# Extensions of modal logic S5 preserving NP-completeness

Stéphane Demri

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**Abstract**

We present a family of multi-modal logics having **NP**-complete satisfiability problems and admitting in the language S5-like modal operators, common knowledge and distributed knowledge operators. Our motivation is to find out interaction conditions between the modal operators that affect the computational complexity of the logics.

**Key-words**: multi-modal logics, interdependent modal operators, computational complexity

## 1. Introduction

In the fields of Artificial Intelligence and Computer Science, the modal logic S5 (see e.g. [11]) has been used in many ways to define logics modelling agents' knowledge (see e.g. [9], [15]). Since such logics often admit in their language a finite *family* of modal operators (in that sense they are *multi-modal* logics) and since the modal operators can be rather expressive (for instance the *common knowledge* operator), it is not surprising that, their satisfiability problems can be **PSPACE**-complete, or even **EXPTIME**-complete (see e.g. [9]). For instance the satisfiability problem for S5 is **NP**-complete [13] and the satisfiability problem for multi-modal logic $S5_n^{\mathsf{C}}$ ($n \geq 2$) with the common knowledge operator is **EXPTIME**-complete [9].

Although in [17], Vardi has defined knowledge logics having **NP**-complete satisfiability problems (using Montague and Scott's semantics), the problem of finding out *conditions between the agents* to collapse the satisfiability problem to **NP** has been seldom considered. That is why in this work, we are mainly interested in finding out assumptions about interactions between the agents that affect the complexity of reasoning about the agents' knowledge.

Although the logics introduced in the paper can be viewed as knowledge logics (it is nowadays highly controversial whether logics admitting the *logical onmiscience principle* are good candidates to model agents' knowledge), our work is rather a contribution to multi-modal logic theory (see e.g. [16]). Hence, in the paper we shall consider the interpretations in terms of knowledge representation as secondary. In the realm of transfer theorems for multi-modal logics (see e.g. [5], [18]), *join* (or *independent*) union between uni-modal logics has been the most common way of combining logics (see also [10], [6]). The difficulty of extending results for uni-modal logics to multi-modal logics where new interactions appear between the different modal operators has thoroughly been described in [2]. This is precisely an instance of this general problem that we tackle in the present work.

The present paper shows the **NP**-completeness of non-join unions of the logic S5 admitting common knowledge and distributed knowledge operators in their language. Indeed, the paper presents a class of multi-modal logics whose language contains a finite family of modal operators $\{\Box_i : i \in \{1, \ldots, n\}\}$, the *common knowledge* operators $\mathtt{C}_X$ and the *distributed knowledge* operators $\mathtt{D}_X$ for $\emptyset \neq X \subseteq \{1, \ldots, n\}$. $\mathtt{C}_X\mathtt{A}$ can be read as "it is common knowledge among the agents in $X$ that $\mathtt{A}$ holds" whereas $\mathtt{D}_X\mathtt{A}$ can be read as "the combined knowledge of the members of $X$ implies $\mathtt{A}$". The main result of the paper is the **NP**-completeness of the satisfiability problem for such logics although similar existing knowledge logics have a **PSPACE**-complete or **EXPTIME**-complete satisfiability problem (see e.g. [9]). In order to state such results, we first prove that the logics considered have the finite model property by using a construction of models extending the construction in [4].

The rest of the paper is organized as follows. In Section 2, a class of multi-modal logics is defined. In Section 3, an original model construction is presented in order to show that every defined logic has the finite model property. Moreover, sufficient conditions are introduced in order to obtain decidable validity problem and **NP**-complete satisfiability problem.

## 2. The class of *S5$^{\mathtt{D}}_{\mathtt{C}}(n)$-logics

The (propositional) *modal language* $\mathtt{L}$ ($n \geq 1$ is fixed) is determined by three sets which are supposed to be pairwise disjoint: a countable set $\mathtt{F}_0$ of *propositional variables*, a set of *propositional operators* $\{\wedge, \neg\}$, and a set of *modal*

*operators* $OP = \{\Box_i : i \in \{1,\ldots,n\}\} \cup \{\mathtt{C}_X, \mathtt{D}_X : \emptyset \neq X \subseteq \{1,\ldots,n\}\}$. The propositional operators $\vee, \Rightarrow, \Leftrightarrow$ are used as abbreviations in the standard way. The set $\mathtt{F}$ of $\mathtt{L}$-*formulae* is the smallest set that satisfies the following conditions: $\mathtt{F}_0 \subseteq \mathtt{F}$, if $\oplus$ is any $k$-ary propositional operator and $\mathtt{A}_1,\ldots,\mathtt{A}_k \in \mathtt{F}$ then $\oplus(\mathtt{A}_1,\ldots,\mathtt{A}_k) \in \mathtt{F}$ and if $\mathtt{A} \in \mathtt{F}$ and $\ominus \in OP$ then $\ominus\mathtt{A} \in \mathtt{F}$. We write $sub(\mathtt{A})$ (resp. $mw(\mathtt{A})$) to denote the set of *subformulae* of the formula $\mathtt{A}$ (resp. the *modal weight* of $\mathtt{A}$, i.e. the number of occurrences of modal operators in $\mathtt{A}$).

As usual, by an $\mathtt{L}$-*model* we understand a structure $(U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$ such that $U$ is a non-empty set of *states*, for all $i \in \{1,\ldots,n\}$, $\mathcal{R}_i$ is a binary relation on $U$ and $V$ is a mapping $\mathtt{F}_0 \to \mathcal{P}(U)$; as usual $\mathcal{P}(U)$ denotes the power set of $U$. Let $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$ be an $\mathtt{L}$-model and $\emptyset \neq U' \subseteq U$. The model $\mathcal{M}_{|U'} = (U', (\mathcal{R}'_i)_{i\in\{1,\ldots,n\}}, V')$ is said to be a *submodel* of $\mathcal{M}$ iff for all $i \in \{1,\ldots,n\}$, $\mathcal{R}'_i = \mathcal{R}_i \cap U' \times U'$ and for all $\mathtt{P} \in \mathtt{F}_0$, $V'(\mathtt{P}) = V(\mathtt{P}) \cap U'$. Let $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$ be an $\mathtt{L}$-model. As usual, we define "the formula $\mathtt{A}$ is *satisfied by the state* $u \in U$ *in* $\mathcal{M}$" (written $\mathcal{M}, u \models \mathtt{A}$) by induction:

- $\mathcal{M}, u \models \mathtt{P}$ iff $u \in V(\mathtt{P})$, for all $\mathtt{P} \in \mathtt{F}_0$,
- $\mathcal{M}, u \models \neg\mathtt{A}$ iff not $\mathcal{M}, u \models \mathtt{A}$,
- $\mathcal{M}, u \models \mathtt{A} \wedge \mathtt{B}$ iff $\mathcal{M}, u \models \mathtt{A}$ and $\mathcal{M}, u \models \mathtt{B}$,
- $\mathcal{M}, u \models \Box_i\mathtt{A}$ iff for all[1] $v \in \mathcal{R}_i(u)$, $\mathcal{M}, v \models \mathtt{A}$,
- $\mathcal{M}, u \models \mathtt{C}_X\mathtt{A}$ iff for all $v \in (\bigcup_{i\in X} \mathcal{R}_i)^*(u)$ $\mathcal{M}, v \models \mathtt{A}$,
- $\mathcal{M}, u \models \mathtt{D}_X\mathtt{A}$ iff for all $v \in (\bigcap_{i\in X} \mathcal{R}_i)(u)$ $\mathcal{M}, v \models \mathtt{A}$.

For any binary relation $R$ on $U$, we write $R^*$ to denote the reflexive and transitive closure of $R$. A formula $\mathtt{A}$ is *true* in an $\mathtt{L}$-model $\mathcal{M}$ (written $\mathcal{M} \models \mathtt{A}$) iff for all $u \in U$, $\mathcal{M}, u \models \mathtt{A}$. In the sequel, by an $^*S5^{\mathtt{D}}_{\mathtt{C}}(n)$-*logic* ($n \geq 1$), we understand a triple $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ such that

1. $\mathcal{S}$ is a non-empty set of $\mathtt{L}$-models closed under submodels
2. for all $\mathtt{L}$-models $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$, if $\mathcal{M} \in \mathcal{S}$ then
   (a) for all $i \in \{1,\ldots,n\}$, $\mathcal{R}_i$ is an equivalence relation and
   (b) for all $u \in U$, there exists a permutation $s_u$ on $\{1,\ldots,n\}$ such that $R_{s_u(1)}(u) \subseteq \ldots \subseteq R_{s_u(n)}(u)$ ($s_u$ may not be unique).
3. $\models_{\mathcal{L}}$ is the restriction of $\models$ to the sets $\mathcal{S}$ and $\mathtt{L}$ (satisfiability relation).

[1] For any binary relation $R$ on the set $U$, for all $u \in U$, $R(u)$ is equal to $\{v \in U : (u, v) \in R\}$.

The condition 1. is a feature shared by numerous non-classical logics *determined* by classes of possible-world structures (see e.g. [12], [18]). The condition 2(b) expresses that *locally* the relations of the models can be linearly ordered. For all models $\mathcal{M} \in \mathcal{S}$, $\mathcal{M}$ is said to be a *model for* $\mathcal{L}$. An $\mathtt{L}$-formula $\mathtt{A}$ is said to be $\mathcal{L}$-*valid* iff $\mathtt{A}$ is true in all $\mathtt{L}$-models of $\mathcal{S}$. An $\mathtt{L}$-formula $\mathtt{A}$ is said to be $\mathcal{L}$-*satisfiable* iff there is $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$, $u \in U$ such that $\mathcal{M}, u \models_{\mathcal{L}} \mathtt{A}$. Observe that for any $^*\mathrm{S5}^{\mathtt{D}}_{\mathtt{C}}(n)$-logic $\mathcal{L}$, for all $i, j \in \{1, \ldots, n\}$, for all $\emptyset \neq X \subseteq \{1, \ldots, n\}$,

(1) $\Box_i\mathtt{P} \wedge \Box_j\mathtt{P} \Leftrightarrow \Box_i\Box_j\mathtt{P} \wedge \Box_j\Box_i\mathtt{P}$,
(2) $\mathtt{C}_X\mathtt{P} \Leftrightarrow \bigwedge_{i\in X} \Box_i\mathtt{P}$ and $\mathtt{D}_X\mathtt{P} \Leftrightarrow \bigvee_{i\in X} \Box_i\mathtt{P}$

are $\mathcal{L}$-valid. By the way, none of the interaction axioms presented in [1], [2] captures the axiom schema (1) above. Moreover, consider the mapping $t : \mathtt{F} \to \mathtt{F}$ such that:

- $t(\mathtt{P}) = \mathtt{P}$ for all $\mathtt{P} \in \mathtt{F}_0$,
- $t(\oplus\mathtt{A}) = \oplus t(\mathtt{A})$ for all $\oplus \in \{\neg, \Box_1, \ldots, \Box_n\}$,
- $t(\mathtt{A} \wedge \mathtt{B}) = t(\mathtt{A}) \wedge t(\mathtt{B})$,
- $t(\mathtt{C}_X\mathtt{A}) = \bigwedge_{i\in X} \Box_i\ t(\mathtt{A})$,
- $t(\mathtt{D}_X\mathtt{A}) = \bigvee_{i\in X} \Box_i\ t(\mathtt{A})$.

It is easy to show that $\mathtt{A} \Leftrightarrow t(\mathtt{A})$ is $\mathcal{L}$-valid. However $t$ is not polynomial. That is why the **NP**-completeness of the $\mathcal{L}$-satisfiability problem might not straightforwardly follow from the possible **NP**-completeness of the satisfiability problem restricted to formulae without the operators $\mathtt{C}_X$ and $\mathtt{D}_X$.

EXAMPLE 2.1. The set of $^*\mathrm{S5}^{\mathtt{D}}_{\mathtt{C}}(n)$-logics contains the logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models\rangle$ (resp. $\mathcal{L}' = \langle \mathtt{L}, \mathcal{S}', \models'\rangle$) where $(U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$ (resp. $(U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}'$) iff $\mathcal{R}_1 \subseteq \ldots \subseteq \mathcal{R}_n$ (resp.[2] for all $u \in U$ and for all $i, j \in \{1, \ldots, n\}$, either $R_i(u) \subseteq R_j(u)$ or $R_j(u) \subseteq R_i(u)$) and the $\mathcal{R}_i$'s are equivalence relations.

For any finite sequence of natural numbers $\sigma$, $set(\sigma)$ (resp. $|\sigma|$) denotes the set of elements occurring in $\sigma$ (resp. the length of $\sigma$). For example $set((1, 2, 3, 3, 4)) = \{1, 3, 2, 4\}$. As usual $\sigma_1.\sigma_2$ denotes the concatenation of two sequences.

[2]This condition is known as the *local agreement* condition [7]. For instance it has been used to define semantical structures for *information logics* [7], [14], [3].

# 3. NP-complete satisfiability problems

In this section we shall show that every *S5$^{\mathtt{D}}_{\mathtt{C}}(n)$-logic $\mathcal{L}$ has the finite model property. Moreover we shall provide sufficient conditions to establish decidability and **NP**-completeness of the $\mathcal{L}$-satisfiability problem.

In the rest of this section $\mathcal{L}$ denotes an *S5$^{\mathtt{D}}_{\mathtt{C}}(n)$-logic $\langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ with $n \geq 2$ unless otherwise stated. The case $n = 1$ is omitted here since it roughly corresponds to the modal logic S5.

PROPOSITION 3.1. *Let $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$ and let $\emptyset \neq U' \subseteq U$. For all $u \in U'$, for all $i, j \in \{1, \ldots, n\}$, if $\mathcal{R}_i(u) \subseteq \mathcal{R}_j(u)$ then $\mathcal{R}'_i(u) \subseteq \mathcal{R}'_j(u)$ with $\mathcal{M}_{|U'} = (U', (\mathcal{R}'_i)_{i\in\{1,\ldots,n\}}, V')$.*

PROOF. By an easy verification. **Q.E.D.**

Proposition 3.2 below states that when the chain of inclusions for the successors of $x$ is known then the chain for $y$ is partially known when $x$ and $y$ are $R_{i_k}$-related for some $i_k \in \{1, \ldots, n\}$.

PROPOSITION 3.2. *Let $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$, $x \in U$ and $\mathcal{R}_{i_1}(x) \subseteq \ldots \subseteq \mathcal{R}_{i_n}(x)$ with $\{1, \ldots, n\} = \{i_1, \ldots, i_n\}$. Assume $(x, y) \in R_{i_k}$ for some $k \in \{1, \ldots, n\}$. Then,*

1. *for all $k' \in \{k, \ldots, n\}$, $R_{i_{k'}}(x) = R_{i_{k'}}(y)$*
2. *for all $k' \in \{1, \ldots, k-1\}$, $R_{i_{k'}}(y) \subseteq R_{i_k}(y)$.*

PROOF. The proof is by an easy verification. **Q.E.D.**

Figure 1 illustrates Proposition 3.2. In order to prove the finite model property for $\mathcal{L}$, the standard filtration construction for modal logics (see e.g. [8]) does not work for the logic $\mathcal{L}$ unless additional assumptions are made. Instead of using the filtration construction, we propose a submodel construction. Let $\mathtt{A}$ be an $\mathtt{L}$-formula, $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$ be a model for $\mathcal{L}$, $w \in U$ such that $\mathcal{M}, w \models \mathtt{A}$. Assume that $\mathcal{R}_{i_1}(w) \subseteq \ldots \subseteq \mathcal{R}_{i_n}(w)$ with $\{1, \ldots, n\} = \{i_1, \ldots, i_n\}$.

We shall construct a set $U' \subseteq U$ such that

- $w \in U'$,
- $U'$ is finite and,
- $\mathcal{M}_{|U'}, w \models \mathtt{A}$.

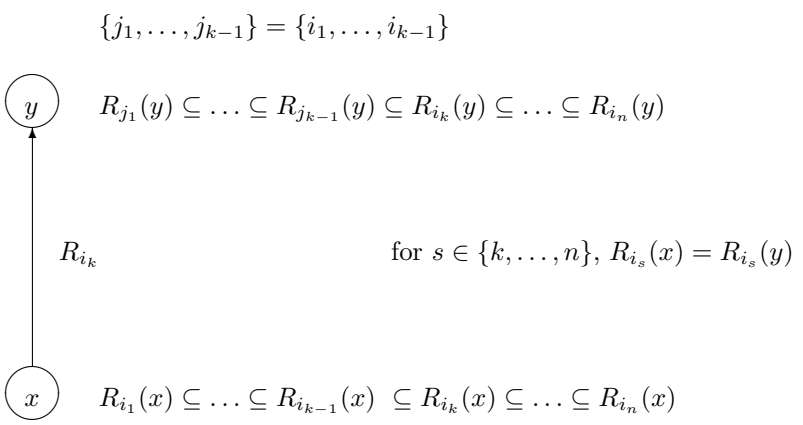


**Fig. 1.** Illustration of Proposition 3.2

*The construction informally described below generalizes the construction in [3] in order to include the operators* $\mathtt{C}_X$ *and* $\mathtt{D}_X$. To build such a set $U'$, we first consider the set $\mathtt{Nec}$ of necessity formulae $\Box_i \mathtt{A}'$ such that $\ominus \mathtt{A}' \in sub(\mathtt{A})$ for some $\ominus \in OP$. Then the construction of $U'$ is done recursively, that is $U' = \bigcup_{0 \leq i \leq N} U_i$ for some $0 \leq N \leq n$ where each $U_i$ is finite. $U_0$ is initialized to $\{w\}$ and then $U_{i+1}$ is defined from $U_i$. Everytime a formula in $\mathtt{Nec}$ is not satisfied at some element of $U_i$, we add a *witness* of this fact in $U_{i+1}$ ($\mathcal{M}, u \not\models \ominus \mathtt{A}'$ iff there exists $u'$ such that $(u, u') \in \mathcal{R}_\ominus$ and $\mathcal{M}, u' \not\models \mathtt{A}'$ where $\mathcal{R}_\ominus$ is the binary relation attached to the modal operator $\ominus$ in $\mathcal{M}$). Moreover, assume that for some $u \in U_I$, $\mathcal{M}, u \not\models \Box_i \mathtt{A}'$ requires a witness and $\mathcal{R}_i(u) \subseteq \mathcal{R}_j(u)$. There exists $u' \in U_{I+1}$ such that $\mathcal{M}, u' \not\models \mathtt{A}'$ and $(u, u') \in \mathcal{R}_i$. We can show that if $\mathcal{M}, u' \not\models \Box_j \mathtt{A}''$ for some $\Box_j \mathtt{A}'' \in \mathtt{Nec}$, there is no need to consider a *new* witness (the set $\mathtt{Nec}_x^\sigma$ defined below contains the necessity formulae that require the introduction of a *new* witness generated from the world $x$ and $\sigma$ encodes some information about $x$ and about some accessibility relations of the model). There exists $u''$ such that $(u', u'') \in \mathcal{R}_j$ and $\mathcal{M}, u'' \not\models \mathtt{A}''$. Since $\mathcal{R}_j$ is an equivalence relation, $(u, u'') \in \mathcal{R}_j$ and therefore $\mathcal{M}, u \not\models \Box_j \mathtt{A}''$. If the set $U_{I+1}$ has been properly built (this point should become clear in the formal definition), there exists

$v \in U_{I+1}$ such that $(u, v) \in \mathcal{R}_j$ and $\mathcal{M}, v \not\models \mathtt{A}''$. Since $(u', v) \in \mathcal{R}_j$, $v$ is already a witness for $\mathcal{M}, u' \not\models \Box_j \mathtt{A}''$. This observation allows us to find $N \leq n$ such that for all $u' \in U_N$, no witness is needed.

We shall give in the sequel the formal definitions. The set $\mathtt{Nec}$ is defined as follows,

$$\mathtt{Nec} = \{\Box_i \mathtt{A}' : \ominus \mathtt{A}' \in sub(\mathtt{A}) \text{ for some } \ominus \in OP, i \in \{1, \ldots, n\}\}.$$

For all $x \in U$ and all sequences $\sigma = (j_1, \ldots, j_k)$ such that $set(\sigma) \subseteq \{1, \ldots, n\}$ and $\mathcal{R}_{j_1}(x) \subseteq \ldots \subseteq \mathcal{R}_{j_k}(x)$, the set $\mathtt{Nec}^{\sigma}_{x}$ is defined as follows:

$$\mathtt{Nec}^{\sigma}_{x} = \{\Box_{j_{k'}} \mathtt{A}' \in \mathtt{Nec} : k' \in \{1, \ldots, |\sigma|\}, \mathcal{M}, x \models \neg \Box_{j_{k'}} \mathtt{A}', \text{ and if } k' \geq 2 \text{ then } \mathcal{M}, x \models \Box_{j_{k'-1}} \mathtt{A}'\}$$

Remember the set $\mathtt{Nec}^{\sigma}_{x}$ allows to reduce the set of witnesses generated from $x$. For each natural number $i \leq n$, we are defining a set $W_i$ of 3-tuples $(u', \sigma, ?)$ where,

- $u' \in U$,
- $\sigma$ is a sequence of elements of $\{1, \ldots, n\}$ without repetition,
- $?$ is either the symbol '$\Lambda$' or some $\Box_p \mathtt{A}' \in \mathtt{Nec}$ with $p \notin set(\sigma)$.

The set $U_i$ shall be later defined as the set $\{u' : (u', (i_1, \ldots, i_n), ?) \in W_i\}$. $W_i$ is an intermediate set that contains some information about the elements of $U_i$. Let $W_0 = \{(w, (i_1, \ldots, i_n), \Lambda)\}$. Assume $W_i$ is defined. We will now define $W_{i+1}$. Initialize $W_{i+1}$ to the empty set $\emptyset$.

For each $(w', \sigma, ?) \in W_i$, for each $\Box_j \mathtt{A}' \in \mathtt{Nec}^{\sigma}_{w'}$,
choose $u \in U$ such that $(w', u) \in \mathcal{R}_j$ and $\mathcal{M}, u \models \neg \mathtt{A}'$.

If $\sigma = (j_1, \ldots, j_k)$ then we write $k'$ to denote the element of $\{1, \ldots, k\}$ such that $j_{k'} = j$. The existence of $k'$ is guaranteed by the definition of $\mathtt{Nec}^{\sigma}_{w'}$. Add the triple $(u, (j'_1, \ldots, j'_{k'-1}), \Box_j \mathtt{A}')$ to $W_{i+1}$ such that $set((j'_1, \ldots, j'_{k'-1}))$ $= set((j_1, \ldots, j_{k'-1}))$ and $\mathcal{R}_{j'_1}(u) \subseteq \ldots \subseteq \mathcal{R}_{j'_{k'-1}}(u)$ – whenever $k' = 1$ the sequence $(j'_1, \ldots, j'_{k'-1})$ is the empty sequence $\Lambda$. Let $W_{i+1}$ be the set augmented this way. There exists $N \in \{0, \ldots, n\}$ such that $W_N \neq \emptyset$ and $W_{N+1} = \emptyset$ since the length of the sequences of natural numbers strictly decreases. Moreover if $(w', \sigma, ?) \in W_i$ then the length of $\sigma$ is at most $n - i$. Let $W$ and $U'$ be the sets defined in the following way:

$$W = \bigcup_{i=0}^{N} W_i \qquad U' = \bigcup_{i=0}^{N} U_i \text{ with } U_i = \{w' : (w', \sigma, ?) \in W_i\}$$

PROPOSITION 3.3. *For all $i \in \{1, \ldots, N\}$ and all $(u_i, (j_1, \ldots, j_k), \Box_p \mathtt{A}') \in W_i$,*

1. *$\mathcal{R}_{j_1}(u_i) \subseteq \ldots \subseteq \mathcal{R}_{j_k}(u_i)$, $\mathcal{R}_{j_k}(u_i) \subseteq \mathcal{R}_p(u_i)$, $\mathcal{M}, u_i \models \neg \mathtt{A}'$, and*
2. *for all $j \in \{1, \ldots, n\} \setminus set((j_1, \ldots, j_k))$, $\mathcal{R}_{j_k}(u_i) \subseteq \mathcal{R}_j(u_i)$.*

Proof. (1) Obvious from the construction of $(u_i, (j_1, \ldots, j_k), \Box_p \mathtt{A}')$.
(2) By induction on $i$ using Proposition 3.2. **Q.E.D.**

Observe that[3] $card(\mathtt{Nec}) \leq n \times mw(\mathtt{A})$, $card(\mathtt{Nec}_x^\sigma) \leq mw(\mathtt{A})$ and $\mathtt{Nec}_x^\sigma = \emptyset$ when $\sigma$ is the empty sequence $\Lambda$. For all $i \in \{0, \ldots, N-1\}$, $card(W_{i+1}) \leq card(W_i) \times mw(\mathtt{A})$ and therefore $card(U') \leq 1 + n \times mw(\mathtt{A})^n$. Proposition 3.4 below states that the set $U'$ contains *enough* states.

PROPOSITION 3.4. *For all $u \in U'$ and for all $\Box_j \mathtt{A}' \in \mathtt{Nec}$, if $\mathcal{M}, u \not\models \Box_j \mathtt{A}'$ then there is $u' \in U'$ such that $(u, u') \in \mathcal{R}_j$ and $\mathcal{M}, u' \not\models \mathtt{A}'$.*

PROOF. By induction on $i$ when $(u, \sigma, ?) \in W_i$. Similar to the proof of Proposition 6 in [3]. **Q.E.D.**

A polynomial bound can be obtained for the size of some model satisfying A.

PROPOSITION 3.5. *An L-formula A is $\mathcal{L}$-satisfiable iff A is satisfiable in a model for $\mathcal{L}$ with at most $1 + n \times mw(\mathtt{A})^n$ states.*

PROOF. Assume there is a model for $\mathcal{L}$, $\mathcal{M} = (U, (\mathcal{R}_i)_{i \in \{1,\ldots,n\}}, V)$, $w \in U$ such that $\mathcal{M}, w \models \mathtt{A}$. Consider the model for $\mathcal{L}$, $\mathcal{M}' = \mathcal{M}_{|U'}$. We then prove that for all states $u' \in U'$ and for all $\mathtt{B} \in sub(\mathtt{A})$, $\mathcal{M}, u' \models \mathtt{B}$ iff $\mathcal{M}', u' \models \mathtt{B}$ (including A). We proceed by induction on the structure of B. The only nontrivial cases are when B has the form $\mathtt{C}_X\mathtt{B}'$, $\mathtt{D}_X\mathtt{B}'$ and $\Box_i \mathtt{B}'$. Only the first case is presented herein. Take $u' \in U'$. If $\mathcal{M}, u' \models \mathtt{C}_X\mathtt{B}'$ then for all $v \in U$ such that $(u', v) \in (\bigcup_{i \in X} \mathcal{R}_i)^*$ we have $\mathcal{M}, v \models \mathtt{B}'$. There is $i_0 \in X$ such that $\mathcal{R}_{i_0}(u') = (\bigcup_{i \in X} \mathcal{R}_i)^*(u')$. So for all $v \in U'$ such that $(u', v) \in \mathcal{R}_{i_0}$, we have $\mathcal{M}, v \models \mathtt{B}'$. Hence for all $v \in U'$ such that $(u', v) \in \mathcal{R}'_{i_0}$ we have $\mathcal{M}, v \models \mathtt{B}'$. By the induction hypothesis, for all $v \in U'$ such that $(u', v) \in \mathcal{R}'_{i_0}$, $\mathcal{M}', v \models \mathtt{B}'$. So $\mathcal{M}', u' \models \Box_{i_0} \mathtt{B}'$. By Proposition

[3]For any finite set $U$, $card(U)$ denotes the cardinality of $U$.

3.1, $\mathcal{M}', u' \models \mathtt{C}_X\mathtt{B}'$. Now assume $\mathcal{M}, u' \not\models \mathtt{C}_X\mathtt{B}'$. There is $i_0 \in X$ such that $\mathcal{M}, u' \models \mathtt{C}_X\mathtt{B}' \Leftrightarrow \Box_{i_0}\mathtt{B}'$. From Proposition 3.4, there exists $v \in U'$ such that $(u', v) \in \mathcal{R}'_{i_0}$ and $\mathcal{M}, v \not\models \mathtt{B}'$. It follows that $(u', v) \in \mathcal{R}'_{i_0}$ and by the induction hypothesis $\mathcal{M}', v \not\models \mathtt{B}'$. Hence $\mathcal{M}', u' \not\models \Box_{i_0}\mathtt{B}'$. By Proposition 3.1, $\mathcal{M}', u' \models \mathtt{C}_X\mathtt{B}' \Leftrightarrow \Box_{i_0}\mathtt{B}'$. Hence $\mathcal{M}', u' \not\models \mathtt{C}_X\mathtt{B}'$. **Q.E.D.**

PROPOSITION 3.6. *Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{M}, \models_{\mathcal{L}} \rangle$ be an $^*S5^{\mathtt{D}}_{\mathtt{C}}(n)$-logic such that the problem: $\ll$Is the finite $\mathtt{L}$-model $(U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$?$\gg$ is decidable. Then the $\mathcal{L}$-satisfiability problem is decidable.*

PROOF. Take any formula $\mathtt{A}$ for which one wishes to know whether $\mathtt{A}$ is $\mathcal{L}$-satisfiable. By Proposition 3.5, $\mathtt{A}$ is $\mathcal{L}$-satisfiable iff there exist an $\mathcal{L}$-model $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$ and $u \in U$ such that $\mathcal{M}, u \models \mathtt{A}$ and $card(U) \leq 1 + n \times mw(\mathtt{A})^n$. So in order to check whether $\mathtt{A}$ is $\mathcal{L}$-satisfiable, enumerate all the $\mathtt{L}$-models $\mathcal{M}$ (modulo the isomorphic copies with respect to $\mathtt{A}$) such that $card(U) \leq 1 + n \times mw(\mathtt{A})^n$ and check whether $(\star)$ $\mathcal{M}, u \models \mathtt{A}$ for some $u \in U$ and $(\star\star)$ $\mathcal{M} \in \mathcal{S}$. $\mathcal{M}' = (U', (R'_i)_{i\in\{1,\ldots,n\}}, V')$ is isomorphic to $\mathcal{M}$ with respect to $\mathtt{A}$ iff there is a 1-1 mapping $g : U \to U'$ such that for all $i \in \{1, \ldots, n\}$, $\{(g(x), g(y)) : (x, y) \in R_i\} = R'_i$ and for all the propositional variables $\mathtt{P}$ occurring in $\mathtt{A}$, $V'(\mathtt{P}) = \{g(x) : x \in V(\mathtt{P})\}$. $(\star)$ can be checked in polynomial-time in $card(U)$ and in the *size* of $A$ (i.e., the length of the representation of $\mathtt{A}$ in any reasonable -unspecified- encoding). Since the set of $\mathtt{L}$-models $\mathcal{M}$ (modulo the isomorphic copies) such that $card(U) \leq 1 + n \times mw(\mathtt{A})^n$ is finite and can be easily generated and $(\star\star)$ is an instance of a decidable problem then the $\mathcal{L}$-satisfiability problem is decidable. **Q.E.D.**

EXAMPLE 3.1. Let $\Sigma$ be a non-empty set of permutations on $\{1, \ldots, n\}$ and $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ be the $^*\mathrm{S5}^{\mathtt{D}}_{\mathtt{C}}(n)$-logic such that for all $\mathtt{L}$-models $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$, $\mathcal{M} \in \mathcal{S}$ iff

(eq) for all $i \in \{1, \ldots, n\}$, $\mathcal{R}_i$ is an equivalence relation and

(la) for all $u \in U$, there exists a permutation $s_u \in \Sigma$ such that $R_{s_u(1)}(u) \subseteq \ldots \subseteq R_{s_u(n)}(u)$ ($s_u$ may not be unique).

Let $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$ be a finite $\mathtt{L}$-model. One can easily check in polynomial-time in $card(U)$ that (eq) holds for $\mathcal{M}$. Moreover whether (la) holds for $\mathcal{M}$ can also be checked in deterministic time $o(n! \times card(U)^5)$ (which is also polynomial in $card(U)$). Hence the problem "Is the finite $\mathtt{L}$-model $(U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$?" is decidable and the $\mathcal{L}$-satisfiability problem is decidable.

PROPOSITION 3.7. *Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{M}, \models_{\mathcal{L}} \rangle$ be an $^*S5^{\mathtt{D}}_{\mathtt{C}}(n)$-logic such that:*

*For any finite* L*-model $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$, one can check in polynomial-time in $card(U)$ whether $\mathcal{M} \in \mathcal{S}$.*

*Then the $\mathcal{L}$-satisfiability problem is* **NP***-complete.*

PROOF. This is immediate from the proof of Proposition 3.6 considering the computational complexity of the following problems:

1. for any finite L-model $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}$ and any $\mathtt{A} \in \mathtt{F}$, deciding whether $\mathcal{M}, u \models \mathtt{A}$ for some $u \in U$ can be known in polynomial-time in $card(U)$ and in the *size* of A
2. For any finite L-model $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$, one can check in polynomial-time in $card(U)$ whether $\mathcal{M} \in \mathcal{S}$. **Q.E.D.**

COROLLARY 3.8. *All the logics $\mathcal{L}$ defined in the Example 1 have an* **NP***-complete $\mathcal{L}$-satisfiability problem.*

For the sake of comparison, remember that for all $k \geq 2$, the satisfiability problem for the multi-modal logics $S5_k$ is **PSPACE**-complete and for $S5^{\mathtt{C}}_k$ the satisfiability problem is **EXPTIME**-complete [9].

Let $\mathcal{S}_0$ be the set of L-models such that for all L-models $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V)$, $\mathcal{M} \in \mathcal{S}_0$ iff for all $i \in \{1, \ldots, n\}$, $\mathcal{R}_i$ is an equivalence relation. Let $\mathcal{C}$ be a condition on the families $(\mathcal{R}_i)_{i\in\{1,\ldots,n\}}$ of binary relations. We write $\mathcal{S}_0^{\mathcal{C}}$ to denote the subset of $\mathcal{S}_0$ such that for all L-models $\mathcal{M} = (U, (\mathcal{R}_i)_{i\in\{1,\ldots,n\}}, V) \in \mathcal{S}_0$, $\mathcal{M} \in \mathcal{S}_0^{\mathcal{C}}$ iff $(\mathcal{R}_i)_{i\in\{1,\ldots,n\}}$ satisfies $\mathcal{C}$. In Figure 2, we present the complexity class of various logics $\langle \mathtt{L}', \mathcal{S}_0^{\mathcal{C}}, \models' \rangle$ for some condition $\mathcal{C}$ and $\mathtt{L}'$ is either L or L without the operators $\mathtt{D}_X$ (written $\mathtt{L} - \{\mathtt{D}\}$) or L without the operators $\mathtt{D}_X$ and $\mathtt{C}_X$ (written $\mathtt{L} - \{\mathtt{C}, \mathtt{D}\}$). The conditions are as follows:

$\mathcal{C}_1 := \mathtt{true}$
$\mathcal{C}_2 := \forall i, j \in \{1, \ldots, n\}, \forall u \in U\ \mathcal{R}_i(u) \subseteq \mathcal{R}_j(u)$ or $\mathcal{R}_j(u) \subseteq \mathcal{R}_i(u)$
$\mathcal{C}_3 := \mathcal{R}_1 \subseteq \ldots \subseteq \mathcal{R}_n$
$\mathcal{C}_4 := \mathcal{R}_1 = \ldots = \mathcal{R}_n$ (uni-modal S5).

| | $\mathcal{C}_1$ | $\mathcal{C}_2$ | $\mathcal{C}_3$ | $\mathcal{C}_4$ |
|---|---|---|---|---|
| $\mathtt{L} - \{\mathtt{C}, \mathtt{D}\}$ | **PSPACE** | **NP** | **NP** | **NP** |
| $\mathtt{L} - \{\mathtt{D}\}$ | **EXPTIME** | **NP** | **NP** | **NP** |
| L | **EXPTIME** | **NP** | **NP** | **NP** |

**Fig. 2.** Interaction and computational complexity

In Figure 2, we assume $n \geq 2$ and the occurrence of the complexity class **C** stands for **C**-completeness.

The results for $\mathcal{C}_2$ and $\mathcal{C}_3$ are due to Corollary 3.8 whereas those for $\mathcal{C}_1$ and $\mathcal{C}_4$ have been proved in [9], [13]. It is clear that for all $i \in \{2, 3, 4\}$, $\mathcal{C}_i$ entails $\mathcal{C}_{i-1}$. The properties above remain valid if for some non-empty set $\Sigma$ of permutations on $\{1, \ldots, n\}$, $\mathcal{C}_3$ is replaced by:

$$\forall u \in U,\ \exists s \in \Sigma\ R_{s(1)}(u) \subseteq \ldots \subseteq R_{s(n)}(u).$$

However it would be desirable to strenghen $\mathcal{C}_2$ (that is to find a condition $\mathcal{C}_2'$ such that $\mathcal{C}_2$ entails $\mathcal{C}_2'$) such that the satisfiability problem for $\langle \mathtt{L}, \mathcal{S}_0^{\mathcal{C}_2'}, \models \rangle$ is still **NP**-complete.

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Laboratoire LEIBNIZ - C.N.R.S.
46 Avenue Félix Viallet
38031 Grenoble Cedex
France
email: demri@imag.fr