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Optimal PI controller rejecting disturbance for ARZ traffic model

Lina Guan, Liguo Zhang and Christophe Prieur

Abstract — Traffic control of congestion regimes is considered in this paper. A perturbed distributed parameter model is used, and a boundary control is designed to reject the perturbations. More precisely an optimal proportional-integral (PI) feedback control law is computed to maximally reject the disturbances, and to stabilize the traffic in congested regime. The disturbance applies at the boundary of the linearized Aw-Rascle-Zhang (ARZ) model. Therefore the disturbance operator is unbounded, rendering the control problem very challenging. In order to analyze and design the optimal PI controller for this infinite-dimensional system, the \( L_2 \) gain is computed to estimate the disturbance rejection. Numerically tractable conditions are computed and written with linear matrix inequalities (LMIs).

As a result, the estimation of an upper bound of the \( L_2 \) gain, from the disturbance to the controlled output, can be formulated as an optimization problem with LMI constraints. The validity of this method is checked on simulations of the nonlinear ARZ model in closed-loop with this optimal PI controller.

I. INTRODUCTION

Traffic congestion usually happens, for instance, when the number of vehicles travelling on a road exceeds the capacity of road transportation during rush hour. Congestion regime happens also when vehicles go through a bottleneck resulting from changed traffic conditions downstream such as a slowly moving oversize vehicle, a speed limit sign, or a narrower road section. Traffic congestion can cause a series of issues including additional travel time, increased fuel consumption and CO\(_2\) emission, higher traffic accident rate and so on.

In order to keep traffic flow stable, the PI feedback control strategy can be used due to its actual superiority of attenuating disturbances in engineering. In control theory, PI feedback control method is a fruitful paradigm for the industrial and real-life applications (see e.g., [1]), and for infinite-dimensional systems. Indeed, dissipative PI boundary conditions have been given for one-dimensional linear hyperbolic systems of balance laws in [3]. Lyapunov stability of hyperbolic systems of conservation laws is achieved for boundary control law with integral action in [7]. For the stability of a cascaded network consisting of several \( 2 \times 2 \) linear hyperbolic systems, PI output feedback controllers are designed to reject disturbance and regulate output to the desired points in [9]. A PI controller only depending on one side measurable angular velocity is used to analyze the stability of inhomogeneous drilling model in [8]. In [11], a PI boundary feedback controller is designed to stabilize the oscillations of the traffic parameters on a freeway segment by using on-ramp metering and variable speed control.

The contribution of this paper is the design of an optimal PI boundary feedback controller maximizing the disturbance rejection, and consequently strengthens the robustness of the traffic dynamics. Computing the value of \( L_2 \) gain from disturbance to output is a classical control method to measure the disturbance rejection capacity (see the recent survey [6]). In the context of congestion regimes with traffic control, it is a difficult and hard problem because it is necessary to consider infinite-dimensional models as the Aw-Rascle-Zhang (ARZ) model [2], [10]. Disturbances apply at the boundary so that the input operator is unbounded, and finite-dimensional approximations may not be sufficient to design optimal controller. In this paper we succeed to design an optimal PI controller computed on a linearized model. Our method yields numerically tractable conditions since convex conditions are derived for the optimal PI controller design. Moreover we check on numerical simulations that the nonlinear ARZ model in closed-loop with this controller is stable.

The paper is organized as follows. In Section II, the linearized ARZ traffic flow model around an uniform steady state is presented and a PI boundary feedback controller with disturbance term is derived. The LMI conditions for finite-gain \( L_2 \) stability and an upper bound of the \( L_2 \) gain are given in Section III. In Section IV, the estimation of an upper bound of the \( L_2 \) gain is addressed in terms of optimization problem with LMI constraints, and the results of the numerical simulation verify the validity of this method. Conclusion is given in Section V.

II. LINEARIZED ARZ TRAFFIC FLOW MODEL

From a macroscopic perspective, the Aw-Rascle-Zhang model describes the traffic flow dynamics as

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0,
\]

\[
\frac{\partial v}{\partial t} + (v - \rho \frac{\partial v}{\partial \rho}) \frac{\partial v}{\partial x} = \frac{G(\rho)-v}{s},
\]

with an independent space variable \( x \) on a finite interval \((0, L)\) and an independent time variable \( t \) in \([0, \infty)\), where...
\( \rho(x, t) \) is the vehicle density, \( v(x, t) \) is the average speed, \( \tau \) is the relaxation time subject to driving behavior, and \( L \) is the length of the considered road segment.

The speed-density relation function \( G(\rho) \) is given by the Greenshields model in [4],

\[
G(\rho) = v_f (1 - \frac{\rho}{\rho_m}),
\]

(3)

where \( v_f \) is the free flow speed, and \( \rho_m \) is the maximum density.

The traffic pressure \( p(\rho) \) is defined as follows,

\[
p(\rho) = v_f - G(\rho) = \frac{v_f}{\rho_m} \rho.
\]

(4)

Let \( \omega = v + \frac{v_f}{\rho_m} \rho \), then (1)-(2) can be rewritten as

\[
\partial_t \left[ \frac{\omega}{v} \right] + \left[ \frac{v}{0} \right] \partial_x \left[ \frac{\omega}{v} \right] = \left[ \frac{v_f - \omega}{v} \right].
\]

(5)

Assume that this system is strictly hyperbolic and lies in the traffic congestion regime, i.e., the sign of the characteristic value \( 2v - \omega \) in equation (5) is negative.

Let \( (\omega^*, v^*)^T \) be an uniform steady state of the system (5), define \( \omega^* = v_f \) and select the constant \( v^* \) such that \( 2v^* - \omega^* < 0 \), then the following linearized ARZ model can be derived,

\[
\partial_t \left[ \frac{\omega}{v} \right] + \left[ \lambda_1 \frac{1}{0} \right] \partial_x \left[ \frac{\omega}{v} \right] = \left[ \lambda_2 \frac{0}{0} \right] \frac{\omega}{v},
\]

(6)

where \( \omega = \omega - \omega^*, \bar{v} = v - v^* \), with the constant characteristic velocities

\[
\lambda_1 = v^*,
\]

\[
\lambda_2 = 2v^* - \omega^*.
\]

(7)

(8)

According to the flow conservation at the inlet (see Fig. 1), we have

\[
p_{in} + \bar{p}(t) + r(t) = \rho(0, t)v(0, t),
\]

(9)

where \( p_{in} \) is the constant traffic demand, \( \bar{p}(t) \) is the fluctuation of the flow rate resulting from physical perturbation, and \( r(t) \) is on-ramp metering. Define nominal flux rate \( r^* \) satisfying the relation \( p_{in} + r^* = \rho^* v^* \), here density \( \rho^* \) is given by \( \rho^* = \frac{v_m}{v_f} (\omega^* - v^*) \).

The following PI boundary feedback control law is introduced, for all \( t \neq 0 \),

\[
r(t) = r^* + k_{P1} (\rho(L, t) - \rho^*)
\]

\[+ k_{I1} \int_0^t (\rho(L, \sigma) - \rho^*) d\sigma,
\]

(10)

\[
v(L, t) = v^* + k_{P2} (v(0, t) - v^*)
\]

\[+ k_{I2} \int_0^t (v(0, \sigma) - v^*) d\sigma,
\]

(11)

where the inflow speed \( v(0, t) \) and the outflow density \( \rho(L, t) \) are measurement outputs, the inlet on-ramp metering \( r(t) \) and the outlet variable speed \( v(L, t) \) are boundary control inputs, and \( k_{P1}, k_{I1}, k_{P2}, k_{I2} \) are proportional and integral tuning gains.

Let \( \bar{\rho} = \rho - \rho^* \), then (10)-(11) can be rewritten as

\[
r(t) = r^* + k_{P1} \bar{\rho}(L, t) + k_{I1} \int_0^t \bar{\rho}(L, \sigma) d\sigma,
\]

(12)

\[
v(L, t) = v^* + k_{P2} \bar{v}(0, t) + k_{I2} \int_0^t \bar{v}(0, \sigma) d\sigma.
\]

(13)

Combining feedback control laws (12)-(13) with (9) and linearizing, the following boundary condition can be attained,

\[
\bar{p}(t) + k_{P1} \bar{\rho}(L, t) + k_{I1} \int_0^t \bar{\rho}(L, \sigma) d\sigma = v^* \bar{v}(0, t) + \rho^* \bar{v}(0, t).
\]

(14)

Getting rid of the high order terms, we deduce from (12) and (14),

\[
\bar{\omega}(0, t) = \bar{v}(0, t) + \frac{v_f}{\rho_m} \bar{\rho}(0, t)
\]

\[
= \frac{v_f \bar{p}(t)}{\rho_m v^*} + (1 - \frac{v_f \rho^*}{\rho_m v^*}) \bar{v}(0, t) + \frac{v_f k_{P1}}{\rho_m v^*} \bar{\rho}(L, t)
\]

\[+ \frac{v_f k_{I1}}{\rho_m v^*} \int_0^t \bar{\rho}(L, \sigma) d\sigma
\]

\[
+ (1 - \frac{v_f \rho^*}{\rho_m v^*} - \frac{k_{P2} k_{I2}}{v^*}) \bar{v}(0, t)
\]

\[
- \frac{k_{P1} k_{I2}}{v^*} + \frac{k_{I1} k_{P2}}{v^*} \int_0^t \bar{v}(0, \sigma) d\sigma.
\]

(15)

Combining (13) and (15) together for the system (6), the following PI boundary feedback controller can be derived,

\[
\left[ \frac{\bar{\omega}(0, t)}{\bar{v}(L, t)} \right] = K_P \left[ \frac{\bar{\omega}(L, t)}{\bar{v}(0, t)} \right] + K_I \int_0^t \left[ \frac{\bar{\omega}(L, \sigma)}{\bar{v}(0, \sigma)} \right] d\sigma + \theta(t),
\]

(16)

where

\[
K_P = \left[ \frac{k_{P1}}{v^*} \frac{1 - \frac{v_f \rho^*}{\rho_m v^*} - \frac{k_{P1} k_{P2}}{v^*}}{k_{P2}} \right]
\]

(17)

and

\[
K_I = \left[ \frac{k_{I1}}{v^*} \frac{-k_{P1} k_{I2} - k_{I1} k_{P2}}{k_{I2}} \right]
\]

(18)
are respectively proportional and integral tuning matrices, and
\[
\theta(t) = \begin{bmatrix} v_f^\omega(t) & -v_f^\nu(t) & 0 \end{bmatrix}^T
\] (19)
is an external disturbance input vector. Assume the tuning parameters \(k_{11}\) and \(k_{12}\) are different to 0, so that the matrix \(K_I\) is invertible.

Letting \(R(x,t) = (\bar{\omega}(x,t), \bar{v}(x,t))^T\), for all \(t \in [0, +\infty)\) and all \(x \in [0, L]\), system (6) and (16) can be rewritten as
\[
\begin{align*}
& \partial_t R(x,t) + \partial_x R(x,t) = MR(x,t), \\
& R_{in}(t) = K_P R_{out}(t) + K_I \int_0^t R_{out}(\sigma) d\sigma + \theta(t),
\end{align*}
\] (20)
where
\[
\begin{align*}
\Lambda &= \text{diag}\{\lambda_1, -\lambda_2\}, \\
M &= \begin{bmatrix} -1 & 0 \\
0 & -1 \end{bmatrix}, \\
R_{in}(t) &= (\bar{\omega}(0,t), \bar{v}(0,t))^T, \\
R_{out}(t) &= (\bar{\omega}(L,t), \bar{v}(L,t))^T.
\end{align*}
\] (23)

The disturbance must be constrained by an upper limit of energy in order to avoid the boundless of the state and output. Assume that the energy of time-varying disturbance vector \(\theta(t)\) is limited in \(L_2\) space by a constant positive value \(\delta\) such that
\[
\|\theta(t)\|_{L_2} + \|\dot{\theta}(t)\|_{L_2} \\
\leq \sqrt{\delta^{-1}} - \frac{1}{\delta}, \quad \forall t \geq 0.
\] (26)

III. FINITE-GAIN \(L_2\) STABLE AND \(L_2\) GAIN

As stated in [5], if there exist nonnegative constants \(k\) and \(g\) such that
\[
\|y\|_{L_2} \leq k \|u\|_{L_2} + g,
\] (27)
for all \(u \in L_2\) space, where \(u\) and \(y\) are respectively the system disturbance input and controlled output, and \(g\) is a function of the initial condition, then the system is finite-gain \(L_2\) stable and has \(L_2\) gain less than or equal to \(k\).

Let \(X(t) = \int_0^t R_{out}(\sigma) d\sigma + K_I^{-1}\theta(t)\),
\[
(28)
\]
where \(t \in [0, \infty)\), then
\[
\dot{X}(t) = R_{out}(t) + K_I^{-1}\dot{\theta}(t),
\] (29) and (21) is rewritten as
\[
R_{in}(t) = K_P R_{out}(t) + K_I X(t).
\] (30)

Consider the initial conditions
\[
\begin{align*}
R(x,0) &= \begin{bmatrix} \bar{\omega}(x,0) \\
\bar{v}(x,0) \end{bmatrix} = R_0(x), \\
X(0) &= K_I^{-1}\theta(0) = K_I^{-1}\theta_0,
\end{align*}
\] (31)
and
\[
\begin{align*}
\Omega &= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{bmatrix} \\
&= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{bmatrix} \\
&= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{bmatrix} \geq 0.
\] (33)

where
\[
\begin{align*}
\Omega_{11} &= -\mu K_P (x) - M^T P_1 (x) - P_1 (x) M, \\
\Omega_{12} &= \Omega_{21} = -P_2 (x), \\
\Omega_{13} &= -\frac{\mu}{2} K_P (x) - M^T P_3 (x), \\
\Omega_{22} &= \frac{1}{L} \int_0^L (K_P^T |\Lambda| P_1 E_3 K_P - |\Lambda| P_1) \frac{1}{\eta} K_P^T K_P, \\
\Omega_{23} &= -\frac{1}{L} \int_0^L (K_P^T |\Lambda| P_1 E_3 K_I + 2K_P^T |\Lambda| E_1 P_3 (0) - 2|\Lambda| E_2 P_3 (0) + \frac{1}{\eta} K_P^T K_I) - P_2, \\
\Omega_{33} &= -\frac{1}{L} \int_0^L (K_I^T |\Lambda| P_1 E_3 K_I + 2K_I^T |\Lambda| E_1 P_3 (0) \\
&\quad + \frac{1}{\eta} K_I^T K_I), \\
\Omega_{34} &= P_2, \\
\Omega_{44} &= \frac{1}{L^2} I_2,
\end{align*}
\] (34) where \(|\Lambda| = \text{diag}\{\lambda_1, \lambda_2\}\), \(P_1 (x) = P_1 \text{diag}\{e^{-\nu(L-x)}, e^{-\nu x}\}\), \(P_3 (x) = P_3 \text{diag}\{e^{-\frac{\nu}{2}(L-x)}, e^{-\frac{\nu}{2} x}\}, E_1 = \text{diag}\{1, e^{\frac{\nu}{2} x}\}\), \(E_2 = \text{diag}\{e^{\frac{\nu}{2} x}, 1\}, E_3 = (E_1 E_2)^{-1}\), null matrix \(O_2\) and identity matrix \(I_2\) are in \(\mathbb{R}^{2 \times 2}\), then for all initial conditions (31) and (32), and \(\theta(t)\) satisfying (26), the system (20), (29)-(30) is finite-gain \(L_2\) stable and the \(L_2\) gain from \((\theta(t), \dot{\theta}(t))^T\) to \(R_{in}\) is less than or equal to \(\sqrt{\eta m}\) for a positive constant \(m\) depending only on \(K_I\).

**Proof:** The following candidate Lyapunov function is defined,
\[
V(R(x,t), X(t)) = \int_0^L \begin{bmatrix} R(x,t) \end{bmatrix}^T P_1 (x) X(t) \begin{bmatrix} \cdot \end{bmatrix} \begin{bmatrix} R(x,t) \end{bmatrix} dx \\
= \int_0^L \begin{bmatrix} R(x,t) \end{bmatrix}^T \begin{bmatrix} P_1 (x) & P_3 (x) \end{bmatrix} \begin{bmatrix} R(x,t) \\
X(t) \end{bmatrix} dx \\
+ \int_0^L \begin{bmatrix} R(x,t) \end{bmatrix}^T P_3 (x) X(t) \\
+ \begin{bmatrix} X^T (t) \end{bmatrix}^T \begin{bmatrix} P_3^T (x) R(x,t) \end{bmatrix} dx + LX^T (t) P_2 X(t). \] (42)
The time derivative of $V$ along the solutions to system (20), (29)-(30) is written as follows,

$$
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3,
$$

with

$$
\dot{V}_1 = \int_0^L [(\partial_t R) P_1(x) R + R^T P_3(x) \partial_t R] dx, \quad (44)
$$

$$
\dot{V}_2 = \int_0^L [(\partial_t R) P_3(x) X + R^T P_3(x) \dot{X}] + X^T P_3(x) R + X^T P_3(x) \partial_t R] dx, \quad (45)
$$

$$
\dot{V}_3 = L \dot{X}^T P_2 X + L X^T P_2 \dot{X}. \quad (46)
$$

Using (20) and performing an integration by parts in (44), the following result is achieved,

$$
\dot{V}_1 = -R^T \Lambda P_1(x) R + \int_0^L R^T [\Lambda P'_1(x) + M^T P_1(x) + P_1(x) M] R dx.
$$

From (24) and (25), previous equation (47) can be rewritten as,

$$
\dot{V}_1 = R^T (0, t) \Lambda P_1(0) R - R^T (L, t) \Lambda P_1(L) R + \int_0^L R^T [\Lambda P'_1(x)] + M^T P_1(x) + P_1(x) M] R dx. \quad (48)
$$

From (20) and (29), the following result can be derived from (45) by using an integration by parts,

$$
\dot{V}_2 = [-R^T \Lambda P_3(x) X - X^T P_3^T (x) \Lambda R]_0^L + \int_0^L [\frac{1}{2} R^T \Lambda P_3(x) X + \frac{\mu}{2} X^T P_3^T (x) \Lambda R
$$

$$
+ R^T P_3(x) R_{out}(t) + R^T P_3(x) (K_I^{-1} \dot{\theta}(t)) + (K_I^{-1} \dot{\theta}(t))^T P_3^T (x) R + R^T M^T P_3(x) X + X^T P_3^T (x) MR] dx.
$$

From (24) and (25), previous equation (49) can be rewritten as,

$$
\dot{V}_2 = 2 R_{out}^T (K_I^T | \Lambda E_2 P_3(0) X
$$

$$
+ 2 X^T K_I^T | \Lambda E_2 P_3(0) X - 2 R_{out}^T (t) | \Lambda E_2 P_3(0) X
$$

$$
+ \int_0^L [\frac{\mu}{2} R^T \Lambda P_3(x) X + \frac{\mu}{2} X^T P_3^T (x) \Lambda R
$$

$$
+ R^T P_3(x) R_{out}(t) + R^T P_3(x) (K_I^{-1} \dot{\theta}(t)) + R_{out}^T (t) P_3^T (x) R + (K_I^{-1} \dot{\theta}(t))^T P_3^T (x) R
$$

$$
+ R^T M^T P_3(x) X + X^T P_3^T (x) MR] dx. \quad (50)
$$

From (29), we can convert (46) into

$$
\dot{V}_3 = \eta R_{out}^T (t) P_2 X + L (K_I^{-1} \dot{\theta}(t))^T P_2 X
$$

$$
+ L X^T P_2 R_{out}(t) + L X^T P_2 (K_I^{-1} \dot{\theta}(t)). \quad (51)
$$

Then from (48), (50) and (51), the following result can be derived,

$$
\dot{V} + \frac{1}{\eta} R_{in}^T (t) R_{in} (t) - \dot{\theta}^T (t) \dot{\theta}(t)
$$

$$
- \dot{\theta}^T (t) (K_I^{-1})^T K_I^{-1} \dot{\theta}(t)
$$

$$
\leq - \int_0^L \left[ \frac{R}{X} \right]^T \Omega \left[ \frac{R}{X} \right] dx, \quad (52)
$$

where $\Omega$ satisfies (33).

Thus, for any $\dot{\theta}(t)$ satisfying (26),

$$
\dot{V} \leq - \frac{1}{\eta} R_{in}^T (t) R_{in} (t) + \dot{\theta}^T (t) \dot{\theta}(t)
$$

$$
+ \dot{\theta}^T (t) (K_I^{-1})^T K_I^{-1} \dot{\theta}(t). \quad (53)
$$

By integrating the both sides of the previous inequality (53) over the interval $[0, t]$, one can derive that

$$
\int_0^t \dot{V}(R(x, \sigma), X(\sigma)) \ d\sigma
$$

$$
\leq - \frac{1}{\eta} \int_0^t R_{in}^T (\sigma) R_{in} (\sigma) \ d\sigma + m \int_0^t \dot{\theta}^T (\sigma) \dot{\theta}(\sigma) \ d\sigma
$$

$$
+ m \int_0^t \dot{\theta}^T (\sigma) \dot{\theta}(\sigma) \ d\sigma, \quad (54)
$$

where $m = \max \{1, \Lambda e^\sigma((K_I^{-1})^T K_I^{-1}) \}$. Then from $V(R(x, t), X(t)) \geq 0$,

$$
\int_0^t R_{in}^T (\sigma) R_{in} (\sigma) \ d\sigma
$$

$$
\leq \eta V(R_0(x), X(0)) + \eta m \int_0^t \dot{\theta}^T (\sigma) \dot{\theta}(\sigma) \ d\sigma
$$

$$
+ \eta m \int_0^t \dot{\theta}^T (\sigma) \dot{\theta}(\sigma) \ d\sigma. \quad (55)
$$

Using the inequality $a^2 + b^2 \leq (a + b)^2$ for nonnegative numbers $a$ and $b$ and taking the square roots, one can obtain that

$$
\| R_{in}(t) \|_{L_2} \leq \sqrt{\eta V(R_0(x), X(0))}
$$

$$
+ \sqrt{\eta m (\| \dot{\theta}(t) \|_{L_2} + \| \dot{\theta}(t) \|_{L_2})}, \quad (56)
$$

$$
\| R_{in}(t) \|_{L_2} \leq \sqrt{\eta V(R_0(x), X(0))}
$$

$$
+ \sqrt{\eta m (\| \dot{\theta}(t) \|_{L_2} + \| \dot{\theta}(t) \|_{L_2})}, \quad (56)
$$

$$
\| R_{in}(t) \|_{L_2} \leq \sqrt{\eta V(R_0(x), X(0))}
$$

$$
+ \sqrt{\eta m (\| \dot{\theta}(t) \|_{L_2} + \| \dot{\theta}(t) \|_{L_2})}, \quad (56)
$$

$$
\| R_{in}(t) \|_{L_2} \leq \sqrt{\eta V(R_0(x), X(0))}
$$

$$
+ \sqrt{\eta m (\| \dot{\theta}(t) \|_{L_2} + \| \dot{\theta}(t) \|_{L_2})}, \quad (56)
$$
where the bias term $\sqrt{\eta V(R_0(x), X(0))}$ depends on the initial conditions $R_0(x)$ and $X(0)$.
Note that if $R_0(x) = 0$ and $\theta_0 = 0$,
$$\|R_m(t)\|_{L_2} \leq \sqrt{m\eta} (\|\theta(t)\|_{L_2} + \|\dot{\theta}(t)\|_{L_2}).$$  \hspace{1cm} (57)

Therefore, the system (20), (29)-(30) is finite-gain $L_2$ stable and has $L_2$ gain which is less than or equal to $\sqrt{m\eta}$.

IV. NUMERICAL ISSUES

A. Optimization

The smaller value of the $L_2$ gain, the stronger the capacity of disturbance rejection is. Based on Theorem 3.1, the following optimization problem can be considered to estimate $L_2$ gain,
$$\min_{P_1, P_2, P_3, \eta} \eta$$  \hspace{1cm} (58)
subject to relations (33).

B. Simulations

In order to seek the optimal values of parameter matrices $K_P, K_I$ through the numerical simulation of optimization problem, we consider a road segment with parameters, $\rho_m = 213.3 \text{veh.}/\text{km}$, $v_f = 160 \text{km}/\text{h}$, $L = 1 \text{km}$, and $\tau = 60 \text{s}$.
The initial conditions are
$$\rho(x, 0) = \rho^* + 0.8 \sin 4\pi x,$$
$$v(x, 0) = v^* + 1.8 \cos 4\pi x,$$
where the steady state $(\rho^*, v^*)^T = (120, 70)^T$ satisfies the ARZ equations (1) and (2). With given $\delta = 0.6$ in (26), the disturbance $\bar{p}(t)$ is given as Fig. 2.

By solving optimization problem (58), we can derive the relation between $\eta_{\min}$ (the minimal value of $\eta$) and $\mu$ in Fig. 3. In particular, for the value of $\eta_{\min} = 1.1113 \times 10^3$, $\mu = 0.09$ should be selected. Moreover, we solve the optimization problem by computing the best tuning parameters $k_{P1}, k_{P2}, k_{I1}, k_{I2}$, and we get $k_{P1} = -20$, $k_{P2} = -0.1$, $k_{I1} = -20$, $k_{I2} = -0.5$, and $m = 14.2518$.
The corresponding parameter matrices $K_P$ and $K_I$ of PI boundary controller in (16) are
$$K_P = \begin{bmatrix} -0.2857 & -0.3143 \\ 0 & -0.1 \end{bmatrix},$$
$$K_I = \begin{bmatrix} -0.2857 & -0.1714 \\ 0 & -0.5 \end{bmatrix}.$$  

Under the optimal values of $K_P, K_I$, the state $(\tilde{\omega}, \tilde{v})^T$ of the linearized ARZ traffic system (20) converges to the zero steady state, as seen in Fig. 4 and Fig. 5. In Fig. 6, the evolution of the two components of $R_m(t)$ given by (30) tends to be stable as time goes. We can observe in Fig. 7 and Fig. 8 that the state $(\rho, v)^T$ of the nonlinear ARZ system (1)-(2) exponentially converges to the steady state $(\rho^*, v^*)^T$.

V. CONCLUSION

This paper addressed the problem of seeking the optimal PI boundary feedback controller to maximize the capacity of
disturbance rejection for the linear traffic flux system. The conditions for finite-gain $L_2$ stability and the computation of the $L_2$ gain were given for the linearized ARZ system. Based on these conditions, we formulated and solved an optimization problem to derive the optimal PI controller.

Theoretical result states the disturbance rejection for the linearized system. Moreover numerical simulation emphasize the interest of this optimal controller for the nonlinear model. Many optimization problems of traffic flux systems remain open. The class of nonlinear traffic flux systems with $L_2$ disturbance or other disturbance measurement methods could be studied to maximize the disturbance rejection capacity. The extension of optimization problem to the network of roads could be also of interest.

REFERENCES