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## On injective tensor powers of $\ell_1$

R. M. Causey, E. M. Galego, and C. Samuel

ABSTRACT. In this paper we prove that the 3-fold injective tensor product  $\ell_1 \widehat{\otimes}_\varepsilon \ell_1 \widehat{\otimes}_\varepsilon \ell_1$  is not isomorphic to any subspace of  $\ell_1 \widehat{\otimes}_\varepsilon \ell_1$ . This result provides a new solution to a problem of Diestel on the projective tensor products of  $c_0$ . Moreover, this result implies that for any infinite countable compact space  $K$ , the 3-fold projective tensor product  $C(K) \widehat{\otimes}_\pi C(K) \widehat{\otimes}_\pi C(K)$  is not isomorphic to any quotient of  $C(K) \widehat{\otimes}_\pi C(K)$ .

### 1. Introduction

For standard Banach space terminology employed throughout the paper the reader is referred to [5] and [7]. For  $n \in \mathbb{N}$ , a tensor norm  $\alpha$ , and a Banach space  $X$ , let  $\widehat{\otimes}_\alpha^n X$  denote the  $n$ -fold  $\alpha$ -tensor product of  $X$  with itself.

Very recently the authors solve a problem attributed to Diestel [3, Theorem 1.3] by proving that  $\widehat{\otimes}_\pi^3 c_0$  is not isomorphic to  $\widehat{\otimes}_\pi^2 c_0$ . In the present paper we consider two natural problems that arise from this result. The first problem is whether this result extends to  $C(K)$  spaces other than  $c_0$ , here the space  $C(K)$  will stand for the Banach space of all continuous, real-valued functions on the compact Hausdorff space  $K$  and equipped with the supremum norm. The first problem can be precisely stated as:

PROBLEM 1.1. *Let  $K$  be an infinite compact Hausdorff space. Is it true that  $\widehat{\otimes}_\pi^3 C(K)$  is not isomorphic to  $\widehat{\otimes}_\pi^2 C(K)$ ?*

The second problem is to know if the dual spaces of  $\widehat{\otimes}_\pi^3 c_0$  and  $\widehat{\otimes}_\pi^2 c_0$  are isomorphic to each other. By using well-known properties of projective and injective tensor products [7] this problem can be rewritten as follows:

PROBLEM 1.2. *Is  $\widehat{\otimes}_\varepsilon^3 \ell_1$  isomorphic to  $\widehat{\otimes}_\varepsilon^2 \ell_1$ ?*

This last problem was proposed to us by Richard M. Aron to whom we are grateful for the interest shown in this research topic.

The main goal of this paper is to present a negative solution to Problem 1.2. This follows directly the following theorem.

THEOREM 1.3.  *$\widehat{\otimes}_\varepsilon^3 \ell_1$  is not isomorphic to any subspace of  $\widehat{\otimes}_\varepsilon^2 \ell_1$ .*

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Observe that if  $K$  is an infinite countable compact metric space, then it is well known that the dual space of  $C(K)$  is isomorphic to  $\ell_1$  [5, p.20]. Therefore it follows from Theorem 1.3 that  $\widehat{\otimes}_\pi^3 C(K)$  is not isomorphic to any quotient of  $\widehat{\otimes}_\pi^2 C(K)$ . In particular, Problem 1.1 has a positive solution when  $K$  is an infinite countable compact metric space.

Theorem 1.3 also provides a new proof that  $\widehat{\otimes}_\varepsilon^2 \ell_1$  is not isomorphic to any subspace of  $\ell_1$  [6, Corollary 2.1]. However we do not know how to solve:

**PROBLEM 1.4.** *Suppose that for some  $m, n \in \mathbb{N}$  with  $m, n \geq 3$ ,  $\widehat{\otimes}_\varepsilon^m \ell_1$  is isomorphic to  $\widehat{\otimes}_\varepsilon^n \ell_1$ . Is it true that  $m = n$ ?*

Of course it would be interesting to know if Problem 1.1 also has a positive solution when  $K$  is the interval of real numbers  $[0, 1]$  or  $K$  is  $\beta\mathbb{N}$ , the Stone-Cech compactification of the discrete set of natural numbers  $\mathbb{N}$ , see [2] to some geometric properties of the spaces  $C([0, 1])\widehat{\otimes}_\pi C([0, 1])$  and  $C(\beta\mathbb{N})\widehat{\otimes}_\pi C(\beta\mathbb{N})$ .

The fundamental property used in [3] concerned  $\ell_2$  upper estimates on the branches of weakly null trees in the 2-fold tensor product  $\widehat{\otimes}_\pi^2 c_0$ . Trees dualize nicely, but the dual property to upper  $\ell_2$  estimates on weakly null trees in some Banach space  $X$  is lower  $\ell_2$  estimates on the branches of weak\* null trees in  $X^*$ . Therefore the result from [3] does not yield that there is no isomorphic embedding of  $\widehat{\otimes}_\varepsilon^3 \ell_1$  into  $\widehat{\otimes}_\varepsilon^2 \ell_1$ , because such an isomorphic embedding need not be weak\*-weak\* continuous. Thus Theorem 1.3 is not a trivial consequence of the result of [3].

Also, the results of [3] were stated in terms of weakly null trees, but the objects produced were weakly null arrays, which can be viewed as a special kind of weakly null tree. Since weakly null arrays are weakly null trees,  $\ell_2$  upper estimates on the branches of weakly null trees implies the same estimates on the branches of weakly null arrays, but the converse need not hold [1]. Therefore the existence of weakly null arrays which do not satisfy a uniform  $\ell_2$  upper estimate is a stronger condition than the existence of weakly null trees. In the current work, we use the fact that [3] produced arrays and not simply sequences, as this allows us to circumvent the difficulty that isomorphic embeddings need not be weak\*-weak\* continuous. The key step is noting that arrays are amendable to a certain differencing procedure, while the same differencing procedure cannot be applied to trees. This differencing is used here to overcome a difficulty not present in [3].

## 2. Proof of Theorem 1.3

For a Banach space  $X$  and  $n \in \mathbb{N}$ , a family  $(x_i^k)_{i=1, k=1}^{\infty, n}$  of  $X$  is called an  $n$ -array. For  $C > 0$ , an  $n$ -array is said to be  $C$ -separated provided that for any  $1 \leq k \leq n$  and any distinct  $i, j \in \mathbb{N}$ ,  $\|x_i^k - x_j^k\| \geq C$ .

For a Banach space  $X$  and  $n \in \mathbb{N}$ , let  $\delta_n(X)$  denote the infimum of  $d > 0$  such that for any  $C > 0$  and any bounded,  $C$ -separated  $n$ -array  $(x_i^k)_{i=1, k=1}^{\infty, n}$  in  $X$ , there exist  $i_1 < j_1 < \dots < i_n < j_n$  such that

$$d \left\| \sum_{k=1}^n (x_{i_k}^k - x_{j_k}^k) \right\| \geq Cn^{1/2}.$$

Obviously if  $X$  is isomorphic to a subspace of  $Y$ , then  $\sup_n \delta_n(X)/\delta_n(Y) < \infty$ . More precisely, if  $X, Z$  are isomorphic Banach spaces and  $d_{BM}$  their Banach-Mazur

distance, then  $\delta_n(X) \leq d_{BM} \delta_n(Z)$  for all  $n$ , and if  $Z$  is a closed subspace of  $Y$ , then  $\delta_n(Z) \leq \delta_n(Y)$  for all  $n \in \mathbb{N}$ . Therefore we will prove Theorem 1.3 by completing the next two lemmas.

LEMMA 2.1. *It holds that*

$$\sup_n \delta_n(\widehat{\otimes}_\varepsilon^2 \ell_1) < \infty.$$

LEMMA 2.2. *It holds that*

$$\inf_n \frac{\delta_n(\widehat{\otimes}_\varepsilon^3 \ell_1)}{\log(n)} > 0.$$

PROOF OF LEMMA 2.1. Let  $(e_i)_i$  be the unit vector basis of  $\ell_1$ ,  $(e_i^*)_i$  the biorthogonal sequence and, for every integer  $k$ ,

$$F_k = \text{span}\{e_i \otimes e_j : \max\{i, j\} = k\}.$$

It was proven in [4] that the sequence of subspaces  $(F_k)_k$  satisfies a lower  $\ell_2$  estimate. That is, there exist a constant  $a > 0$  such that for any  $0 = q_0 < q_1 < \dots$ , any  $n \in \mathbb{N}$ , and any  $(y_i)_{i=1}^n \in \prod_{i=1}^n \text{span}\{F_j : q_{i-1} < j \leq q_i\}$ ,

$$a^2 \left\| \sum_{i=1}^n y_i \right\|^2 \geq \sum_{i=1}^n \|y_i\|^2.$$

Fix  $C > 0$ ,  $n \in \mathbb{N}$ , and a  $C$ -separated, bounded  $n$ -array  $(x_i^k)_{i=1, k=1}^{\infty, n}$  in  $\widehat{\otimes}_\varepsilon^2 \ell_1$ . By passing to subsequences  $n$  times and relabeling, we may assume that for each  $1 \leq k \leq n$  and each  $(p, q) \in \mathbb{N} \times \mathbb{N}$ ,  $\lim_i \langle e_p^* \otimes e_q^*, x_i^k \rangle$  exists. Then for  $\varepsilon > 0$  and some appropriately chosen  $i_1 < j_1 < \dots < i_n < j_n$ ,  $(x_{i_k}^k - x_{j_k}^k)_{k=1}^n$  will be a small perturbation of a block sequence with respect to the blocking  $(F_j)_{j=1}^\infty$  and will satisfy

$$a \left\| \sum_{k=1}^n (x_{i_k}^k - x_{j_k}^k) \right\| \geq \left( \sum_{k=1}^n \|x_{i_k}^k - x_{j_k}^k\|^2 \right)^{1/2} - \varepsilon \geq Cn^{1/2} - \varepsilon.$$

From this it follows that  $\sup_n \delta_n(\widehat{\otimes}_\varepsilon^2 \ell_1) \leq a$ . □

PROOF OF LEMMA 2.2. For  $1 < n \in \mathbb{N}$  and  $1 \leq k \leq n$ , define

$$t_k^n = \sum_{n+1-k \neq j=1}^n \frac{1}{n+1-j-k} e_j.$$

Define

$$g_k^n = \frac{1}{2^n} \sum_{i=1}^{2^k} \sum_{j=(i-1)2^{n-k}+1}^{i2^{n-k}} (-1)^i e_j \in S_{\ell_1}.$$

Note that there exists a constant  $0 < \beta$  (independent of both  $n$  and  $k$ ) such that

$$\beta \log(n) \leq \|t_k^n\|_{\ell_1}.$$

It was shown in [3] that there exists a constant  $\tau < \infty$  (independent of  $n$ ) such that

$$\left\| \sum_{k=1}^n e_k \otimes t_k^n \otimes g_k^n \right\|_{\widehat{\otimes}_\varepsilon^3 \ell_1} \leq \tau n^{1/2}.$$

There the norm was computed with  $T = \sum_{k=1}^n e_k \otimes t_k^n \otimes g_k^n$  treated as a member of  $(\widehat{\otimes}_\pi^3 c_0)^*$ , but this is equivalent to the norm in  $\widehat{\otimes}_\varepsilon^3 \ell_1$ .

Define the array  $(x_i^k)_{i=1, k=1}^{\infty, n}$  by letting  $x_i^k = e_i \otimes t_k^n \otimes g_k^n$ . By 1-unconditionality of the  $\ell_1$  basis, for any  $1 \leq k \leq n$  and any distinct  $i, j \in \mathbb{N}$ ,

$$\|x_i^k - x_j^k\| \geq \|x_i^k\| = \|e_i\| \|t_k^n\| \|g_k^n\| \geq \beta \log(n).$$

Therefore the array  $(x_i^k)_{i=1, k=1}^{\infty, n}$  is  $C = \beta \log(n)$ -separated.

By 1-subsymmetry of the  $\ell_1$  basis, it follows that for any  $i_1 < i_2 < \dots < i_n < j_n$ ,

$$\begin{aligned} \left\| \sum_{k=1}^n x_{i_k}^k - x_{j_k}^k \right\| &\leq \left\| \sum_{k=1}^n x_{i_k}^k \right\| + \left\| \sum_{k=1}^n x_{j_k}^k \right\| \\ &= 2 \left\| \sum_{k=1}^n x_k^k \right\| = 2 \left\| \sum_{k=1}^n e_k \otimes t_k^n \otimes g_k^n \right\| \leq 2\tau n^{1/2}. \end{aligned}$$

Therefore  $d2\tau \geq \beta \log(n)n^{1/2}$ .

From this it follows that  $\delta_n(\widehat{\otimes}_\varepsilon^3 \ell_1) \geq \frac{\beta \log(n)}{2\tau}$ . Since neither  $\beta$  nor  $\tau$  depends on  $n$ , we are done. □

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MIAMI UNIVERSITY, DEPARTMENT OF MATHEMATICS, OXFORD, OH 45056, USA  
*Email address:* `causeyrm@miamioh.edu`

UNIVERSITY OF SÃO PAULO, DEPARTMENT OF MATHEMATICS, IME, RUA DO MATÃO 1010,  
SÃO PAULO, BRAZIL  
*Current address:* Department of Mathematics and Statistics,  
*Email address:* `eloi@ime.usp.br`

AIX MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE  
*Email address:* `christian.samuel@univ-amu.fr`