So in order to check whether A is $\mathcal{L}$-satisfiable, enumerate all the L-models $\mathcal{M}$ (modulo the isomorphic copies with respect to A) such that $\operatorname{card}(U) \leq 1+n \times m w(\mathrm{~A})^{n}$ and check whether $(\star) \mathcal{M}, u=\mathrm{A}$ for some $u \in U$ and $(\star \star) \mathcal{M} \in \mathcal{S} . \mathcal{M}^{\prime}=\left(U^{\prime},\left(R_{i}^{\prime}\right)_{i \in\{1, \ldots, n\}}, V^{\prime}\right)$ is isomorphic to $\mathcal{M}$ with respect to A iff there is a 1-1 mapping $g: U \rightarrow U^{\prime}$ such that for all $i \in\{1, \ldots, n\},\left\{(g(x), g(y)):(x, y) \in R_{i}\right\}=R_{i}^{\prime}$ and for all the propositional variables P occurring in $\mathrm{A}, V^{\prime}(\mathrm{P})=\{g(x): x \in V(\mathrm{P})\}$. ( $\star$ ) can be checked in polynomial-time in $\operatorname{card}(U)$ and in the size of $A$ (i.e., the length of the representation of A in any reasonable -unspecified- encoding). Since the set of L-models $\mathcal{M}$ (modulo the isomorphic copies) such that $\operatorname{card}(U) \leq 1+n \times m w(\mathrm{~A})^{n}$ is finite and can be easily generated and ( $(\star$ ) is an instance of a decidable problem then the $\mathcal{L}$-satisfiability problem is decidable. Q.E.D.
Example 3.1. Let $\Sigma$ be a non-empty set of permutations on $\{1, \ldots, n\}$ and $\mathcal{L}=\left\langle\mathrm{L}, \mathcal{S}, \models_{\mathcal{L}}\right\rangle$ be the ${ }^{*} \mathrm{~S}_{\mathrm{C}_{\mathrm{C}}^{\mathrm{D}}}^{\mathrm{D}}(n)$-logic such that for all L-models $\mathcal{M}=$ $\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right), \mathcal{M} \in \mathcal{S}$ iff
(eq) for all $i \in\{1, \ldots, n\}, \mathcal{R}_{i}$ is an equivalence relation and
(la) for all $u \in U$, there exists a permutation $s_{u} \in \Sigma$ such that $R_{s_{u}(1)}(u) \subseteq$ $\ldots \subseteq R_{s_{u}(n)}(u)$ ( $s_{u}$ may not be unique).
Let $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$ be a finite L-model. One can easily check in polynomial-time in $\operatorname{card}(U)$ that (eq) holds for $\mathcal{M}$. Moreover whether (la) holds for $\mathcal{M}$ can also be checked in deterministic time $o\left(n!\times \operatorname{card}(U)^{5}\right)$ (which is also polynomial in $\operatorname{card}(U)$ ). Hence the problem "Is the finite L-model $\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}$ ?" is decidable and the $\mathcal{L}$-satisfiability problem is decidable.

Proposition 3.7. Let $\mathcal{L}=\left\langle\mathrm{L}, \mathcal{M},=_{\mathcal{L}}\right\rangle$ be an ${ }^{*} S 5_{\mathrm{C}}^{\mathrm{D}}(n)$-logic such that:
For any finite L-model $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$, one can check in polynomial-time in $\operatorname{card}(U)$ whether $\mathcal{M} \in \mathcal{S}$.
Then the $\mathcal{L}$-satisfiability problem is $\mathbf{N P}$-complete.
Proof. This is immediate from the proof of Proposition 3.6 considering the computational complexity of the following problems:

1. for any finite L-model $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}$ and any $\mathrm{A} \in$ F , deciding whether $\mathcal{M}, u \models \mathrm{~A}$ for some $u \in U$ can be known in polynomial-time in $\operatorname{card}(U)$ and in the size of A
2. For any finite L-model $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right)$, one can check in polynomial-time in $\operatorname{card}(U)$ whether $\mathcal{M} \in \mathcal{S}$. Q.E.D.

Corollary 3.8. All the logics $\mathcal{L}$ defined in the Example 1 have an NPcomplete $\mathcal{L}$-satisfiability problem.

For the sake of comparison, remember that for all $k \geq 2$, the satisfiability problem for the multi-modal logics $\mathrm{S} 5_{k}$ is PSPACE-complete and for $\mathrm{S} 5{ }_{k}^{\mathrm{C}}$ the satisfiability problem is EXPTIME-complete [9].

Let $\mathcal{S}_{0}$ be the set of L-models such that for all L-models $\mathcal{M}=$ $\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right), \mathcal{M} \in \mathcal{S}_{0}$ iff for all $i \in\{1, \ldots, n\}, \mathcal{R}_{i}$ is an equivalence relation. Let $\mathcal{C}$ be a condition on the families $\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}$ of binary relations. We write $\mathcal{S}_{0}^{\mathcal{C}}$ to denote the subset of $\mathcal{S}_{0}$ such that for all L-models $\mathcal{M}=\left(U,\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}, V\right) \in \mathcal{S}_{0}, \mathcal{M} \in \mathcal{S}_{0}^{\mathcal{C}}$ iff $\left(\mathcal{R}_{i}\right)_{i \in\{1, \ldots, n\}}$ satisfies $\mathcal{C}$. In Figure 2, we present the complexity class of various logics $\left\langle L^{\prime}, \mathcal{S}_{0}^{\mathcal{C}}, \models^{\prime}\right\rangle$ for some condition $\mathcal{C}$ and $\mathrm{L}^{\prime}$ is either L or L without the operators $\mathrm{D}_{X}$ (written $\mathrm{L}-\{\mathrm{D}\})$ or L without the operators $\mathrm{D}_{X}$ and $\mathrm{C}_{X}($ written $\mathrm{L}-\{\mathrm{C}, \mathrm{D}\})$. The conditions are as follows:

$$
\begin{aligned}
& \mathcal{C}_{1}:=\text { true } \\
& \mathcal{C}_{2}:=\forall i, j \in\{1, \ldots, n\}, \forall u \in U \mathcal{R}_{i}(u) \subseteq \mathcal{R}_{j}(u) \text { or } \mathcal{R}_{j}(u) \subseteq \mathcal{R}_{i}(u) \\
& \mathcal{C}_{3}:=\mathcal{R}_{1} \subseteq \ldots \subseteq \mathcal{R}_{n} \\
& \mathcal{C}_{4}:=\mathcal{R}_{1}=\ldots=\mathcal{R}_{n} \text { (uni-modal S5). }
\end{aligned}
$$

|  | $\mathcal{C}_{1}$ | $\mathcal{C}_{2}$ | $\mathcal{C}_{3}$ | $\mathcal{C}_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{~L}-\{\mathrm{C}, \mathrm{D}\}$ | PSPACE | NP | NP | NP |
| $\mathrm{L}-\{\mathrm{D}\}$ | EXPTIME | NP | NP | NP |
| L | EXPTIME | NP | NP | NP |

Fig. 2. Interaction and computational complexity

In Figure 2, we assume $n \geq 2$ and the occurrence of the complexity class $\mathbf{C}$ stands for $\mathbf{C}$-completeness.

The results for $\mathcal{C}_{2}$ and $\mathcal{C}_{3}$ are due to Corollary 3.8 whereas those for $\mathcal{C}_{1}$ and $\mathcal{C}_{4}$ have been proved in [9], [13]. It is clear that for all $i \in\{2,3,4\}$, $\mathcal{C}_{i}$ entails $\mathcal{C}_{i-1}$. The properties above remain valid if for some non-empty set $\Sigma$ of permutations on $\{1, \ldots, n\}, \mathcal{C}_{3}$ is replaced by:

$$
\forall u \in U, \exists s \in \Sigma R_{s(1)}(u) \subseteq \ldots \subseteq R_{s(n)}(u)
$$

However it would be desirable to strenghen $\mathcal{C}_{2}$ (that is to find a condition $\mathcal{C}_{2}^{\prime}$ such that $\mathcal{C}_{2}$ entails $\mathcal{C}_{2}^{\prime}$ ) such that the satisfiability problem for $\left\langle\mathrm{L}, \mathcal{S}_{0}^{\mathcal{C}_{2}^{\prime}}, \models\right\rangle$ is still NP-complete.

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[^0]:    ${ }^{1}$ For any binary relation $R$ on the set $U$, for all $u \in U, R(u)$ is equal to $\{v \in U$ : $(u, v) \in R\}$.

[^1]:    ${ }^{2}$ This condition is known as the local agreement condition [7]. For instance it has been used to define semantical structures for information logics [7], [14], [3].

[^2]:    ${ }^{3}$ For any finite set $U, \operatorname{card}(U)$ denotes the cardinality of $U$.

