



HAL
open science

Adaptive observer design for a class of Lipschitz nonlinear systems with multirate outputs and uncertainties: Application to attitude estimation with gyro bias

Tristan Bonargent, Tomas Menard, Olivier Gehan, Eric Pigeon

► To cite this version:

Tristan Bonargent, Tomas Menard, Olivier Gehan, Eric Pigeon. Adaptive observer design for a class of Lipschitz nonlinear systems with multirate outputs and uncertainties: Application to attitude estimation with gyro bias. *International Journal of Robust and Nonlinear Control*, 2021, 31 (8), pp.3137-3162. 10.1002/rnc.5439 . hal-03191378

HAL Id: hal-03191378

<https://hal.science/hal-03191378>

Submitted on 7 Apr 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RESEARCH ARTICLE - ACCEPTED VERSION

Adaptive observer design for a class of Lipschitz nonlinear systems with multi-rate outputs and uncertainties - Application to attitude estimation with gyro bias

Tristan Bonargent^{*1,2} | Tomas Ménard¹ | Olivier Gehan¹ | Eric Pigeon¹

¹Normandie Univ, Unicaen, Ensicaen, LAC (EA 7478), 6 Bd du Maréchal Juin, Caen, France.

²Kad!, Espace Conquéran, 3 Place Jean Nouzille, 14000 Caen, France

Correspondence

*Corresponding author: Tristan Bonargent.
Email: tristan.bonargent@unicaen.fr

Summary

The design of an adaptive observer for a class of multi-input Multi-output (MIMO) Lipschitz nonlinear systems with uncertainties and multi-rated outputs is proposed in this paper. The sampling instants of the different outputs can be asynchronous and irregular. The exponential convergence of the proposed observer is proved with the help of a Lyapunov approach. The stability is ensured, provided that some conditions, including a persistent excitation condition and a maximum value condition on the sample rate of each output, are verified. The performances of the observer are illustrated through simulations of the attitude estimation problem using sensors from an inertial measurement unit and considering gyro bias.

KEYWORDS:

Adaptive observer, multi-rate, uncertainties, attitude estimation

1 | INTRODUCTION

Adaptive observers have received a lot of attention during the last decades. These observers are used to simultaneously estimate the state of a system and some unknown parameters. Their various applications such as fault detection, adaptive control or signal transmission show the importance of the given problem (see for instance [2, 23]).

Most of the works dedicated to adaptive observer design consider systems whose output is available continuously with respect to time. Early contributions have been concerned with linear invariant systems such as in [15] and [17], and thereafter with linear time-varying systems such as in [27, 26]. A consequent number of papers has then dealt with non-linear systems with linear parametrization. A first idea has been to seek for new coordinates such that the error dynamics are linear. Such changes of coordinates are usually related to some canonical forms (see [3, 19, 18]). Using Lyapunov functions, some other approaches do not require to linearize the systems. For example, delayed observers have been used in [22] or high gain design in [25, 4, 8]. Then the consideration of a larger class of system with non-linear parametrization has received much attention. In fact, in order to deal with this problem, several approaches have been considered in the literature. Indeed, a min-max optimization algorithm has been used in [16] to overcome the limits of the gradient descent method for adaptive systems with concave/convex parametrization. Similarly, an optimization approach for adaptive control has been used in [14]. Some high-gain based observers have been proposed for systems with a triangular structure. Indeed, Single-Input Single-Output (SISO) systems have been considered in [10] and [20] and MIMO systems in [9]. An observer based on the concept of weakly attracting sets has also been presented in [24], but which does not guarantee the asymptotic stability.

While most of the works consider continuous-time measurements, for real systems, the sensor measurements are usually available only at discrete instants. Following this paradigm, a full discrete-time approach which allows to reconstruct both the

state and the unknown parameters in discrete time has been used for linear systems in [11] and for nonlinear systems in [6, 21], for example. Another approach is to consider a continuous estimation using the continuous-time model which is fed by the discrete-time measurements. This allows to obtain global convergence in many cases, contrary to a full discrete-time approach which usually lead to only semi-global convergence [1]. Therefore, several observers which estimate continuously the state from discrete-time measurements have been proposed. Indeed, a class of state affine MIMO systems with linear parametrization has been considered in [12] and state-affine SISO systems with non-linear parametrization in [7]. Similarly, an adaptive observer for another class of MIMO systems has been designed in [28].

In all the aforementioned works, convergence has been obtained for large classes of systems, however no uncertainty on the dynamics is considered while it is rather common in practice. Furthermore, the outputs are supposed to be synchronized, which means that the measurements of the different outputs are all obtained at the same sampling instants. This may not be the case in practice, since the considered sensors might not be synchronized or have different sampling rates. Thus, one considers in this paper, the design of an adaptive observer for a class of uniformly observable non-linear MIMO systems with a triangular structure, dynamic uncertainties and multi-rated outputs. In particular, the outputs can be asynchronous and can have irregular sampling instants. Using a high-gain approach, the proposed adaptive observer estimates continuously the state of the system from discrete time measurements. The exponential convergence of the observer is proved with a Lyapunov approach under a persistent excitation condition and provided that for each output, the corresponding sampling periods are over-valued by a given bound. An advantage of the followed approach is that it is fairly easy to tune since there are only a few parameters to set.

The remaining of the paper is organized as follows. First, the considered model is presented in Section 2. The proposed observer and the convergence result are given in Section 3. Simulations of an attitude estimation system using sampled sensors commonly found in inertial measurement units are provided in Section 4 in order to illustrate the proposed approach. Finally, Section 5 concludes the article.

In the rest of the paper, one respectively denotes $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ the minimum and maximum eigenvalues of M , where M is a square symmetric matrix. For a continuous matrix valued function $t \mapsto M(t)$ where for each $t \geq 0$, $M(t)$ is a square symmetric matrix, one denotes $\bar{\lambda}_{\max}(M) = \sup_{t \geq 0} \lambda_{\max}(M(t))$ and $\underline{\lambda}_{\min}(M) = \inf_{t \geq 0} \lambda_{\min}(M(t))$.

2 | PROBLEM STATEMENT

One considers, in this paper, the class of nonlinear systems whose dynamics are given by

$$\dot{x}(t) = Ax(t) + \Psi(u(t), x(t))\rho + g(u(t), x(t), \rho) + B\varepsilon(t) \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $\rho \in \mathbb{R}^p$ is the vector of unknown parameters and $\varepsilon \in \mathbb{R}^q$ the unknown time varying uncertainties.

In particular, one assumes that the state x , the unknown parameters ρ and the uncertainties ε can be partitioned as

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{bmatrix} \text{ with } x^i = \begin{bmatrix} x^i_1 \\ x^i_2 \\ \vdots \\ x^i_{\lambda_i} \end{bmatrix}, \quad x^i_j \in \mathbb{R}^{n_i}, \quad i = 1, \dots, q, \quad j = 1, \dots, \lambda_i, \quad (2)$$

$$\rho = \begin{bmatrix} \rho^1 \\ \vdots \\ \rho^q \end{bmatrix}, \quad \rho^i = \begin{bmatrix} \rho^i_1 \\ \vdots \\ \rho^i_{p_i} \end{bmatrix} \in \mathbb{R}^{p_i}, \quad i = 1, \dots, q, \quad \varepsilon = \begin{bmatrix} \varepsilon^1 \\ \vdots \\ \varepsilon^q \end{bmatrix} \text{ with } \varepsilon^i \in \mathbb{R}^{n_i}, \quad (3)$$

and that the dynamics of x^i , $i = 1, \dots, q$ are given by:

$$\dot{x}^i = \begin{cases} A^i x^i + \Psi^i(u, x)\rho^i + g^i(u, x, \rho) + B^i \varepsilon^i, & \text{if } \rho^i \text{ is not empty} \\ A^i x^i + g^i(u, x, \rho) + B^i \varepsilon^i, & \text{if } \rho^i \text{ is empty} \end{cases} \quad (4)$$

with

$$A^i = \begin{bmatrix} 0_{n_i} & I_{n_i} & 0_{n_i} & \dots & 0_{n_i} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0_{n_i} \\ \vdots & & & \ddots & I_{n_i} \\ 0_{n_i} & \dots & \dots & \dots & 0_{n_i} \end{bmatrix}, \quad B^i = \begin{bmatrix} 0_{n_i} \\ \vdots \\ 0_{n_i} \\ I_{n_i} \end{bmatrix}, \quad (5)$$

$$\Psi^i = \begin{bmatrix} \Psi_{1,1}^i & \dots & \Psi_{1,p_i}^i \\ \vdots & & \vdots \\ \Psi_{\lambda_i,1}^i & \dots & \Psi_{\lambda_i,p_i}^i \end{bmatrix} \text{ with } \Psi_{j,l}^i \in \mathbb{R}^{n_i}, \quad g^i = \begin{bmatrix} g_1^i \\ \vdots \\ g_{\lambda_i}^i \end{bmatrix} \text{ with } g_j^i \in \mathbb{R}^{n_i}, \quad (6)$$

Remark 1. It should be noted that ρ^i contains the sub-components of ρ that were not present in the dynamics of x^1, \dots, x^{i-1} and that first appear in the dynamics of x^i . Then, some ρ^i may be empty if no new unknown parameter appears in the dynamics of x^i . Furthermore, when a new unknown parameter appears in the dynamics of x^i , it is supposed to appear in a linear way.

2.1 | Structure of the nonlinearities

Let $i \in \{1, \dots, q\}$, one now details the structure of the nonlinear functions Ψ^i and g^i contained in the dynamics of x^i . In particular, one specifies the dependence of Ψ^i and g^i with respect to both the state x and the unknown parameters ρ .

First, the functions Ψ^i and g^i are supposed to have a triangular structure according to the state x both between the sub-states x^1, \dots, x^q and for each sub-state x^i between $x_1^i, \dots, x_{\lambda_i}^i$, that is, for $i = 1, \dots, q, j = 1, \dots, \lambda_i, l = 1, \dots, p_i$:

$$g_j^i(u, x, \rho) = g_j^i(u, x^1, \dots, x^{i-1}, x_1^i, \dots, x_j^i, \rho) \quad (7)$$

$$\Psi_{j,l}^i(u, x) = \Psi_{j,l}^i(u, x^1, \dots, x^{i-1}, x_1^i, \dots, x_j^i) \quad (8)$$

In order to state the structure of g^i according to the unknown parameters ρ , one first introduces the indices $v_l^i \in \mathbb{N}, l = 1, \dots, p_i$, representing the sub-components $x_{v_l^i}^i$ where $\rho_{v_l^i}^i$ appears for the first time, that is such that

$$\Psi_{v_l^i, l}^i \neq 0 \text{ and } \Psi_{j, l}^i \equiv 0 \text{ if } j \neq v_l^i \quad (9)$$

Similarly to the structure of the nonlinearities with respect to the state, the function g^i is supposed to have a triangular structure between the sub-components ρ^1, \dots, ρ^q and a special structure for each sub-component between $\rho_1^i, \dots, \rho_{p_i}^i$, that is, for $i = 1, \dots, q, j = 1, \dots, \lambda_i, l = 1, \dots, p_i$:

$$\frac{\partial g_j^i}{\partial \rho^k}(u, x, \rho) \equiv 0 \text{ if } k > i \quad \text{and} \quad \frac{\partial g_j^i}{\partial \rho_l^i}(u, x, \rho) \equiv 0 \text{ if } v_l^i \geq j \quad (10)$$

Remark 2. The first condition of (10) means that g_j^i cannot depend on sub-components of ρ that did not appear linearly for the first time in the dynamics of x^1, \dots, x^i while the second condition means that g_j^i cannot depend on sub-components of ρ^i that did not appear linearly for the first time in the dynamics of x_1^i, \dots, x_{j-1}^i .

2.2 | Measurements of the considered class of systems

One assumes that only the first sub-components x_1^i of each sub-state x^i are measured. That is, the measured outputs $y^i \in \mathbb{R}^{n_i}$ are given by

$$y^i = x_1^i = C^i x^i, \text{ with } C^i = [I_{n_i} \ 0_{n_i} \ \dots \ 0_{n_i}] \quad (11)$$

Moreover, each output is supposed to be sampled at independent and possibly irregular sampling instants, that is:

$$y^1(t_1^1) = C^1 x^1(t_1^1) \quad (12)$$

\vdots

$$y^q(t_1^q) = C^q x^q(t_1^q) \quad (13)$$

for $l \in \mathbb{N}$, and the sampling instants are assumed to verify

$$t_0^l < t_1^l < t_2^l < \dots, \quad \lim_{l \rightarrow +\infty} t_l^l = +\infty \text{ and } |t_{l+1}^l - t_l^l| < \tau_M^l \quad (14)$$

⋮

$$t_0^q < t_1^q < t_2^q < \dots, \quad \lim_{l \rightarrow +\infty} t_l^q = +\infty \text{ and } |t_{l+1}^q - t_l^q| < \tau_M^q \quad (15)$$

3 | MAIN RESULTS

3.1 | Observer design

In order to design the observer that both reconstruct the state and the unknown parameters of system (1), one first needs to introduce some notations.

Let us define first a sequence of integers $(\delta_k)_{k=1, \dots, q}$ that will be used for the design of the observer:

- $\delta_k \in \mathbb{N}$, $k = 1, \dots, q$ is defined such that $\delta_q = 1$ and $\delta_k \geq \lambda_{k+1} \delta_{k+1}$.

Let us further define

- $\Delta_i(\theta) = \text{diag} \left(I_{n_i}, \frac{1}{\theta^{\delta_i}} I_{n_i}, \dots, \frac{1}{\theta^{\delta_i(\lambda_i-1)}} I_{n_i} \right)$
- $\Omega_i(\theta) = \text{diag} \left(\frac{1}{\theta^{\delta_i \nu_i}}, \dots, \frac{1}{\theta^{\delta_i \nu_i}} \right)$

where θ is a parameter of the observer.

One now states some identities that will be used in the following for the proof of convergence. The proof of these identities are given in the appendices.

Proposition 1. The following identities hold true for $i = 1, \dots, q$:

- (i) $C^i \Delta_i^{-1}(\theta) = C^i$,
- (ii) $\Delta_i(\theta) A^i \Delta_i^{-1}(\theta) = \theta^{\delta_i} A^i$,
- (iii) $\Delta_i(\theta) B^i = \frac{1}{\theta^{\delta_i(\lambda_i-1)}} B^i$,
- (iv) $\Delta_i(\theta) \Psi^i \Omega_i^{-1}(\theta) = \theta^{\delta_i} \Psi^i$.

Define the observer state $\hat{x} = [\hat{x}^1 T \dots \hat{x}^q T]^T \in \mathbb{R}^n$ with $\hat{x}^i \in \mathbb{R}^{\lambda_i n_i}$, $i = 1, \dots, q$. Then, for $i = 1, \dots, q$ and $t \in [t_k^i, t_{k+1}^i)$

- if ρ^i is not empty, the dynamics of \hat{x}^i is given by

$$\begin{aligned} \dot{\hat{x}}^i(t) = & A^i \hat{x}^i(t) + \Psi^i(u(t), \hat{x}(t)) \hat{\rho}^i(t) + g^i(u(t), \hat{x}(t), \hat{\rho}(t)) \\ & - \theta^{\delta_i} \Delta_i^{-1}(\theta) (S_i^{-1} + \Upsilon^i(t) P_i(t) \Upsilon^{iT}(t)) C^{iT} \rho^i(t) \end{aligned} \quad (16)$$

$$\dot{\hat{\rho}}^i(t) = -\theta^{\delta_i} \Omega_i^{-1}(\theta) P_i(t) \Upsilon^{iT} C^{iT} \rho^i(t) \quad (17)$$

$$\dot{\rho}^i(t) = -\theta^{\delta_i} C^i (S_i^{-1} + \Upsilon^i(t) P_i(t) \Upsilon^{iT}(t)) C^{iT} \rho^i(t) \quad (18)$$

$$\dot{\Upsilon}^i(t) = \theta^{\delta_i} (A^i - S_i^{-1} C^{iT} C^i) \Upsilon^i(t) + \theta^{\delta_i} \Psi^i(u(t), \hat{x}(t)) \quad (19)$$

$$\dot{P}_i(t) = -\theta^{\delta_i} P_i(t) \Upsilon^{iT}(t) C^{iT} C^i \Upsilon^i(t) P_i(t) + \theta^{\delta_i} P_i(t) \quad (20)$$

with $\rho_i(t_k^i) = (\hat{x}_1^i(t_k^i) - y^i(t_k^i))$, $\Upsilon^i(0) = 0$, $P_i(0) = P_i^T(0) > 0$, $\hat{x}^i(0) \in \mathbb{R}^{\lambda_i n_i}$ and $\hat{\rho}^i(0) \in \mathbb{R}^{\rho_i}$ which can be chosen arbitrarily.

- if ρ^i is empty, the dynamics of \hat{x}^i is given by

$$\dot{\hat{x}}^i(t) = A^i \hat{x}^i(t) + g^i(u(t), \hat{x}(t), \hat{\rho}(t)) - \theta^{\delta_i} \Delta_i^{-1}(\theta) \Gamma^i e^{-\theta^{\delta_i} \Gamma_1^i (t-t_k^i)} (\hat{x}_1^i(t_k^i) - y^i(t_k^i)) \quad (21)$$

with $\hat{x}^i(0) \in \mathbb{R}^{\lambda_i n_i}$, $\hat{\rho}^i(0) \in \mathbb{R}^{\rho_i}$ which can be chosen arbitrarily

The matrix S_i is the Symmetric Definite Positive Matrix solution of

$$S_i + S_i A^{iT} + S_i A^i - C^{iT} C^i = 0 \quad (22)$$

Finally, Γ^i is define by $\Gamma^i = S_i^{-1} C^{iT}$. It can be shown that Γ^i can be written in the following form:

$$\Gamma^i = \begin{bmatrix} \Gamma_1^i I_{n_i} \\ \vdots \\ \Gamma_{\lambda_i}^i I_{n_i} \end{bmatrix} \quad (23)$$

with $\Gamma_k^i = \binom{\lambda_i}{k} = \frac{\lambda_i!}{k!(\lambda_i-k)!} \in \mathbb{R}$ the binomial coefficient.

Remark 3. Equations (16) and (21) can look quite different, but it can actually be noted that if $\Psi^i \equiv 0$ in (16), then equations (16) and (21) are strictly equivalent. Indeed, given the structure of Γ_i , it can be shown that if $\Psi^i \equiv 0$, then $\phi^i(t) = -\theta^{\delta_i} \Gamma_1^i \phi^i(t)$ and then $\phi^i(t) = e^{-\theta^{\delta_i} \Gamma_1^i (t-t_k^i)} (\hat{x}_1^i(t_k^i) - y^i(t_k^i))$ for $t \in [t_k^i, t_{k+1}^i)$.

3.2 | Convergence result

In order to obtain the convergence of the proposed observer, one first needs to make some assumptions about system (1) and the observability of the unknown parameters of system (1).

Assumption 1. The state x , the control input u and the unknown parameters ρ are bounded, that is, there exists some compact sets $\mathbf{X} \subset \mathbb{R}^n$, $\mathbf{U} \subset \mathbb{R}^m$ and $\Lambda \subset \mathbb{R}^p$ such that for all $t \geq 0$, $x(t) \in \mathbf{X}$, $u(t) \in \mathbf{U}$ and $\rho \in \Lambda$.

Assumption 2. The functions $\Psi^i(u, x)$ are continuous over $\mathbf{U} \times \mathbf{X}$ and bounded outside $\mathbf{U} \times \mathbf{X}$.

Assumption 3. The functions $g^i(u, x, \rho)$ are Lipschitz with respect to (x, ρ) uniformly with respect to u . Their Lipschitz constant are denoted L_g^i .

Assumption 4. The functions $\Psi^i(u, x)$ are Lipschitz with respect to x uniformly with respect to u . Their Lipschitz constant are denoted L_Ψ^i .

Assumption 5. The uncertainties are bounded, that is, for $i = 1, \dots, q$, there exists $\kappa_\varepsilon^i \geq 0$ such that

$$\|\varepsilon^i(t)\| \leq \kappa_\varepsilon^i, \quad \forall t \geq 0. \quad (24)$$

Assumption 6. For $i = 1, \dots, q$ there exists $\gamma_i > 0$ (independent of θ) such that for all $t \geq 0$ and $\theta > 0$

$$\lambda_{\min}(P_i^{-1}(t)) \geq \frac{\gamma_i}{\alpha_i(\theta)} \quad (25)$$

where $\alpha_i(\theta)$ is a positive function satisfying for all $\theta > 0$

$$\alpha_i(\theta) \geq 1 \text{ and } \lim_{\theta \rightarrow \infty} \alpha_i(\theta) / \theta^{2\delta_i} = 0 \quad (26)$$

Remark 4. As it was shown in [9, 5], the assumption 6 is satisfied under a classical persistent excitation condition as the following one. For $i = 1, \dots, q$ such that ρ^i is not empty, the input u is such that for any trajectory of system (16)-(20) starting from $(\hat{x}(0), \hat{\rho}(0)) \in \mathbf{X} \times \Lambda$, there exists $\bar{\gamma}^i > 0$ (independent of θ) and $T > 0$ such that for all $t \geq 0$ and $\theta > 0$

$$\int_t^{t+\frac{T}{\theta^{\delta_i}}} \Upsilon^{iT}(s) C^{iT} C^i \Upsilon^i(s) ds \geq \frac{\bar{\gamma}^i}{\alpha_i(\theta)} I_{n_i} \quad (27)$$

One can also note that the assumption 6 can be checked online by computing the minimum of the eigenvalues of P_i^{-1} .

Remark 5. In the rest of the paper, the following notations will be used.

$$\Psi_{\max}^i = \sup_{t \geq 0} \|\Psi^i(u(t), \hat{x}(t))\| \quad (28)$$

$$\rho_{\max}^i = \max_{l=1, \dots, p_i} |\rho_l^i| \quad (29)$$

The result of the observer convergence proposed in this chapter can now be established.

Theorem 1. Consider the system (4) under the assumptions 1 to 6. Assume that $\theta \geq 1$ satisfies

$$\frac{\theta^{\delta_i}}{4\sqrt{\alpha_i(\theta)}} \geq \frac{\sqrt{\lambda_{\max}(S_i)}L_g^i\sqrt{\lambda_i(1+(\Upsilon_{\max}^i)^2)}}{\sqrt{\min(\lambda_{\min}(S_i), \gamma_i)}} + \frac{\sqrt{\lambda_{\max}(S_i)}L_{\Psi}^i\rho_{\max}^i\sqrt{\lambda_i}\max(1, \Upsilon_{\max}^i)}{\sqrt{\min(\lambda_{\min}(S_i), \gamma_i)}} \quad (30)$$

with λ_i the number of sub-components of the i -th block state x^i , Υ_{\max}^i the upper bound of Υ^i , L_g^i the Lipschitz constant of g^i , L_{Ψ}^i the Lipschitz constant of Ψ^i , ρ_{\max}^i the maximum value of $|\rho^i|$ and γ_i defined in (25). Then, if, for each block $i = 1, \dots, q$, the upper bound on the sampling periods τ_M^i is chosen such that

$$\tau_M^i < \frac{8 + \min(\lambda_{\min}(S_i), \gamma_i)}{\sqrt{\alpha_i(\theta)} \left(\theta^{\delta_i}(\max(1, \Psi_{\max}^i + \Upsilon_{\max}^i)) + L_g^i(1 + \Upsilon_{\max}^i) + L_{\Psi}^i\rho_{\max}^i \max(1, \Upsilon_{\max}^i) \right) \sqrt{2}} \quad (31)$$

with Ψ_{\max}^i the upper bound of Ψ^i , then there exists $\alpha^i(\theta) \geq 0$ and $\beta^i \geq 0$, with β^i independent of θ and the uncertainties, such that the following inequality holds true

$$\left(\|\tilde{x}^i(t)\| + \|\tilde{\rho}^i(t)\| \right) \leq \alpha^i(\theta)e^{-\frac{\theta^{\delta_i}}{8}t} + \theta^{-\delta_i}\beta^i(\kappa_{\varepsilon}^1 + \dots + \kappa_{\varepsilon}^i) \quad (32)$$

with $\tilde{x}^i = \hat{x}^i - x^i$, $\tilde{\rho}^i = \hat{\rho}^i - \rho^i$ and κ_{ε}^i an upper bound of ε^i .

Before proceeding to the proof of this theorem, lemmas necessary for the proof of convergence must first be presented. The proof of both lemmas are given in the appendices.

Lemma 1. For each $i = 1, \dots, q$ such that ρ^i is not empty, the corresponding vector Υ^i is norm bounded, that is

$$\Upsilon_{\max}^i = \sup_{t \geq 0} \|\Upsilon_i(u(t), \hat{x}(t))\| < +\infty \quad (33)$$

Lemma 2. Let $w : [-\tau, +\infty) \rightarrow \mathbb{R}$ be a C^1 function verifying the following inequality

$$\frac{d}{dt}(w^2(t)) \leq -aw^2(t) + b \int_{t-\tau}^t w^2(s)ds + ce^{-dt} + k \quad (34)$$

with $a > 0, b \geq 0, \tau > 0, c \geq 0, d > 0, k \geq 0$ and $\tau < \min\left(\frac{\sqrt{2}-1}{2}\frac{a}{b}, \frac{1}{\sqrt{2a}}\right)$. Then there exists $\bar{\alpha} \geq 0$ such that the following inequality holds true

$$w^2(t) \leq \bar{\alpha}e^{-\bar{\beta}t} + \frac{2k}{a} \quad (35)$$

where $\bar{\beta} = \min\left(\frac{a}{2}, d\right)$.

Let us now move on to the proof of the main result.

Proof of Theorem 1. The proof of convergence is split into three parts. First, some adapted coordinates are defined and their corresponding dynamics are derived. In a second part, some candidate Lyapunov functions are defined and some over-valuations involving their derivatives along the error dynamics are obtained. Finally, an induction reasoning is applied in order to obtain the convergence.

Part 1: Equation error

Let $t \geq 0$ and denote for $i = 1, \dots, q$: $\tilde{x}^i = \hat{x}^i - x^i$, $\tilde{\rho}^i = \hat{\rho}^i - \rho^i$ and $k_i = \max\{k \in \mathbb{N} | t_k^i \leq t\}$, then

$$\begin{aligned} \dot{\tilde{x}}^i &= A^i \tilde{x}^i + \tilde{g}^i(u, \hat{x}, \hat{\rho}, x, \rho) + \Psi^i(u, \hat{x})\tilde{\rho}^i - \Psi^i(u, x)\rho^i - B^i \varepsilon^i(t) \\ &\quad - \theta^{\delta_i} \Delta_i^{-1}(\theta) (S_i^{-1} + \Upsilon^i P_i \Upsilon^{iT}) C^{iT} \rho^i(t) \end{aligned} \quad (36)$$

$$\dot{\tilde{\rho}}^i = -\theta^{\delta_i} \Omega_i^{-1}(\theta) P_i \Upsilon^{iT} C^{iT} \rho^i \quad (37)$$

where $\tilde{g}^i(u, \hat{x}, \hat{\rho}, x, \rho) = g^i(u, \hat{x}, \hat{\rho}) - g^i(u, x, \rho)$. In order to simplify the notations in the proof, one does not make difference in writing between the case ρ^i empty and ρ^i non empty. Indeed, according to remark 3, it is direct to see that the error equation for the case ρ^i empty can also be written as equations (36)-(37) by considering that $\Psi^i = 0$ and $\tilde{\rho}^i = 0$.

Denote now $\bar{x}^i = \Delta_i(\theta)\bar{x}^i$ and $\bar{\rho}^i = \Omega_i(\theta)\bar{\rho}^i$, then

$$\begin{aligned} \dot{\bar{x}}^i &= \theta^{\delta_i} \bar{A}^i \bar{x}^i + \Delta_i(\theta) \bar{g}^i(u, \hat{x}, \hat{\rho}, x, \rho) + \Delta_i(\theta) \tilde{\Psi}^i(u, \hat{x}, x) \rho^i + \Delta_i(\theta) \Psi(u, \hat{x}) \Omega_i^{-1}(\theta) \bar{\rho}^i \\ &\quad - \theta^{\delta_i} (S_i^{-1} + \Upsilon^i P_i \Upsilon^{iT}) C^{iT} z^i - \theta^{\delta_i} \Upsilon^i P_i \Upsilon^{iT} C^{iT} C^i \bar{x}^i - \theta^{-(\lambda_i-1)\delta_i} B^i \varepsilon^i(t) \end{aligned} \quad (38)$$

$$\dot{\bar{\rho}}^i = -\theta^{\delta_i} P_i \Upsilon^{iT} C^{iT} C^i \bar{x}^i - \theta^{\delta_i} P_i \Upsilon^{iT} C^{iT} z^i \quad (39)$$

where $\bar{A}^i = (A^i - S_i^{-1} C^{iT} C^i)$, $z^i = (\rho^i(t) - C^i \bar{x}^i(t))$ and $\tilde{\Psi}^i(u, \hat{x}, x) = \Psi^i(u, \hat{x}) - \Psi(u, x)$.

Finally, one obtains

$$\begin{aligned} \dot{\bar{x}}^i &= \theta^{\delta_i} \bar{A}^i \bar{x}^i + \Delta_i(\theta) [\bar{g}^i(u, \hat{x}, \hat{\rho}, x, \rho) + \tilde{\Psi}^i(u, \hat{x}, x) \rho^i] \\ &\quad + \theta^{\delta_i} \Psi^i(u, \hat{x}) \bar{\rho}^i - \theta^{\delta_i} S_i^{-1} C^{iT} z^i + \Upsilon^i \dot{\bar{\rho}}^i - \theta^{-(\lambda_i-1)\delta_i} B^i \varepsilon^i(t) \end{aligned} \quad (40)$$

Define $\eta^i = \bar{x}^i - \Upsilon^i \bar{\rho}^i$, then

$$\dot{\eta}^i = \dot{\bar{x}}^i - \dot{\Upsilon}^i \bar{\rho}^i - \Upsilon^i \dot{\bar{\rho}}^i \quad (41)$$

$$= \theta^{\delta_i} \bar{A}^i \eta^i + \Delta_i(\theta) [\bar{g}^i(u, \hat{x}, \hat{\rho}, x, \rho) + \tilde{\Psi}^i(u, \hat{x}, x) \rho^i] - \theta^{\delta_i} S_i^{-1} C^{iT} z^i - \theta^{-(\lambda_i-1)\delta_i} B^i \varepsilon^i(t) \quad (42)$$

Part 2: Lyapunov function

Let us now define the candidate Lyapunov functions $V^i(\eta^i, \bar{\rho}^i) = V_1^i(\eta^i) + V_2^i(\bar{\rho}^i)$ with $V_1^i(\eta^i) = \eta^{iT} S_i \eta^i$ and $V_2^i(t, \bar{\rho}^i) = \bar{\rho}^{iT} P_i^{-1}(t) \bar{\rho}^i$. These functions are valid candidate Lyapunov functions since each S_i and $P_i^{-1}(t)$ are symmetric definite positive matrices and thanks to assumption 6, the eigenvalues of $P_i^{-1}(t)$ are lower bounded independently of t . Then, the following equalities hold

$$\begin{aligned} \dot{V}_1^i &= \theta^{\delta_i} (\eta^{iT} (A^i - S_i^{-1} C^{iT} C^i)^T S_i + S_i (A^i - S_i^{-1} C^{iT} C^i) \eta^i) \\ &\quad + 2\theta^{\delta_i} \eta^{iT} C^{iT} z^i + 2\eta^{iT} S_i (\Delta_i(\theta) \bar{g}^i) + 2\eta^{iT} S_i (\Delta_i(\theta) \tilde{\Psi}^i \rho^i) - 2\theta^{-(\lambda_i-1)\delta_i} \eta^{iT} S_i (B^i \varepsilon^i(t)) \end{aligned} \quad (43)$$

$$\begin{aligned} &= -\theta^{\delta_i} \eta^{iT} S_i \eta^i - \theta^{\delta_i} \eta^{iT} C^{iT} C^i \eta^i \\ &\quad + 2\theta^{\delta_i} \eta^{iT} C^{iT} z^i + 2\eta^{iT} S_i (\Delta_i(\theta) \bar{g}^i) + 2\eta^{iT} S_i (\Delta_i(\theta) \tilde{\Psi}^i \rho^i) - 2\theta^{-(\lambda_i-1)\delta_i} \eta^{iT} S_i (B^i \varepsilon^i(t)) \end{aligned} \quad (44)$$

where equality (44) is obtained by using equation (22). Now, by using the Cauchy-Schwartz inequality, the fact that $\|C^i\| = \|B^i\| = 1$ and assumption 5, one further has

$$\begin{aligned} \dot{V}_1^i(\eta^i) &\leq -\theta^{\delta_i} \eta^{iT} S_i \eta^i - \theta^{\delta_i} \eta^{iT} C^{iT} C^i \eta^i + 2\sqrt{\lambda_{\max}(S_i)} \sqrt{V_1^i(\eta^i)} \|\Delta_i(\theta) \bar{g}^i\| \\ &\quad + 2\sqrt{\lambda_{\max}(S_i)} \sqrt{V_1^i(\eta^i)} \|\Delta_i(\theta) \tilde{\Psi}^i \rho^i\| + 2 \frac{\theta^{\delta_i}}{\sqrt{\lambda_{\min}(S_i)}} \sqrt{V_1^i(\eta^i)} \|z^i\| \\ &\quad + 2\theta^{-(\lambda_i-1)\delta_i} \kappa_\varepsilon^i \sqrt{\lambda_{\max}(S_i)} \sqrt{V_1^i(\eta^i)} \end{aligned} \quad (45)$$

Using the following facts, whose proof is given in the appendices:

Fact 1. $\|\Delta_i(\theta) \bar{g}^i\| \leq \frac{\sqrt{\alpha_i(\theta)} C_1^i}{\sqrt{\lambda_{\max}(S_i)}} \sqrt{V^i(\eta^i, \bar{\rho}^i)} + \frac{C_2^i}{\sqrt{\lambda_{\max}(S_i)}} \sum_{k=1}^{i-1} (\|\tilde{x}^k\| + \|\tilde{\rho}^k\|)$, with

$$C_1^i = \frac{\sqrt{\lambda_{\max}(S_i)} L_g^i \sqrt{\lambda_i (1 + (\Upsilon_{\max}^i)^2)}}{\sqrt{\min(\lambda_{\min}(S_i), \gamma_i)}} \quad (46)$$

$$C_2^i = L_g^i \sqrt{\lambda_i} \sqrt{\lambda_{\max}(S_i)} \quad (47)$$

Fact 2. $\|\Delta_i(\theta) \tilde{\Psi}^i \rho^i\| \leq \frac{\sqrt{\alpha_i(\theta)} C_3^i}{\sqrt{\lambda_{\max}(S_i)}} \sqrt{V^i(\eta^i, \bar{\rho}^i)} + \frac{C_4^i}{\sqrt{\lambda_{\max}(S_i)}} \sum_{k=1}^{i-1} (\|\tilde{x}^k\| + \|\tilde{\rho}^k\|)$ with

$$C_3^i = \frac{\sqrt{\lambda_{\max}(S_i)} L_\Psi^i \rho_{\max}^i \sqrt{\lambda_i} \max(1, \Upsilon_{\max}^i)}{\sqrt{\min(\lambda_{\min}(S_i), \gamma_i)}} \quad (48)$$

$$C_4^i = L_\Psi^i \rho_{\max}^i \sqrt{\lambda_i} \sqrt{\lambda_{\max}(S_i)} \quad (49)$$

Fact 3. $\|z^i\| \leq C_5^i(\theta) \int_{t-\tau_M^i}^t \sqrt{V^i(s)} ds + C_6^i \sum_{k=1}^{i-1} \int_{t-\tau_M^i}^t (\|\tilde{x}^k(s)\| + \|\tilde{\rho}^k(s)\|) ds$ with

$$C_5^i(\theta) = \frac{\sqrt{\alpha_i(\theta)\theta^{\delta_i}(\max(1, \Psi_{\max}^i + \Upsilon_{\max}^i)) + L_g^i(1 + \Upsilon_{\max}^i) + L_{\Psi}^i \rho_{\max}^i \max(1, \Upsilon_{\max}^i)}}{\min(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i})} \sqrt{2} \quad (50)$$

$$C_6^i = (L_g^i + L_{\Psi}^i \rho_{\max}^i) \quad (51)$$

one obtains

$$\begin{aligned} \dot{V}_1^i(\eta^i) &\leq -\theta^{\delta_i} \eta^{iT} S_i \eta^i - \theta^{\delta_i} \eta^{iT} C^{iT} C^i \eta^i + 2\sqrt{\alpha_i(\theta)}(C_1^i + C_3^i) V^i(\eta^i, \bar{\rho}^i) \\ &\quad + 2(C_2^i + C_4^i) \sqrt{V_1^i(\eta^i)} \sum_{k=1}^{i-1} (\|\tilde{x}^k\| + \|\tilde{\rho}^k\|) + 2\theta^{\delta_i} \frac{C_5^i(\theta)}{\sqrt{\lambda_{\min}(S_i)}} \sqrt{V_1^i(\eta^i)} \int_{t-\tau_M^i}^t \sqrt{V^i(s)} ds \\ &\quad + 2\theta^{\delta_i} \frac{C_6^i}{\sqrt{\lambda_{\min}(S_i)}} \sqrt{V_1^i(\eta^i)} \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} (\|\tilde{x}^k(s)\| + \|\tilde{\rho}^k(s)\|) ds \\ &\quad + 2\theta^{-(\lambda_i-1)\delta_i} \kappa_{\varepsilon}^i \sqrt{\lambda_{\max}(S_i)} \sqrt{V_1^i(\eta^i)} \end{aligned} \quad (52)$$

Now, let us obtain an over-valuation of the derivative of V_2^i . One has

$$\dot{V}_2^i = \ddot{\rho}^{iT} P_i^{-1} \bar{\rho}^i + \bar{\rho}^{iT} P_i^{-1} \dot{\rho}^i - \bar{\rho}^{iT} P_i^{-1} \dot{P}_i P_i^{-1} \bar{\rho}^i \quad (53)$$

$$\begin{aligned} &= -\theta^{\delta_i} \bar{x}^{iT} C^{iT} C^i \Upsilon^i P_i P_i^{-1} \bar{\rho}^i - \theta^{\delta_i} \bar{\rho}^{iT} P_i^{-1} P_i \Upsilon^{iT} C^{iT} C^i \bar{x}^i + 2\theta^{\delta_i} \bar{\rho}^{iT} P_i^{-1} P_i \Upsilon^{iT} C^{iT} z_i \\ &\quad + \theta^{\delta_i} \bar{\rho}^{iT} P_i^{-1} (P_i \Upsilon^{iT} C^{iT} C^i \Upsilon^i P_i - P_i) P_i^{-1} \bar{\rho}^i \end{aligned} \quad (54)$$

$$\begin{aligned} &= -\theta^{\delta_i} \eta^{iT} C^{iT} C^i \Upsilon^i \bar{\rho}^i - \theta^{\delta_i} \bar{\rho}^{iT} \Upsilon^{iT} C^{iT} C^i \Upsilon^i \bar{\rho}^i - \theta^{\delta_i} \bar{\rho}^{iT} \Upsilon^{iT} C^{iT} C^i \eta^i - \theta^{\delta_i} \bar{\rho}^{iT} \Upsilon^{iT} C^{iT} C^i \Upsilon^i \bar{\rho}^i \\ &\quad + 2\theta^{\delta_i} \|\Upsilon^i\| \|\bar{\rho}^i\| \|z^i\| + \theta^{\delta_i} \bar{\rho}^{iT} \Upsilon^{iT} C^{iT} C^i \Upsilon^i \bar{\rho}^i - \theta^{\delta_i} \bar{\rho}^{iT} P_i^{-1} \dot{\rho}^i \end{aligned} \quad (55)$$

$$\begin{aligned} &\leq -\theta^{\delta_i} \eta^{iT} C^{iT} C^i \Upsilon^i \bar{\rho}^i - \theta^{\delta_i} \bar{\rho}^{iT} \Upsilon^{iT} C^{iT} C^i \eta^i - \theta^{\delta_i} \bar{\rho}^{iT} \Upsilon^{iT} C^{iT} C^i \Upsilon^i \bar{\rho}^i \\ &\quad + 2\theta^{\delta_i} \sqrt{V_2^i(\bar{\rho}^i)} \left(\frac{C_5^i(\theta)}{\sqrt{\gamma_i}} \int_{t-\tau_M^i}^t \sqrt{V^i(s)} ds + \frac{C_6^i}{\sqrt{\gamma_i}} \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} (\|\tilde{x}^k(s)\| + \|\tilde{\rho}^k(s)\|) ds \right) - \theta^{\delta_i} V_2^i(\bar{\rho}^i) \end{aligned} \quad (56)$$

with $C_5^i(\theta)$ and C_6^i given by fact 3 and using assumption 6.

An over-valuation of the derivative of V^i can then be obtained as follows

$$\begin{aligned} \dot{V}^i &\leq -\theta^{\delta_i} V^i - \theta^{\delta_i} (C^i \eta^i + C^i \Upsilon^i \bar{\rho}^i)^T (C^i \eta^i + C^i \Upsilon^i \bar{\rho}^i) + 2\sqrt{\alpha_i(\theta)} (C_1^i + C_3^i) V^i \\ &\quad + 2\theta^{\delta_i} C_7^i(\theta) \sqrt{V^i} \int_{t-\tau_M^i}^t \sqrt{V^i(s)} ds + \sqrt{V^i} \sum_{k=1}^{i-1} (C_2^i + C_4^i) (\|\tilde{x}^k\| + \|\tilde{\rho}^k\|) \\ &\quad + 2\theta^{\delta_i} \sqrt{V^i} \sum_{k=1}^{i-1} C_8^i \int_{t-\tau_M^i}^t (\|\tilde{x}^k(s)\| + \|\tilde{\rho}^k(s)\|) ds + 2\theta^{-(\lambda_i-1)\delta_i} \kappa_{\varepsilon}^i \sqrt{\lambda_{\max}(S_i)} \sqrt{V^i} \end{aligned} \quad (57)$$

with $C_7^i(\theta) = \frac{C_5^i(\theta)}{\min(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i})}$ and $C_8^i = \frac{C_6^i}{\min(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i})}$.

Finally, it is direct to see that, thanks to assumption 6, there exists $\theta_i^* \geq 1$ such that for any $\theta \geq \theta_i^*$ one has

$$\theta^{\delta_i} - 2\sqrt{\alpha_i(\theta)}(C_1^i + C_3^i) \geq \frac{\theta^{\delta_i}}{2} \quad (58)$$

which is equivalent to

$$\frac{\theta^{\delta_i}}{\sqrt{\alpha_i(\theta)}} \geq 4(C_1^i + C_3^i) \quad (59)$$

This leads to

$$\begin{aligned} \frac{d}{dt}(\sqrt{V^i}) &\leq -\frac{\theta^{\delta_i}}{4}\sqrt{V^i} + \theta^{\delta_i}C_7^i(\theta) \int_{t-\tau_M^i}^t \sqrt{V^i(s)}ds + (C_2^i + C_4^i) \sum_{k=1}^{i-1} (\|\tilde{x}^k\| + \|\tilde{\rho}^k\|) \\ &\quad + \theta^{\delta_i}C_8^i \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} (\|\tilde{x}^k(s)\| + \|\tilde{\rho}^k(s)\|) ds + \theta^{-(\lambda_i-1)\delta_i} \kappa_\varepsilon^i \sqrt{\lambda_{\max}(S_i)} \end{aligned} \quad (60)$$

Part 3: Induction reasoning

Let us now proceed to an induction reasoning on $i = 1, \dots, q$, that is, let us prove that the exponential convergence of $(\|\tilde{x}^1\| + \|\tilde{\rho}^1\|), \dots, (\|\tilde{x}^{i-1}\| + \|\tilde{\rho}^{i-1}\|)$ to a ball, whose radius depends linearly over the uncertainties bounds $\kappa_\varepsilon^1, \dots, \kappa_\varepsilon^{i-1}$ and θ with a negative power, when t tends to infinity implies the exponential convergence of $(\|\tilde{x}^i\| + \|\tilde{\rho}^i\|)$ to a ball whose radius depends linearly on $\kappa_\varepsilon^1, \dots, \kappa_\varepsilon^i$. More precisely, one will show that for $i = 1, \dots, q$, there exists $C_9^i, C_{10}^i \geq 0$, with C_{10}^i independent of θ , such that $(\|\tilde{x}^i\| + \|\tilde{\rho}^i\|)$ verifies the following inequality:

$$(\|\tilde{x}^i(s)\| + \|\tilde{\rho}^i(s)\|) \leq C_9^i(\theta)e^{-\frac{\theta^{\delta_i}}{8}s} + \theta^{-\delta_i}C_{10}^i(\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^i) \quad (61)$$

For $i = 1$, if θ is taken such as $\theta \geq \theta_1^*$, one has the following over-valuation implying the derivative of $\sqrt{V^1}$

$$\frac{d}{dt}(\sqrt{V^1}) \leq -\frac{\theta^{\delta_1}}{4}\sqrt{V^1} + \theta^{\delta_1}C_7^1(\theta) \int_{t-\tau_M^1}^t \sqrt{V^1(s)}ds + \theta^{-(\lambda_1-1)\delta_1} \kappa_\varepsilon^1 \sqrt{\lambda_{\max}(S_1)} \quad (62)$$

Applying lemma 2 with $w^2 = \sqrt{V^1}$, $a = \frac{\theta^{\delta_1}}{4}$, $b = \theta^{\delta_1}C_7^1(\theta)$, $c = 0$, $d \in \mathbb{R}$ arbitrary, $k = \theta^{-(\lambda_1-1)\delta_1} \sqrt{\lambda_{\max}(S_1)} \kappa_\varepsilon^1$ and $\tau = \tau_M^1$, shows that there exists $\bar{C}_9^1 \geq 0$ such that $\sqrt{V^1}$ verifies the following inequality

$$\sqrt{V^1} \leq \bar{C}_9^1 e^{-\frac{\theta^{\delta_1}}{8}t} + \theta^{-\lambda_1\delta_1} \bar{C}_{10}^1 \kappa_\varepsilon^1 = \bar{C}_9^1 e^{-\frac{\theta^{\delta_1}}{8}t} + \theta^{-\lambda_1\delta_1} \bar{C}_{10}^1 \kappa_\varepsilon^1 \quad (63)$$

with $\bar{C}_{10}^1 = 2\sqrt{\lambda_{\max}(S_1)}$, provided that

$$\tau_M^1 < \min\left(\frac{1}{4C_7^1(\theta)}, \frac{8}{\theta^{\delta_1}}\right) \leq \frac{8 + \min(\lambda_{\min}(S_1), \gamma_1)}{\sqrt{\alpha_1(\theta)} \left(\theta^{\delta_1}(\max(1, \Psi_{\max}^1 + \Upsilon_{\max}^1)) + L_g^1(1 + \Upsilon_{\max}^1) + L_\Psi^1 \rho_{\max}^1 \max(1, \Upsilon_{\max}^1)\right) \sqrt{2}} \quad (64)$$

Then, since $(\|\tilde{x}^1\| + \|\tilde{\rho}^1\|) \leq \theta^{(\lambda_1-1)\delta_1} \frac{(1+\Upsilon_{\max}^1)}{\min(\sqrt{\lambda_{\min}(S_1)}, \sqrt{\gamma_1})} \sqrt{V^1}$, one obtains

$$\left(\|\tilde{x}^1\| + \|\tilde{\rho}^1\|\right) \leq C_9^1(\theta)e^{-\frac{\theta^{\delta_1}}{8}t} + C_{10}^1 \theta^{-\delta_1} \kappa_\varepsilon^1 \quad (65)$$

with $C_9^1 = \theta^{(\lambda_1-1)\delta_1} \frac{(1+\Upsilon_{\max}^1)}{\min(\sqrt{\lambda_{\min}(S_1)}, \sqrt{\gamma_1})} \bar{C}_9^1$ and $C_{10}^1 = \frac{(1+\Upsilon_{\max}^1)}{\min(\sqrt{\lambda_{\min}(S_1)}, \sqrt{\gamma_1})} \bar{C}_{10}^1$.

Now, let us assume that $(\|\tilde{x}^1\| + \|\tilde{\rho}^1\|), \dots, (\|\tilde{x}^{i-1}\| + \|\tilde{\rho}^{i-1}\|)$ verify inequality (61) and let us prove that $(\|\tilde{x}^i\| + \|\tilde{\rho}^i\|)$ do as well.

If θ is taken such that $\theta \geq \theta_i^*$, using inequality (60) and the induction hypothesis, implies that the derivative of $\sqrt{V^i}$ verifies the

following inequality:

$$\begin{aligned} \frac{d}{dt}(\sqrt{V^i}) &\leq -\frac{\theta^{\delta_i}}{4}\sqrt{V^i} + \theta^{\delta_i}C_7^i(\theta) \int_{t-\tau_M^i}^t \sqrt{V^i(s)}ds + \theta^{-(\lambda_i-1)\delta_i}\kappa_\varepsilon^i \sqrt{\lambda_{\max}(S_i)} \\ &\quad + (C_2^i + C_4^i) \sum_{k=1}^{i-1} \left[C_9^k e^{-\frac{\theta^{\delta_k}}{8}t} + \theta^{-\delta_k} C_{10}^k (\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^k) \right] \\ &\quad + \theta^{\delta_i} C_8^i \sum_{k=1}^{i-1} \int_{t-\tau_M^i}^t \left[C_9^k e^{-\frac{\theta^{\delta_k}}{8}s} + \theta^{-\delta_k} C_{10}^k (\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^k) \right] ds \end{aligned} \quad (66)$$

$$\begin{aligned} &\leq -\frac{\theta^{\delta_i}}{4}\sqrt{V^i} + \theta^{\delta_i}C_7^i(\theta) \int_{t-\tau_M^i}^t \sqrt{V^i(s)}ds + \theta^{-(\lambda_i-1)\delta_i}\kappa_\varepsilon^i \sqrt{\lambda_{\max}(S_i)} \\ &\quad + \sum_{k=1}^{i-1} \left[(C_2^i + C_4^i)C_9^k + C_8^i C_9^k 8 \left(\frac{e^{\tau_M^i \theta^{\delta_i}} - 1}{\theta^{\delta_i}} \right) \right] e^{-\frac{\theta^{\delta_i}}{8}t} \\ &\quad + \theta^{\delta_i - \delta_{i-1}} \sum_{k=1}^{i-1} \left[(C_2^i + C_4^i)C_{10}^k + C_8^i \tau_M^i C_{10}^k \right] (\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^{i-1}) \end{aligned} \quad (67)$$

where the latter inequality is obtained since (δ_k) is decreasing.

Thus, applying lemma 2 with $w^2 = \sqrt{V^i}$, $a = \frac{\theta^{\delta_i}}{4}$, $b = \theta^{\delta_i}C_7^i(\theta)$, $c = \sum_{k=1}^{i-1} \left[(C_2^i + C_4^i)C_9^k + C_8^i C_9^k 8 \left(\frac{e^{\tau_M^i \theta^{\delta_i}} - 1}{\theta^{\delta_i}} \right) \right]$, $d = \frac{\theta^{\delta_i}}{8}$ and $k = \theta^{\delta_i - \delta_{i-1}} \sum_{k=1}^{i-1} \left[(C_2^i + C_4^i)C_{10}^k + C_8^i \tau_M^i C_{10}^k \right] (\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^{i-1}) + \theta^{-(\lambda_i-1)\delta_i}\kappa_\varepsilon^i \sqrt{\lambda_{\max}(S_i)}$, and using the fact that $\delta_{i-1} \geq \lambda_i \delta_i$, shows that there exists $\bar{C}_9^i \geq 0$ such that

$$\sqrt{V^i} \leq \bar{C}_9^i e^{-\frac{\theta^{\delta_i}}{8}t} + \theta^{-\lambda_i \delta_i} \bar{C}_{10}^i (\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^i) \quad (68)$$

with $\bar{C}_{10}^i = \max \left(\left[\sum_{k=1}^{i-1} (C_1^{i,k} + C_2^{i,k})C_8^k + C_6^{i,k} \tau_M^i C_8^k \right], \sqrt{\lambda_{\max}(S_i)} \right)$, providing that

$$\tau_M^i < \min \left(\frac{1}{4C_7^i(\theta)}, \frac{8}{\theta^{\delta_i}} \right) \leq \frac{8 + \min(\lambda_{\min}(S_i), \gamma_i)}{\sqrt{\alpha_i(\theta)} \left(\theta^{\delta_i} (\max(1, \Psi_{\max}^i + \Upsilon_{\max}^i)) + L_g^i (1 + \Upsilon_{\max}^i) + L_{\Psi}^i \rho_{\max}^i \max(1, \Upsilon_{\max}^i) \right) \sqrt{2}} \quad (69)$$

Since $(\|\tilde{x}^i\| + \|\tilde{p}^i\|) \leq \theta^{(\lambda_i-1)\delta_i} \frac{(1+Y_{\max}^i)}{\min(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i})} \sqrt{V^i}$, one obtains

$$(\|\tilde{x}^i\| + \|\tilde{p}^i\|) \leq C_9^i(\theta) e^{-\frac{\theta^{\delta_i}}{8}t} + \theta^{-\delta_i} C_{10}^i (\kappa_\varepsilon^1 + \dots + \kappa_\varepsilon^i) \quad (70)$$

with $C_9^i(\theta) = \theta^{(\lambda_i-1)\delta_i} \frac{(1+Y_{\max}^i)}{\min(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i})} \bar{C}_9^i$ and $C_{10}^i = \frac{(1+Y_{\max}^i)}{\min(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i})} \bar{C}_{10}^i$.

This ends the proof. \square

4 | EXAMPLE: ATTITUDE ESTIMATION WITH GYRO BIAS

4.1 | Model of the system

One considers here the problem of the simultaneous estimation of the attitude of a rigid body and the gyro bias from the measurements given by an IMU unit. Particularly, the measurements are provided by three commonly used sensors: an accelerometer, a magnetometer and a gyroscope. The latter is assumed to be corrupted by a constant bias.

The attitude of a rigid body represents the rotation between an inertial reference frame $\{I\}$ and a body fixed frame $\{B\}$. One then defines the orientation at each time instant $t \geq 0$ by a rotation matrix $R(t) \in SO(3)$ from $\{B\}$ to $\{I\}$. For the remains of this study, $R(t)$ will be in $\mathbb{R}^{3 \times 3}$, indeed, the particular structure of $SO(3)$ is not used in the proposed estimation scheme.

The first two measurements (the accelerometer and the magnetometer) can be seen as the result of the rotation between $\{B\}$ and $\{I\}$ of two non co-linear constant vectors. On one side, the measured acceleration $a(t)$ is the resultant of the rotation of the

standard gravity g_0 , that is

$$a(t) = R^T(t)g_0 \quad (71)$$

where $g_0 = [0 \ 0 \ G]^T$. On the other side the measured magnetic field $m(t)$ is the resultant of the rotation of the earth magnetic field m_0 , that is

$$m(t) = R^T(t)m_0 \quad (72)$$

where $m_0 = [m_a \ m_b \ m_c]^T$. Another physical quantity can be induced from these measurements, namely, the cross product between a and m . This cross product is defined as follows:

$$c(t) = a(t) \times m(t) \quad (73)$$

The following assumption is made on $c(t)$:

Assumption 7. For all $t \geq 0$, $a(t)$ and $m(t)$ are non-colinear and then $c(t) \neq 0$.

A vector of observations is obtained from a concatenation of the three previously defined vectors. It is given by

$$v(t) = \begin{bmatrix} a(t) \\ m(t) \\ c(t) \end{bmatrix} \in \mathbb{R}^{9 \times 1} \quad (74)$$

The last measurement is the biased angular velocity $\omega_b(t)$ of the considered rigid body expressed in $\{B\}$ and is given by

$$\omega_b(t) = \omega(t) + b \quad (75)$$

where $\omega(t) \in \mathbb{R}^{3 \times 1}$ is the unbiased angular velocity and $b \in \mathbb{R}^{3 \times 1}$ the bias. The bias is considered constant and then $\dot{b} = 0$.

The kinematics of the attitude is thus given by the following differential equation involving the angular velocity:

$$\dot{R}(t) = R(t)(\omega(t))_{\times} \quad (76)$$

where ω_{\times} is the associated skew-matrix of $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ and is defined by $\omega_{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$.

Finally, the dynamics of the observation vector can be defined by

$$\dot{v}(t) = -S^3(\omega(t))v(t) = -S^3(\omega_b(t))v(t) + S^3(b)v(t) \quad (77)$$

and then:

$$\dot{v}(t) = -S^3(\omega_b(t))v(t) - S^9(v(t))b \quad (78)$$

where $S^3(\omega) = \text{diag}(\omega_{\times}, \omega_{\times}, \omega_{\times}) \in \mathbb{R}^{9 \times 9}$ and $S^9(v) = \begin{bmatrix} a_{\times} \\ m_{\times} \\ c_{\times} \end{bmatrix} \in \mathbb{R}^{9 \times 3}$.

4.2 | Observer design

In order to apply the proposed observer to the model, one now writes the dynamics in the form of the equation (4) and with its notations. The system is split in two subsystems. The first one describes the behavior of the angular velocity. In fact, the angular jerk is considered as an unknown uncertainty. The mathematical model can then be written as follow:

$$\begin{cases} \dot{x}_1^1 &= x_2^1 \\ \dot{x}_2^1 &= x_3^1 \\ \dot{x}_3^1 &= \epsilon^1 \\ y^1 &= x_1^1 = \omega_b \end{cases} \quad (79)$$

The second subsystem describes the kinematics of the attitude as previously shown in equation (78). By denoting the gyroscope bias b as ρ^2 , on can write

$$\begin{cases} \dot{x}_1^2 &= -S^3(x_1^1)x_1^2 - S^9(x_1^2)\rho^2 \\ y^2 &= x_1^2 = v \end{cases} \quad (80)$$

As it can be noticed, the first state block does not contain unknown parameters. The dynamics of the estimated angular velocity \hat{x}^1 can thus be written with one equation as in (21). This is not true for the second state block since ρ^2 has to be estimated. Then, the complete observer system is given by

$$\begin{cases} \dot{\hat{x}}^1(t) = \begin{bmatrix} \hat{x}_2^1 \\ \hat{x}_3^1 \\ 0 \end{bmatrix} - \theta^{\delta_1} \Delta_1^{-1}(\theta) \Gamma^1 e^{-\theta^{\delta_1} \Gamma_1^1 (t-t_k^1)} (\hat{y}^1(t) - y^1(t)) \\ \hat{y}^1(t) = \hat{x}_1^1(t) \\ \dot{\hat{x}}^2(t) = -S^3(\hat{x}_1^1(t)) \hat{x}_1^2(t) - S^9(\hat{x}_1^2) \hat{\rho}^2 - \theta^{\delta_2} \Delta_2^{-1}(\theta) (I_9 + Y^2(t) P_2(t) Y^{2T}(t)) \rho^2(t) \\ \dot{\hat{\rho}}^2(t) = -\theta^{\delta_2} P_2(t) Y^{2T}(t) \rho^2(t) \\ \dot{\hat{\rho}}^2(t) = -\theta^{\delta_2} (I_9 + Y^2(t) P_2(t) Y^{2T}(t)) \rho^2(t) \\ \dot{Y}^2(t) = -\theta^{\delta_2} Y^2(t) - \theta^{\delta_2} S^9(\hat{x}_1^2(t)) \\ \dot{P}_2(t) = -\theta^{\delta_2} P_2(t) Y^{2T}(t) Y^2(t) P_2(t) + \theta^{\delta_2} P_2(t) \\ \hat{y}^2(t) = \hat{x}_1^2(t) \end{cases} \quad (81)$$

where $\Gamma^1 = \begin{bmatrix} 3I_3 \\ 3I_3 \\ I_3 \end{bmatrix}$, $\delta_1 \geq 1$, $\delta_2 = 1$, $\Delta_1(\theta) = \text{diag} \left(I_3, \frac{1}{\theta^{\delta_1}} I_3, \frac{1}{\theta^{2\delta_1}} I_3 \right)$ and $\Delta_2(\theta) = I_9$.

Finally, once the observation vector is estimated, the estimated rotation matrix $\hat{R}(t)$ can be deduced by using

$$\hat{R}(t) = C_2^{-1} \hat{x}_1^2(t) \quad (82)$$

where C_2 is a constant and non singular matrix, involving the constant vectors g_0 and m_0 , defined by

$$C_2 = \begin{bmatrix} 0 & 0 & GI_3 \\ m_a I_3 & m_b I_3 & m_c I_3 \\ -Gm_b I_3 & Gm_a I_3 & 0 \end{bmatrix} \quad (83)$$

4.3 | Simulation results

Simulations of the reconstruction of the attitude with the observer given in equation (81) have been carried out with generated measurements. The rotation dynamic is defined by the angular velocity ω depicted on figure 1. The sampling periods of the two outputs have been taken such that $\tau_m^1 \leq t_{k+1}^1 - t_k^1 \leq \tau_M^1$ and $\tau_m^2 \leq t_{k+1}^2 - t_k^2 \leq \tau_M^2$ with $\tau_m^1 = 0.1s$, $\tau_M^1 = 0.4s$, $\tau_m^2 = 0.2s$ and $\tau_M^2 = 0.6s$. The observer's parameters to be set up, θ and δ_1 , have been chosen to $\theta = 2$ and $\delta_1 = 3$. The sampling periods are presented in figure 2. Finally, the constant vectors have been taken to $g_0 = [0 \ 0 \ 9.81]^T$ and $m_0 = [0.434 \ -0.0091 \ -0.9008]^T$.

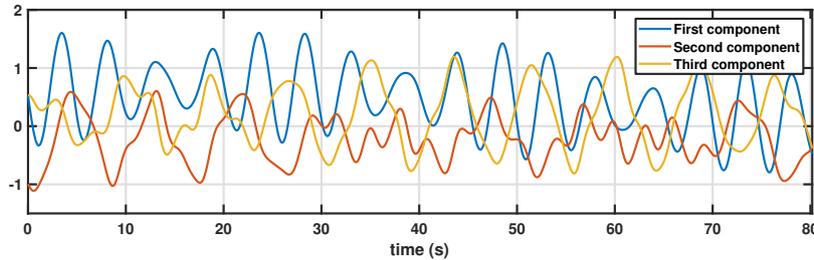


FIGURE 1 Angular velocity

Two simulations are exposed. The first one is performed in the noise-free case. On the contrary, in the second simulation, white Gaussian noises are added on the accelerometer and on the magnetometer. Their standard deviations are equal to $\sigma_a = 0.16m/s^2$ for the accelerometer and $\sigma_m = 0.04\mu T$ for the magnetometer. These two signals are depicted on figure 3.

It can be noted that good performances are obtained for both simulations. Indeed, in the noise free case, the estimated attitude converges very closely to its real value, as it can be seen on figure 4 where the Euler angles corresponding to R and \hat{R} are depicted.

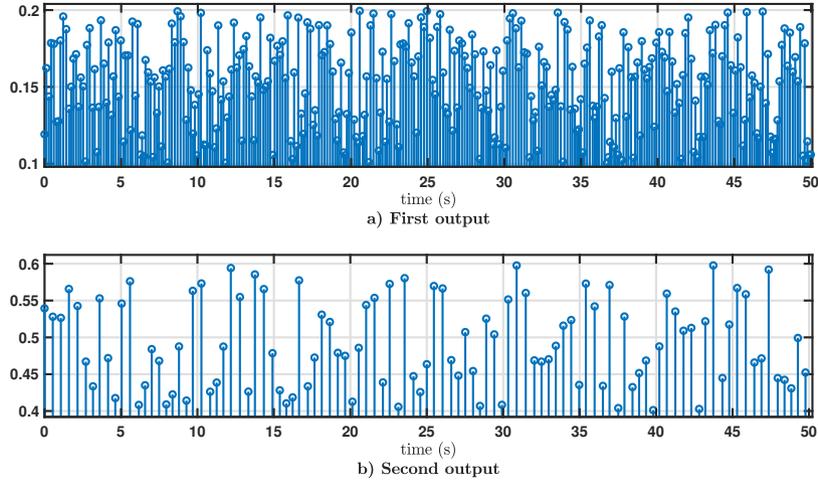


FIGURE 2 Sampling instants for both outputs

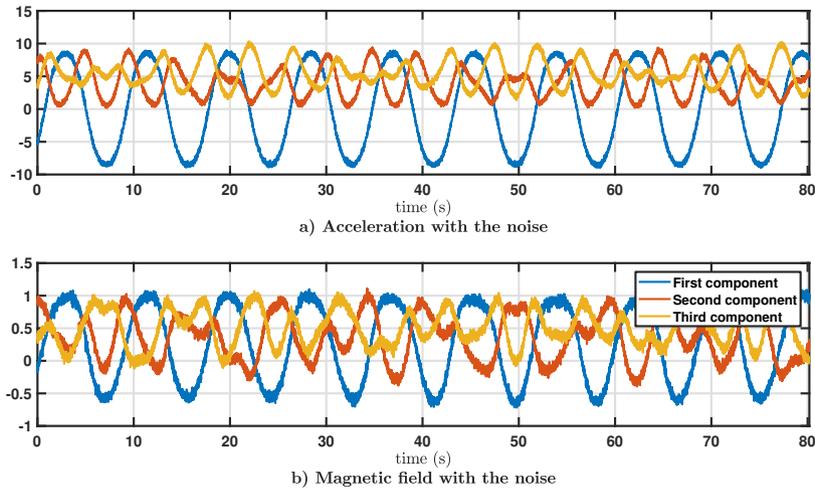


FIGURE 3 Accelerometer and magnetometer with the added white Gaussian noise

In fact, as expected from the theoretical results, the uncertainties considered on the first subsystem prevent the estimated attitude to match exactly the generated one. The errors between the estimated outputs and the real outputs and the error $\|\hat{R}(t) - R(t)\|$, presented respectively on figures 6 and 8, confirm these performances. When the noise is added, the performances are still decent, but errors are higher, as shown in figures 7 and 9. Indeed, one can see that the Euler angles corresponding to R and \hat{R} , and represented on figure 5 are not as close as in the noise free case. Regarding the estimated bias, depicted on figures 10 and 11, they are estimated correctly for both simulations, but with less error in the noise-free case.

5 | CONCLUSION

An adaptive high gain observer has been proposed in this paper, for a class of non-linear systems with uncertainties and multi-rated outputs. One of the main features of the proposed scheme is that the sampling periods can be irregular and asynchronous between different outputs. Another interesting feature is that thanks to its high-gain structure, only a few observer parameters have to be set: the high-gain parameter θ and the powers δ_k associated to each block.

Sufficient conditions have been provided to ensure that the estimation error converges inside a ball whose radius depends on the

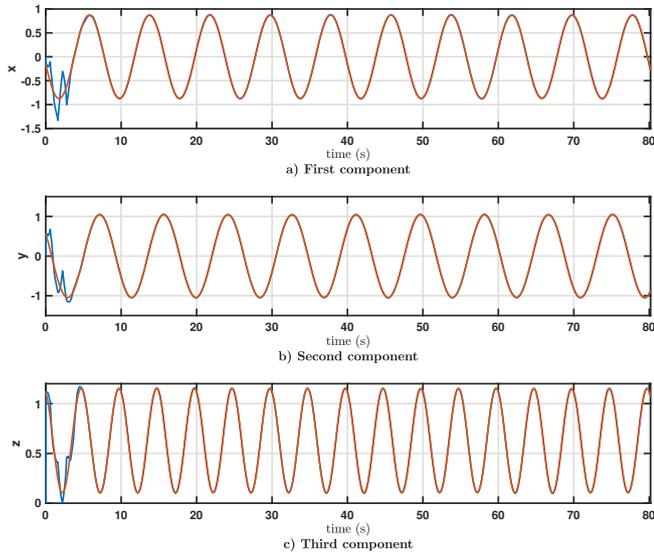


FIGURE 4 Euler angles for the first simulation

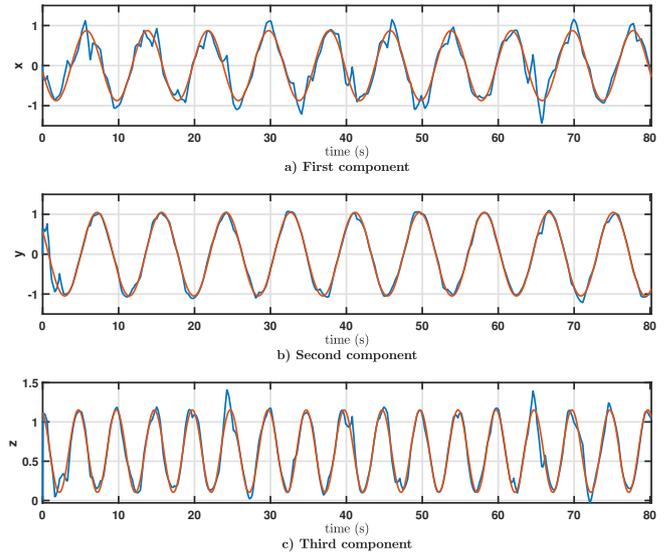


FIGURE 5 Euler angles for the second simulation

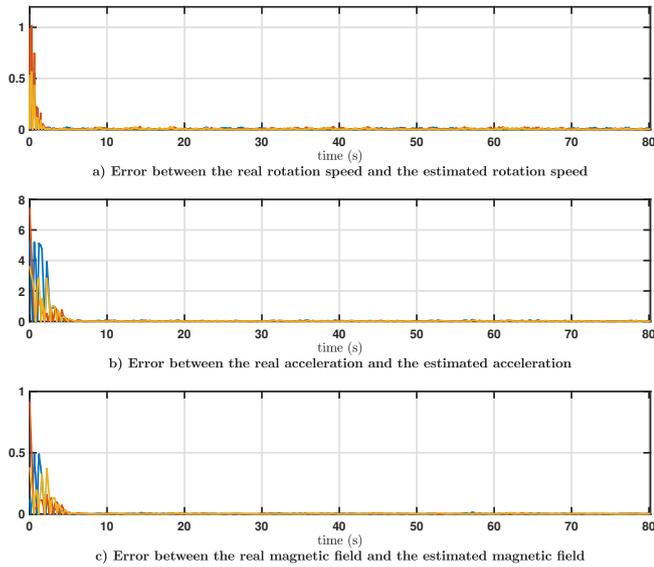


FIGURE 6 Errors between real and estimated outputs for the first simulation

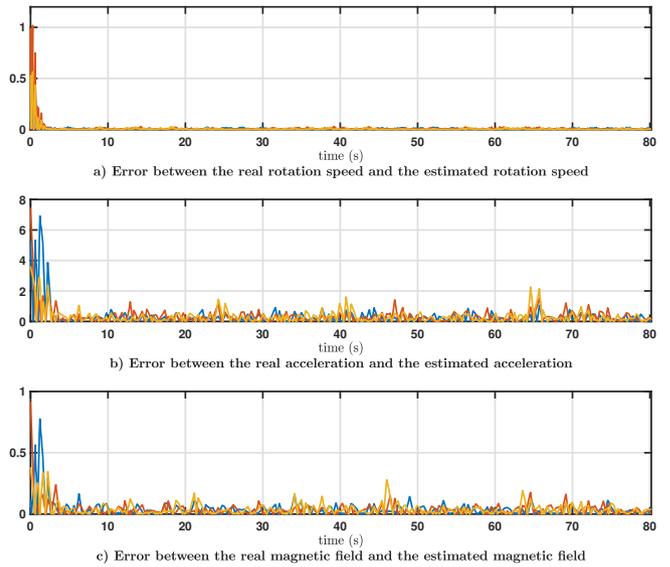


FIGURE 7 Errors between real and estimated outputs for the second simulation

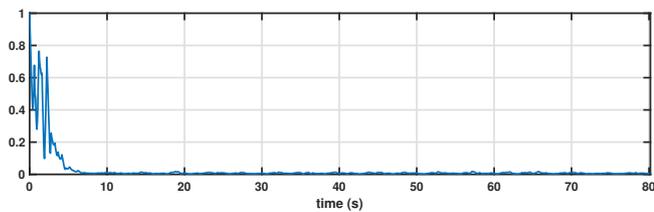


FIGURE 8 Errors $\|\hat{R}(t) - R(t)\|$ for the first simulation

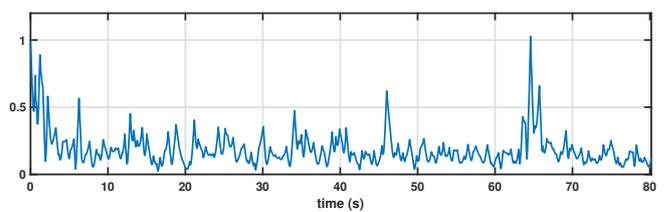


FIGURE 9 Errors $\|\hat{R}(t) - R(t)\|$ for the second simulation

external disturbances and can be rendered as small as desired by properly setting θ . Moreover, the performances of the observer have been validated on a practical problem, namely, the simultaneous estimation of the attitude of a rigid body and the gyro bias.

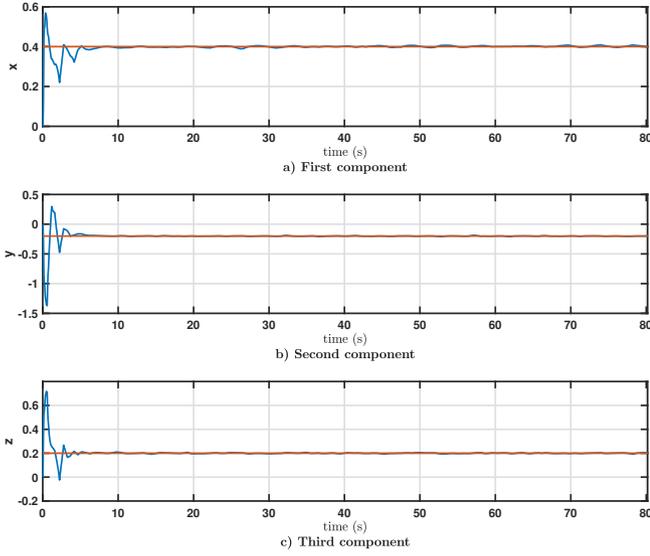


FIGURE 10 Estimated and real gyroscope bias for the first simulation



APPENDIX

A PROOFS OF THE IDENTITIES

The identities to prove are:

1. $C^i \Delta_i^{-1}(\theta) = C^i$,
2. $\Delta_i(\theta) A^i \Delta_i^{-1}(\theta) = \theta^{\delta_i} A^i$,
3. $\Delta_i(\theta) B^i = \frac{1}{\theta^{\delta_i(\lambda_i-1)}} B^i$,
4. $\Delta_i(\theta) \Psi^i \Omega_i^{-1}(\theta) = \theta^{\delta_i} \Psi^i$.

Let us prove them:

1. One has

$$C^i \Delta_i^{-1}(\theta) = \begin{bmatrix} I_{n_i} & 0_{n_i} & \dots & \dots & 0_{n_i} \\ 0_{n_i} & \frac{1}{\theta^{-\delta_i}} I_{n_i} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0_{n_i} \\ 0_{n_i} & \dots & \dots & 0_{n_i} & \frac{1}{\theta^{-\delta_i(\lambda_i-1)}} I_{n_i} \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0_{n_i} & \dots & 0_{n_i} \end{bmatrix} = C^i$$

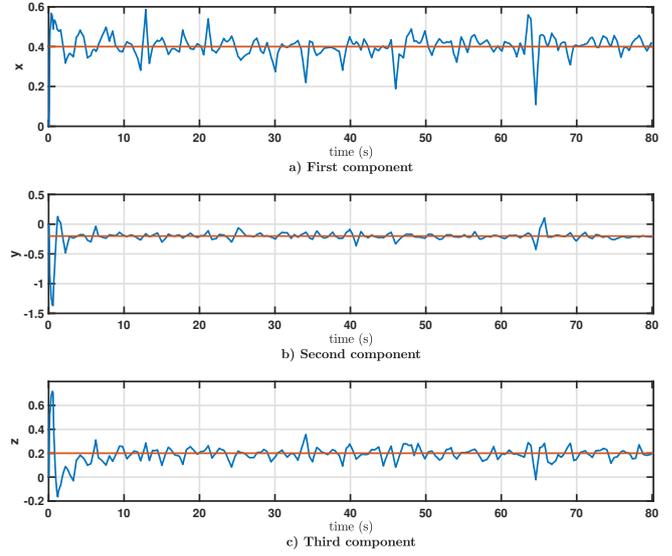


FIGURE 11 Estimated and real gyroscope bias for the second simulation

2. By proceeding in a similar way, one has:

$$\begin{aligned} \Delta_i(\theta)A^i\Delta_i^{-1}(\theta) &= \begin{bmatrix} I_{n_i} & 0_{n_i} & \cdots & \cdots & 0_{n_i} \\ 0_{n_i} & \frac{1}{\theta^{\delta_i}}I_{n_i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & 0_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & 0_{n_i} & \frac{1}{\theta^{\delta_i(\lambda_i-1)}}I_{n_i} \end{bmatrix} \begin{bmatrix} 0_{n_i} & I_{n_i} & 0_{n_i} & \cdots & 0_{n_i} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & I_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & \cdots & 0_{n_i} \end{bmatrix} \begin{bmatrix} I_{n_i} & 0_{n_i} & \cdots & \cdots & 0_{n_i} \\ 0_{n_i} & \theta^{\delta_i}I_{n_i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & 0_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & 0_{n_i} & \theta^{\delta_i(\lambda_i-1)}I_{n_i} \end{bmatrix} \\ &= \begin{bmatrix} 0_{n_i} & I_{n_i} & 0_{n_i} & \cdots & 0_{n_i} \\ \vdots & \ddots & \frac{1}{\theta^{\delta_i}}I_{n_i} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \frac{1}{\theta^{\delta_i(\lambda_i-2)}}I_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & 0_{n_i} & \frac{1}{\theta^{\delta_i(\lambda_i-1)}}I_{n_i} \end{bmatrix} \begin{bmatrix} I_{n_i} & 0_{n_i} & \cdots & \cdots & 0_{n_i} \\ 0_{n_i} & \theta^{\delta_i}I_{n_i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & 0_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & 0_{n_i} & \theta^{\delta_i(\lambda_i-1)}I_{n_i} \end{bmatrix} \\ &= \theta^{\delta_i}A^i \end{aligned}$$

3. One has

$$\Delta_i(\theta)B^i = \begin{bmatrix} I_{n_i} & 0_{n_i} & \cdots & \cdots & 0_{n_i} \\ 0_{n_i} & \frac{1}{\theta^{\delta_i}}I_{n_i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & 0_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & 0_{n_i} & \frac{1}{\theta^{\delta_i(\lambda_i-1)}}I_{n_i} \end{bmatrix} \begin{bmatrix} 0_{n_i} \\ \vdots \\ 0_{n_i} \\ I_{n_i} \end{bmatrix} = \begin{bmatrix} 0_{n_i} \\ \vdots \\ 0_{n_i} \\ \frac{1}{\theta^{\delta_i(\lambda_i-1)}}I_{n_i} \end{bmatrix} = \frac{1}{\theta^{\delta_i(\lambda_i-1)}}B^i$$

4. The product can be written as:

$$\begin{aligned} \Delta_i(\theta)\Psi^i\Omega_i^{-1}(\theta) &= \begin{bmatrix} I_{n_i} & 0_{n_i} & \cdots & \cdots & 0_{n_i} \\ 0_{n_i} & \frac{1}{\theta^{\delta_i}}I_{n_i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & 0_{n_i} & \vdots \\ 0_{n_i} & \cdots & \cdots & 0_{n_i} & \frac{1}{\theta^{\delta_i(\lambda_i-1)}}I_{n_i} \end{bmatrix} \begin{bmatrix} \Psi_{1,1}^i & \cdots & \Psi_{1,p_i}^i \\ \vdots & & \vdots \\ \Psi_{\lambda_i,1}^i & \cdots & \Psi_{\lambda_i,p_i}^i \end{bmatrix} \begin{bmatrix} \theta^{\delta_i v_1^i} & 0 & \cdots & \cdots & 0 \\ 0 & \theta^{\delta_i v_2^i} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & \theta^{\delta_i v_{p_i}^i} \end{bmatrix} \\ &= \begin{bmatrix} \theta^{\delta_i v_1^i} \Psi_{1,1}^i & \cdots & \theta^{\delta_i v_1^i} \Psi_{1,l}^i & \cdots & \theta^{\delta_i v_{p_i}^i} \Psi_{1,p_i}^i \\ \vdots & & \vdots & & \vdots \\ \theta^{\delta_i(v_1^i-j+1)} \Psi_{j,1}^i & \cdots & \theta^{\delta_i(v_1^i-j+1)} \Psi_{j,l}^i & \cdots & \theta^{\delta_i(v_{p_i}^i-j+1)} \Psi_{j,p_i}^i \\ \vdots & & \vdots & & \vdots \\ \theta^{\delta_i(v_1^i-\lambda_i+1)} \Psi_{\lambda_i,1}^i & \cdots & \theta^{\delta_i(v_1^i-\lambda_i+1)} \Psi_{\lambda_i,l}^i & \cdots & \theta^{\delta_i(v_{p_i}^i-\lambda_i+1)} \Psi_{\lambda_i,p_i}^i \end{bmatrix} \end{aligned}$$

Thus, using the equation (9) that for $i = 1, \dots, q$, $j = 1, \dots, \lambda_i$, $l = 1, \dots, p_i$ one obtains:

- if $j = v_l^i$ then $\theta^{\delta_i(v_l^i-j+1)}\Psi_{j,l}^i = \theta^{\delta_i}\Psi_{j,l}^i$
- otherwise if $j \neq v_l^i$, then $\Psi_{j,l}^i = 0_{n_i \times 1}$ and thus $\theta^{\delta_i(v_l^i-j+1)}\Psi_{j,l}^i = 0_{n_i \times 1} = \theta^{\delta_i}\Psi_{j,l}^i$

Which prove that $\Delta_i(\theta)\Psi^i\Omega_i^{-1}(\theta) = \theta^{\delta_i}\Psi^i$.

B PROOF OF LEMMA 1

Consider the change of variable such that

$$\bar{Y}^i(t) = Y^i\left(\frac{t}{\theta^{\delta_i}}\right). \quad (\text{B1})$$

Thus, the dynamic of \bar{Y}^i is given by

$$\dot{\bar{Y}}^i(t) = (A^i - S_i^{-1}C^{iT}C^i)\bar{Y}^i(t) + \Psi^i\left(u\left(\frac{t}{\theta^{\delta_i}}\right), \hat{x}\left(\frac{t}{\theta^{\delta_i}}\right)\right). \quad (\text{B2})$$

This permits to show that

$$\tilde{Y}^i(t) = e^{\bar{A}^i t} \tilde{Y}^i(0) + \int_0^t e^{\bar{A}^i(t-s)} \Psi^i \left(u \left(\frac{s}{\theta^{\delta_i}} \right), \hat{x} \left(\frac{s}{\theta^{\delta_i}} \right) \right) ds \quad (\text{B3})$$

where $\bar{A}^i = A^i - S_i^{-1} C^{iT} C^i$. Since, according to assumption 2, the function Ψ^i is norm bounded and since the matrix \bar{A}^i is Hurwitz, \tilde{Y}^i is bounded and its upper bound is independent of θ . This concludes the proof.

C PROOF OF LEMMA 2

Let us first deal with the case $b = 0$. In this case, one has

$$\frac{d}{dt}(w^2(t)) \leq -aw^2(t) + ce^{-dt} + k \quad (\text{C4})$$

Define now the following candidate Lyapunov function

$$\Theta(w) = w^2(t) \quad (\text{C5})$$

Its derivate can be overbounded as follow

$$\dot{\Theta}(w) \leq -\frac{a}{2}\Theta(w) + ce^{-dt} + k \quad (\text{C6})$$

Solving the equation $\dot{y}(t) = -\frac{a}{2}y(t) + ce^{-dt} + k$ and applying the comparison lemma [13, lemma 3.4 p102] show the result for $b = 0$.

Let $b > 0$, and let us denote $\alpha = \frac{a}{\sqrt{2}}$, $\beta = \frac{b}{a} \left(\frac{e^{\alpha\tau} - 1}{\alpha} \right)$ and $\gamma = 1 - \beta$, then, one has

$$0 < \beta \leq \frac{b}{a} \left(\frac{1 + \sqrt{2}\alpha\tau - 1}{\alpha} \right) \quad (\text{C7})$$

$$\leq \frac{b}{a} \sqrt{2}\tau \quad (\text{C8})$$

$$\leq 1 - \frac{1}{\sqrt{2}} \quad (\text{C9})$$

Inequality (C7) is obtained by using the fact that

$$e^x \leq 1 + \sqrt{2}x, \quad \forall x \in \left(0, \frac{1}{2}\right) \quad (\text{C10})$$

and since τ is chosen such that $\tau \leq \frac{1}{\sqrt{2}a}$, this ensures that $\alpha\tau \in \left(0, \frac{1}{2}\right)$. Inequality (C9) is obtained by using the fact that τ is chosen such that $\tau \leq \frac{\sqrt{2}-1}{2} \frac{a}{b}$.

One has proven until now that $\beta \in \left(0, 1 - \frac{1}{\sqrt{2}}\right)$, which directly implies that $\gamma \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

Let us now define the following candidate Lyapunov function

$$\Theta(w_t) = w^2(t) + b \int_0^\tau \int_{t-s}^t e^{\gamma\alpha(v-t+s)} w^2(v) dv ds \quad (\text{C11})$$

where $w_t(s) = w(t+s)$, $s \in [-\tau, 0]$. Then, its derivative can be over-valued in the following way

$$\dot{\Theta}(w_t) = \frac{d}{dt}(w^2(t)) - b\gamma\alpha \int_0^\tau \int_{t-s}^t e^{\gamma\alpha(v-t+s)} w^2(v) dv ds + b \int_0^\tau e^{\gamma\alpha s} w^2(t) - w^2(t-s) ds \quad (C12)$$

$$\leq \left(-aw^2(t) + b \int_{t-\tau}^t w^2(s) ds + ce^{-dt} + k \right) - \gamma\alpha (\Theta(w_t) - w^2(t)) + b \left(\frac{e^{\gamma\alpha\tau} - 1}{\gamma\alpha} \right) w^2(t) - b \int_{t-\tau}^t w^2(s) ds \quad (C13)$$

$$\leq -\gamma\alpha\Theta(w_t) - aw^2(t) + \gamma\alpha w^2(t) + b \left(\frac{e^{\alpha\tau} - 1}{\alpha} \right) w^2(t) + ce^{-dt} + k \quad (C14)$$

$$\leq -\frac{a}{2}\Theta(w_t) + \left(-a + \gamma \frac{a}{\sqrt{2}} + a(1-\gamma) \right) w^2(t) + ce^{-dt} + k \quad (C15)$$

$$\leq -\frac{a}{2}\Theta(w_t) - \left(1 - \frac{1}{\sqrt{2}} \right) \gamma a w^2(t) + ce^{-dt} + k \quad (C16)$$

$$\leq -\frac{a}{2}\Theta(w_t) + ce^{-dt} + k \quad (C17)$$

Inequality (C14) is obtained by using the fact that the function $x \rightarrow \left(\frac{e^{x\tau} - 1}{x} \right)$ is continuous on $(0, +\infty)$ and increasing, and $\gamma \in (0, 1)$. Inequality (C15) is obtained by using the fact that $\beta = 1 - \gamma$ and $\gamma \in \left(\frac{1}{\sqrt{2}}, 1 \right)$.

Solving the equation $\dot{y}(t) = -\frac{a}{2}y(t) + ce^{-dt} + k$ and applying the comparison lemma [13, lemma 3.4 p102] concludes the proof.

D PROOF OF FACT 1

One has

$$\|\Delta_i(\theta)\bar{g}^i\|^2 = \sum_{j=1}^{\lambda_i} \theta^{-2(j-1)\delta_i} \|\bar{g}_j^i\|^2 \quad (D18)$$

$$\leq \sum_{j=1}^{\lambda_i} (L_g^i)^2 \theta^{-2(j-1)\delta_i} \left[\left(\sum_{k=1}^{i-1} (\|\bar{x}^k\|^2 + \|\bar{\rho}^k\|) \right) + \left(\sum_{k=1}^j \|\bar{x}_k^i\|^2 \right) + \left(\sum_{l=1, v_l^i < j}^{p_i} \|\bar{\rho}_l^i\|^2 \right) \right] \quad (D19)$$

$$\leq \sum_{j=1}^{\lambda_i} (L_g^i)^2 \left(\sum_{k=1}^{i-1} (\|\bar{x}^k\|^2 + \|\bar{\rho}^k\|) \right) + \sum_{j=1}^{\lambda_i} (L_g^i)^2 \left(\sum_{k=1}^j \theta^{-2(j-1)\delta_i + 2(k-1)\delta_i} \|\bar{x}_k^i\|^2 \right) + \sum_{j=1}^{\lambda_i} (L_g^i)^2 \left(\sum_{l=1, v_l^i < j}^{p_i} \theta^{-2(j-1)\delta_i + 2v_l^i\delta_i} \|\bar{\rho}_l^i\|^2 \right) \quad (D20)$$

$$\leq (L_g^i)^2 \sum_{j=1}^{\lambda_i} \left[\left(\sum_{k=1}^{i-1} (\|\bar{x}^k\|^2 + \|\bar{\rho}^k\|) \right) + (\|\bar{x}^i\|^2 + \|\bar{\rho}^i\|) \right] \quad (D21)$$

$$\leq (L_g^i)^2 \lambda_i \left[\left(\sum_{k=1}^{i-1} (\|\bar{x}^k\|^2 + \|\bar{\rho}^k\|) \right) + (\|\bar{n}^i\|^2 + (1 + (Y_{\max}^i)^2) \|\bar{\rho}^i\|^2) \right] \quad (D22)$$

where inequality (D19) is due to the structure of g given by (7) and (10) and inequality (D22) is obtained by using the fact that $\bar{x}^i = \eta^i + \Upsilon^i \bar{\rho}^i$ and lemma 1. Finally, one obtains

$$\|\Delta_i(\theta) \tilde{g}^i\|^2 \leq (L_g^i)^2 \lambda_i \left(\sum_{k=1}^{i-1} (\|\tilde{x}^k\|^2 + \|\tilde{\rho}^k\|^2) \right) + (L_g^i)^2 \lambda_i \left(\frac{V_1^i(\eta^i)}{\lambda_{\min}(S_i)} + \frac{(1 + (\Upsilon_{\max}^i)^2) V_2^i(\bar{\rho}^i)}{\lambda_{\min}(P_i^{-1})} \right) \quad (D23)$$

$$\leq (L_g^i)^2 \lambda_i \left(\sum_{k=1}^{i-1} (\|\tilde{x}^k\|^2 + \|\tilde{\rho}^k\|^2) \right) + (L_g^i)^2 \lambda_i \left(\frac{V_1^i(\eta^i)}{\lambda_{\min}(S_i)} + \frac{\alpha_i(\theta)(1 + (\Upsilon_{\max}^i)^2) V_2^i(\bar{\rho}^i)}{\gamma_i} \right) \quad (D24)$$

$$\leq (L_g^i)^2 \lambda_i \left(\sum_{k=1}^{i-1} (\|\tilde{x}^k\|^2 + \|\tilde{\rho}^k\|^2) \right) + \frac{\alpha_i(\theta)(L_g^i)^2 \lambda_i (1 + (\Upsilon_{\max}^i)^2)}{\min(\lambda_{\min}(S_i), \gamma_i)} V^i(\eta^i, \bar{\rho}^i) \quad (D25)$$

where inequality (D24) is obtained by using assumption 6 and remark 4.

E PROOF OF FACT 2

Similarly to the proof of fact 1, one has

$$\|\Delta_i(\theta) \tilde{\Psi}^i \rho^i\|^2 \leq \sum_{j=1}^{\lambda_i} \theta^{-2(j-1)\delta_i} \|\tilde{\Psi}_j^i\|^2 \|\rho^i\|^2 \quad (E26)$$

$$\leq \sum_{j=1}^{\lambda_i} (L_{\Psi}^i)^2 \left[\left(\sum_{k=1}^{i-1} \|\tilde{x}^k\|^2 \right) + \left(\sum_{k=1}^j \theta^{-2(j-1)\delta_i} \|\tilde{x}_k^i\|^2 \right) \right] (\rho_{\max}^i)^2 \quad (E27)$$

$$\leq (L_{\Psi}^i)^2 (\rho_{\max}^i)^2 \sum_{j=1}^{\lambda_i} \left[\left(\sum_{k=1}^{i-1} \|\tilde{x}^k\|^2 \right) + \|\bar{x}^i\|^2 \right] \quad (E28)$$

$$\leq (L_{\Psi}^i)^2 (\rho_{\max}^i)^2 \lambda_i \left[\left(\sum_{k=1}^{i-1} \|\tilde{x}^k\|^2 \right) + (\|\eta^i\|^2 + (\Upsilon_{\max}^i)^2 \|\bar{\rho}^i\|^2) \right] \quad (E29)$$

where inequality (E27) is obtained by using the structure of Ψ^i given by (8) and the notation introduced in remark 5 and inequality (E29) is obtained by using the fact that $\bar{x}^i = \eta^i + \Upsilon^i \bar{\rho}^i$ and lemma 1. Finally, one obtains

$$\|\Delta_i(\theta) \tilde{\Psi}^i \rho^i\|^2 \leq (L_{\Psi}^i)^2 (\rho_{\max}^i)^2 \lambda_i \left[\left(\sum_{k=1}^{i-1} \|\tilde{x}^k\|^2 \right) + \left(\frac{V_1^i(\eta^i)}{\lambda_{\min}(S_i)} + \frac{(\Upsilon_{\max}^i)^2 V_2^i(\bar{\rho}^i)}{\lambda_{\min}(P_i^{-1})} \right) \right] \quad (E30)$$

$$\leq (L_{\Psi}^i)^2 (\rho_{\max}^i)^2 \lambda_i \left[\left(\sum_{k=1}^{i-1} \|\tilde{x}^k\|^2 \right) + \left(\frac{V_1^i(\eta^i)}{\lambda_{\min}(S_i)} + \frac{\alpha_i(\theta)(\Upsilon_{\max}^i)^2 V_2^i(\bar{\rho}^i)}{\gamma_i} \right) \right] \quad (E31)$$

$$\leq (L_{\Psi}^i)^2 (\rho_{\max}^i)^2 \lambda_i \left(\sum_{k=1}^{i-1} \|\tilde{x}^k\|^2 \right) + \frac{\alpha_i(\theta)(L_{\Psi}^i)^2 (\rho_{\max}^i)^2 \lambda_i \max(1, (\Upsilon_{\max}^i)^2)}{\min(\lambda_{\min}(S_i), \gamma_i)} V^i(\eta^i, \bar{\rho}^i) \quad (E32)$$

where inequality (E31) is obtained by using assumption 6 and remark 4.

F PROOF OF FACT 3

The derivative of z^i is given by

$$\dot{z}^i = \dot{\rho}^i - C^i \tilde{x}^i(t) \quad (F33)$$

$$= -C^i (\theta^{\delta_i} A^i \tilde{x}^i + \Delta_i(\theta)[\tilde{g}^i + \tilde{\Psi}^i \rho^i] + \theta^{\delta_i} \Psi^i \bar{\rho}^i) \quad (F34)$$

Then, noticing that $z^i(t_{k_i}^i) = 0$ yields

$$z^i(t) = - \int_{t_{k_i}^i}^t C^i (\theta^{\delta_i} A^i \bar{x}^i(s) + \Delta_i(\theta)[\bar{g}^i(s) + \tilde{\Psi}^i(s)\rho^i] + \theta^{\delta_i}\Psi^i(s)\bar{\rho}^i) ds \quad (\text{F35})$$

$$\|z^i(t)\| \leq \int_{t-\tau_M^i}^t \left\| C^i (\theta^{\delta_i} A^i \bar{x}^i(s) + \Delta_i(\theta)[\bar{g}^i(s) + \tilde{\Psi}^i(s)\rho^i] + \theta^{\delta_i}\Psi^i(s)\bar{\rho}^i) \right\| ds \quad (\text{F36})$$

Proceeding as for fact 1 and 2, using the fact that $\|C^i A^i\| = 1$ and using the notation Ψ_{\max}^i and ρ_{\max}^i as in remark 5 yields

$$\begin{aligned} \|z^i(t)\| &\leq \int_{t-\tau_M^i}^t \theta^{\delta_i} (\|\bar{x}^i(s)\| + \Psi_{\max}^i \|\bar{\rho}^i(s)\|) + \left(L_g^i (\|\bar{x}^i(s)\| + \|\bar{\rho}^i(s)\|) + L_{\Psi}^i \rho_{\max}^i \|\bar{x}^i(s)\| \right) ds \\ &\quad + \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} \left[L_g^i (\|\bar{x}^k(s)\| + \|\bar{\rho}^k(s)\|) + L_{\Psi}^i \rho_{\max}^i \|\bar{x}^k(s)\| \right] ds \end{aligned} \quad (\text{F37})$$

Using the fact that $\bar{x}^i = \eta^i + \Upsilon^i \bar{\rho}^i$ and using lemma 1 gives $\|x^i\| \leq \|\eta^i\| + \Upsilon_{\max}^i \|\bar{\rho}^i\|$, and then

$$\begin{aligned} \|z^i(t)\| &\leq \int_{t-\tau_M^i}^t \left(\theta^{\delta_i} + (L_g^i + L_{\Psi}^i \rho_{\max}^i) \right) \|\eta^i(s)\| ds \\ &\quad + \int_{t-\tau_M^i}^t \left(\theta^{\delta_i} (\Psi_{\max}^i + \Upsilon_{\max}^i) + (L_g^i (1 + \Upsilon_{\max}^i) + L_{\Psi}^i \rho_{\max}^i \Upsilon_{\max}^i) \right) \|\bar{\rho}^i(s)\| ds \\ &\quad + \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} \left[(L_g^i + L_{\Psi}^i \rho_{\max}^i) \|\bar{x}^k(s)\| + L_g^i \|\bar{\rho}^k(s)\| \right] ds \end{aligned} \quad (\text{F38})$$

$$\begin{aligned} &\leq \int_{t-\tau_M^i}^t \frac{\left(\theta^{\delta_i} + L_g^i + L_{\Psi}^i \rho_{\max}^i \right) \sqrt{V_1^i(s)} + \left(\theta^{\delta_i} (\Psi_{\max}^i + \Upsilon_{\max}^i) + (L_g^i (1 + \Upsilon_{\max}^i) + L_{\Psi}^i \rho_{\max}^i \Upsilon_{\max}^i) \right) \sqrt{V_2^i(s)}}{\sqrt{\lambda_{\min}(S_i)}} ds \\ &\quad + \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} (L_g^i + L_{\Psi}^i \rho_{\max}^i) [\|\bar{x}^k(s)\| + \|\bar{\rho}^k(s)\|] ds \end{aligned} \quad (\text{F39})$$

$$\begin{aligned} &\leq \int_{t-\tau_M^i}^t \frac{\left(\theta^{\delta_i} + L_g^i + L_{\Psi}^i \rho_{\max}^i \right) \sqrt{V_1^i(s)} + \frac{\sqrt{\alpha_i(\theta)} \left(\theta^{\delta_i} (\Psi_{\max}^i + \Upsilon_{\max}^i) + (L_g^i (1 + \Upsilon_{\max}^i) + L_{\Psi}^i \rho_{\max}^i \Upsilon_{\max}^i) \right)}{\sqrt{\gamma_i}} \sqrt{V_2^i(s)}}{\sqrt{\lambda_{\min}(S_i)}} ds \\ &\quad + (L_g^i + L_{\Psi}^i \rho_{\max}^i) \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} [\|\bar{x}^k(s)\| + \|\bar{\rho}^k(s)\|] ds \end{aligned} \quad (\text{F40})$$

where inequality (F40) is obtained by using assumption 6.

Using the inequality $\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$ for $a, b \geq 0$, one obtains

$$\|z^i(t)\| \leq \frac{\sqrt{\alpha_i(\theta)} \left(\theta^{\delta_i} (\max(1, \Psi_{\max}^i + \Upsilon_{\max}^i)) + L_g^i (1 + \Upsilon_{\max}^i) + L_{\Psi}^i \rho_{\max}^i \max(1, \Upsilon_{\max}^i) \right)}{\min \left(\sqrt{\lambda_{\min}(S_i)}, \sqrt{\gamma_i} \right)} \sqrt{2} \int_{t-\tau_M^i}^t \sqrt{V^i(s)} ds$$

$$+ \left(L_g^i + L_{\Psi}^i \rho_{\max}^i \right) \int_{t-\tau_M^i}^t \sum_{k=1}^{i-1} [\|\tilde{x}^k(s)\| + \|\tilde{p}^k(s)\|] ds \quad (\text{F41})$$

References

- [1] M. Arcak and D. Nešić. A framework for nonlinear sampled-data observer design via approximate discrete-time models and emulation. *Automatica*, 40(11):1931–1938, Nov. 2004.
- [2] P. Baldi, M. Blanke, P. Castaldi, N. Mimmo, and S. Simani. Fault diagnosis for satellite sensors and actuators using nonlinear geometric approach and adaptive observers. *International Journal of Robust and Nonlinear Control*, 29(16):5429–5455, 2019.
- [3] G. Bastin and M. Gevers. Stable adaptive observers for nonlinear time-varying systems. *IEEE Transactions on Automatic Control*, 33(7):650–658, July 1988.
- [4] G. Besançon, J. D. León-Morales, and O. Huerta-Guevara. On adaptive observers for state affine systems. *International Journal of Control*, 79(6):581–591, June 2006.
- [5] A. M. Boker and H. K. Khalil. Nonlinear observers comprising high-gain observers and extended Kalman filters. *Automatica*, 49(12):3583–3590, Dec. 2013.
- [6] F. Caccavale, F. Pierri, and L. Villani. Adaptive Observer for Fault Diagnosis in Nonlinear Discrete-Time Systems. *Journal of Dynamic Systems, Measurement, and Control*, 130(2), Mar. 2008.
- [7] M. Farza, I. Bouraoui, T. Ménard, R. Ben Abdennour, and M. M’Saad. Adaptive observers for a class of uniformly observable systems with nonlinear parametrization and sampled outputs. *Automatica*, 50(11):2951–2960, Nov. 2014.
- [8] M. Farza, M. M’Saad, T. Maatoug, and M. Kamoun. Adaptive observers for nonlinearly parameterized class of nonlinear systems. *Automatica*, 45(10):2292–2299, Oct. 2009.
- [9] M. Farza, M. M’Saad, T. Ménard, A. Ltaief, and T. Maatoug. Adaptive observer design for a class of nonlinear systems. Application to speed sensorless induction motor. *Automatica*, 90:239–247, Apr. 2018.
- [10] H. F. Grip, A. Saberi, and T. A. Johansen. Estimation of states and parameters for linear systems with nonlinearly parameterized perturbations. *Systems & Control Letters*, 60(9):771–777, Sept. 2011.
- [11] A. Guyader and Q. Zhang. Adaptive Observer for Discrete Time Linear Time Varying Systems. *IFAC Proceedings Volumes*, 36(16):1705–1710, Sept. 2003.
- [12] C. A. B. Hann and T. Ahmed-Ali. Continuous adaptive observer for state affine sampled-data systems. *International Journal of Robust and Nonlinear Control*, 24(4):669–681, 2014.
- [13] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.
- [14] A. Kojić and A. M. Annaswamy. Adaptive control of nonlinearly parameterized systems with a triangular structure. *Automatica*, 38(1):115–123, Jan. 2002.
- [15] G. Kreisselmeier. Adaptive observers with exponential rate of convergence. *IEEE Transactions on Automatic Control*, 22(1):2–8, Feb. 1977.

-
- [16] A.-P. Loh, A. Annaswamy, and F. Skantze. Adaptation in the presence of a general nonlinear parameterization: an error model approach. *IEEE Transactions on Automatic Control*, 44(9):1634–1652, Sept. 1999.
- [17] G. Luders and K. Narendra. An adaptive observer and identifier for a linear system. *IEEE Transactions on Automatic Control*, 18(5):496–499, Oct. 1973.
- [18] R. Marino, G. Santosuosso, and P. Tomei. Robust adaptive observers for nonlinear systems with bounded disturbances. *IEEE Transactions on Automatic Control*, 46(6):967–972, June 2001.
- [19] R. Marino and P. Tomei. Global adaptive observers and output-feedback stabilization for a class of nonlinear systems. *Foundations of Adaptive Control*, pages 455–493, 1991.
- [20] T. Menard, A. Maouche, B. Targui, I. Bouraoui, M. Farza, and M. M’Saad. Adaptive high gain observer for uniformly observable systems with nonlinear parametrization. *2014 European Control Conference (ECC)*, pages 1735–1740, June 2014.
- [21] K. Srinivasarengan, J. Ragot, C. Aubrun, and D. Maquin. An adaptive observer design approach for discrete-time nonlinear systems. *International Journal of Applied Mathematics and Computer Science*, 28, Mar. 2018.
- [22] Ø. N. Starnes, O. M. Aamo, and G.-O. Kaasa. Redesign of adaptive observers for improved parameter identification in nonlinear systems. *Automatica*, 47(2):403–410, Feb. 2011.
- [23] A. G. Stefanopoulou, O. F. Storset, and R. Smith. Pressure and temperature-based adaptive observer of air charge for turbocharged diesel engines. *International Journal of Robust and Nonlinear Control*, 14(6):543–560, 2004.
- [24] I. Y. Tyukin, E. Steur, H. Nijmeijer, and C. van Leeuwen. Adaptive observers and parameter estimation for a class of systems nonlinear in the parameters. *Automatica*, 49(8):2409–2423, Aug. 2013.
- [25] A. Xu and Q. Zhang. Nonlinear system fault diagnosis based on adaptive estimation. *Automatica*, 40(7):1181–1193, July 2004.
- [26] Q. Zhang. Adaptive observer for multiple-input-multiple-output (MIMO) linear time-varying systems. *IEEE Transactions on Automatic Control*, 47(3):525–529, Mar. 2002.
- [27] Q. Zhang and A. Clavel. Adaptive observer with exponential forgetting factor for linear time varying systems. In *Proceedings of the 40th IEEE Conference on Decision and Control*, volume 4, pages 3886–3891 vol.4, Dec. 2001.
- [28] G. Zhao and C. Hua. Continuous–discrete-time adaptive observers for nonlinear systems with sampled output measurements. *International Journal of Systems Science*, 48(12):2599–2609, Sept. 2017.