A Class of Decidable Information Logics

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Abstract

For a class of propositional information logics defined from Pawlak's information systems, the validity problem is proved to be decidable using a significant variant of the standard filtration technique. Decidability is proved by showing that each logic has the strong finite model property and by bounding the size of the models. The logics in the scope of this paper are characterized by classes of Kripke-style structures with interdependent relations pairwise satisfying the Gargov's local agreement condition and closed under the so-called restriction operation. They include Gargov's data analysis logic with local agreement and Nakamura's logic of graded modalities. The last part of the paper is devoted to the definition of complete Hilbert-style axiomatizations for subclasses of the introduced logics, thus providing evidence that such logics are subframe logics in Wolter's sense.

Keywords: information system, multimodal logic, local agreement condition, finite model property, decidability

1 Introduction

During the last decade, the information logics derived from Pawlak's *information systems* [Paw81] have been the object of active research (see for example [Orło84, OP84, Orło85, FdCO85, Kon87, Orło88, Vak91b, Nak93, Bal96, Kon97]). The information systems have been proposed for the representation of knowledge by the introduction of the concept of a *rough set* leading to the notion of approximation of sets of objects by means of equivalence relations. The rough sets are based on the notion of *indiscernibility relations* that are binary relations identifying objects having the same description with respect to a given set of attributes. The indiscernibility relations are equivalence relations, and so the logics of indiscernibility relations can be viewed as multimodal logics for which the modal operators behave as the S5-modalities, *except* that they are usually interdependent. Numerous logics of this type have been studied in the past (see e.g., [Orło85, FdCO85, Orło89]) whereas

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the works [Paw82, Orł084] provide the logical foundations of the knowledge representation by means of the modal logics.

An *information system* (see e.g. [Paw91]) can be seen as a structure

$$(OB, AT, \{Val_{at} : at \in AT\}, f)$$

such that

- OB is a non-empty set of *objects*,
- AT is a non-empty set of *attributes*,
- for each $at \in AT$, Val_{at} is a non-empty set of *values* of at and,
- f is a mapping $OB \times AT \to \bigcup_{at \in AT} Val_{at}$ such that for all $(x, at) \in OB \times AT$, $f(x, at) \in Val_{at}$.

In that setting, two objects o_1, o_2 are said to be *indiscernible* with respect to a set of attributes $A \subseteq AT$ (in short o_1 $ind(A)$ o_2) iff

for all $at \in A$, $f(o_1, at) = f(o_2, at)$.

Different generalizations of the notion of information system (for instance by changing the profile of f with $\emptyset \neq f(o, at) \subseteq Val_{at}$) and various other relations between the objects (similarity, weak indiscernibility, ...) can be found for instance in [Vak91a]. The modal logics obtained from the information systems are multimodal logics such that the relations in the Kripke-style semantical structures correspond to relations between objects in the underlying information systems. Hence the relations are interdependent; for instance, if $B \subseteq A \subseteq AT$, then $ind(A) \subseteq ind(B)$. The decidability of the validity problem for various information logics has been an issue of interest in the past (see for example the valuable Vakarelov's contributions in [Vak91b, Vak91a, Vak97]). The aim of this paper is to prove that various information logics derived from Pawlak's information systems have a decidable validity problem by defining an original construction (see e.g. [Gab72, Seg71]). The decidability is proved by showing that each logic has the strong finite model property and by bounding the size of the models. The logics defined in [Nak93, Gar86] are used to illustrate the general construction.

The paper is structured as follows. In section 2, the class of LA-logics is defined by refining the *local agreement condition* defined in [Gar86]. Section 3 contains various filtration constructions for logics determined by classes of frames satisfying the local agreement condition. In Section 4, an original construction is presented in order to show that every LA-logic has the strong finite model property and we provide sufficient conditions so that the validity problem is decidable. In Section 5, we show how to apply the results of the previous section to logics defined in [Gar86, Nak93]. As a side-effect of our work, a sound and complete axiomatization is defined for the logic introduced in [Nak93]. In Section 6, complete Hilbert-style axiomatizations are defined for a particular class of LA-logics that happens to contain only *subframe logics* in Wolter's sense [Wol93]. In section 7, possible extensions of the present work are briefly discussed.

This paper is a corrected and full version of [Dem96a]. Section 3 and Section 6 are not contained in the short version.

2 The LA-logics

In the sequel, any (propositional) *modal language* $\mathtt{L}$ is determined by three sets which are supposed to be pairwise disjoint: the fixed countable set $\mathtt{F}_0 = \{p, q, \ldots\}$ of *propositional variables*, the set $\mathtt{M}$ of *modal expressions* and the set of *propositional operators* composed of the unary $\neg$ and the binary $\Leftrightarrow$, $\Rightarrow$, $\vee$, $\wedge$. The set $\mathtt{F}$ of *L-formulae* is the smallest set that satisfies the following conditions: $\mathtt{F}_0 \subseteq \mathtt{F}$; if $\ominus$ is any n-ary propositional operator and $A_1, \ldots, A_n \in \mathtt{F}$ then $\ominus(A_1, \ldots, A_n) \in \mathtt{F}$ and if $a \in \mathtt{M}$ and $A \in \mathtt{F}$ then $\{\Box_a A, \Diamond_a A\} \subseteq \mathtt{F}$ -$\Box_a$ and $\Diamond_a$ are called *modal operators*. Let $\mathtt{L}$ be a modal language[1]. We write $sub(A)$ (resp. $mw(A)$) to denote the set of *subformulae* of the formula A (resp. the *modal weight* of A, i.e. the number of occurrences of modal operators in A). We also write $sub_{\mathtt{M}}(A)$ to denote the set of modal expressions occurring in the formula A:

$$sub_{\mathtt{M}}(A) = \{a \in \mathtt{M} : \Diamond_a B \in sub(A)\} \cup \{a \in \mathtt{M} : \Box_a B \in sub(A)\}$$

As usual, by an *L-frame* we understand a pair $(W, (R_a)_{a \in \mathtt{M}})$ such that W is a non-empty set and for all $a \in \mathtt{M}$, R_a is a binary relation on W. We write $R_{|W'}$ to denote the restriction of the binary relation R to the set W', that is $R \cap (W' \times W')$. The set of $\mathtt{L}$-frames is denoted by $X^{\mathcal{F}}_{\mathtt{L}}$. An $\mathtt{L}$-frame $\mathcal{F}' = (W', (R'_a)_{a \in \mathtt{M}})$ is said to be a *subframe* of the $\mathtt{L}$-frame $(W, (R_a)_{a \in \mathtt{M}})$ iff $W' \subseteq W$ and for all $a \in \mathtt{M}$, $R'_a = (R_a)_{|W'}$. As usual, by an *L-model* we understand a triple $(W, (R_a)_{a \in \mathtt{M}}, V)$ such that $\mathcal{F} = (W, (R_a)_{a \in \mathtt{M}})$ is an $\mathtt{L}$-frame and V is a mapping $\mathtt{F}_0 \to \mathcal{P}(W)$, the power set of W. $\mathcal{M}$ is said to be *based on* $\mathcal{F}$. The class of $\mathtt{L}$-models is denoted by $mod_{\mathtt{L}}$. Let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V)$ be an $\mathtt{L}$-model. We say that a formula A is *satisfied by the object* $u \in W$ *in* $\mathcal{M}$ (denoted by $\mathcal{M}, u \models A$) when the following conditions are satisfied.

- $\mathcal{M}, u \models p$ iff $u \in V(p)$, for all $p \in \mathtt{F}_0$,
- $\mathcal{M}, u \models \neg A$ iff not $\mathcal{M}, u \models A$,
- $\mathcal{M}, u \models A \wedge B$ iff $\mathcal{M}, u \models A$ and $\mathcal{M}, u \models B$,
- $\mathcal{M}, u \models \Box_a A$ iff for all $v \in R_a(u)$, $\mathcal{M}, v \models A$ where $R_a(u) = \{v \in W : (u, v) \in R_a\}$,
- $\mathcal{M}, u \models \Diamond_a A$ iff there is $v \in R_a(u)$ such that $\mathcal{M}, v \models A$.

The conditions for the other logical operators correspond to their standard interpretation. Since the interpretation of $\Diamond_a$ can be *defined* in terms of $\Box_a$ (as usual, $\Diamond_a p \Leftrightarrow \neg\Box_a\neg p$ is satisfied in any model), in the sequel only the operators of the form $\Box_a$ are used when it is possible. A formula A is *true* in an $\mathtt{L}$-model $\mathcal{M}$ (denoted by $\mathcal{M} \models A$) iff for all $u \in W$, $\mathcal{M}, u \models A$.

In the sequel, by a *logic* $\mathcal{L}$, we understand a triple[2] $\langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ such that $\mathtt{L}$ is a modal language, $\mathcal{S} \subseteq mod_{\mathtt{L}}$ and $\models_{\mathcal{L}}$ is the restriction of $\models$ to the sets $\mathcal{S}$ and $\mathtt{L}$ (satisfiability

[1] In the rest of the paper, the symbols A, B (possibly decorated) are used to denote formulae, the symbols a, b (possibly decorated) are used to denote modal expressions, the symbols Γ, X (possibly decorated) are used to denote sets of formulae and the symbol Y (possibly decorated) is used to denote sets of modal expressions. Moreover the symbol $\mathcal{M}$ (possibly decorated) is used to denote models, the symbol $\mathcal{F}$ (possibly decorated) is used to denote frames, the symbol $\mathcal{L}$ (possibly decorated) is used to denote logics, the symbol $\mathcal{S}$ (possibly decorated) is used to denote sets of models and the symbol $X^{\mathcal{F}}$ (possibly decorated) is used to denote sets of frames.

[2] It is possible to define a logic in terms of $\mathtt{L}$-frames but the definition of logic used in the paper is sufficient for our needs.

relation). For all models $\mathcal{M} \in \mathcal{S}$, $\mathcal{M}$ is said to be a *model for* $\mathcal{L}$. An $\mathtt{L}$-formula A is said to be $\mathcal{L}$-*valid* iff A is true in all $\mathtt{L}$-models of $\mathcal{S}$. An $\mathtt{L}$-formula A is said to be $\mathcal{L}$-*satisfiable* iff there exist $\mathcal{M} = (W, (R_a)_{a\in\mathtt{M}}, V) \in \mathcal{S}$ and $u \in W$ such that $\mathcal{M}, u \models_{\mathcal{L}} A$. A logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ has the *strong finite model property* iff there is an effective procedure $g : \mathtt{F} \to \omega$ such that for every $\mathcal{L}$-satisfiable formula A, there exist $\mathcal{M} = (W, (R_a)_{a\in\mathtt{M}}, V) \in \mathcal{S}$ and $w \in W$ such that W is finite, $\mathcal{M}, w \models_{\mathcal{L}} A$ and $card(W) \leq g(A)$. As usual, an *instance of the validity (resp. satisfiability) problem for* $\mathcal{L}$ consists in the question: is the $\mathtt{L}$-formula A $\mathcal{L}$-valid (resp. $\mathcal{L}$-satisfiable)? It is immediate that the validity problem for $\mathcal{L}$ is decidable iff the satisfiability problem for $\mathcal{L}$ is decidable.

Definition 2.1. A logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ is said to be an *LA-logic* iff there is a set of linear[3] orders over $\mathtt{M}$, say $lo(\mathcal{L})$, such that for all $\mathtt{L}$-models $\mathcal{M} = (W, (R_a)_{a\in\mathtt{M}}, V)$, $\mathcal{M} \in \mathcal{S}$ iff

1. for all $a \in \mathtt{M}$, R_a is an equivalence relation and,
2. for all $u \in W$, there is $\underline{\leq} \in lo(\mathcal{L})$ such that for all $a, b \in \mathtt{M}$, if $a\underline{\leq}b$ then $R_a(u) \subseteq R_b(u)$.

The set $lo(\mathcal{L})$ is said to be the set of *local agreements* of $\mathcal{L}$. ∇

By the condition 1, the relations of the models can be interpretated as indiscernibility relations between objects of some information systems. Hence the modal operators behave as in the modal logic S5. The condition 2. is trickier since it states that *locally* the relations in the family $(R_a)_{a\in\mathtt{M}}$ can be linearly ordered with respect to the set inclusion $\subseteq$. However the different possible ways of ordering are fixed for each LA-logic. Condition 2. can also be interpretated in terms of indiscernibility relations. Let $(OB, AT, \{Val_{at} : at \in AT\}, f)$ be an information system and $\mathtt{AT} \subseteq \mathcal{P}(AT)$ such that for all $\underline{a}, \underline{b} \in \mathtt{AT}$, either $\underline{a} \subseteq \underline{b}$ or $\underline{b} \subseteq \underline{a}$. By writing $\underline{a\leq b}$ to denote $\underline{b} \subseteq \underline{a}$, we have:

for all $o \in OB$, if $\underline{a\leq b}$ then $ind(\underline{a})(o) \subseteq ind(\underline{b})(o)$

As a straightforward consequence of Definition 2.1, each non-empty set lo of linear orders over $\mathtt{M}$ defines a *unique* LA-logic $\mathcal{L}$ such that $lo(\mathcal{L}) = lo$.

Example 2.1. Let $\mathcal{L}' = \langle \mathtt{L}', \mathcal{S}', \models_{\mathcal{L}'} \rangle$ be the LA-logic such that $\mathtt{M}' = \{\mathtt{1}, \mathtt{2}\}$ and for all $\mathtt{L}'$-models $\mathcal{M} = (W, (R_i)_{i\in\{\mathtt{1},\mathtt{2}\}}, V)$, $\mathcal{M} \in \mathcal{S}'$ iff $R_{\mathtt{1}}$ and $R_{\mathtt{2}}$ are equivalence relations and $R_{\mathtt{1}} \subseteq R_{\mathtt{2}}$. $\mathcal{L}'$ is an example of LA-logic where $lo(\mathcal{L}')$ is a singleton consisting of a single linear order $\underline{\leq}$ such that $\mathtt{1}\underline{\leq}\mathtt{2}$. The set $\mathcal{S}'$ can be related to the set of information systems as follows. Let $(OB, AT, \{Val_{at} : at \in AT\}, f)$ be an information system and $\emptyset \neq AT' \subseteq AT$. The $\mathtt{L}'$-model $(OB, (R_i)_{i\in\{\mathtt{1},\mathtt{2}\}}, V)$ with

$(\star)$ $R_{\mathtt{1}} = ind(AT)$ and $R_{\mathtt{2}} = ind(AT')$

belongs to $\mathcal{S}'$. Moreover for all $\mathcal{M} = (W, (R_i)_{i\in\{\mathtt{1},\mathtt{2}\}}, V) \in \mathcal{S}'$, there exist an information system $(OB, AT, \{Val_{at} : at \in AT\}, f)$ and $\emptyset \neq AT' \subseteq AT$ such that $(\star)$ holds. Actually, take

[3] A linear order $\underline{\leq}$ is a binary relation over W such that $\underline{\leq}$ is reflexive, transitive, totally connected (for all $x, y \in W$ either $(x, y) \in \underline{\leq}$ or $(y, x) \in \underline{\leq}$) and antisymmetric (for all $x, y \in W$ if $(x, y) \in \underline{\leq}$ and $(y, x) \in \underline{\leq}$ then $x = y$).

- $OB = W$, $AT = \{at, at'\}$,
- $Val_{at} = \{R_1(x) : x \in W\}$, $Val_{at'} = \{R_2(x) : x \in W\}$,
- for all $x \in W$, $f(x, at) = R_1(x)$, $f(x, at') = R_2(x)$ and $AT' = \{at'\}$.

The term 'LA-logic' refers to the local agreement condition defined in [Gar86]. Two relations R and S on a set W are said to be in *local agreement* (LA) iff

for all $u \in W$ either $R(u) \subseteq S(u)$ or $S(u) \subseteq R(u)$

It is easy to show that for any LA-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$, for any model $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V) \in \mathcal{S}$, for all $a, b \in \mathtt{M}$, R_a and R_b are in local agreement. The property stated in Proposition 2.1 might explain why the local agreement condition has been introduced in [Gar86].

Proposition 2.1. Let R and S be two equivalence relations on a set W. Then the following conditions are equivalent:

1. R and S are in local agreement.
2. $R \cup S$ is transitive (i.e., $R \cup S$ is an equivalence relation).

Proof: (1) $\rightarrow$ (2). Assume $(x, y) \in R$ and $(y, z) \in S$. If $R(x) \subseteq S(x)$ then by transitivity of S, $(x, z) \in S$. Now assume $S(x) \subseteq R(x)$. If $S(y) \subseteq R(y)$ then by transitivity of R, $(x, z) \in R$. Now assume $R(y) \subseteq S(y)$. Since R and S are equivalence relations, $R(x) = R(y)$, $S(y) = S(z)$ and therefore $S(x) \subseteq R(x) = R(y) \subseteq S(y) = S(z)$. So $(x, z) \in S$. The case $(x, y) \in S$ and $(y, z) \in R$ is symmetric. Since R and S are transitive, $R \cup S$ is therefore transitive.

$\neg$ (1) $\rightarrow \neg$ (2). Assume there are $x_0, y_0, z_0 \in W$ such that $(x_0, y_0) \in R$, $(x_0, y_0) \notin S$ (hence $R(x_0) \not\subseteq S(x_0)$), $(x_0, z_0) \in S$ and $(x_0, z_0) \notin R$ (hence $S(x_0) \not\subseteq R(x_0)$). It can be shown that $(y_0, z_0) \notin R \cup S$ (and therefore $R \cup S$ is not transitive since $(y_0, x_0) \in R$ and $(x_0, z_0) \in S$). Suppose $(y_0, z_0) \in R$. By transitivity of R, $(x_0, z_0) \in R$ which leads to a contradiction. Now suppose $(y_0, z_0) \in S$. By symmetry and transitivity of S, $(x_0, y_0) \in S$ which also leads to a contradiction. **Q.E.D.**

The property stated in Fact 2.2 below shall be needed in the sequel.

Fact 2.2. Let $(R_i)_{i \in \{1,\ldots,n\}}$ be a finite family of relations on the set W such that the relations are pairwise in local agreement. Then the following conditions are satisfied:

1. for all $i \in \{1, \ldots, n\}$, for all $W' \subseteq W$,
   (a) if R_i is reflexive (resp. symmetrical, transitive) then $(R_i)_{|W'}$ is reflexive (resp. symmetrical, transitive).
   (b) for all $j \in \{1, \ldots, n\}$, for all $x \in W'$, if $R_i(x) \subseteq R_j(x)$ then $(R_i)_{|W'}(x) \subseteq (R_j)_{|W'}(x)$
2. for all $x \in W$, there is a permutation s of $\{1, \ldots, n\}$ such that $R_{s(1)}(x) \subseteq \ldots \subseteq R_{s(n)}(x)$.

3 Standard Filtrations and the Local Agreement Condition

In this section, we shall give some hints in order to understand why the standard filtration constructions cannot be applied straightforwardly to the LA-logics.

3.1 The Filtration Technique

Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ be a logic (not necessarily an LA-logic), let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V)$ be an $\mathtt{L}$-model and Γ be a set of formulae closed under subformulae (that is $\bigcup\{sub(B) : B \in \Gamma\} = \Gamma$). As usual, the relation $\equiv_\Gamma$ on W is defined by:

$$\forall x, y \in W,\ x \equiv_\Gamma y \ iff\ \forall B \in \Gamma,\ \mathcal{M}, x \models B \ iff\ \mathcal{M}, y \models B$$

$\equiv_\Gamma$ is an equivalence relation. For each equivalence relation $\equiv \subseteq \equiv_\Gamma$ on W, we write $|x|$ to denote the set $\{y \in W : x \equiv y\}$. The model $\mathcal{M}' = (W', (R'_a)_{a \in \mathtt{M}}, V')$ is said to be a $\Gamma(\equiv)$-*filtration* of $\mathcal{M}$ (see e.g. [Seg71, HC84, Gol92]) iff

1. $W' = \{|x| : x \in W\}$, $V'(p) = \{|x| : x \in V(p)$ *and* $p \in \Gamma\}$ for all $p \in \mathtt{F}_0$,
2. for all $a \in \mathtt{M}$,
   (a) if xR_ay then $|x|R'_a|y|$,
   (b) if $|x|R'_a|y|$ and $\mathcal{M}, x \models \Box_a B$ for some $\Box_a B \in \Gamma$ then $\mathcal{M}, y \models B$.

When the equivalence relation $\equiv$ is equal to $\equiv_\Gamma$ we use the standard term 'Γ-filtration'. Proposition 3.1 slightly generalizes the standard result about filtrations (see e.g. [Gol92]):

Proposition 3.1. Let Γ be a set of formulae closed under formulae and let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V)$ be an $\mathtt{L}$-model. Then for each equivalence relation $\equiv \subseteq \equiv_\Gamma$ on W, for all $\Gamma(\equiv)$-filtrations $\mathcal{M}' = (W', (R'_a)_{a \in \mathtt{M}}, V')$ of $\mathcal{M}$,

$$\forall B \in \Gamma,\ \forall x \in W,\ \mathcal{M}, x \models B \ iff\ \mathcal{M}', |x| \models B$$

The proof is by induction of the size of the formulae. The filtration technique has been extensively used to prove the strong finite model property since if Γ is finite so is W' ($card(W') \leq 2^{card(\Gamma)}$)[4] when $\mathcal{M}' = (W', (R'_a)_{a \in \mathtt{M}}, V')$ is a Γ-filtration of $\mathcal{M}$. For instance take the logic $S5_k$ for some $k \geq 1$ ($S5_k$ has k distinct necessity modal operators). Let $\mathcal{M} = (W, (R_i)_{i \in \{1,\ldots,k\}}, V)$ be an $S5_k$-model, $x_0 \in W$ and A be a formula such that $\mathcal{M}, x_0 \models A$. Take $\Gamma = sub(A)$ and $\mathcal{M}' = (W', (R'_i)_{i \in \{1,\ldots,k\}}, V')$ with W' and V' defined as above with $\equiv_\Gamma$ and for all $i \in \{1, \ldots, k\}$, $|x|R'_i|y|$ iff for all $\Box_i B \in \Gamma$, $\mathcal{M}, x \models \Box_i B$ iff $\mathcal{M}, y \models \Box_i B$. It can be shown that the R'_i's are equivalence relations and $\mathcal{M}'$ is a Γ-filtration of $\mathcal{M}$. As a consequence, $S5_k$ has the strong finite model property and the validity problem for $S5_k$ is decidable.

It would be nice to use this construction to prove the decidability of the validity problem for the LA-logics. However the following example definitely invalidates our first hope since *the local agreement condition is not preserved by the filtration construction for* $S5_k$. Let $A = \Box_a p \wedge \Box_b q$, $\Gamma = sub(A)$, and consider the model $\mathcal{M} = (\{1, 2, 3\}, (R_{a'})_{a' \in \mathtt{M}}, V)$ such that

[4] For any finite set U, $card(U)$ denotes the cardinality of U.

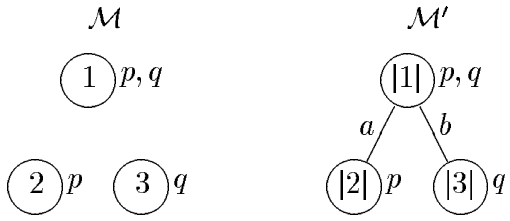


Figure 1: A simple filtration construction not preserving the local agreement condition

- $R_a = R_b = \{(1,1),(2,2),(3,3)\}$,
- $V(p) = \{1,2\}$, $V(q) = \{1,3\}$.

Obviously, $\mathcal{M}, 1 \models A$ and R_a and R_b are in local agreement. Using the filtration construction for $S5_k$, we get the Γ-filtration $\mathcal{M}' = (\{|1|,|2|,|3|\}, (R'_{a'})_{a' \in \mathtt{M}}, V')$ of $\mathcal{M}$ with

- $R'_a = \{(|1|,|1|),(|2|,|2|),(|3|,|3|),(|1|,|2|),(|2|,|1|)\}$,
- $R'_b = \{(|1|,|1|),(|2|,|2|),(|3|,|3|),(|1|,|3|),(|3|,|1|)\}$,
- $V'(p) = \{|1|,|2|\}$ and $V'(q) = \{|1|,|3|\}$.

Although $\mathcal{M}', |1| \models A$ ($\mathcal{M}'$ is a Γ-filtration of $\mathcal{M}$), R'_a and R'_b are *not* in local agreement (see Figure 1 -the reflexive closure of the relations is omitted in the figure). One can easily find examples where the local agreement condition is not preserved with the set Γ defined as follows:

$$\Gamma = sub(A) \cup \{\Box_a B : \Box_b B \in sub(A), a \in sub_{\mathtt{M}}(A)\}$$

In order to prove the decidability of the LA-logics, modifications might be operated about either the definition of Γ or the construction of the R'_a's or the definition of $|x|$ or another type of construction has to be introduced. The last possibility is developed in Section 4 whereas we have been unsuccessful with the first one. Before presenting our restriction construction, we would like to emphasize that the very problem lies in the fact that the relations of the models are equivalence relations being pairwise in local agreement. When we use the local agreement condition with weaker conditions on the relations (for instance with only reflexivity or symmetry) the usual filtration construction can be adequately adapted which is shown in Section 3.2 below. For instance, consider the multimodal logic $T_{\mathtt{M}}$ that extends the modal logic T (see e.g. [HC68]) such that $\mathtt{M}$ is the set of modal expressions and the relations in the $T_{\mathtt{M}}$-models are in local agreement. Then the *minimal* filtration construction suffices to show the decidability of the satisfiability problem for $T_{\mathtt{M}}$. *However the results in Section 3.2 do not apply to the LA-logics.*

3.2 Minimal and Maximal Filtrations

Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models \rangle$ be a *monomodal* logic ($card(\mathtt{M}) = 1$) *determined* by the set of *frames* $X^{\mathcal{F}}$, that is for all $\mathtt{L}$-models $\mathcal{M} = (W, R, V)$, $\mathcal{M} \in \mathcal{S}$ iff $(W, R) \in X^{\mathcal{F}}$ (following the terminology

in [HC84] for instance). So, we do not necessarily assume that the binary relations of the models are equivalence relations. $\mathcal{L}$ denotes a monomodal logic in the rest of the section.

The logic $\mathcal{L}' = \langle \mathtt{L}', \mathcal{S}', \models' \rangle$ is said to be an *LA-$\mathcal{L}$-logic* iff there is a set $lo(\mathcal{L}')$ of linear orders over $\mathtt{M}'$ such that for all $\mathtt{L}'$-models $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}'}, V)$, $\mathcal{M} \in \mathcal{S}'$ iff

1. for all $a \in \mathtt{M}'$, $(W, R_a) \in X^{\mathcal{F}}$ (the condition 1. in Definition 2.1 is replaced by the present one),
2. for all $u \in W$, there is $\leq \in lo(\mathcal{L}')$ such that for all $a, b \in \mathtt{M}'$, if $a \leq b$ then $R_a(u) \subseteq R_b(u)$.

It is immediate that the class of LA-logics is the class of LA-S5-logics. Different monomodal logics satisfying the hypothesis in Proposition 3.2 below can be found in [HC84] (K,T,B).

Proposition 3.2. Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models \rangle$ be a monomodal logic determined by the class of frames $X^{\mathcal{F}}$ such that it is decidable whether $(W, R) \in X^{\mathcal{F}}$ for all finite frames (W, R). We assume that for all sets of formulae Γ closed under subformulae, for all $\mathcal{M} \in \mathcal{S}$, the minimal[5] (resp. the maximal[6]) $\Gamma(\equiv)$-filtration $\mathcal{M}'$ of $\mathcal{M}$ belongs to $\mathcal{S}$ where $\equiv$ is an equivalence relation included in $\equiv_\Gamma$. Then every LA-$\mathcal{L}$-logic $\mathcal{L}' = (\mathtt{L}', \mathcal{S}', \models')$ having a finite set $\mathtt{M}'$ of modal expressions has the strong finite model property and the validity problem is decidable.

The hypothesis of Proposition 3.2 are crucial. For instance the modal logic S5 does not satisfy the hypothesis since neither the maximal filtration construction nor the minimal filtration construction preserves systematically reflexivity, symmetry and transitivity. Hence the LA-logics are not in the scope of Proposition 3.2.

Proof: Let $\mathtt{M}' = \{a_1, \ldots, a_n\}$ and Σ be the set of permutations on $\mathtt{M}'$. Let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}'}, V) \in \mathcal{S}'$, $A \in \mathtt{F}$, $x_0 \in W$ be such that $\mathcal{M}, x_0 \models A$. We write Γ to denote the set

$$\Gamma = \{\Box_a B : \exists\, \Box_b B \in sub(A), a \in \mathtt{M}'\} \cup sub(A)$$

closed under subformulae. We also write $\equiv_0$ to denote the binary relation on W such that for all $x, y \in W$, $x \equiv_0 y$ iff $x \equiv_\Gamma y$ and

$$\{s \in \Sigma : R_{s(a_1)}(x) \subseteq \ldots \subseteq R_{s(a_n)}(x)\} = \{s \in \Sigma : R_{s(a_1)}(y) \subseteq \ldots \subseteq R_{s(a_n)}(y)\}$$

Then $\equiv_0$ is an equivalence relation, and we write $|x|$ to denote the equivalence class that contains $x \in W$. Let $\mathcal{M}' = (W', (R'_a)_{a \in \mathtt{M}'}, V')$ be the minimal (resp. the maximal) $\Gamma(\equiv_0)$-filtration of $\mathcal{M}$. Observe that W' is finite,

$$card(W') \leq 2^{(card(sub(A)) \times card(\mathtt{M}'))+n!} \tag{1}$$

Now we prove that

($\star$) for all $a, b \in \mathtt{M}'$ and all $x \in W$, if $R_a(x) \subseteq R_b(x)$ then $R'_a(|x|) \subseteq R'_b(|x|)$.

[5] For all $x, y \in W$, $a \in \mathtt{M}$, $|x| R'_a |y|$ iff $\exists x' \in |x|, y' \in |y|$ such that $x' R_a y'$.

[6] For all $x, y \in W$, $a \in \mathtt{M}$, $|x| R'_a |y|$ iff for all $\Box_a A \in \Gamma$, if $\mathcal{M}, x \models \Box_a A$ then $\mathcal{M}, y \models A$.

Remember that for all $x \in W$, there is $\leq_x \in lo(\mathcal{L}')$ such that for all $a, b \in \mathtt{M}'$, if $a \leq_x b$ then $R_a(x) \subseteq R_b(x)$. By proving $(\star)$ we establish that for all $|x| \in W'$, for all $a, b \in \mathtt{M}'$, if $a \leq_x b$ then $R'_a(|x|) \subseteq R'_b(|x|)$. Moreover for all $a \in \mathtt{M}'$, (W', R'_a) belongs to $X^{\mathcal{F}}$ by the property of the filtration in the monomodal case. So $\mathcal{M}' \in \mathcal{S}'$, W' is finite and $\mathcal{M}', |x_0| \models A$ (by Proposition 3.1). Hence $\mathcal{L}'$ has the strong finite model property and the validity problem for $\mathcal{L}'$ is decidable. Indeed $card(W')$ is bounded by (1) and it is decidable whether $\mathcal{M}' \in \mathcal{S}'$ for all finite $\mathtt{L}'$-models $\mathcal{M}'$.

It remains to prove $(\star)$. Assume $R_a(x) \subseteq R_b(x)$ and suppose there is $|y| \in W'$ such that $|x| R'_a |y|$ and not $|x| R'_b |y|$. If the minimal filtration is used to build $\mathcal{M}'$ then there exist no $x' \in |x|$ and no $y' \in |y|$ such that $x' R_b y'$. Since $|x| R'_a |y|$, there exist $x'_0 \in |x|$ and $y'_0 \in |y|$ such that $x'_0 R_a y'_0$. Since $|x'_0| = |x|$ then $R_a(x'_0) \subseteq R_b(x'_0)$ and $x'_0 R_b y'_0$ which leads to a contradiction. In case the maximal filtration is used to build $\mathcal{M}'$, there is $\Box_b B \in \Gamma$ such that $\mathcal{M}, x \models \Box_b B$ and not $\mathcal{M}, y \models B$. Since $R_a(x) \subseteq R_b(x)$, then $\mathcal{M}, x \models \Box_a B$. By construction of Γ, $\Box_a B \in \Gamma$, so $|x| R'_a |y|$ and $\mathcal{M}, y \models B$ which leads to a contradiction. **Q.E.D.**

Simple counterexamples can be found in order to show that Proposition 3.2 does not hold when the *transitive filtration* or the filtration for S5 are used which reduces our chance to use the standard filtration construction for the LA-logics. However, a sufficient condition to extend Proposition 3.2 to infinite $\mathtt{M}'$ would be to prove that for each LA-$\mathcal{L}$-logic $\mathcal{L}' = \langle \mathtt{L}', \mathcal{S}', \models' \rangle$ ($\mathcal{L}$ satisfying the hypothesis of Proposition 3.2) the following holds: for each finite set $Y = \{a_1, \ldots, a_n\} \subseteq \mathtt{M}'$, for all structures $(W, (R_a)_{a \in Y})$, if for all $u \in W$,

$$\exists \leq \in lo(\mathcal{L}'), \forall a, b \in Y \; if \; (a(\leq_{|Y})b) \; then \; R_a(u) \subseteq R_b(u)$$

and for all $a \in Y$, $(W, R_a) \in X^{\mathcal{F}}$ then there exists $\mathcal{M}' = (W, (R'_a)_{a \in \mathtt{M}'}, V) \in \mathcal{S}'$ such that for all $a \in Y$, $R'_a = R_a$ (to be related to Definition 4.2). We were only able to prove this property when $card(lo(\mathcal{L})) = 1$ or when $\mathcal{L}'$ is either an LA-K-logic or an LA-KT-logic.

4 A Restriction Construction for the LA-logics

The aim of this section is to show that every LA-logic has the strong finite model property. Although the modal operators for each LA-logic behave as modal operators for S5, the usual filtration construction for the multimodal logics $S5_k$ cannot be used straightforwardly for the LA-logics (see Section 3). Instead of defining equivalence classes of worlds (as done in the standard filtration constructions), restrictions of models are used. In the literature restrictions are defined for instance in [Cer94, HM92]. With such a construction, no new *arrow* is added, that is if x and y are in the relation R'_a of the restricted model then x and y are in the relation R_a of the initial model. This does not always hold when filtrations are involved.

In the rest of this section $\mathcal{L}$ denotes an LA-logic $\langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ unless otherwise stated. Let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V) \in \mathcal{S}$ and $\emptyset \neq W' \subseteq W$. The *restriction*[7] of $\mathcal{M}$ to W', denoted by $\mathcal{M}_{|W'}$, is the $\mathtt{L}$-model $(W', (R'_a)_{a \in \mathtt{M}}, V')$ such that for all $a \in \mathtt{M}$, $R'_a = (R_a)_{|W'}$ and for all $p \in \mathtt{F}_0$, $V'(p) = V(p) \cap W'$. Proposition 4.1 below states that the class of models for an LA-logic is closed under the restriction operation.

[7] $\mathcal{M}_{|W'}$ is also called a *submodel* of $\mathcal{M}$.

Proposition 4.1. For all $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V) \in \mathcal{S}$ and $\emptyset \neq W' \subseteq W$, $\mathcal{M}_{|W'} \in \mathcal{S}$.

Proof: Direct consequence of Fact 2.2. **Q.E.D.**

Proposition 4.2 below states that in a model, if $R_{a_k}(x) = R_{a_k}(y)$ then the linear orders associated to x and y are not independent.

Proposition 4.2. Let $\{a_1, \ldots, a_n\} \subseteq \mathtt{M}$, $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V) \in \mathcal{S}$, $x \in W$ be such that $R_{a_1}(x) \subseteq \ldots \subseteq R_{a_n}(x)$. Assume $(x, y) \in R_{a_k}$ for some $k \in \{1, \ldots, n\}$. Then,

1. for all $k' \in \{k, \ldots, n\}$, $R_{a_{k'}}(x) = R_{a_{k'}}(y)$.
2. for all $k' \in \{1, \ldots, k-1\}$, $R_{a_{k'}}(y) \subseteq R_{a_k}(y)$.

Proof: (1) Since $R_{a_k}(x) \subseteq R_{a_{k'}}(x)$ then $(x, y) \in R_{a_{k'}}$ and $R_{a_{k'}}(x) = R_{a_{k'}}(y)$ since $R_{a_{k'}}$ is an equivalence relation. (2) Similar to (1). **Q.E.D.**

For all $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V) \in \mathcal{S}$, for all L-formulae A and all $w \in W$, if for some $a \in sub_{\mathtt{M}}(A)$, for all $b \in sub_{\mathtt{M}}(A)$, $R_b(w) \subseteq R_a(w)$ then

$$\mathcal{M}, w \models A \; iff \; \mathcal{M}_{|R_a(w)}, w \models A$$

Although all the LA-logics satisfy the property above, there exist logics that do not satisfy it. For the LA-logics, it is easy to check that

$$(\bigcup_{b \in sub_{\mathtt{M}}(A)} R_b)^*(w) = R_a(w)$$

when for some $a \in sub_{\mathtt{M}}(A)$, for all $b \in sub_{\mathtt{M}}(A)$, $R_b(w) \subseteq R_a(w)$.

For any finite sequence of natural numbers σ, we write $set(\sigma)$ (resp. $|\sigma|$) to denote the set of elements occurring in σ (resp. the length of σ). For example $set((1,2,3,3,4)) = \{1,3,2,4\}$. As usual, $\sigma_1.\sigma_2$ denotes the concatenation of two sequences.

4.1 The Construction

Let A be an L-formula, $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V) \in \mathcal{S}$, $w \in W$ such that $\mathcal{M}, w \models A$. Assume that $sub_{\mathtt{M}}(A) = \{a_1, \ldots, a_n\}$ with $R_{a_1}(w) \subseteq \ldots \subseteq R_{a_n}(w)$ (see Fact 2.2(2)). We shall construct a set $W' \subseteq W$ such that

- $w \in W'$,
- W' is finite and,
- $\mathcal{M}_{|W'}, w \models A$.

For the sake of clarity of exposition, first the construction is informally discussed and then the formal definitions follow (a simple example shall be also given).

To build such a set W', we first consider the set Nec of necessity formulae $\Box_b A'$ such that $\Box_a A' \in sub(A)$ for some modal expression b occurring in A. Then the construction of W' is done recursively, that is $W' = \bigcup_{0 \leq i \leq \alpha} W'_i$ for some $0 \leq \alpha \leq n$ where each W'_i is finite. W'_0 is initialized to $\{w\}$ and then W'_{i+1} is defined from W'_i. If a formula in Nec is not satisfied at some element of W'_i, we add a *witness* of this fact in W'_{i+1} ($\mathcal{M}, u \not\models \Box_a A'$

iff there exists u' such that $(u,u') \in R_a$ and $\mathcal{M}, u' \not\models A'$). Moreover, if $\mathcal{M}, u \not\models \Box_{a_i} A'$, $\mathcal{M}, u \not\models \Box_{a_j} A'$ and $R_{a_i}(u) \subseteq R_{a_j}(u)$ then a *single* witness u' needs to be considered satisfying $(u,u') \in R_{a_i}$ and $\mathcal{M}, u' \models A'$. The set of necessity formulae $\mathtt{Nec}_u^{(\ldots,i,\ldots,j,\ldots)}$ is introduced and it contains only the elements of $\mathtt{Nec}$ that require a witness for u. This is an optimization of the construction.

The last point that deserves to be explained is how to end the construction (for the multi-modal logics $S5_k$, $k \geq 2$, we would not know when to terminate the process). Assume that for some $u \in W'_l$, $\mathcal{M}, u \not\models \Box_{a_i} A'$ requires a witness and $R_{a_i}(u) \subseteq R_{a_j}(u)$. There exists $u' \in W'_{l+1}$ such that $\mathcal{M}, u' \not\models A'$ and $(u,u') \in R_{a_i}$. We can show that if $\mathcal{M}, u' \not\models \Box_{a_j} A''$ for some $\Box_{a_j} A'' \in \mathtt{Nec}$, there is no need to consider a *new* witness. Indeed, there exists u'' such that $(u',u'') \in R_{a_j}$ and $\mathcal{M}, u'' \not\models A''$. Since R_{a_j} is an equivalence relation, $(u,u'') \in R_{a_j}$ and therefore $\mathcal{M}, u \not\models \Box_{a_j} A''$. If the set W'_{l+1} has been properly built (this point should become clear in the formal definition), there exists $v \in W'_{l+1}$ such that $(u,v) \in R_{a_j}$ and $\mathcal{M}, v \not\models A''$. Since $(u',v) \in R_{a_j}$, v is *already* a witness for $\mathcal{M}, u' \not\models \Box_{a_j} A''$. This observation allows us to find $\alpha \leq n$ such that for all $u' \in W'_\alpha$, no witness is needed. The crucial point in the development above is the hypothesis $R_{a_i}(u) \subseteq R_{a_j}(u)$. In case $R_{a_j}(u) \subset R_{a_i}(u)$ (strict inclusion), we may have to introduce a new witness for $\mathcal{M}, u' \not\models \Box_{a_j} A''$.

We shall give in the sequel the formal definitions. The set $\mathtt{Nec}$ (which was related to the construction of the set Γ in Section 3.2) is defined by

$$\mathtt{Nec} = \{\Box_b A' : \exists\, \Box_a\ A' \in sub(A), \exists b \in sub_{\mathtt{M}}(A)\}$$

For all $x \in W$, for all sequences $\sigma = (j_1, \ldots, j_k)$ such that $set(\sigma) \subseteq \{1, \ldots, n\}$ and $R_{a_{j_1}}(x) \subseteq \ldots \subseteq R_{a_{j_k}}(x)$, the set $\mathtt{Nec}_x^\sigma$ is defined as follows,

$$\mathtt{Nec}_x^\sigma = \{\Box_{a_{j'_k}} A' \in \mathtt{Nec} : \exists k' \in \{1, \ldots, k\},\ \mathcal{M}, x \models \neg\, \Box_{a_{j'_k}}\ A',\ and \\ if\ k' \geq 2\ then\ \mathcal{M}, x \models \Box_{a_{j_{k'-1}}} A'\}$$

Observe that $card(\mathtt{Nec}) \leq n \times mw(A)$, $card(\mathtt{Nec}_x^\sigma) \leq mw(A)$ and $\mathtt{Nec}_x^\sigma = \emptyset$ when σ is the empty sequence -denoted by Λ. Roughly speaking, $\mathtt{Nec}$ is the set of *necessity* formulae $\Box_a A'$ occurring in A with their copies $\Box_b A'$ for all the indices b occurring in A. The set $\mathtt{Nec}_x^\sigma$ contains the elements $\Box_a A'$ of $\mathtt{Nec}$ such that there is $u \in W$ with $\mathcal{M}, u \not\models A'$ and $(x,u) \in R_a$ (there is $i \in set(\sigma)$ such that $a_i = a$ and i is *minimal* in σ). In that way, if $a, b \in \mathtt{M}$, $\mathcal{M}, x \not\models \Box_a A'$ and $\mathcal{M}, x \not\models \Box_b A'$ then $card(\mathtt{Nec}_x^\sigma \cap \{\Box_a A', \Box_b A'\}) \leq 1$. In particular if $\{\Box_a A', \Box_b A'\} \cap \mathtt{Nec} = \emptyset$ then $card(\mathtt{Nec}_x^\sigma \cap \{\Box_a A', \Box_b A'\}) = 0$. For each natural number $i \leq n$, we are defining a set W_i of triples $(w', \sigma, ?)$ where

- $w' \in W$,
- σ is a sequence of elements of $\{1, \ldots, n\}$ without repetition,
- $?$ is either the symbol 'Λ' or some $\Box_{a_p} A' \in \mathtt{Nec}$ with $p \notin set(\sigma)$.

Each set W'_i shall be later defined as the set $\{w' : (w', \sigma, ?) \in W_i\}$. The set W_i is an intermediate set that contains some information about the elements of W'_i. We let $W_0 = \{(w, (1, \ldots, n), \Lambda)\}$. Assume W_i is defined. We will now define W_{i+1}. Initialize W_{i+1} to the empty set $\emptyset$.

For each $(w', \sigma, ?) \in W_i$,

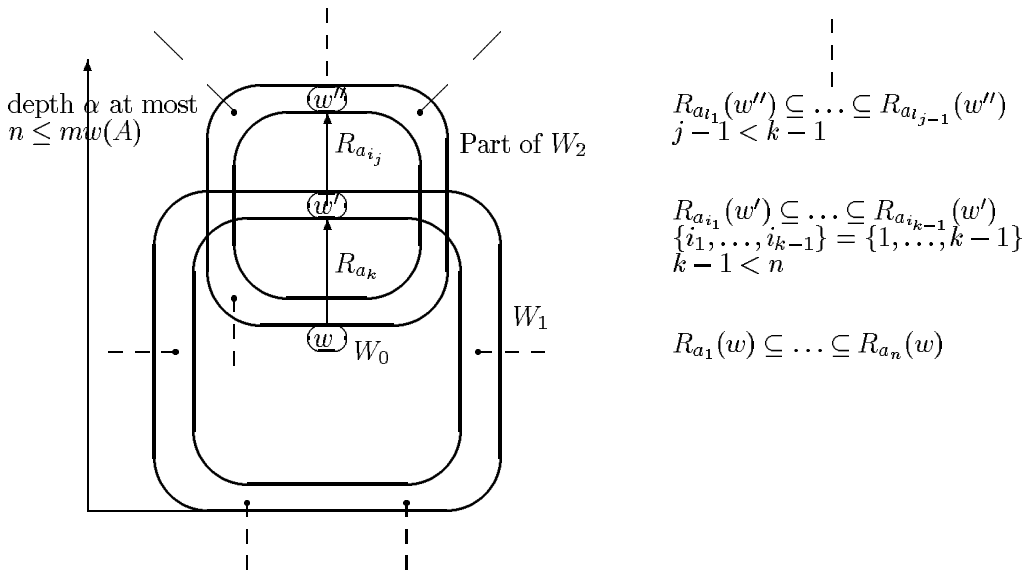


Figure 2: Illustration of the construction

for each $\Box_{a_j} A' \in \mathtt{Nec}^{\sigma}_{w'}$,

choose one $u \in W$ with the property $(w', u) \in R_{a_j}$ and $\mathcal{M}, u \models \neg A'$.

If $\sigma = (j_1, \ldots, j_k)$ then we write k' to denote the element of $\{1, \ldots, k\}$ such that $j_{k'} = j$. The existence of k' is guaranteed by the definition of $\mathtt{Nec}^{\sigma}_{w'}$. Add the 3-tuple

$$(u, (j'_1, \ldots, j'_{k'-1}), \Box_{a_j} A')$$

to the set W_{i+1} such that

- $set((j'_1, \ldots, j'_{k'-1})) = set((j_1, \ldots, j_{k'-1}))$ and
- $R_{a_{j'_1}}(u) \subseteq \ldots \subseteq R_{a_{j'_{k'-1}}}(u)$ -whenever $k' = 1$ the sequence $(j'_1, \ldots, j'_{k'-1})$ is empty.

Let W_{i+1} be the set augmented this way. There exists $\alpha \in \{0, \ldots, n\}$ such that $W_\alpha \neq \emptyset$ and $W_{\alpha+1} = \emptyset$ as the length of the sequences of natural numbers strictly decreases. Moreover if $(w', \sigma, ?) \in W_i$ then $|\sigma| \leq n - i$. Let U and W' be the sets defined in the following way:

$$U = \bigcup_{i=0}^{\alpha} W_i \quad W' = \bigcup_{i=0}^{\alpha} W'_i \ \ with\ W'_i = \{w' \in W : (w', \sigma, ?) \in W_i\}$$

Figure 4.1 illustrates the first steps of the construction.

Proposition 4.3 below states the main properties of the construction. In particular, for each $(w', \sigma, ?) \in U$ and for $i \in \{1, \ldots, n\} \setminus set(\sigma)$ the elements in $R_{a_i}(w')$ are partly known.

Proposition 4.3. For all $i \in \{0, \ldots, \alpha\}$ and all $(w_i, (j_1, \ldots, j_k), ?) \in W_i$,

1. $R_{a_{j_1}}(w_i) \subseteq \ldots \subseteq R_{a_{j_k}}(w_i)$,
2. If '?' $= \Box_{a_p} A'$ then $R_{a_{j_k}}(w_i) \subseteq R_{a_p}(w_i)$ and $\mathcal{M}, w_i \models \neg A'$,
3. for all $j \in \{1, \ldots, n\} \setminus set((j_1, \ldots, j_k))$, $R_{a_{j_k}}(w_i) \subseteq R_{a_j}(w_i)$.

Proof: (1),(2) Immediate by construction of W_i. (3) By induction on i.
Base case $(i = 0)$: Obvious.
Induction step: let $(w_i, \sigma.(j_{k'}, \ldots, j_k), ?) \in W_i$ with

- $set(\sigma) = \{j_1, \ldots, j_{k'-1}\}$ and $|\sigma| = k' - 1$ (no repetition),
- $(w_i, w_{i+1}) \in R_{a_{j_{k'}}}$ and,
- $(w_{i+1}, (j_1, \ldots, j_{k'-1}), \Box_{a_{j_{k'}}} A') \in W_{i+1}$ (w_{i+1} is a witness for $\mathcal{M}, w_i \not\models \Box_{a_{j_{k'}}} A'$).

By the induction hypothesis, for all $j \in \{1, \ldots, n\} \setminus set((j_1, \ldots, j_k))$, $R_{a_{j_k}}(w_i) \subseteq R_{a_j}(w_i)$. By Proposition 4.3(1),

$$R_{a_{j_{k'}}}(w_i) \subseteq \ldots \subseteq R_{a_{j_k}}(w_i)$$

So for all $j \in (\{1, \ldots, n\} \setminus set((j_1, \ldots, j_k))) \cup \{j_{k'+1}, \ldots, j_k\}$, $R_{a_{j_k'}}(w_i) \subseteq R_{a_j}(w_i)$. By Proposition 4.2, for all $j \in \{1, \ldots, n\} \setminus set((j_1, \ldots, j_{k'-1}))$,

$$R_{a_j}(w_{i+1}) = R_{a_j}(w_i)$$

Since $R_{a_{j_{k'-1}}}(w_{i+1}) \subseteq R_{a_{j_{k'}}}(w_{i+1})$ by Proposition 4.3(1), for all $j \in \{1, \ldots, n\} \setminus set((j_1, \ldots, j_{k'-1}))$, $R_{a_{j_{k'-1}}}(w_{i+1}) \subseteq R_{a_j}(w_{i+1})$. **Q.E.D.**

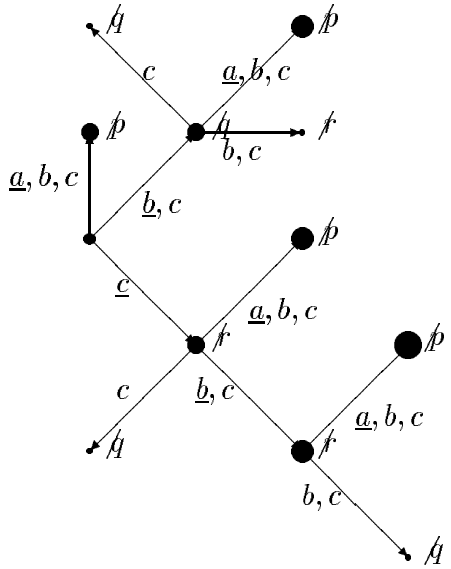
For all $i \in \{0, \ldots, \alpha - 1\}$, $card(W_{i+1}) \leq card(W_i) \times mw(A)$ and therefore $card(W') \leq 1 + n \times mw(A)^n$. This construction is more general than the construction defined in [HM92] to prove the **NP**-completeness of the satisfiability problem for the propositional modal logic S5. Indeed for $n = 1$ the construction in [HM92] (see also [FHMV95]) and ours are identical. Observe that S5 can be seen as an LA-logic with a unique modal expression (then the local agreement condition is trivially satisfied). In Figure 3, an example of construction with some finite model is given.

4.2 How the Construction Captures Enough Worlds

Now assume that $\mathcal{M}, w \models A$ for some L-formula A, $\mathcal{M} = (W, (R_a)_{a \in \mathbb{M}}, V) \in \mathcal{S}$, $w \in W$ and the distinct modal expressions occurring in A are exactly $a_1, \ldots, a_n$ with $R_{a_1}(w) \subseteq \ldots \subseteq R_{a_n}(w)$. The rest of the section is partly devoted to showing that

$$\mathcal{M}, w \models A \; iff \; \mathcal{M}_{|W'}, w \models A$$

Proposition 4.4 below states that the set W' contains *enough* worlds:



- Element of W_0
- Element of W_1
- Element of W_2
- Element of W_3

The relations of the model are the reflexive, symmetric and transitive closure of the relations presented in the figure.

Some modal expressions are underlined to specify the used modal expressions of the construction.

$\not p$: the propositional variable p is not satisfied in the world.

$W' = \bigcup_{0 \leq i \leq 3} W'_i \qquad card(\mathtt{Nec}) = 9$

Figure 3: An $\mathcal{L}$-model and a construction of the set W' for $\neg(\Box_a p \vee \Box_b q \vee \Box_c r)$

Proposition 4.4. For all $w' \in W'$ and $\Box_{a_j} A' \in \mathtt{Nec}$, if $\mathcal{M}, w' \models \neg \Box_{a_j} A'$ then there is $w'' \in W'$ such that $(w', w'') \in R_{a_j}$ and $\mathcal{M}, w'' \not\models A'$.

Proof: By induction on i for $(w', \sigma, ?) \in W_i$.
Base case ($i = 0$, $w' = w$): assume that $\mathcal{M}, w \models \neg \Box_{a_j} A'$. Then there exist $\Box_{a_k} A' \in \mathtt{Nec}_w^{(1,\ldots,n)}$ and $(u_1, (j_1, \ldots, j_{k-1}), \Box_{a_k} A') \in W_1$ such that $(w, u_1) \in R_{a_k}$ and $\mathcal{M}, u_1 \not\models A'$. Since $R_{a_k}(w) \subseteq R_{a_j}(w)$ by the definition of $\mathtt{Nec}_w^{(1,\ldots,n)}$, we have $(w, u_1) \in R_{a_j}$.
Induction step: assume $\mathcal{M}, w_{i+1} \models \neg \Box_{a_j} A''$, $(w_{i+1}, (j_1, \ldots, j_{k'-1}), \Box_{a_{j_{k'}}} A') \in W_{i+1}$.

If $j \in set((j_1, \ldots, j_{k'-1}))$ then a new object in W_{i+2} is built. There exist $\Box_{a_k} A'' \in \mathtt{Nec}_{w_{i+1}}^{(j_1,\ldots,j_{k'-1})}$ and $(u_{i+2}, (j'_1, \ldots, j'_{k''-1}), \Box_{a_k} A'') \in W_{i+2}$ such that $(w_{i+1}, u_{i+2}) \in R_{a_k}$ and $\mathcal{M}, u_{i+2} \models \neg A''$. By definition of $\mathtt{Nec}_{w_{i+1}}^{(j_1,\ldots,j_{k'-1})}$ and Proposition 4.3(1), we have $R_{a_k}(w_{i+1}) \subseteq R_{a_j}(w_{i+1})$ and therefore $(w_{i+1}, u_{i+2}) \in R_{a_j}$. Now assume $j \notin set((j_1, \ldots, j_{k'-1}))$ (it is the most interesting case), whence $w \neq w_{i+1}$. There exists

$$(u_i, \sigma'.(j_{k'}, \ldots, j_k), ?) \in W_i$$

such that $(u_i, w_{i+1}) \in R_{a_{j_{k'}}}$, $set(\sigma') = \{j_1, \ldots, j_{k'-1}\}$ and $|\sigma'| = k' - 1$. By Proposition 4.3(3), $R_{a_{j_{k'}}}(u_i) \subseteq R_{a_j}(u_i)$. Hence $(u_i, w_{i+1}) \in R_{a_j}$ and therefore $\mathcal{M}, u_i \models \neg \Box_{a_j} A''$ since R_{a_j} is transitive. By the induction hypothesis, there is $u \in W'$ such that $(u_i, u) \in R_{a_j}$, $\mathcal{M}, u \not\models A''$, whence $(w_{i+1}, u) \in R_{a_j}$ since R_{a_j} is an equivalence relation. **Q.E.D.**

Proposition 4.5 below provides a bound for the size of the models.

Proposition 4.5. An L-formula A is $\mathcal{L}$-satisfiable iff it is satisfiable in a model for $\mathcal{L}$ with at most $1 + n \times mw(A)^n$ objects, where $n = card(sub_{\mathtt{M}}(A))$.

The proof of Proposition 4.5 follows the lines of the proof of Lemma 6.1 in [Lad77].
Proof: Assume there exist $\mathcal{M} = (W, (R_a)_{a\in\mathtt{M}}, V) \in \mathcal{S}$ and $w \in W$ such that $\mathcal{M}, w \models A$. Let $\mathcal{M}'$ be $\mathcal{M}_{|W'}$. We can show that for all objects $u' \in W'$ and for all $B \in sub(A)$, $\mathcal{M}, u' \models B$ iff $\mathcal{M}', u' \models B$ (so we prove $\mathcal{M}', u \models A$). We proceed by induction on the structure of B. The only nontrivial case is when B is of the form $\Box_a B'$. Take $u' \in W'$. If $\mathcal{M}, u' \models \Box_a B'$, then for all $v \in W$ such that $(u', v) \in R_a$ we have $\mathcal{M}, v \models B'$. In particular, for all $v \in W'$ such that $(u', v) \in R'_a$ we have $\mathcal{M}, v \models B'$. By the induction hypothesis, for all $v \in W'$, $\mathcal{M}', v \models B'$. Thus $\mathcal{M}', u' \models \Box_a B'$. Now assume $\mathcal{M}, u' \not\models \Box_a B'$. By Proposition 4.4, there exists $v \in W'$ such that $(u', v) \in R_a$ and $\mathcal{M}, v \not\models B'$. In consequence, $(u', v) \in R'_a$ and by the induction hypothesis $\mathcal{M}', v \not\models B'$. Hence $\mathcal{M}', u' \not\models \Box_a B'$. **Q.E.D.**

Corollary 4.6. $\mathcal{L}$ has the strong finite model property.

The construction in this section generalizes the technique used in [Dem96b] to the set of LA-logics. Corollary 4.6 takes advantage of the fact that, for any LA-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$, $\mathcal{S}$ is closed under submodels and any two binary relations R, S of a model for $\mathcal{L}$ are in local agreement.

In the sequel we shall provide sufficient conditions to prove the decidability of LA-logics. For any set LO of linear orders over $\mathtt{M}$ and for any subset Y of $\mathtt{M}$, we write $LO \uparrow Y$ to denote the set of linear orders $LO \uparrow Y = \{\leq_{|Y} : \leq \in LO\}$.

Definition 4.1. An LA-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ is said to be (resp. *polynomially*) *lo-decidable* iff it is decidable whether for all finite sets $Y \subseteq \mathtt{M}$ and all linear orders $\leq$ over Y, $\leq \in lo(\mathcal{L}) \uparrow Y$ (resp. and it can be checked in polynomial-time with respect to $card(Y)$). ∇

For instance for an LA-logic such that $lo(\mathcal{L})$ is finite and it is decidable for $\leq \in lo(\mathcal{L})$ and $a, b \in \mathtt{M}$ whether $(a, b) \in \leq$, $\mathcal{L}$ is lo-decidable.

Definition 4.2. An LA-logic $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ is said to be (resp. *polynomially*) *lo-complete* iff it is decidable whether for all finite sets $Y \subseteq \mathtt{M}$ and all structures $(W, (R_a)_{a\in Y})$ such that

1. W is a non-empty set,
2. for $a \in Y$, R_a is an equivalence relation over W

there is an $\mathcal{L}$-model $(W, (R'_a)_{a\in\mathtt{M}}, V)$ such that for all $a \in Y$, $R'_a = R_a$ (resp. and it can be checked in polynomial-time with respect to $card(Y)$ and $card(W)$). ∇

Proposition 4.7 provides sufficient conditions for the decidability of LA-logics.

Proposition 4.7. Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ be an LA-logic such that $\mathcal{L}$ is lo-decidable and lo-complete. Then the $\mathcal{L}$-satisfiability problem is decidable.

Proof: Take any formula A for which one wishes to know whether A is $\mathcal{L}$-satisfiable. By Proposition 4.5, A is $\mathcal{L}$-satisfiable iff there exist an $\mathcal{L}$-model $\mathcal{M} = (W, (R_a)_{a\in\mathtt{M}}, V)$ and $w \in W$ such that $\mathcal{M}, w \models A$ and $card(W) \leq 1 + n \times mw(A)^n$ where $n = card(sub_{\mathtt{M}}(A))$.

So in order to check whether A is $\mathcal{L}$-satisfiable, enumerate all the structures $\mathcal{M}' = (W, (R_a)_{a \in sub_{\mathtt{M}}(A)}, V)$ (modulo the isomorphic[8] copies with respect to A) where

1. $W = \{w_1, \ldots, w_l\}$ is a finite non-empty set such that $l \leq 1 + n \times mw(A)^n$
2. $(R_a)_{a \in sub_{\mathtt{M}}(A)}$ is a family of binary relations over W
3. V is mapping $V : \mathtt{F}_0(A) \rightarrow \mathcal{P}(W)$ where $\mathtt{F}_0(A)$ denotes the set of propositional variables occurring in A.

and check whether

($\star$) $\mathcal{M}', w \models A$ for some $w \in W$

($\star\star$) for $a, b \in sub_{\mathtt{M}}(A)$, R_a and R_b are equivalence relations in local agreement.

($\star\star\star$) For all $i \in \{1, \ldots, l\}$, there is a linear order $\underline{\leq}$ over $sub_{\mathtt{M}}(A)$ such that for all $a, b \in sub_{\mathtt{M}}(A)$ if $a \underline{\leq} b$ then $R_a(w_i) \subseteq R_b(w_i)$ and $\underline{\leq} \in lo(\mathcal{L}) \uparrow sub_{\mathtt{M}}(A)$

($\star\star\star\star$) there is an $\mathcal{L}$-model $\mathcal{M} = (W, (R'_a)_{a \in \mathtt{M}}, V')$ such that for all $a \in sub_{\mathtt{M}}(A)$, $R'_a = R_a$ and the restriction of V' to $\mathtt{F}_0(A)$ is V.

When ($\star$) – ($\star\star\star\star$) hold A is $\mathcal{L}$-satisfiable. ($\star$) can be checked in polynomial-time with respect to l and to the *size* of A (i.e., the length of the representation of A in any reasonable -unspecified- encoding). ($\star\star$) can be checked in $O(n! \times l^5)$. ($\star\star\star$) and ($\star\star\star\star$) are instances of decidable problems since $\mathcal{L}$ is lo-decidable and lo-complete. Since the set of structures $(W, (R_a)_{a \in sub_{\mathtt{M}}(A)}, V)$ (modulo the isomorphic copies) such that $card(U) \leq 1 + n \times mw(A)^n$ is finite, the decidability of the $\mathcal{L}$-satisfiability problem follows. **Q.E.D.**

Proposition 4.7 is a weak version of Corollary 8 in [Dem96a] since the initial proof (related to decidability) contains a flaw. It is however an open question whether every LA-logic has a decidable validity problem.

Example 4.1. Let $\mathcal{L}$ be an LA-logic such that $lo(\mathcal{L})$ is the set of all the linear orders over $\mathtt{M}$ (resp. $lo(\mathcal{L}) = \{\underline{\leq}\}$ is a singleton and it is decidable whether $(a, b) \in \underline{\leq}$ for $a, b \in \mathtt{M}$). It can be shown that $\mathcal{L}$ is lo-decidable and lo-complete. Then the $\mathcal{L}$-satisfiability problem is decidable.

More can be said when $\mathtt{M}$ is finite.

Proposition 4.8. Let $\mathcal{L} = \langle \mathtt{L}, \mathcal{S}, \models_{\mathcal{L}} \rangle$ be an LA-logic such that $\mathtt{M}$ is finite. Then,

1. $\mathcal{L}$ is polynomially lo-decidable.
2. for all finite $\mathtt{L}$-models $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V)$ one can check that $\mathcal{M} \in \mathcal{S}$ in polynomial-time with respect to $card(W)$.
3. The $\mathcal{L}$-satisfiability problem is **NP**-complete.

[8] $(W, (R_a)_{a \in sub_{\mathtt{M}}(A)}, V)$ and $(W', (R'_a)_{a \in sub_{\mathtt{M}}(A)}, V')$ are isomorphic with respect to A iff there is a 1-1 mapping $g : W \rightarrow W'$ such that for all $a \in sub_{\mathtt{M}}(A)$, $\{(g(x), g(y)) : (x, y) \in R_a\} = R'_a$ and for all propositional variables p occurring in A, $V'(p) = \{g(x) : x \in V(p)\}$.

Proof: (1) is immediate from Definition 4.1. (2) Direct calculations lead to the conclusion that $\mathcal{M} \in \mathcal{S}$ can be checked in $O(card(M)! \times card(W)^5)$. (3) is a direct consequence of Proposition 4.5 and (2). **Q.E.D.**

For the sake of comparison, for all $k \geq 2$, the satisfiability problem for the multimodal logics $S5_k$ is **PSPACE**-complete [HM92], whereas the **PSPACE**-complete satisfiability problem for S4 can be reduced in linear-time to the satisfiability problem for any LA-S4-logic with a non-empty set of modal expressions.

5 Applications to Logics from the Literature

Below we relate the LA-logics to some logics from the literature.

5.1 Gargov's Data Analysis Logic with Local Agreement

The logic DALLA defined in [Gar86] (originally called DAL) restricts the class of models of the logic DAL [FdCO85] by requiring that any two indiscernibility relations of a model are in (LA). A complete axiomatization of DALLA is given in [Gar86]. The decidability of the validity problem for the logic DALLA is open, as mentioned in [Gar86]. The logic DALLA $= \langle \mathtt{L}_D, \mathcal{S}_D, \models_{DALLA} \rangle$ is defined as follows. $\mathtt{L}_D$ has a countable set of modal constants $\mathtt{M}_{0D}$, and the operators $\cap$ and $\cup^*$ interpretated as the intersection[9] and the transitive closure of union respectively. The set of modal expressions (resp. $\mathtt{L}_D$-formulae) is denoted by $\mathtt{M}_D$ (resp. $\mathtt{F}_D$) and $\mathtt{M}_D$ is the smallest set such that $\mathtt{M}_{0D} \subseteq \mathtt{M}_D$ and if $a, b \in \mathtt{M}_D$ and $\oplus \in \{\cap, \cup^*\}$ then $a \oplus b \in \mathtt{M}_D$.

For all $\mathtt{L}_D$-models $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}_D}, V)$, $\mathcal{M} \in \mathcal{S}_D$ iff for all $a, b \in \mathtt{M}_D$, for any $\oplus \in \{\cap, \cup^*\}$,

- R_a is an equivalence relation,
- $R_{a \oplus b} = R_a \oplus R_b$,
- R_a and R_b are in local agreement.

Consider the Hilbert-style system `dalla` containing the following axiom schemes and inference rules ($A, B \in \mathtt{F}_D$, $a, a_1, a_2 \in \mathtt{M}_D$):

P. All formulae having the form of a classical propositional tautology

K. $\Box_a(A \Rightarrow B) \Rightarrow (\Box_a A \Rightarrow \Box_a B)$,

T. $\Box_a A \Rightarrow A$,

5. $\Diamond_a A \Rightarrow \Box_a \Diamond_a A$,

U. $\Box_{a_1 \cup^* a_2} A \Leftrightarrow \Box_{a_1} A \wedge \Box_{a_2} A$,

I. $\Box_{a_1 \cap a_2} A \Leftrightarrow \Box_{a_1} A \vee \Box_{a_2} A$

MP. From A and $A \Rightarrow B$ infer B,

NR. From A infer $\Box_a A$.

[9] For the sake of simplicity, the operators and the relational operations are denoted by the same symbols.

For all $\mathtt{L}_D$-formulae A, A is DALLA-valid iff A is a theorem of `dalla` [Gar86]. Although DALLA is not *stricto sensu* an LA-logic (because of the condition $R_{a\oplus b} = R_a \oplus R_b$), there exists a simple translation between DALLA and an LA-logic DALLA$'$ defined below. Consider the LA-logic[10] DALLA$' = \langle \mathtt{L}_{D'}, \mathcal{S}_{D'}, \models_{D'} \rangle$, where $\mathtt{L}_{D'}$ is the subset of $\mathtt{L}_D$ without the modal operators $\{\cap, \cup^*\}$ and $lo(DALLA')$ is the set of all the linear orders over $\mathtt{M}_{0D}$. Consider the mapping $\mathtt{T}$ of formulae from $\mathtt{L}_D$ into $\mathtt{L}_{D'}$ defined as follows:

- $\mathtt{T}(p) = p$ for all $p \in \mathtt{F}_0$, $\mathtt{T}(\Box_a A_1) = \Box_a \mathtt{T}(A_1)$ for all $a \in \mathtt{M}_{0D}$,
- $\mathtt{T}(A_1 \wedge A_2) = \mathtt{T}(A_1) \wedge \mathtt{T}(A_2)$, $\mathtt{T}(\neg A_1) = \neg \mathtt{T}(A_1)$,
- $\mathtt{T}(\Box_{a_1 \cap a_2} A_1) = \mathtt{T}(\Box_{a_1} A_1) \vee \mathtt{T}(\Box_{a_2} A_1)$ and $\mathtt{T}(\Box_{a_1 \cup^* a_2} A_1) = \mathtt{T}(\Box_{a_1} A_1) \wedge \mathtt{T}(\Box_{a_2} A_1)$.

Proposition 5.1. For all $\mathtt{L}_D$-formulae A, A is DALLA-valid iff $\mathtt{T}(A)$ is DALLA-valid iff $\mathtt{T}(A)$ is DALLA$'$-valid.

The proof is immediate considering the replacement of equivalents in `dalla`, completeness of `dalla` with respect to the DALLA-validity, and the fact that for all $A \in \mathtt{L}_{D'}$, A is DALLA-valid iff A is DALLA$'$-valid. Actually, for all $\mathtt{L}_{D'}$-formulae A, for all $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}_D}, V) \in \mathcal{S}_D$, $w \in W$, $\mathcal{M}, w \models A$ iff $\mathcal{M}', w \models A$ where $\mathcal{M}' = (W, (R'_a)_{a \in \mathtt{M}_{0D}}, V)$ with $R_a = R'_a$ for all $a \in \mathtt{M}_{0D}$. DALLA has therefore the strong finite model property and the validity problem for DALLA is decidable (see Example 4.1 for the decidability of DALLA$'$). This section has reproduced the main arguments from [Dem96b].

5.2 Nakamura's Logic of Graded Modalities

The logic of graded[11] modalities (LGM $= (\mathtt{L}_{LGM}, \mathcal{S}_{LGM}, \models_{LGM})$ for short) introduced in [Nak93] (see also [Nak92]) is based on the graded equivalence relations, i.e. the graded similarity in Zadeh's meaning [Zad71]. Although the decidability of LGM is proved in [Nak93] using the rectangle method developed in [HC68], we prove that LGM has the strong finite model property. The set of modal expressions of $\mathtt{L}_{LGM}$ is the closed interval $[0, 1]$ of the real line. For all $\mathtt{L}_{LGM}$-models $\mathcal{M} = (W, (R_\lambda)_{\lambda \in [0,1]}, V)$, $\mathcal{M} \in \mathcal{S}_{LGM}$ iff there is $\mu : W \times W \to [0, 1]$ such that

1. for all $x \in W$, $\mu(x, x) = 1$,
2. for all $x, y \in W$, $\mu(x, y) = \mu(y, x)$,
3. for all $x, z \in W$, $\mu(x, z) \geq lub\{min(\mu(x, y), \mu(y, z)) : y \in W\}$ ($lubX$: least upper bound of the set X) and
4. for all $\lambda \in [0, 1]$, $R_\lambda = \{(x, y) \in W \times W : \mu(x, y) \geq \lambda\}$.

The binary relations R_λ in a model for LGM are therefore equivalence relations. Moreover we can easily show that LGM is not an LA-logic. However the LA-logic LGM$'$ (defined below) is strongly related to LGM as shown below. So consider the unique LA-logic LGM$' = (\mathtt{L}_{LGM}, \mathcal{S}_{LGM'}, \models_{LGM'})$ such that

[10] In [DO97] (see also [Bal97]) a complete axiomatization of DALLA$'$ is given: it corresponds to `dalla` where the axiom schemes **U.** and **I.** are deleted and the axiom schema $(\Box_{a_1}\, \Box_{a_2}\, A \wedge \Box_{a_2}\, \Box_{a_1}\, A) \Leftrightarrow (\Box_{a_1} A \wedge \Box_{a_2} A)$ for all $a_1, a_2 \in \mathtt{M}_D$ is added.

[11] The term 'graded' should not be confused with its use, for instance, in [HM91, Cer94, MMN95, Mik95]. Its present use is similar to that in [Cha94].

$lo(LGM') = \{\geq\}$, where $\geq$ is the usual linear order on $[0,1]$.

Then for all $\mathcal{M} = (W, (R_\lambda)_{\lambda\in[0,1]}, V) \in \mathcal{S}_{LGM'}$,

for all $\lambda, \lambda' \in [0,1]$, $\lambda \geq \lambda'$ implies $R_\lambda \subseteq R_{\lambda'}$.

By Corollary 4.6, LGM$'$ has the strong finite model property, and the validity problem for LGM$'$ is decidable (see Example 4.1). In the sequel we show that for all $A \in \mathtt{L}_{LGM}$, A is LGM-valid iff A is LGM$'$-valid, although $\mathcal{S}_{LGM} \neq \mathcal{S}_{LGM'}$.

Proposition 5.2. Let $\mathcal{M} = (W, (R_\lambda)_{\lambda\in[0,1]}, V) \in \mathcal{S}_{LGM}$. Then $\mathcal{M} \in \mathcal{S}_{LGM'}$ and for all $A \in \mathtt{L}_{LGM}$, $w \in W$, $\mathcal{M}, w \models_{LGM} A$ iff $\mathcal{M}, w \models_{LGM'} A$.

The proof is by an easy verification. Proposition 5.3 below states the converse result.

Proposition 5.3. Let $\mathcal{M} = (W, (R_\lambda)_{\lambda\in[0,1]}, V) \in \mathcal{S}_{LGM'}$, $w \in W$ and let A be an $\mathtt{L}_{LGM}$-formula. Let denote the set of real numbers being indices of modal operators that occur in A by $\{\lambda_1, \ldots, \lambda_n\}$ (in increasing order). Let $\mathcal{M}'$ be $\mathcal{M}_{|R_{\lambda_1}(w)} = (R_{\lambda_1}(w), (R'_\lambda)_{\lambda\in[0,1]}, V')$, and let $\mathcal{M}'' = (R_{\lambda_1}(w), (R''_\lambda)_{\lambda\in[0,1]}, V')$ be a structure such that,

1. for all $\lambda \in [0, \lambda_1]$ $R''_\lambda = R'_{\lambda_1}$,
2. for all $i \in \{1, \ldots, n-1\}$, $\lambda \in]\lambda_i, \lambda_{i+1}]$ (right-closed interval of the real line) $R''_\lambda = R'_{\lambda_{i+1}}$, and
3. for all $\lambda \in]\lambda_n, 1]$, $R''_\lambda = R'_1$.

Then $\mathcal{M}'' \in \mathcal{S}_{LGM}$ and for all $w' \in R_{\lambda_1}(w)$,

$$\mathcal{M}, w' \models_{LGM'} A \text{ iff } \mathcal{M}', w' \models_{LGM'} A \text{ iff } \mathcal{M}'', w' \models_{LGM} A.$$

Proof: First it is clear that $\mathcal{M}'' \in \mathcal{S}_{LGM'}$. For all $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$, $R'_\lambda = R''_\lambda$ so for all $w' \in R_{\lambda_1}(w)$, $\mathcal{M}', w' \models A$ iff $\mathcal{M}'', w' \models A$. Moreover, considering the remark following the proof of Proposition 4.2, $\mathcal{M}, w' \models A$ iff $\mathcal{M}', w' \models A$ (remember that $R_{\lambda_n} \subseteq \ldots \subseteq R_{\lambda_1}$). Now we prove that $\mathcal{M}'' \in \mathcal{S}_{LGM}$.

Consider the function $\mu : R_{\lambda_1}(w) \times R_{\lambda_1}(w) \to [0,1]$ such that

for all $(x,y) \in R_{\lambda_1}(w) \times R_{\lambda_1}(w)$, $\mu(x,y) = lub\{\lambda \in [0,1] : (x,y) \in R''_\lambda\}$.

This definition is correct since the latter set is not empty (it contains λ_1) and the least upper bound of $\{\lambda \in [0,1] : (x,y) \in R''_\lambda\}$ always exists. The possible values for $lub\{\lambda : (x,y) \in R''_\lambda\}$ are in the set $\{\lambda_1, \ldots, \lambda_n, 1\}$. Hence by construction, for all $(x,y) \in R_{\lambda_1}(w) \times R_{\lambda_1}(w)$, $\mu(x,y) = max\{\lambda : (x,y) \in R''_\lambda\}$.
(1) Since R''_1 is reflexive, then for all $x \in R_{\lambda_1}(w)$, $(x,x) \in R''_1$ whence $\mu(x,x) = 1$.
(2) For all $x, y \in R_{\lambda_1}(w)$, $\mu(x,y) = max\{\lambda : (x,y) \in R''_\lambda\} = max\{\lambda : (y,x) \in R''_\lambda\}$ (by symmetry of the relations R''_λ) whence $\mu(x,y) = \mu(y,x)$.
(3) Take $x, y, z \in R_{\lambda_1}(w)$. We write κ to denote $min(\mu(x,y), \mu(y,z))$. By definition of μ, $\kappa = min(max\{\lambda : (x,y) \in R''_\lambda\}, max\{\lambda : (y,z) \in R''_\lambda\})$. It follows that $max\{\lambda : (x,y) \in R''_\lambda\} \geq \kappa$. There is $\kappa' \geq \kappa$ such that $(x,y) \in R''_{\kappa'}$; so $(x,y) \in R''_\kappa$ since $R''_{\kappa'} \subseteq R''_\kappa$. In a similar way it can be shown that $(y,z) \in R''_\kappa$. By transitivity, $(x,z) \in R''_\kappa$, whence $\mu(x,z) \geq \kappa$ by definition of μ. Thus for all $x, y, z \in R_{\lambda_1}(w)$, $\mu(x,z) \geq min(\mu(x,y), \mu(y,z))$.

In consequence for all $x, z \in R_{\lambda_1}(w)$, $\mu(x,z) \geq lub\{min(\mu(x,y), \mu(y,z)) : y \in R_{\lambda_1}(w)\}$.
(4) By construction of $\mathcal{M}''$, $(x,y) \in R''_{\lambda'}$ iff $\lambda' \in \{\lambda : (x,y) \in R''_\lambda\}$ iff $\lambda' \leq max\{\lambda : (x,y) \in R''_\lambda\}$ iff $\lambda' \leq \mu(x,y)$. Hence for all $x, y \in R_{\lambda_1}(w)$, $\lambda \in [0,1]$, $R''_\lambda = \{(x,y) : \mu(x,y) \geq \lambda\}$. This terminates the proof. **Q.E.D.**

Proposition 5.2 implies that $\mathcal{S}_{LGM} \subseteq \mathcal{S}_{LGM'}$ but $\mathcal{S}_{LGM'} \not\subseteq \mathcal{S}_{LGM}$. For instance in a model for LGM′, R_0 may not be the universal relation although in each model for LGM R_0 is the universal relation. Proposition 5.2 and 5.3 imply that for all $A \in \mathtt{L}_{LGM}$, A is LGM-valid iff A is LGM′-valid. Hence LGM has the strong finite model property, and the validity problem for LGM is decidable.

Although LGM is not an LA-logic (see Proposition 4.5), one can state:

Corollary 5.4. An L-formula A is LGM-satisfiable iff it is satisfiable in an LGM-model with at most $1 + n \times mw(A)^n$ objects where n is the number of real numbers (viewed as modal expressions) that occur in A.

As mentioned in [Nak92], the axiomatization of LGM is an interesting open problem. As a side-effect of our work, we define a sound and complete axiomatization of LGM using standard techniques for modal logics. Consider a Hilbert-style system lgm′ containing the following axiom schemes and inference rules ($A, B \in \mathtt{L}_{LGM}$, $\lambda_1, \lambda_2 \in [0,1]$):

P. All formulae having the form of a classical propositional tautology,

K′. $\Box_{\lambda_1}(A \Rightarrow B) \Rightarrow (\Box_{\lambda_1} A \Rightarrow \Box_{\lambda_1} B)$,

T′. $\Box_{\lambda_1} A \Rightarrow A$,

5′. $\Diamond_{\lambda_1} A \Rightarrow \Box_{\lambda_1} \Diamond_{\lambda_1} A$,

<. $\Box_{\lambda_1} A \Rightarrow \Box_{\lambda_2} A$ when $\lambda_1 < \lambda_2$,

MP. + RN.

By using the canonical model construction (see e.g [Mak66]), it is a standard task to prove that for all $A \in \mathtt{L}_{LGM}$, A is a theorem of lgm′ iff A is LGM′-valid. As a consequence, lgm′ is a sound and complete system for the logic LGM.

The logics LGM′ and DALLA′ correspond to two extreme cases of LA-logics: $lo(LGM')$ is a singleton, whereas $lo(DALLA')$ contains all the linear orders on the set of modal expressions. Although the concepts of rough sets and fuzzy sets are different (see e.g. a discussion in [Paw85]), the technique developed in Section 4 can be applied to logics derived either from the notion of rough sets (DALLA for instance), or from the notion of fuzzy sets (LGM for instance). However, our technique does not seem to be applicable for instance to the modal fuzzy logics presented in [Yin88] since the value of any formula in a model is a fuzzy set, which is not the case with LGM. Indeed the fuzziness in LGM is relative to the binary relations but not to the valuation functions.

6 Hilbert-Style Axiomatization

In this section, we define complete Hilbert-style systems for each LA-logic such that $lo(\mathcal{L})$ is *finite* (finite M is a particular case). As usual, by an *L-normal system* we understand

a subset $\mathtt{X}$ of $\mathtt{F}$ which contains the axioms of classical logic together with the formulae $\Box_a(p \Rightarrow q) \Rightarrow (\Box_a p \Rightarrow \Box_a q)$, and is closed under *Modus Ponens*, the *Necessitation Rule* (from A infer $\Box_a A$ for any $a \in \mathtt{M}$) and the *Substitution Rule*.

Let $\mathcal{L}$ be an LA-logic such that $lo(\mathcal{L})$ is finite (say $card(lo(\mathcal{L})) = M$) and let $\mathtt{X}_\mathcal{L}$ be the set of $\mathcal{L}$-valid formulae

$$\mathtt{X}_\mathcal{L} = \{A : \forall \mathcal{M} \in \mathcal{S},\ \mathcal{M} \models A\}$$

$\mathcal{L}$ is fixed in the rest of the section, unless otherwise stated. A formula A is *true* in an $\mathtt{L}$-frame $\mathcal{F} = (W, (R_a)_{a \in \mathtt{M}})$ (written $\mathcal{F} \models A$) iff for all $\mathtt{L}$-models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M} \models A$. For any set $X^\mathcal{F}$ of $\mathtt{L}$-frames, we write $Th(X^\mathcal{F})$ to denote the set of $\mathtt{L}$-formulae below:

$$Th(X^\mathcal{F}) = \{A \in \mathtt{F} : \forall \mathcal{F} \in X^\mathcal{F},\ \mathcal{F} \models A\}$$

For any set X of $\mathtt{L}$-formulae, we write $Fr(X)$ to denote the set of $\mathtt{L}$-frames

$$Fr(X) = \{\mathcal{F} \in X^\mathcal{F}_\mathtt{L} : \forall A \in X,\ \mathcal{F} \models A\}$$

Proposition 6.1. $\mathtt{X}_\mathcal{L}$ is an $\mathtt{L}$-normal system.

The proof is immediate and it does not depend on the cardinality of $lo(\mathcal{L})$. Let

$$X^\mathcal{F}_\mathcal{L} = \{\mathcal{F} \in X^\mathcal{F}_\mathtt{L} : \exists \mathcal{M} \in \mathcal{S} \textit{ based on } \mathcal{F}\}$$

be the set of $\mathtt{L}$-frames on which the models of $\mathcal{S}$ are based on. It is easy to show that $Th(X^\mathcal{F}_\mathcal{L}) = \mathtt{X}_\mathcal{L}$. Although it is clear that $X^\mathcal{F}_\mathcal{L}$ is closed under subframes (see Proposition 4.1), we shall show that $Fr(\mathtt{X}_\mathcal{L})$ is also closed under subframes. Until now, we can only state that $X^\mathcal{F}_\mathcal{L} \subseteq Fr(\mathtt{X}_\mathcal{L})$. Indeed take $\mathcal{F} = (W, (R_a)_{a \in \mathtt{M}}) \in X^\mathcal{F}_\mathcal{L}$. All the models based on $\mathcal{F}$ belong to $\mathcal{S}$. Hence for all $B \in \mathtt{X}_\mathcal{L} = \{A : \forall\ \mathcal{M} \in \mathcal{S},\ \mathcal{M} \models A\}$, $\mathcal{F} \models B$. So $\mathcal{F} \in Fr(\mathtt{X}_\mathcal{L})$.

Each linear order $\underline{\leq}$ over a finite set of modal expressions $\{a_1, \ldots, a_n\}$ shall be denoted by $(a_{i_1}, \ldots, a_{i_n})$ where $set((i_1, \ldots, i_n)) = \{1, \ldots, n\}$ and for all $j \in \{1, \ldots, n-1\}$, $a_{i_j} \underline{\leq} a_{i_{j+1}}$. Let $LO = \{\underline{\leq}_1, \ldots, \underline{\leq}_m\}$ be a finite set of linear orders such that for all $j \in \{1, \ldots, m\}$, $\underline{\leq}_j = (a_{i^j_1}, \ldots, a_{i^j_n})$. We write A^{LO} to denote the $\mathtt{L}$-formula

$$\bigvee_{k \in \{1,\ldots,m\}} \Big(\bigwedge_{j \in \{2,\ldots,n\}} (\Box_{a_{i^k_j}} p_k \Rightarrow \Box_{a_{i^k_{j-1}}} p_k) \Big)$$

where the p_k's are propositional variables, and $k \neq k'$ implies $p_k \neq p_{k'}$.

Proposition 6.2. Let $\mathcal{F} = (W, (R_a)_{a \in \mathtt{M}})$ be an $\mathtt{L}$-frame. Then

($\star$) $\mathcal{F} \models \{A^{lo(\mathcal{L}) \uparrow Y} : Y \subseteq \mathtt{M}, 2 \leq card(Y) \leq 2 \times card(lo(\mathcal{L}))\}$ iff,

($\star\star$) for every $u \in W$ there exist $\underline{\leq} \in lo(\mathcal{L})$ such that for all $a, b \in \mathtt{M}$, $a \underline{\leq} b$ implies $R_a(u) \subseteq R_b(u)$.

Proof: Let $lo(\mathcal{L}) = \{\underline{\leq}_1, \ldots, \underline{\leq}_M\}$. First assume ($\star$) holds. Suppose that there is $x \in W$ such that for all $k \in \{1, \ldots, M\}$, there are $a_k, a'_k \in \mathtt{M}$ with $a_k \underline{\leq}_k a'_k$ and $R_{a_k}(x) \not\subseteq R_{a'_k}(x)$. Hence for all $k \in \{1, \ldots, M\}$, there is $x^k_0 \in W$ such that $(x, x^k_0) \in R_{a_k}$ and $(x, x^k_0) \notin R_{a'_k}$. We write LO to denote the set $\{\underline{\leq}'_1, \ldots, \underline{\leq}'_{m'}\} = lo(\mathcal{L}) \uparrow Y_0$ with

$$Y_0 = \bigcup_{k=1}^{M} \{a_k, a'_k\}$$

and $card(Y_0) = n' \leq 2 \times M$. For all $i \in \{1, \ldots, m'\}$, we write $\underline{\leq}'_i = (a^i_1, \ldots, a^i_{n'})$. We have

$$A^{LO} \in \{A^{lo(\mathcal{L})\uparrow Y} : Y \subseteq \mathtt{M}, 2 \leq card(Y) \leq 2 \times M\}$$

since $2 \leq card(Y_0) \leq 2 \times M$. For all $k \in \{1, \ldots, m'\}$, we write $\alpha(k)$ to denote an element of $\{1, \ldots, M\}$ such that $(\underline{\leq}_{\alpha(k)})_{|Y_0} = \underline{\leq}'_k$ ($\alpha(k)$ may not be unique).

Let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V)$ be a model based on $\mathcal{F}$ such that

$$\forall k \in \{1, \ldots, m'\},\ V(p_k) = R_{a'_{\alpha(k)}}(x) \tag{1}$$

Since $\mathcal{M}, x \models A^{LO}$ (by hypothesis $\mathcal{F} \models A^{LO}$) there is $k_0 \in \{1, \ldots, m'\}$ such that $\mathcal{M}, x \models \bigwedge_{j \in \{2, \ldots, n'\}} \Box_{a^{k_0}_j} p_{k_0} \Rightarrow \Box_{a^{k_0}_{j-1}} p_{k_0}$. Since

$$\{a'_{\alpha(k_0)}, a_{\alpha(k_0)}\} \subseteq Y_0 = \{a^{k_0}_j : j \in \{1, \ldots, n'\}\}$$

and

$$(a_{\alpha(k_0)}, a'_{\alpha(k_0)}) \in \underline{\leq}'_{k_0} = (\underline{\leq}_{\alpha(k_0)})_{|Y_0}$$

there exist $j_1, j_2 \in \{1, \ldots, n'\}$ such that $j_1 < j_2$, $a^{k_0}_{j_1} = a_{\alpha(k_0)}$ and $a^{k_0}_{j_2} = a'_{\alpha(k_0)}$ and for all $j \in \{j_1 + 1, \ldots, j_2\}$, $\mathcal{M}, x \models \Box_{a^{k_0}_j} p_{k_0} \Rightarrow \Box_{a^{k_0}_{j-1}} p_{k_0}$. So $\mathcal{M}, x \models \Box_{a'_{\alpha(k_0)}} p_{k_0} \Rightarrow \Box_{a_{\alpha(k_0)}} p_{k_0}$. We have $\mathcal{M}, x \models \Box_{a'_{\alpha(k_0)}} p_{k_0}$ by (1). Therefore $\mathcal{M}, x \models \Box_{a_{\alpha(k_0)}} p_{k_0}$ and $\mathcal{M}, x^{\alpha(k_0)}_0 \models p_{k_0}$ since $(x, x^{\alpha(k_0)}_0) \in R_{a_{\alpha(k_0)}}$. But $(x, x^{\alpha(k_0)}_0) \notin R_{a'_{\alpha(k_0)}}$ so by (1), $\mathcal{M}, x^{\alpha(k_0)}_0 \not\models p_{k_0}$ which leads to a contradiction.

Now assume ($\star\star$). Let $\mathcal{M} = (W, (R_a)_{a \in \mathtt{M}}, V)$ be a model based on $\mathcal{F}$. Let $Y \subseteq \mathtt{M}$ be such that $card(Y) = n' \leq 2 \times M$ and $n' \geq 2$. We write LO to denote the set $lo(\mathcal{L}) \uparrow Y$. For all $u \in W$, there is $\underline{\leq} = (a_1, \ldots, a_{n'}) \in LO$ such that $R_{a_1}(u) \subseteq \ldots \subseteq R_{a_{n'}}(u)$. Take any $k \in \{1, \ldots, n'\}$. It is easy to show that for all $k' \in \{1, \ldots, k\}$ and all $p \in \mathtt{F}_0$, $\mathcal{M}, u \models \Box_{a_k} p \Rightarrow \Box_{a_{k'}} p$. Hence $\mathcal{M}, u \models A^{LO}$ since for all $p \in \mathtt{F}_0$, $\mathcal{M}, u \models \bigwedge_{j \in \{2, \ldots, n'\}} \Box_{a_j} p \Rightarrow \Box_{a_{j-1}} p$. **Q.E.D.**

In a similar way we can prove the particular case:

Proposition 6.3. Let $\mathcal{F} = (W, (R_a)_{a \in \mathtt{M}})$ be an $\mathtt{L}$-frame. If $\mathtt{M}$ is finite then

($\star$) $\mathcal{F} \models A^{lo(\mathcal{L})}$ iff,

($\star\star$) for every $u \in W$ there exists $\underline{\leq} \in lo(\mathcal{L})$ such that for all $a, b \in \mathtt{M}$, $a \underline{\leq} b$ implies $R_a(u) \subseteq R_b(u)$.

Propositions 6.2 and 6.3 can be viewed as correspondence results (see e.g. [Ben84]) about the local agreement condition. It should be observed that in Proposition 6.2, the set $X^* = \{A^{lo(\mathcal{L})\uparrow Y} : Y \subseteq \mathtt{M}, 2 \leq card(Y) \leq 2 \times card(lo(\mathcal{L}))\}$ is infinite if $\mathtt{M}$ is infinite. Moreover for all finite subsets Y, Y' of $\mathtt{M}$, if $Y \subseteq Y'$ then $\mathcal{F} \models A^{lo(\mathcal{L})\uparrow Y'} \Rightarrow A^{lo(\mathcal{L})\uparrow Y}$. Thus the set X^* can also be replaced in Proposition 6.2, by $\{A^{lo(\mathcal{L})\uparrow Y} : Y \subseteq \mathtt{M}, card(Y) = 2 \times card(lo(\mathcal{L}))\}$.

Corollary 6.4. For any $\mathtt{L}$-normal system $\mathtt{X}$, if

$$\{A^{lo(\mathcal{L})\uparrow Y} : Y \subseteq \mathtt{M}, 2 \leq card(Y) \leq 2 \times card(lo(\mathcal{L}))\} \subseteq \mathtt{X}$$

then for all $\mathcal{F} \in Fr(\mathtt{X})$ the relations in $\mathcal{F}$ are pairwise in local agreement with the local agreements in $lo(\mathcal{L})$, i.e. for each $\mathcal{F} = (W, (R_a)_{a \in \mathtt{M}}) \in Fr(\mathtt{X})$, for all $w \in W$, there is $\underline{\leq} \in lo(\mathcal{L})$ such that for all $a, b \in \mathtt{M}$, $a \underline{\leq} b$ implies $R_a(w) \subseteq R_b(w)$.

From now on, $\mathtt{X}'_{\mathcal{L}}$ denotes the smallest $\mathtt{L}$-normal system that contains the axiom schemata:

T. $\Box_a p \Rightarrow p$,

B. $p \Rightarrow \Box_a \neg \Box_a \neg p$,

4. $\Box_a p \Rightarrow \Box_a \Box_a p$,

LA $A^{lo(\mathcal{L})\uparrow Y}$ for all finite $Y \subseteq \mathtt{M}$ such that $2 \leq card(Y) \leq 2 \times card(lo(\mathcal{L}))$

As usual, a set $X \subseteq \mathtt{F}$ is said to be $\mathtt{X}'_{\mathcal{L}}$*-consistent* iff there is no finite subset $\{A_1, \ldots, A_k\} \subseteq X$ such that $\neg(A_1 \wedge \ldots \wedge A_k) \in \mathtt{X}'_{\mathcal{L}}$. $X \subseteq \mathtt{F}$ is called a *maximal* $\mathtt{X}'_{\mathcal{L}}$*-consistent* set iff X is $\mathtt{X}'_{\mathcal{L}}$-consistent and for all $A \in \mathtt{F}$ either $A \in X$ or $\neg A \in X$. We write $\Box_a X$ to denote the set $\{A : \Box_a A \in X\}$, where $X \subseteq \mathtt{F}$ and $a \in \mathtt{M}$.

Proposition 6.5. $\mathtt{X}'_{\mathcal{L}} \subseteq Th(X^{\mathcal{F}}_{\mathcal{L}})$.

The proof is standard, considering the correspondence result of Proposition 6.2 as well as the correspondences between **T**, **B** and **4** and the property of reflexivity, symmetry and transitivity (see e.g. [Ben84]). We use the standard construction of the canonical model (see e.g., [Mak66]). The *canonical model* for $\mathtt{X}'_{\mathcal{L}}$ is the triple $\mathcal{M}^c = (W^c, (R^c_a)_{a \in \mathtt{M}}, V^c)$ where

- W^c is the family of all the maximal $\mathtt{X}'_{\mathcal{L}}$-consistent sets,
- For all $X, X' \in W^c$ and all $a \in \mathtt{M}$, $(X, X') \in R^c_a$ iff $\Box_a X \subseteq X'$,
- $V^c(p) = \{X \in W^c \mid p \in X\}$ for all $p \in \mathtt{F}_0$.

Proposition 6.6. $\mathcal{M}^c \in \mathcal{S}$.

Proof: One can prove that the relations in $\mathcal{M}^c$ are equivalence relations. The proof, being quite standard is omitted here. Let $lo(\mathcal{L}) = \{\leq_1, \ldots, \leq_M\}$. In the sequel, we prove that for all $X \in W^c$, there is $\leq \in lo(\mathcal{L})$ such that for all $a, b \in \mathtt{M}$, $a \leq b$ implies $R^c_a(X) \subseteq R^c_b(X)$. Suppose there is $X_0 \in W^c$ such that for all $k \in \{1, \ldots, M\}$, there exist $a_k, a'_k \in \mathtt{M}$ such that $a_k \leq_k a'_k$ and $R^c_{a_k}(X_0) \not\subseteq R^c_{a'_k}(X_0)$. Hence for all $k \in \{1, \ldots, M\}$, there is $X^k_0 \in W^c$ such that $(X_0, X^k_0) \in R^c_{a_k}$ and there is $A_k \in \mathtt{F}$ such that $\Box_{a'_k} A_k \in X_0$ and $A_k \notin X^k_0$.

We write LO to denote the set $LO = \{\leq'_1, \ldots, \leq'_{m'}\} = lo(\mathcal{L}) \uparrow Y_0$ with $Y_0 = \bigcup_{k=1}^{M} \{a_k, a'_k\}$ and $card(Y_0) = n'$ $(2 \leq n' \leq 2 \times M)$. For all $i \in \{1, \ldots, m'\}$, we write $\leq'_i = (a^i_1, \ldots, a^i_{n'})$ and $\alpha(i)$ is an element of $\{1, \ldots, M\}$ such that $\leq'_i = (\leq_{\alpha(i)})_{|Y_0}$. We write A to denote the formula obtained from A^{LO} by simultaneously replacing each p_i in it by $A_{\alpha(i)}$. It is easy to show that $A \in \mathtt{X}'_{\mathcal{L}}$. Thus $A \in X_0$ (every maximal $\mathtt{X}'_{\mathcal{L}}$-consistent set contains the elements of $\mathtt{X}'_{\mathcal{L}}$) and there is $k_0 \in \{1, \ldots, m'\}$ such that $\bigwedge_{j \in \{2, \ldots, n'\}} \Box_{a^{k_0}_j} A_{\alpha(k_0)} \Rightarrow \Box_{a^{k_0}_{j-1}} A_{\alpha(k_0)} \in X_0$ $(B \vee B' \in X_0$ iff $B \in X_0$ or $B' \in X_0$ for all $B, B' \in \mathtt{F})$. Since $a_{\alpha(k_0)} \leq_{\alpha(k_0)} a'_{\alpha(k_0)}$ and $\Box_{a'_{\alpha(k_0)}} A_{\alpha(k_0)} \in X_0$, then $\Box_{a_{\alpha(k_0)}} A_{\alpha(k_0)} \in X_0$ by Modus Ponens. Thus $A_{\alpha(k_0)} \in X^{\alpha(k_0)}_0$, since $(X_0, X^{\alpha(k_0)}_0) \in R^c_{a_{\alpha(k_0)}}$, which leads to a contradiction. **Q.E.D.**

One can prove in a standard way that for all $A \in \mathtt{F}$ and all $X \in W^c$, $A \in X$ iff $\mathcal{M}^c, X \models A$ (see e.g. [HC84]). Basing on this, one can establish completeness of $\mathtt{X}'_{\mathcal{L}}$ with respect to $\mathcal{L}$.

Proposition 6.7. $Th(X^{\mathcal{F}}_{\mathcal{L}}) \subseteq \mathtt{X}'_{\mathcal{L}}$.

Proof: Assume that $A \in Th(X^{\mathcal{F}}_{\mathcal{L}})$ and suppose $A \notin \mathtt{X}'_{\mathcal{L}}$. Hence there is a maximal $\mathtt{X}'_{\mathcal{L}}$-consistent set, say X_0, such that $\neg A \in X_0$ (every $\mathtt{X}'_{\mathcal{L}}$-consistent set can be extended[12] to a maximal $\mathtt{X}'_{\mathcal{L}}$-consistent set). Then $\mathcal{M}^c \not\models A$. By the remark above, since $(W^c, (R^c_a)_{a\in\mathtt{M}}) \in X^{\mathcal{F}}_{\mathcal{L}}$ (because $\mathcal{M}^c \in \mathcal{S}$ by Proposition 6.6) this yields $A \notin Th(X^{\mathcal{F}}_{\mathcal{L}})$, which leads to a contradiction. **Q.E.D.**

Proposition 6.8. The following propositions hold:

1. $\mathtt{X}_{\mathcal{L}} = \mathtt{X}'_{\mathcal{L}}$.
2. $\mathtt{X}_{\mathcal{L}}$ is a subframe logic (see [Wol93]), that is $Fr(\mathtt{X}_{\mathcal{L}})$ is closed under subframes.
3. $Th(Fr(\mathtt{X}_{\mathcal{L}})) = \mathtt{X}_{\mathcal{L}}$.

Proof: (1) By Proposition 6.5 and 6.7, $Th(X^{\mathcal{F}}_{\mathcal{L}}) = \mathtt{X}'_{\mathcal{L}}$. Since $Th(X^{\mathcal{F}}_{\mathcal{L}}) = \mathtt{X}_{\mathcal{L}}$, then $\mathtt{X}_{\mathcal{L}} = \mathtt{X}'_{\mathcal{L}}$, whence $\mathtt{X}'_{\mathcal{L}}$ provides a complete Hilbert-style axiomatization for $\mathcal{L}$.
(2) By Corollary 6.4, $Fr(\mathtt{X}_{\mathcal{L}}) = X^{\mathcal{F}}_{\mathcal{L}}$. Since $X^{\mathcal{F}}_{\mathcal{L}}$ is closed under subframes, then $Fr(\mathtt{X}_{\mathcal{L}})$ is also closed under subframes.
(3) Direct consequence of (1) and (2). **Q.E.D.**

It is now a routine task to find complete axiomatizations for the LA-K-logics, the LA-S4-logics, the LA-T-logics (and for some other ones) when $lo(\mathcal{L})$ is finite. Moreover the axiomatization of LGM' ($card(lo(LGM')) = 1$) turns out to be a particular case of the present construction. When $\mathtt{M}$ is finite, **LA** can be replaced by $A^{lo(\mathcal{L})}$ in the definition of $\mathtt{X}'_{\mathcal{L}}$ and the Propositions 6.5-6.8 still hold true.

7 Concluding Remarks

Among the classes of logics defined in the paper, the LA-logics play a special role, not only because of their relationships with the indiscernibility relations in information systems but also because in their cases the standard filtration construction fails. This class includes for instance some logics strongly related to the logics DALLA and LGM, respectively, introduced in [Gar86, Nak92]. It has been proved herein that each LA-logic $\mathcal{L}$ has the strong finite model property (see Proposition 4.5). Moreover the $\mathcal{L}$-validity problem is decidable when $\mathcal{L}$ is lo-decidable and lo-complete (see Proposition 4.7), and the satisfiability problem is **NP**-complete under the hypothesis that the set of modal expressions is finite (see Proposition 4.8). As a side-effect of our work, we have defined a simple complete axiomatization for LGM which has been until now an open problem stated in [Nak92] (see Section 5.2). In Section 6, complete axiomatizations have been defined for LA-logics characterized by a *finite* set of local agreements (a particular case is when the set of modal expressions is finite). Although the construction introduced in Section 4.1 seems to be limited to the set of LA-logics it provides an elegant construction strongly guided by the properties of relations satisfying the local agreement condition. This technique cannot be applied in a straightforward way to the family of logics $DALD^{i'}_i$ defined in [DO97].

[12] It holds even if $\mathtt{M}$ is uncountable.

Further investigations of logics determined by classes of frames with relations agreeing locally are possible. Concerning computational complexity questions, we believe that the following open problems are worth investigating:

1. What is the complexity class of the satisfiability problem for the decidable LA-logics? We have answered this question in the case when $\mathtt{M}$ is finite but for instance the case of infinite $\mathtt{M}$ and finite $lo(\mathcal{L})$ is open.
2. What is the complexity class of the satisfiability problem for logics sharing the same language with the LA-logics but not requiring equivalence relations in the semantical structures (while preserving the local agreement condition)? The standard filtration constructions are enough to prove the finite model property for some of these logics (for instance, when only reflexivity of the relations is required).
3. The complexity class of the satisfiability problem for the logics defined in [Bal97, Waa] similar in some aspects to the LA-logics would be also worth investigating.

Some other questions that remain unanswered:

1. Is there a filtration-like construction that can prove the strong finite model property for the LA-S4-logics, in case this property holds? Proposition 3.2 and the results in Section 4 do not cover either the LA-S4-logics or the LA-K4-logics.
2. Is it true that if a monomodal logic $\mathcal{L}$ (characterized by a given class of frames) has the strong finite model property, then so does every LA-$\mathcal{L}$-logic?
3. Two equivalence relations R and S are said to be *permutable* (see e.g. [Pal95]) iff $R;S = S;R$ (where ';' is the composition operator) iff $R;S$ is an equivalence relation. If the equivalence relations R and S are in local agreement then they are also permutable -but not conversely. It is an open question whether the results in the present work can be extended to the case when the local agreement condition is generalized to permutability.

   Actually the question has been open for me until Maarten Marx has communicated me recently that the logic determined by all the frames $(W, (R_i)_{i\in\{1,2,3\}})$ such that the R_i's are equivalence relations and for all $i, j \in \{1, 2, 3\}$, $R_i; R_j = R_j; R_i$ has an *undecidable* satisfiability problem due to a result proved by Roger Maddux in [Mad80]. So assume that the *finite* set $\mathtt{M}$ of modal expressions has at least three elements. For all $\oplus \in \{\cup, \cap, ;\}$, we write $\mathrm{S5}^{\oplus}$ to denote the logic determined by all the frames $(W, (R_a)_{a\in\mathtt{M}})$ such that

   - for all $a \in \mathtt{M}$, R_a is an equivalence relation and,
   - for all $a, b \in \mathtt{M}$, $R_a \oplus R_b$ is transitive.

   Hence,

   (a) The satisfiability problem for $\mathrm{S5}^{\cup}$ is decidable and it is **NP**-complete (see Section 4.2).

   (b) The satisfiability problem for $\mathrm{S5}^{\cap}$ is decidable and it is **PSPACE**-complete (see e.g. [HM92]).

   (c) The satisfiability problem for $\mathrm{S5}^{;}$ is undecidable (see [Mad80]).

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