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Deepest Voting: a new way of electing

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Abstract

Classical approval voting processes suffer from well-known impossibility theorems. We consider grading-based voting, where voters give a grade rather than a preference to each candidate. Numerous voting processes have been proposed in this framework, such as majority judgment or range voting, showing interesting properties. This article aims to present a unified framework for grading-based voting processes. Our statement is that a voting process can be seen as the finding of the innermost point of a multivariate scatterplot. Introducing depth functions in voting theory enables to associate each voting process to a depth notion, and enlarges the scope of procedures. We focus on a given family of depth functions, the wL^p depths, to highlight the behavior of the depth-based approach.

Keywords. voting process, grade modeling, depth functions

1 Introduction

Social choice theory aims at determining which decision or candidate, from a set of possibilities, is chosen by a set of voters. There exists a lot of election mechanisms trying to aggregate individual preferences of voters into a collective preference. Finding a method (usually called *social decision function* or *voting process*) that makes the choice satisfying for most of the voters is a challenging issue.

In classical voting processes each voter’s input message is a rank-ordering of the candidates. A social decision function gives an output (a rank-ordering) for any inputs that have been proposed. However, social decision functions have theoretical weakness. They can’t satisfy simultaneously a small set of desirable properties, which is known since [1] as “impossibility theorem”. There is no satisfactory method for electing or ranking candidates in such a framework. The *model* introduced in [2] is an interesting framework which is not based on rank-ordering of the candidates but on a grading of the candidates by the voters. It gives more nuanced information and thus escapes important drawbacks of the *classical model* involving only rankings. Approval voting [5], majority judgment [2] and range voting [16] are well-known examples of methods based on the grading model.

This article aims to present a unified framework for grading-based voting processes, and to extend the scope of voting processes. This family of social decision functions is based on the statistical notions of depth functions and their related deepest points. Let us consider in the following that we have n voters and d candidates. Each voter gives a grade to each candidate. Each voter can then be assimilated to a point in \mathbb{R}^d , whose coordinates are the grades for each candidate. The set of all voters’ grades can hence be seen as a scatterplot. The key idea is to consider the *most central* voter in this scatterplot. This innermost (possibly imaginary) voter according to his grades can be seen as the most representative of all voters, and his preferences should meet a large consensus among the others voters. Therefore, the presented social decision function returns simply the candidate which has the maximum grade of this innermost voter. We refer to any voting process based on the use of such notion of *most central* voter as a “deepest voting” process.

The paper is organized as follows. Section 2 recalls the definition of Balinski and Laraki’s grading model [2]. In Section 3, we recall the statistical notion of depth function which will help us to determine the innermost voter, knowing that there is an infinity of ways of choosing the center of a scatterplot. Each way of choosing it corresponds to a new member of the family of social decision functions. We next give the definition of the deepest voting process. We study some of its interesting properties for a given family of depth functions in Section 4. All the proofs are given in the Appendix.

2 Voting framework

2.1 Voting process

Voting is a common way to choose an alternative or a candidate from a set of different alternatives or candidates. It is for example widely used in political affairs to choose a president, a major or a deputy, at least in democratic countries. A voting process can be seen as a mathematical function and therefore voting processes have been widely studied in a mathematical point of view since the early works of Borda and Condorcet at the end of the XVIIIth century (see [8] for a review). This research field is known as “social choice theory”. Theoretical voting processes are generally based on the assumption that each voter is able to rank the candidates in a strict order from the best one, or the preferred one, to the worst one. Several properties have been proposed to characterize a good voting process. These properties often refer to unwanted situations that might appear, and are then generally known as “paradoxes”. Any voting process can be theoretically analyzed in terms of satisfied and non-satisfied properties. Then, an axiomatic approach of social choice aims

at determining which voting process satisfies a set of desirable properties. But the well-known impossibility theorems due to Arrow [1] or Gibbart [9] and Satterthwaite [15] demonstrate that no voting process can satisfy simultaneously a minimal set of desirable properties. Therefore the choice of a voting process appears as a matter of compromise between pro and cons arguments. One can refer to [8] for a complete review of the properties/paradoxes satisfied by the most popular voting processes.

Another way is to change the paradigm of the voting situation, so that voters do to not only rank the candidates but also rate them.

2.2 Grade modeling

Grading candidates better than ranking them allows to use specific voting process out of the framework of Arrow's theorem. If voters are supposed to grade the candidates, then the voting process consists in finding the best candidate considering all grades. Approval voting [5] is the most simple example of such a grading-based voting process, where grades are 0 or 1 and the chosen candidate is the one with the greatest number of 1. The majority judgment [2] is another example of grading-based voting process, using discrete or continuous grades. We propose in the following to formalize the use of grades in voting process as in [2].

Consider that we have n voters and d candidates. Suppose that each voter v_j , $j = 1, \dots, n$ grades each candidate c_i , $i = 1, \dots, d$. We denote $\Phi(i, j)$ the corresponding grade. Let Λ be the set of possible grades $\Phi(i, j)$. As pointed out by [2], the set of grades Λ needs to be strictly ordered but may be finite or an interval of the real numbers. We suppose without loss of generality that $\Lambda \subseteq [0, 1]$. The grading is summarized in a $d \times n$ grading matrix $\Phi = \{\Phi(i, j), i = 1, \dots, d, j = 1, \dots, n\} \in \Lambda^{d \times n}$. Any collection of d grades is called a *profile* and, in particular, every column $\Phi(., j)$ is a profile.

We distinguish three cases depending on the nature of the set Λ :

Binary set $\Lambda = \{0, 1\}$. Approval voting is a voting process based on a binary set of grades. Classical model may correspond to the case where $\Phi(i, j) = 1$ for one and only one candidate c_i for each voter v_j , whereas approval voting gives no constraint on the number of 0 or 1 by voter.

Discrete set $\Lambda = \{0, 1/N, 2/N, \dots, 1\}$, with $N > 1$. This case includes *e.g.* grading with finite words from bad to excellent, letters from E to A, etc, where a numerical ranking is applied. A usual example in everyday life is the evaluation process of a product or of a service, where each consumer is asked to put a mark between 0 and 5.

Continuous set. The set Λ is a real interval. Let $\Lambda = [0, 1]$ without loss of generality. In practice, this kind of generalization can be proceed by putting an horizontal segment in front of each candidate's name:

Photo and Name of Candidate 1:	To Reject	_____	Excellent
Photo and Name of Candidate 2:	To Reject	_____	Excellent
⋮			
Photo and Name of Candidate d:	To Reject	_____	Excellent

A voter is asked to put a vertical line according to his grade for each candidate. This generalization gives a more nuanced information with respect to a finite set of grades. Moreover, this intuitive procedure should allow to open the mechanism to illiterate people by adding a photo of the candidate and a smiling face (respectively a sad face) instead of the comment “Excellent” (respectively to the comment “To Reject”).

Example Table 1 illustrates on an example the different types of grading. Consider 15 voters and 2 candidates. The continuous grading allows each voter to give any grade between 0 and 1 to any candidate. The discrete grading (here on 6 levels, *i.e.* $N = 5$) can be seen as a rounding of continuous grading. Binary grading just corresponds to a rounding of continuous grading or to set the maximal grade to 1 and others to 0 for majority voting.

voter	continuous		discrete		binary		voter	continuous		discrete		binary	
	c_1	c_2	c_1	c_2	c_1	c_2		c_1	c_2	c_1	c_2	c_1	c_2
v_1	0.59	0.67	0.4	0.6	0	1	v_9	0.43	0.78	0.4	0.6	0	1
v_2	0.49	0.79	0.4	0.6	0	1	v_{10}	0.48	0.63	0.4	0.6	0	1
v_3	0.45	0.73	0.4	0.6	0	1	v_{11}	0.95	0.13	0.8	0.0	1	0
v_4	0.43	0.66	0.4	0.6	0	1	v_{12}	0.95	0.17	0.8	0.0	1	0
v_5	0.46	0.79	0.4	0.6	0	1	v_{13}	0.92	0.14	0.8	0.0	1	0
v_6	0.44	0.71	0.4	0.6	0	1	v_{14}	0.91	0.15	0.8	0.0	1	0
v_7	0.54	0.79	0.4	0.6	0	1	v_{15}	0.95	0.10	0.8	0.0	1	0
v_8	0.59	0.67	0.4	0.6	0	1							

Table 1: Example of grades given by 15 voters on 2 candidates with continuous, discrete or binary scales

2.3 Grading-based voting process

A grading-based voting process can then be seen as a function G , called *method of grading* in [2], that assigns a profile to any matrix Φ . The function G is defined from $\Lambda^{d \times n}$ with values in the subsets of $[0, 1]^d$. Remark that the set of possible grades of the profiles given by function G may differs from Λ since one may associate for example the mean of initial grades and obtain a continuous set from a discrete set Λ .

Balinski and Laraki [2] propose some conditions (which they call *axioms*) that a *method of grading* should satisfy:

Condition 1 G is *neutral*: G gives the same result by permuting the rows of Φ (*i.e.* by permuting the candidates).

Condition 2 G is *anonymous*: G gives the same result by permuting the columns of Φ (*i.e.* by permuting the voters).

Condition 3 G is *unanimous*: if a candidate is given an identical grade α by every voter, then G assigns him the grade α .

Condition 4 G is *monotonic*: if $\Phi = \Phi'$ except that one or more voters give higher grades to candidate c_i in Φ than in Φ' , then $G(\Phi)(c_i)$ is higher than $G(\Phi')(c_i)$

Condition 4 bis G is *strictly monotonic*: if $\Phi = \Phi'$ except that one or more voters give strictly higher grades to candidate c_i in Φ than in Φ' , then $G(\Phi)(c_i)$ is strictly higher than $G(\Phi')(c_i)$

Condition 5 G is *independent of irrelevant alternatives (IIA)*: if the grades assigned by the voters to a candidate c_i in two profiles Φ and $\tilde{\Phi}$ are the same, then $G(\Phi)(c_i) = G(\tilde{\Phi})(c_i)$.

Note that in [2], Conditions 4 and 4 bis are embedded in a unique condition. Jennings [11] shows that a *strictly monotonic* method of grading can't avoid rewarding voter dishonesty.

These conditions are similar to the ones used in Arrow's theorem [1]. Therefore, in the classical framework of preference orders, it is impossible to find a voting process that satisfies all these conditions. This has led to the proposal of grading-based voting process by [2] to overcome these impossibility.

The simplest way to compute a grading-based voting process is to aggregate the grades given by the voters on each candidate independently. Note that such a process supposes that condition 5 (IIA) is satisfied. In such a case, we introduce the aggregation function $g : \Lambda^n \mapsto [0, 1]$. The function g summarizes the n grades received by a candidate (a row of Φ) in one grade. One has $G(\Phi)(c_i) = g(\Phi(i, \cdot))$. Many aggregation functions are available to sum up n grades into a unique one – see [4] or [10] for a review of aggregation functions. Some of them have been studied in the specific framework of grading-based voting process, taking into account the fact that votes often take place in a political background.

Balinski and Laraki [2] propose the *majority-grade* voting as aggregation function. For a given candidate c_i , let rank the n grades $\{\Phi(i, j), j = 1, \dots, n\}$ as $r_1 \leq r_2 \leq \dots \leq r_n$. The majority-grade voting, denoted g^{maj} is defined as follows:

$$g^{maj}(r_1, \dots, r_n) = \begin{cases} r_{(n+1)/2} & \text{if } n \text{ odd,} \\ r_{(n+2)/2} & \text{if } n \text{ even.} \end{cases}$$

$g^{maj}(r_1, \dots, r_n)$ can be interpreted as a median of (r_1, \dots, r_n) (note that tie-breaking rules have been proposed for example in [3] or [6]). A *majority-ranking* can be deduced from the majority grade, obviously noting that a candidate receiving a higher majority grade than another will be ranked higher.

Smith [16] proposes another example of aggregation function. Smith introduces the *Range Voting* g^{rv} as follows:

$$\forall (r_1, \dots, r_n) \in \Lambda^n, g^{rv}(r_1, \dots, r_n) = \frac{1}{n} \sum_{j=1}^n r_j,$$

with similar notations than above. $g^{rv}(r_1, \dots, r_n)$ is the mean of the grades r_1, \dots, r_n .

Grading-based voting processes use more information than just a preference order on the candidates since they are based on quantitative grades on the candidates. This supplementary piece of information permits to overcome classical impossibility theorems. We propose in the following a unified framework for grading-based voting process that enlighten the specificity of both Majority Voting and Range Voting within a wide range of new voting processes.

3 Deepest voting

Our statement is that depth functions enable to consider classical grading models and continuous grading models in a uniform way. It expands to numerous voting processes. We first recall what is a depth function and we present on a second hand how it applies in voting processes.

3.1 What is a depth function?

Quoting R. Serfling: “Associated with a given distribution F on \mathbb{R}^d , a depth function is designed to provide a F -based center-outward ordering (and thus a ranking) of points x in \mathbb{R}^d . High depth corresponds to *centrality*, low depth to *outlyingness*” [19].

In other words, a depth function takes high (positive) values at the middle of a scatterplot and vanishes out of it.

Zuo and Serfling [19] give a rigorous definition of a depth function. Denote by \mathcal{F} the class of distributions on the Borel sets of \mathbb{R}^d and F_X the distribution of a given random vector X :

Definition 1. Let the mapping $D : \mathbb{R}^d \times \mathcal{F} \mapsto \mathbb{R}$ be bounded, nonnegative and satisfying:

- (P1) $D(Ax + b; F_{AX+b}) = D(x; F_X)$ for any random vector $X \in \mathbb{R}^d$, any $d \times d$ nonsingular matrix A and any d -vector b .
- (P2) $D(\theta; F) = \sup_{x \in \mathbb{R}^d} D(x, F)$ (i.e. θ is the deepest point with respect to D and F) for any $F \in \mathcal{F}$ having a center θ .
- (P3) for any $F \in \mathcal{F}$ having deepest point θ , $D(x; F) \leq D(\theta + \alpha(x - \theta); F)$ for any $0 \leq \alpha \leq 1$.
- (P4) $D(x; F) \rightarrow 0$ as $\|x\| \rightarrow \infty$ for each $F \in \mathcal{F}$.

Then $D(\cdot; F)$ is called a **statistical depth function**.

Applications of depth techniques include for example robust estimation, center-outward ordering of multivariate observations, data exploration and multivariate confidence regions. Several measures of data depth have been proposed in nonparametric statistics as multidimensional generalizations of the ranks and of the median, each attempts to maintain certain robustness properties.

In this work, the deepest point is a location estimator of the preferences of the voters. The coordinates of the deepest point are the grades the innermost (possibly imaginary) voter would give.

Let give some examples of depth functions.

The weighted L^p depths [18] of a point $x \in \mathbb{R}^d$, $wL^pD(x)$, given a set of n points $\Phi(\cdot, 1), \dots, \Phi(\cdot, n)$ in \mathbb{R}^d is defined by

$$wL^pD(x, F_n) = \frac{1}{1 + \frac{1}{n} \sum_{j=1}^n \omega(\|\Phi(\cdot, j) - x\|_p)}$$

where $p > 0$, ω is a non-decreasing and continuous function on $[0, \infty)$ with $\omega(\infty) = \infty$ and $\|x - x'\|_p = \left(\sum_{j=1}^d |x_j - x'_j|^p\right)^{1/p}$.

If $\omega : x \mapsto x^p$, then

$$wL^p D(x, F_n) = \frac{1}{1 + \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^d |\Phi(i, j) - x_i|^p}. \quad (1)$$

For $p = \infty$, let define similarly

$$wL^\infty D(x, F_n) = \frac{1}{1 + \frac{1}{n} \sum_{j=1}^n \max_{i=1, \dots, d} |\Phi(i, j) - x_i|}. \quad (2)$$

The simplicial depth [13] of a point $x \in \mathbb{R}^d$, $SD_n(x)$, given a set of n points $\Phi = \{\Phi(\cdot, 1), \dots, \Phi(\cdot, n)\} \subset \mathbb{R}^d$ is defined by

$$D_n(x) = P(x \in S[x_1, \dots, x_{d+1}]).$$

It associates to a point x the probability that x is inside a random simplex whose vertices are given by $d + 1$ independent random observations $x_1, \dots, x_{d+1} \in \Phi$.

The halfspace depth [17] of a point $x \in \mathbb{R}^d$, $HD(x)$, given a set of n points $\Phi(\cdot, 1), \dots, \Phi(\cdot, n)$ in \mathbb{R}^d is defined by

$$HD_n(x) := \text{minimum proportion of voters in a halfspace H including } x.$$

Many others depth functions (Mahalanobis depth, projection depth, zonoid depth... see *e.g.* [14]) exist, leading to as many corresponding social decision functions. Let's explicit now the link between a depth function and its associated social decision function.

3.2 Deepest Voting

In the following, the distribution of the grades Φ of the n voters will be defined as Φ_n , giving a weight $\frac{1}{n}$ at the independent profiles $\Phi(\cdot, j) = (\Phi(1, j), \dots, \Phi(d, j)) \in \mathbb{R}^d$, $j = 1, \dots, n$. Each profile can be seen as a point of \mathbb{R}^d and hence depth functions can be applied to points $\{\Phi(\cdot, j), j = 1, \dots, n\}$.

Figure 1 illustrates the behavior of four classical depth functions on the example of Section 2.2. Namely it displays the values of the Tukey's Halfspace depth [17], the Liu's Simplicial depth [13], the weighted L^1 , L^2 , L^3 , and L^∞ depths [18]. It highlights the diversity of depth measures.

Definition 2 (Deepest Voting). *Consider a grading matrix Φ , the associated empirical distribution Φ_n and a given depth function D . Denote*

$$\mathcal{X}_D^* := \operatorname{argsup}_{x \in [0, 1]^d} D(x, \Phi_n)$$

the set of deepest points of Φ_n with respect to D . Let $G_D : \Phi_n \mapsto \mathcal{X}_D^$ be the method of grading with respect to the depth D .*

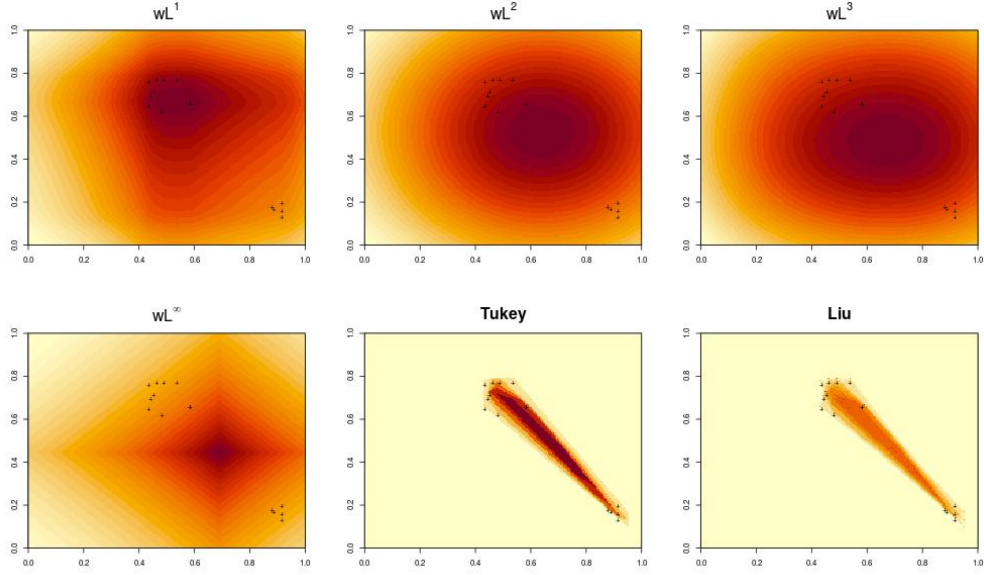


Figure 1: Examples of classical depth functions on the example of Section 2.2. Horizontal axes gives the grade for candidate c_1 and vertical axes for candidate c_2 . Each cross corresponds to a voter.

Let

$$i_D := \operatorname{argmax}_{1 \leq i \leq d} \{x_{D,i}^* \in \mathcal{X}_D^*\}, \text{ with } x_D^* = (x_{D,1}^*, \dots, x_{D,d}^*).$$

The deepest voting process with respect to the depth D is defined as the function v^D which maps $\{\Phi(i, j), i = 1, \dots, d, j = 1, \dots, n\}$ to $i_D \subseteq \{1, \dots, d\}$.

If i_D is unique, then the winner of the election is the candidate c_{i_D} . If i_D is not unique, there is no unique winner of the election.

It is worth noticing that even if \mathcal{X}_D^* does not contain a unique element, the deepest voting i_D may contain only one element. If i_D contains several elements a tie-breaking rule should therefore be proposed. Such a rule can refer to the deepest space, *e.g.* by reducing \mathcal{X}_D^* to its unique center of gravity, or refer directly to the candidates, *e.g.* by electing the oldest candidate. Note that such tie-breaking rules are necessary in any voting process.

Figure 2 displays some deepest voting results obtained on the grades given in the example of Section 2.2 for some depth functions. The transposed grades matrix Φ is represented as a scatterplot and deepest points are given on the figure. Note that deepest point is not unique for Tukey's depth, it was obtained by taking the center of gravity of the deepest set with respect to the Euclidean distance.

It can be seen in Figure 2 that deepest points do not have the same coordinates and that they depend of the depth. Table 2 gives the coordinates of the deepest points. Note that the deepest points of wL^1 and Liu's depths give the second candidate winner while the other deepest points choose the first candidate.

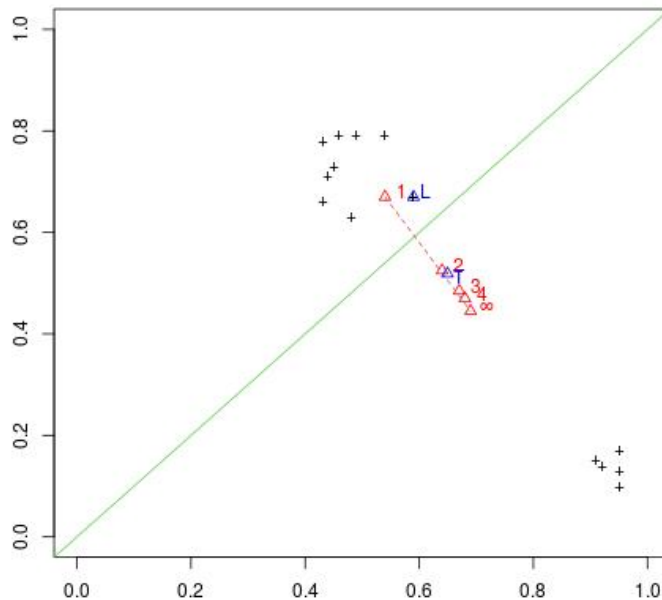


Figure 2: Examples of deepest points on the example of Section 2.2. Horizontal axes gives the grade for candidate c_1 and vertical axes for candidate c_2 . Each cross corresponds to a voter. Triangles give the deepest points. Deepest points for the weighted L^p depths, for $p \in \{1, 2, 3, 4, \infty\}$, are displayed with labels p . Deepest points for Tukey's halfspace depth and Liu's simplicial depth are displayed with respective labels T and L .

	Candidate c_1	Candidate c_2
wL^1 depth	0.54	0.67
wL^2 depth	0.64	0.53
wL^3 depth	0.67	0.48
wL^4 depth	0.68	0.47
wL^∞ depth	0.69	0.45
Tukey's halfspace depth	0.65	0.51
Liu's simplicial depth	0.59	0.67

Table 2: Coordinates of the deepest points of Figure 2.

4 Properties of weighted L^p deepest voting family

In the following, we focus on the family of *weighted L^p depths*. This choice is motivated by the fact that some classical social decision functions are members of this family. We suppose that $\omega : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \omega(x) = x^p$ for a given $p > 1$. We assume that we dispose of a set of n points $\Phi(\cdot, 1), \dots, \Phi(\cdot, n)$ in \mathbb{R}^d . As described above, the weighted L^p depth [18] of a point x , $wL^pD(x, \Phi_n)$,

is defined by (1) and (2). The wL^p deepest voting is the function taking in argument the grading matrix Φ and returning the coordinates of the point maximizing the weighted L^p depth function applied to Φ .

We can first check that wL^p deepest voting indeed defines a decision function in the sense of [2].

Proposition 3. *For a $p \geq 1$, let's consider the associated wL^p deepest voting and denote by D the depth function. Then the method of grading G_D satisfies the Conditions 1-5 exposed in the subsection 2.3.*

Next propositions deal with the characterization of the *weighted L^p deepest voting* for different choices of p .

Proposition 4. *For all $p > 1$, the set of wL^p depth has a unique element.*

The case $p < 1$ should be avoided because the uniqueness of the deepest point is not ensured. It can also be shown that if p tends to 0, each voter is a deepest point!

The class of wL^p deepest voting includes three usual voting processes, which are Majority Judgment [2], approval voting [5] and range voting [16].

Proposition 5. *Not considering the tie-breaking procedure, the Majority Judgment is one point of the wL^1 deepest voting set. Analogously, the Range Voting and the approval Voting are elements of the wL^2 deepest voting.*

wL^1 deepest voting leads to the Majority Judgment. This method has many advantages (see [2] for example). Nevertheless, this method presents several drawbacks too (see [12] or [7]). A weird property of the wL^1 deepest voting is that the value of the decision function may not be included in the convex hull of the n voters' grades, *i.e.*, \mathcal{X}_D^* is not included in the convex hull of Φ . Let consider *e.g.* the situation with 3 candidates and 3 voters and the associated matrix Φ :

$$\begin{aligned}\Phi(1, \cdot) &= (a, 0, 0) \\ \Phi(2, \cdot) &= (0, b, 0) \\ \Phi(3, \cdot) &= (0, 0, c)\end{aligned}$$

with $0 < a < b < c \leq 1$. The wL^1 deepest point is $x^* = (0, 0, 0)$, which is not included in the convex hull of the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

Another case where we can explicit the deepest point is wL^∞ deepest voting.

Proposition 6. *wL^∞ deepest voting maps to the point whose coordinates are the middle of the most extreme coordinates component-wise, that is, it attributes to each candidate the mean of his best and worst grades.*

The choice of p in wL^p deepest voting is of course critical. Roughly speaking, the greater p is, the more importance is given to immoderate grades. For p large enough, the deepest point only depends component-wise on most immoderate grades. So the wL^p Deepest Voting is very vulnerable to strategic voting for large p . These considerations meet the intuition of Balinski and

Laraki arguing that Majority Judgment (wL^1 Deepest Voting) is more resistant to strategic voting than Range Voting (wL^2 Deepest Voting).

As pointed out above, any voting process suffers from unwanted properties generally named paradoxes (see [8] for a more detailed description of voting paradoxes). As other voting processes, *weighted L^p deepest voting* seems to suffer from paradoxes. We propose in the following to focus specifically on four classical properties, and to show how the paradigm changes due to the use of a grading model. The four studied properties are:

- the *Condorcet winner paradox*: a candidate c is not elected despite the fact that c is preferred by a majority of the voters over each of the remaining candidates;
- the *Condorcet loser paradox*: a candidate c is elected despite the fact that a majority of voters prefer each of the remaining candidates to c ;
- the *reinforcement paradox*: if c is elected in each of several disjoint electorates, it is possible that c is not elected if all electorates are combined into a single electorate;
- the *no-show paradox*: a voter may obtain a more preferable outcome if he decides not to participate in an election than if he decides to participate in the election and vote sincerely for his preferences.

Proposition 7. *For all $p \geq 1$, the wL^p deepest voting suffers from the Condorcet winner and the Condorcet loser paradoxes.*

Proposition 8. *wL^p deepest voting suffers from reinforcement and no-show properties, for $p \in [1, \infty] \setminus \{2\}$. These properties do not hold for wL^2 deepest voting.*

Many properties which appear as paradoxes in classical voting processes, with binary Λ , may not be seen as drawbacks under a more complex grading-based model, that is, for discrete or continuous sets Λ . The four properties studied above well illustrate our statement.

Consider for example the Condorcet winner paradox. Suppose that a candidate c_1 obtains a score 0.80 for 51% of voters and 0.01 for 49% of voters and a candidate c_2 obtains a score 0.79 for 51% of voters and 0.90 for 49% of voters. In this configuration, the Condorcet winner is c_1 . Yet, as we have a quantification of the preference of voters, we can see that candidate c_2 is less divisive in the sense that no voters reject this candidate with very low grades. The fact that a voting process elects c_2 is acceptable in this configuration.

Now consider the no-show paradox. Suppose that two candidates c_1 and c_2 obtained the following grades for 3 voters:

	Candidate c_1	Candidate c_2
voter v_1	0.5	1
voter v_2	0.5	ε
voter v_3	0	ε

Then candidate c_2 is elected with voters $\{v_1, v_2\}$. But for some $\varepsilon > 0$ sufficiently small, the candidate c_2 is not elected with voters $\{v_1, v_2, v_3\}$, considering wL^p -deepest voting with $p < 2$ (see

the proof of Proposition 8). In this configuration, the fact that voter v_3 votes will make c_2 loose. wL^p depth functions with $p < 2$ favor candidates with less dispersion, which are more consensual. The no-show property results from the fact that v_3 does not approve candidate c_2 and that his preference with respect to candidate c_1 is not significant.

When $p > 2$, a configuration where no-show property holds is the following:

	Candidate c_1	Candidate c_2
voter v_1	0	$0.5+\varepsilon$
voter v_2	1	$0.5+\varepsilon$
voter v_3	0	ε

with $\varepsilon > 0$ sufficiently small (see the proof of Proposition 8). It is due to the low grade given by v_3 to c_2 and the fact that the difference with the grade for c_1 is not significant. wL^p depth functions with $p > 2$ here favor candidates with highest grades. In both situations, the fact that no-show property occurs is acceptable since even if voter v_3 prefers candidate c_2 , the grades show that he does not support him yet.

These examples illustrate that the amount of quantity available through grading is likely to change the result of a voting process. It enables to explain some results which may be seen as paradoxes with preferential grades. A drawback is that it may give more weight to extremes opinions for determining the final winner of elections.

5 Conclusion

Grading-based voting offers a richer information than classical model. It is based on a discrete or continuous evaluation of candidates rather than a single preference. Grading widens the scope of decision processes available.

In this paper, we introduce a new viewpoint on voting procedures, introducing a parallelism with the statistical notion of depth. Depth functions give a unified way to define voting processes, with finite or continuous grading. We define the notion of deepest voting, which includes classical voting procedures such as majority judgment, approval voting and range voting.

Deepest voting related to the family of wL^p depths satisfies Balinski and Laraki's axioms. We studied some basic paradoxes, namely Condorcet's, no-show and reinforcement paradoxes.

A large area of new voting procedures is opened with deepest voting, for example considering Tukey's halfspace depth or Liu's simplicial depth. Depth functions have encountered much interest in statistical literature and their attractive properties (in particular robustness) may be useful in social decision theory. It will be the object of future research.

Appendix. Proofs

Proof of Proposition 3

Denote $x^* := (x_1^*, \dots, x_d^*)$ a deepest point for wL^p depth. Maximizing the weighted wL^p depth function is equivalent to minimizing the quantity $\sum_{j=1}^n (\sum_{i=1}^d |\Phi(i, j) - x_i|^p)$. Since all the quantities $\sum_{i=1}^d |\Phi(i, j) - x_i|^p$ are positive, it follows that

$$\forall i \in 1, \dots, d, \quad x_i^* = \operatorname{argmin}_{x \in \mathbb{R}} \sum_{j=1}^n |\Phi(i, j) - x|^p.$$

The optimization is made component-wise so Conditions 1 and 5 are satisfied. Condition 2 is also true since the sum is order free. If for a given i_0 , $\forall j \in 1, \dots, n$, $\Phi(i_0, j) = \alpha$, then $x \mapsto \sum_{j=1}^n |\Phi(i_0, j) - x|^p = \sum_{j=1}^n |\alpha - x|^p$ is minimal for $x = \alpha$ (as a sum of positive terms which is equal to zero if and only if $x = \alpha$) so Condition 3 holds. For Condition 4 and Condition 4 bis, see the proof in [11], p.44.

Proof of Proposition 4

As stated above, each coordinate of a deepest point $x^* := (x_1^*, \dots, x_d^*)$ is given by the optimization problem:

$$\forall i \in 1, \dots, d, \quad x_i^* = \operatorname{argmin}_{x \in \mathbb{R}} \sum_{j=1}^n |\Phi(i, j) - x|^p.$$

Since the function $x \mapsto x^p$ is strictly convex for all $p > 1$, it results that x_i^* exists and is unique for all $i = 1, \dots, d$.

Proof of Proposition 5

The proof is trivial since it is well-known that the quantity x minimizing $\sum_{j=1}^n |\Phi(i_0, j) - x|^2$ (resp. $\sum_{j=1}^n |\Phi(i_0, j) - x|$) is the quadratic mean (resp. the median) of $(\Phi(i_0, 1), \dots, \Phi(i_0, n))$. The same result holds for approval Voting, since it corresponds to Range Voting when the possible grades are restricted to 0 and 1.

Proof of Proposition 6

Consider the ordered grades $r_1 \leq r_2 \leq \dots \leq r_n$ of a given candidate c_i , $i = 1, \dots, d$. We want to prove that the point $x_0 = \frac{r_1 + r_n}{2}$ minimizes the function $g : x \mapsto \max_{j=1, \dots, n} |r_j - x|$. Remark that as we ordered the values, $g(x) = \max\{|r_1 - x|, |r_n - x|\}$, and thus

$$g(x) = \begin{cases} r_n - x > r_n - x_0 & \text{if } x < x_0 \\ x_0 - r_1 = r_n - x_0 & \text{if } x = x_0 \\ x - r_1 > x_0 - r_1 & \text{if } x > x_0. \end{cases}$$

Proof of Proposition 7

We present configurations such that

- a Condorcet winner can be not elected by wL^p Deepest Voting,
- a Condorcet loser can be elected by wL^p Deepest Voting.

We distinguish the two cases $p = 1$ and $p > 1$.

- Case $p=1$.

Let consider the following configuration with 9 voters and 3 candidates:

Number of voters	Grade for c_1	Grade for c_2	Grade for c_3
4	0.5	0.1	0.4
1	0.5	0.6	0.4
4	1	0.6	0.7

The corresponding scatterplot is given in Figure 3 taking $p_0 = 1/9$. As the median of grades for candidate c_2 is higher than the median for the other candidates, candidate c_2 is elected by wL^1 Deepest voting. However, c_1 is the Condorcet winner since an absolute majority of voters prefer c_1 to c_2 and c_3 . In the same time, c_2 is elected even if he is the Condorcet loser, as c_1 and c_3 are preferred to c_2 by a majority of voters.

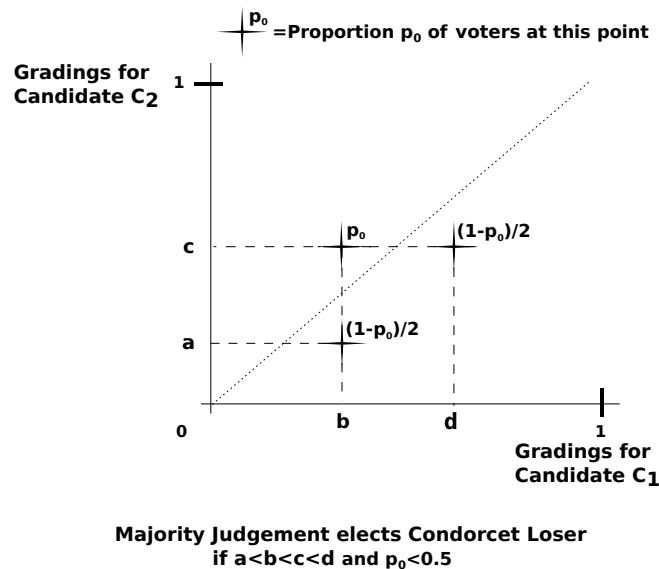


Figure 3: Configuration of grades considered in the proof of Proposition 7 in the case $p = 1$.

- Case $p > 1$.

Let consider the following profiles for 3 candidates and n voters:

Number of voters	Grade for c_1	Grade for c_2	Grade for c_3
$n - 1$	$0.5 + \varepsilon_{n,p}$	0.5	$0.5 + \varepsilon_{n,p}/2$
1	0	1	0

with $0 < \varepsilon_{n,p} \leq \min(0.5; (n - 1)^{\frac{-1}{p-1}})$. The corresponding scatterplot is given in Figure 4, taking $p_0 = 1/n$.

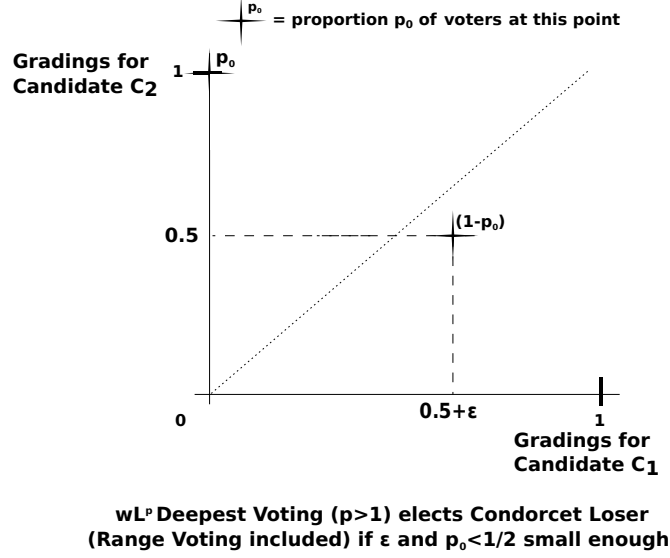


Figure 4: Configuration of grades considered in the proof of Proposition 7 in the case $p > 1$.

The coordinates of the deepest point $x_p^* = (x_{p,1}^*, x_{p,2}^*, x_{p,3}^*)$ are solutions of the following optimization problems:

$$x_{p,1}^* = \arg \min_{x \in [0,1]} (n-1)|x - (0.5 + \varepsilon_{n,p})|^p + x^p$$

$$x_{p,2}^* = \arg \min_{x \in [0,1]} (n-1)|x - 0.5|^p + |1 - x|^p$$

$$x_{p,3}^* = \arg \min_{x \in [0,1]} (n-1)|x - (0.5 + \varepsilon_{n,p}/2)|^p + x^p.$$

Using the first order conditions of the optimization problems, we get:

$$x_{p,1}^* = \frac{(0.5 + \varepsilon_{n,p})(n-1)^{\frac{1}{p-1}}}{(n-1)^{\frac{1}{p-1}} + 1}$$

$$x_{p,2}^* = \frac{1 + 0.5(n-1)^{\frac{1}{p-1}}}{1 + (n-1)^{\frac{1}{p-1}}}$$

$$x_{p,3}^* = \frac{(0.5 + \varepsilon_{n,p}/2)(n-1)^{\frac{1}{p-1}}}{(n-1)^{\frac{1}{p-1}} + 1}.$$

When $0 < \varepsilon_{n,p} \leq (n-1)^{\frac{-1}{p-1}}$, it follows that $x_{p,3}^* < x_{p,1}^* < x_{p,2}^*$. Consequently, candidate c_2 (who is the Condorcet loser) is elected by wL^p -deepest voting even if the Condorcet winner is c_1 .

Proof of Proposition 8

- Case $p = 1$

For an example proving the vulnerability of the Majority Judgment to reinforcement (resp. no-show) paradox, see Felsenthal and Machover, 2008, p. 327 (resp. p. 329) [8].

- Case $1 < p < 2$

Consider the following configuration of grades:

	Candidate c_1	Candidate c_2
Voter v_1	0.5	1
Voter v_2	0.5	ε
Voter v_3	0	ε

with $0 < \varepsilon < 1$. If only voters v_1 and v_2 are voting, then candidate c_2 wins, whatever $\varepsilon > 0$. Moreover, voter v_3 prefers candidate c_2 . Hence, to prove that no-show and reinforcement paradoxes hold, we may establish that considering voters $\{v_1, v_2, v_3\}$, candidate c_2 may loose. Denote $x^* := (x_1^*, x_2^*)$ the deepest point obtained for the wL^p depth with voters (v_1, v_2, v_3) . We aim at proving that $x_2^* < x_1^*$ for some $0 < \varepsilon < 0.5$.

For $j = 1, 2$, components of the deepest point x^* are obtained by

$$x_j^* = \operatorname{argmin}_{x \in \mathbb{R}} \sum_{i=1}^n |\Phi(i, j) - x|^p.$$

First order conditions for $j = 1$ lead to

$$2|x_1^* - 0.5|^{p-1} \operatorname{sign}(x_1^* - 0.5) + |x_1^*|^{p-1} \operatorname{sign}(x_1^*) = 0$$

Consequently, $0 < x_1^* < 0.5$ and we deduce that $x_1^* = 0.5 \frac{2^{1/(p-1)}}{1+2^{1/(p-1)}}$.

For $j = 2$ we get

$$2|\varepsilon - x_2^*|^{p-1} \operatorname{sign}(\varepsilon - x_2^*) + |1 - x_2^*|^{p-1} \operatorname{sign}(1 - x_2^*) = 0$$

Similarly, we deduce that $x_2^* = \frac{1+2^{1/(p-1)}\varepsilon}{1+2^{1/(p-1)}}$.

Thus

$$x_2^* < x_1^* \iff \varepsilon < 0.5 - 2^{-1/(p-1)}$$

Since the right-hand side is positive for $p < 2$, we deduce that there exists $\varepsilon > 0$ such that $x_2^* < x_1^*$. This concludes the proof.

- Case $p = 2$

wL^2 -depth deepest point is the point with coordinates equal to the mean of votes. Let \bar{x}_1 and \bar{y}_1 be the mean score obtained respectively by two candidate c_1 and c_2 on a population of voters of size n_1 and let \bar{x}_2 and \bar{y}_2 be the mean score obtained respectively by the two candidates c_1 and c_2 on a population of size n_2 . If $\bar{x}_1 < \bar{y}_1$ and $\bar{x}_2 < \bar{y}_2$, then $n_1\bar{x}_1 + n_2\bar{x}_2 < n_1\bar{y}_1 + n_2\bar{y}_2$. Hence, if candidate c_2 wins in the two sub-populations, he also wins on the total population. We can deduce that reinforcement paradox does not hold. Next, taking $n_2 = 1$, it implies that no-show paradox does not hold either.

- Case $2 < p$

For $0 < \varepsilon < 0.5$, consider the following grades:

	Candidate c_1	Candidate c_2
voter v_1	0	$0.5 + \varepsilon$
voter v_2	1	$0.5 + \varepsilon$
voter v_3	0	ε

Then if only voters v_1 and v_2 are voting, candidate c_2 wins because $0.5 + \varepsilon > 0.5$. Since v_3 prefers c_2 , to prove that no-show and reinforcement paradoxes hold, let's establish that considering voters $\{v_1, v_2, v_3\}$, candidate c_2 may loose.

Denote $x^* := (x_1^*, x_2^*)$ the deepest point obtained for the wL^p depth with voters (v_1, v_2, v_3) .

We aim at proving that $x_2^* < x_1^*$ for some $0 < \varepsilon < 0.5$.

For $j = 1, 2$, components of the deepest point x^* are obtained by

$$x_j^* = \operatorname{argmin}_x \sum_{i=1}^n |\Phi(i, j) - x|^p.$$

Differentiating the criterion with respect to x leads to

$$\sum_{i=1}^n |\Phi(i, j) - x|^{p-1} \operatorname{sign}(\Phi(i, j) - x).$$

First order conditions for $j = 1$ imply

$$2 |x_1^*|^{p-1} \operatorname{sign}(x_1^*) + |1 - x_1^*|^{p-1} \operatorname{sign}(1 - x_1^*) = 0.$$

Hence $0 < x_1^* < 1$ and we obtain that $x_1^* = \frac{1}{1+2^{1/(p-1)}}$.

First order conditions for $j = 2$ lead to

$$2 |0.5 + \varepsilon - x_2^*|^{p-1} \operatorname{sign}(0.5 + \varepsilon - x_2^*) + |\varepsilon - x_2^*|^{p-1} \operatorname{sign}(\varepsilon - x_2^*) = 0.$$

We deduce analogously that $x_2^* = 0.5 \frac{2^{1/(p-1)}}{1+2^{1/(p-1)}} + \varepsilon$.

Thus

$$\begin{aligned} x_2^* < x_1^* &\iff \varepsilon < \frac{1}{1+2^{1/(p-1)}} - \frac{1}{2} \frac{2^{1/(p-1)}}{1+2^{1/(p-1)}} \\ &\iff \varepsilon < \frac{2 - 2^{1/(p-1)}}{2(1+2^{1/(p-1)})}. \end{aligned}$$

Since the right-hand side is positive for $p > 2$, we deduce that there exists $\varepsilon > 0$ such that $x_2^* < x_1^*$. This concludes the proof.

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