Absence of WARM percolation in the very strong reinforcement regime

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ABSENCE OF WARM PERCOLATION IN THE VERY STRONG REINFORCEMENT REGIME [§]

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We study a class of reinforcement models involving a Poisson process on the vertices of certain infinite graphs G. When a vertex fires, one of the edges incident to that vertex is selected. The edge selection is biased towards edges that have been selected many times previously, and a parameter α governs the strength of this bias.

We show that for various graphs (including all graphs of bounded degree), if $\alpha \gg 1$ (the very strong reinforcement regime) then the random subgraph consisting of edges that are ever selected by this process does not percolate (all connected components are finite).

Combined with results appearing in a companion paper, this proves that on these graphs, with α sufficiently large, all connected components are in fact trees. If the Poisson firing rates are constant over the vertices, then these trees are of diameter at most 3.

The proof of non-percolation relies on coupling with a percolation-type model that may be of interest in its own right.

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CONTENTS

1. Introduction. Pólya-type urn models are random processes where balls are repeatedly sampled from an urn, and additional balls are added depending on the colours of the sampled balls. Since their introduction in 1931 [20], there have been many generalisations of Pólya urn models (see e.g. [18, 15]), and many more examples of random processes with reinforcement (see e.g. [19]). Some of these models take place on (hyper-)graphs and have been introduced as toy models for market competition or for neuronal connections in the brain (see e.g. [1, 2, 9]).

In this paper we study a version of the so-called WARMs introduced in [9], but defined on infinite graphs. These models involve a parameter α, which in this paper will always be larger than 1, and typically much larger. For finite graphs $\alpha > 1$ has been studied in [9, 10]. Situations (infinite or finite graphs) with $\alpha = 1$ and $\alpha < 1$ are studied in [11] and [3] respectively.

The fact that the underlying graphs $G = (V, E)$ are infinite means that the definition of the model is more technical. Time $t \in [0, \infty)$ in this paper is continuous (note that $t \in \mathbb{Z}_+$ in [9, 10]), and $N_t(e) \in \mathbb{N}$ denotes the edge count of edge $e \in E$ at time t. Starting with edge counts $N_0(e) = 1$ for each $e \in E$, the dynamics is induced by Poisson-based firings with rates $\boldsymbol{\lambda}_V := (\lambda_v)_{v \in V}$ at the vertices V as follows:

1. When a firing occurs at $v \in V$, choose an edge from those incident to v with probability proportional to the current count raised to the power α, i.e., choose $e \sim v$ with probability proportional to $N_{\cdot}(e)^\alpha$. (If there are no edges incident to v, do not choose an edge).
2. Increment the count of the chosen edge (if one was chosen).

When the graph G is finite and all vertices fire at the same rate, the *jump process* of our model is the discrete-time WARM process studied in [9, 10]. In general in the infinite graph setting, because of possibly infinite dependencies, some restrictions (see Definition 1 below) are required on $G, \boldsymbol{\lambda}_V$ to even ensure that the process is well-defined.

In this paper we are interested in the random subset of edges $\mathcal{N} = \{e \in E : \sup_{t\geq 0} N_t(e) = 1\}$ that are never reinforced, or more precisely it's complement $\mathcal{N}^c = E \setminus \mathcal{N}$.

Given G, α, and $\boldsymbol{\lambda}_V$, let $\mathbb{P}_{G,\alpha,\boldsymbol{\lambda}_V}$ denote a probability measure on a measureable space under which $\boldsymbol{N}_G := (N_t(e))_{t\geq 0, e\in E}$ has the law of a WARM on G with firing rates $\boldsymbol{\lambda}_V$ and reinforcement parameter α.

The following is straightforward to prove.

LEMMA 1. *Let $G = (V, E)$ be a graph on which the process is well defined, and $\alpha > 1$. Then $\mathbb{P}_{G,\alpha,\boldsymbol{\lambda}_V}(e \in \mathcal{N}) = 0 \iff e$ is incident to a leaf of G.*

This shows that except on star graphs, $\mathcal{N}$ is non-empty with positive probability. Our main result is that on various natural infinite graphs, when α is sufficiently large, all connected components of $\mathcal{N}^c$ are finite. In preparation for that result we define the following.

For a graph $G = (V, E)$, let $d_x = d_x(G)$ denote the degree of $x \in V$. A graph G is said to have *bounded degree* if $\partial = \partial(G) := \sup_{x\in V} d_x$ is finite. Standard examples include $\mathbb{Z}^d$ (where $\partial = 2d$).

In this paper $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes a random graph with law ν. Examples will include Galton-Watson (G-W) trees and the so-called Gilbert spatial graph. The latter is defined as follows. Let Φ be a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\mu > 0$. That is, the expected number of points in a region of volume 1 is μ. Then, the *Gilbert spatial graph* is the graph $\mathcal{G}$ with vertex set $\mathcal{V} = \Phi$ and edge set $\mathcal{E} = \{(v, y) : v, y \in \mathcal{V}, |v - y| < 1\}$.

We will assume throughout this paper that the firing rates satisfy the following condition.

CONDITION 1. *There exists a constant $L > 0$ such that ν-almost surely, $0 < \lambda_v \leq L$ for each $v \in \mathcal{V}$.*

Condition 1 is of course implied by the following.

CONDITION 2. *$\lambda_v = 1$ for each $v \in \mathcal{V}$, ν-almost surely.*

Our main result is the following.

THEOREM 1. *Let $\boldsymbol{\lambda}_{\mathcal{V}}$ satisfy Condition 1, where $\mathcal{G}$ is one of the following:*

(a) a G-W tree with offspring distribution having finite mean, or
(b) any random (connected) graph for which the maximal degree is at most a constant $\partial < \infty$, ν-almost surely, or

(c) a Gilbert spatial graph.

Then ν-almost surely, for any $\alpha > 0$ the WARM process on $\mathcal{G}$ is well defined. Moreover, there exists $\alpha_0 > 1$ such that for every $\alpha > \alpha_0$: for ν-almost every $\mathcal{G}$, all connected components of $\mathcal{G} \setminus \mathcal{N}$ are finite, $\mathbb{P}_{\mathcal{G},\alpha,\boldsymbol{\lambda}_{\mathcal{V}}}$-almost surely.

Examining the proof of Theorem 1 reveals that the parameter α_0 can be taken to only depend on: (a) the offspring distribution; (b) the degree bound ∂; and (c) the spatial dimension and the intensity of the Poisson point process, respectively. Note that if $G = \mathbb{Z}$ then $\alpha_0 = 1$ (see Lemma 8 below).

In a companion paper [7], the finite clusters of (finite or infinite) graphs are studied. As a consequence of Theorem 1 and the results of [7, 10], we have the following corollary, in which $\mathcal{E}_\infty = \{e \in \mathcal{E} : \sup_{t>0} N_t(e) = \infty\}$.

Corollary 1. *Let $\mathcal{G}$, $\boldsymbol{\lambda}_{\mathcal{V}}$ be as in Theorem 1.*

(i) If $\alpha > \max(\alpha_0, 2)$, then for ν-almost every $\mathcal{G}$, all connected components of $\mathcal{E}_\infty$ are (finite) trees, $\mathbb{P}_{\mathcal{G},\alpha,\boldsymbol{\lambda}_{\mathcal{V}}}$-a.s.
(ii) If Condition 2 holds, then for $\alpha > \max(\alpha_0, 25)$, for ν-almost every $\mathcal{G}$ all connected components of $\mathcal{E}_\infty$ are of diameter at most 3, $\mathbb{P}_{\mathcal{G},\alpha,\boldsymbol{\lambda}_{\mathcal{V}}}$-a.s.

The conclusion of (ii) fails in general if Condition 2 does not hold, see [10, Example 1].

Remarks. To say much more than what we have proved requires more careful analysis. In particular we don't know the natural general condition on a graph and firing rates that makes the conclusion of Theorem 1 true.

The first conclusion of the theorem is that the process is well-defined. Note that for graphs whose volume (number of vertices within graph distance n of a fixed vertex) grows too quickly, there *are* infinite descending chains of dependencies when Condition 2 holds (i.e. $\boldsymbol{\lambda} \equiv 1$). The same is true if the firing rates (rather than the volume) grow too quickly. Formally speaking, it does not mean that the process is *not* well-defined, but it does motivate the following.

Open Problem 1. Is there an example of a graph $G = (V, E)$ equipped with intensities $\boldsymbol{\lambda}_V$ satisfying Condition 1, for which the process is not well defined? For instance, such that the limit of WARM processes on finite subgraphs exhausting G does not exist?

We believe that for any $\alpha > 2$ there exists a WARM process on a bounded-degree graph with bounded firing rates that does contain an infinite connected component of infinitely reinforced edges. This illustrates the non-triviality of the conditions of the theorem, since the main conclusion (finite connected components) cannot be extended to all large α on all bounded degree graphs.

To be precise, we believe that for any $\alpha > 2$ one can find a regular (so bounded degree) rooted tree $T(\alpha)$ with firing rates $(\lambda_v)_{v \in T(\alpha)} \leq 1$ for which $\mathcal{E}_\infty$ almost surely contains an infinite component. The firing rates in this example decrease exponentially with distance from the root, so in particular the rates are not bounded away from 0. Moreover, by joining such trees $T(\alpha_i)$ for various values of α_i together (gluing them along an integer line), we believe that the aforementioned construction will also give rise to a tree T' (with finite but unbounded degrees) with bounded firing rates $(\lambda_v)_{v \in T'} \leq 1$ for which the WARM process is still well-defined, but with the property that $\mathcal{E}_\infty$ almost surely contains an infinite component for *any* $\alpha > 2$.

These examples motivate the following open problem(s).

Open Problem 2. Let G be a graph on which the WARM process with constant firing rates ($\lambda_v = 1$) is well defined for each $\alpha > 1$. Now consider a WARM process on G with $\varepsilon < \lambda_v < \varepsilon^{-1}$ for some $\varepsilon > 0$ and for every $v \in V$ (i.e. that the firing rates are bounded away from 0 and ∞). For this process, does it follow that there exists $\alpha_{G,\varepsilon}$ such that all connected components of $\mathcal{E}_\infty$ are finite for $\alpha > \alpha_{G,\varepsilon}$?

Open Problem 3. In the setting of Open Problem 2 when can one take $\alpha_{G,\varepsilon} = 1$?

In this paper we are primarily concerned with the set $\mathcal{N}$ of edges that are never reinforced, and its complement. This is important because (conditional on $\mathcal{N}$) the connected components of $\mathcal{N}^c$ behave independently of each other, and this fact is exploited in [7]. One can of course also ask about other sets of edges, such as the set of edges reinforced linearly over time. The latter is the primary topic of interest in [7], and in particular it is proved therein that if an edge is used infinitely often then it used linearly over time.

The rest of the article is structured as follows. In Section 2 we show that the WARM process with bounded firing rates on any graph that "does not grow too fast" is well defined, and we show that ν-almost surely our graphs $\mathcal{G}$ in Theorem 1 do not grow too fast. In Section 3 we couple the cluster $\mathcal{N}^c$ of a WARM process on a graph G with a percolation-type model whose open clusters dominate $\mathcal{N}^c$. In Section 4 we use this coupling/domination to prove Theorem 1.

2. Construction of the process. In this section, we fix a a graph $G = (V,E)$ with vertices of finite degrees, and give an explicit construction of a probability space on which the WARM process on G exists. Our construction is more elaborate than what is required to prove existence, however the additional complexity is used e.g. in the proof of Theorem 1. Since disjoint components of G do not interact, we lose no generality in assuming that G is connected in this section.

Let $\mathbf{M} = \{(X_n, T_n)\}_{n\geq 1}$ be a Poisson point process on G (to be precise, on $V \times [0,\infty)$) with intensity field $\boldsymbol{\lambda}_V$ indicating the vertex-specific firing rates. Here, we use an arbitrary enumeration of the Poisson point process, and note that the times $\{T_n\}_{n\geq 1}$ are not increasing in n. The construction of the WARM process relies on the concept of descending chains [4, 14].

Definition 1. Let $\mathbf{M}$ be a Poisson point process on G, and $m \geq 2$.

A sequence $\{(X_{n_i}, T_{n_i})\}_{i\in[m]}$ such that for every $i \in [m-1]$,

(i) X_{n_i} is adjacent to $X_{n_{i+1}}$, and

(ii) $T_{n_i} > T_{n_{i+1}}$,

is called *a descending chain of length* m.

$\mathbf{M}$ *admits infinite descending chains* if there exists an infinite sequence $\{(X_{n_i}, T_{n_i})\}_{i\geq 1}$ such that (i) and (ii) above hold for every $i \geq 1$.

A graph G is *good* if it has vertices of finite degrees, and the associated Poisson point process with $\lambda_v = 1$ for every $v \in V$ a.s. does not admit infinite descending chains.

Note that for any G with vertices of finite degrees, and $\boldsymbol{\lambda}_V$ bounded, G is good if and only if a Poisson point process $\mathbf{M}$ on G with firing rates $\boldsymbol{\lambda}_V$ does not admit infinite descending chains.

Let $E_v = \{(v,v') \in E\}$ denote the set of edges incident to $v \in V$. For each $x \in V$ and $\mathbf{n}_x = \{n(e)\}_{e\in E_x}$ we first define a total ordering $\overset{\mathbf{n}_x}{\prec}$ of the edges incident to the vertex x. This ordering depends on an initial edge ordering $\overset{x}{\prec}_0$ and the edge counts $(n(e))_{e\in E_x}$. It is defined by imposing that

$$e \overset{\mathbf{n}_x}{\prec} e' \text{if } n(e) < n(e'), \text{ or if } n(e) = n(e') \text{ and } e \overset{x}{\prec}_0 e'.$$

That is, fatter edges are preferred. We then define the *selection function*

$$\mathsf{sel}_x\big(\cdot; (n(e))_{e\in E_x}\big) : [0,1] \to E_x$$

such that $\mathsf{sel}_x\big(u; (n(e))_{e\in E_x}\big)$ is the uniquely determined edge $e \in E_x$ satisfying

$$u \in \left[\frac{\sum_{e' \overset{\mathbf{n}_x}{\prec} e} n(e')^\alpha}{\sum_{e'\in E_x} n(e')^\alpha}, \frac{n(e)^\alpha + \sum_{e' \overset{\mathbf{n}_x}{\prec} e} n(e')^\alpha}{\sum_{e'\in E_x} n(e')^\alpha}\right).$$

That is, higher values of u correspond to choosing fatter edges. This property is convenient for coupling constructions appearing in Lemma 3.

LEMMA 2. *Let G be good, $\alpha > 0$, and suppose that there exists $L > 0$ such that $\lambda_v \le L$ for every $v \in V$. Then the $\mathbb{P}_{G,\alpha,\boldsymbol{\lambda}_V}$-WARM process exists.*

PROOF. Fix G, a good graph. Since nothing happens when there is a firing at an isolated vertex $v \in V$, by removing such vertices we may assume that G has no isolated vertices (i.e. every vertex $v \in V$ has an edge incident to it).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space on which $\mathbf{M} = \{(X_n, T_n)\}_{n\ge1}$ is a Poisson point process on $V \times [0,\infty)$ with intensity $\boldsymbol{\lambda}_V$, and $\mathbf{U} = \{U_m(x)\}_{m\in\mathbb{Z}_+, x\in V}$ is a family of i.i.d. standard uniform random variables that are independent of $\mathbf{M}$. Since $\lambda_v \le L$ for every $v \in V$, $\mathbf{M}$ does not admit infinite descending chains.

We construct a family of approximations $(N_{t;i}(e))_{t\ge0, i\in\mathbb{Z}_+}$ and first set $N_{t;0}(e) = 1$ for all $e \in E$. Then, letting $\mathcal{V}_{X_m}$ denote the set of vertices consisting of X_m and all adjacent vertices, the initial layer

$$\mathcal{L}_1 = \{(X_m, T_m) \in \mathbf{M} : \mathbf{M} \cap (\mathcal{V}_{X_m} \times [0, T_m)) = \emptyset\}$$

consists of all firing events such that no firing event has occurred earlier either at the considered vertex or at one of its neighbouring vertices. Then, $\mathcal{L}_1$ is non-empty because there are no infinite descending chains.

For every $(X_m, T_m) \in \mathcal{L}_1$ and $e \in E_{X_m}$ define

$$N_{t;1}(e) = N_{t;0}(e) + \mathbb{1}_{\left\{t\ge T_m,\, \mathsf{sel}_{X_m}\left(U_0(X_m);(N_{T_m-;0}(e'))_{e'\in E_{X_m}}\right)=e\right\}}.$$

For other edges, we put $N_{t;1}(e) = N_{t;0}(e)$.

For $i \ge 1$ we proceed recursively and define the $(i+1)$th layer

$$\mathcal{L}_{i+1} = \{(X_m, T_m) \in \mathbf{M} : \mathbf{M} \cap \big(\mathcal{V}_{X_m} \times [0, T_m)\big) \subset \mathcal{L}_i\}$$

as the family of all firing events such that all earlier firing events at this or adjacent vertices are in layer $\mathcal{L}_i$. Let $Q(x)$ denote the event that an edge in E_x has been reinforced before the first firing time of x and

$$\hat{U}_m(x) = \begin{cases} U_m(x), & \text{if } m \ge 2 \\ U_1(x)\mathbb{1}_{Q(x)} + U_0(x)\mathbb{1}_{Q(x)^c}, & \text{if } m = 1. \end{cases}$$

Note that $\hat{\mathbf{U}} = (\hat{U}_m(x))_{x\in V, m\ge1}$ are i.i.d. since $\mathbf{U}$ was an i.i.d. collection, and that $Q(x)$ is independent of $U_1(x), U_0(x)$.

For every $(X_m, T_m) \in \mathcal{L}_{i+1} \setminus \mathcal{L}_i$ and $e \in E_{X_m}$ we put

$$N_{t;i+1}(e) = N_{t;i}(e) + \mathbb{1}_{\left\{t \geq T_m,\, \mathsf{sel}_{X_m}\left(\hat{U}_{S_m(X_m)}(X_m);(N_{T_m-;i}(e'))_{e' \in E_{X_m}}\right)=e\right\}},$$

where $S_m(x) := \#\{(T_k, X_k) \in \mathbf{M} : T_k \leq T_m, X_k = x\}$ denotes the number of times that x has fired up to (and including) time T_m. Again, for other edges there are no changes. Note that, we use the $U_0(X_m)$ variable to determine the chosen edge at time T_m if and only if $N_{T_m-;i}(e') = 1$ for every $e' \in E_{X_m}$ (otherwise we use $U_{S_m}(X_m)$).

Since $N_{t;i}(e)$ is increasing in i, and $N_{t;i}(e) \leq \#\{(X_m, T_m) : T_m \leq t, X_m \sim e\}$, the limiting count

$$N_t(e) := \lim_{i \to \infty} N_{t;i}(e)$$

is well-defined, and a.s. finite. Finally, as the Poisson point process does not admit infinite descending chains, we have $\cup_{i \geq 0} \mathcal{L}_i = \mathbf{M}$. That is, every Poisson firing event is indeed accounted for in this dynamics. ∎

Based on the above construction, we now provide an upper bound on the probability that a firing at a vertex leads to a new edge being selected. Let

$$T_n(x) = \inf\{t \geq 0 : \#(\mathbf{M} \cap (\{x\} \times [0, t])) = n\}$$

be the nth firing time of the vertex $x \in V$, for $n \in \mathbb{N}$. Then,

$$\mathcal{F}_{x,n} := \sigma\big(((X_m, T_m) : T_m \leq T_n(x)), \mathbf{U} \setminus U_n(x)\big)$$

denotes the σ-algebra generated by $\mathbf{M}$ up to time $T_n(x)$ and all selection variables except for $U_n(x)$. Moreover, $K_n(x) = \#\{e \in E_x : N_{T_n(x)-}(e) = 1\}$ denotes the number of edges adjacent to x that are not reinforced before time $T_n(x)$. Finally define,

$$W_{x,n} := \mathbb{P}_{G,\alpha,\boldsymbol{\lambda}_V}(N_{T_n(x)}(e) = 2 \neq N_{T_n(x)-}(e) \text{ for some } e \in E_x | \mathcal{F}_{x,n}).$$

LEMMA 3. *Fix G, $\boldsymbol{\lambda}_V$ so that the process is well defined. Let $n \geq 1$ and $x \in V$ be arbitrary, and $\alpha > 0$. Then,*

$$W_{x,n} \leq \frac{d_x - 1}{d_x - 1 + \max_{e \in E_x} N_{T_n(x)-}(e)^\alpha}, \qquad \textit{a.s. on the event } \{K_n(x) < d_x\}.$$

PROOF. Fix x, n and let $\kappa_n = \{e \in E_x : N_{T_n(x)-}(e) = 1\}$. Letting $m \geq 1$ be such that $T_n(x) = T_m$, the construction described above shows that on the event $\{K_n(x) < d_x\}$ we have $\hat{U}_n(x) = U_n(x)$, which is independent of $\mathcal{F}_{x,n}$. Therefore, almost surely on $\{K_n(x) < d_x\}$,

$$\begin{aligned} W_{x,n} &= \mathbb{P}_{G,\alpha,\boldsymbol{\lambda}_V}\left(\mathsf{sel}_x\big(U_n(x);(N_{T_n(x)-}(e))_{e\in E_x}\big) \in \kappa_n \big| \mathcal{F}_{x,n}\right) \\ &\le \frac{K_n(x)}{K_n(x) + \sum_{e\in E_x\setminus\kappa_n} N_{T_n(x)-}(e)^\alpha} \\ &\le \frac{K_n(x)}{K_n(x) + \max_{e\in E_x} N_{T_n(x)-}(e)^\alpha} \\ &\le \frac{d_x - 1}{d_x - 1 + \max_{e\in E_x} N_{T_n(x)-}(e)^\alpha}. \end{aligned}$$

■

For a graph $G = (V, E)$, let $c_{n,x}$ denote the number of self-avoiding walks of length $n \ge 1$ in G started from $x \in V$. We will use the following lemma to show that all of the graphs in Theorem 1 are good, ν-almost surely.

LEMMA 4. *Let $G = (V, E)$ be a graph for which there exist $r > 1$ and non-negative constants $\{C_x\}_{x\in V}$ such that*

$$c_{n,x} \le C_x r^n, \qquad \text{for all } n \ge 1 \text{ and } x \in V.$$

Then G is good.

PROOF. The fact that $c_{n,x}$ is finite for every x implies that G has finite degrees.

Next, note that for a fixed finite set $V_0 \subset V$ the probability that there is an infinite descending chain $\{(X_{n_i}, T_{n_i})\}_{i\ge1}$ with $X_{n_i} \in V_0$ for each i is 0. Since G is countable, the collection of finite subsets of G is also countable, from which it follows that almost surely no infinite descending chain can be found on a finite subset of G. Thus, (almost surely) an infinite descending chain exists if and only if an infinite descending chain $\{(X_{n_i}, T_{n_i})\}_{i\ge1}$ with $\{X_{n_i}\}_{i\ge1}$ *all distinct* exists.

Let $t > 0$ and $v \in V$ be arbitrary. From the above, it suffices to show that the expected number of descending chains of length $n \ge 1$ starting at v before time t *consisting of n distinct vertices*, tends to 0 as $n \to \infty$. Let $\gamma = \langle v_1, \ldots, v_n\rangle$ be a fixed self-avoiding path of length n in G starting from $v_1 = v$. Then, we let $\mathcal{L}_\gamma$ denote the number of n-tuples $\{(X_{k_i}, T_{k_i})\}_{i\in[n]}$ of points from $\mathbf{M}$ such that: $\langle X_{k_0}, \ldots, X_{k_{n-1}}\rangle = \gamma$ and $t > T_{k_i} > T_{k_{i+1}}$ for every $i \in [n-1]$. By the multivariate Mecke formula [16, Theorem 4.4], applied to the Poisson point process $\mathbf{M}$ and the mapping

$$((x_1, t_1), \ldots, (x_n, t_n)) \mapsto \mathbb{1}_{\{\{x_i = v_i \text{ for every } i \le n\}\}} \mathbb{1}_{\{\{t_1 > t_2 > \cdots > t_n\}\}}$$

we arrive at

$$\mathbb{E}_{G,\alpha,\boldsymbol{\lambda}_V}[\mathcal{L}_\gamma] = \prod_{i=1}^{n} \lambda_{v_i} \int_{[0,t]^n} \mathbb{1}_{\{t_1>t_2>\cdots>t_n\}} \mathrm{d}t_1 \cdots \mathrm{d}t_n \leq \frac{(tL)^n}{n!}.$$

Thus, the expected number of descending chains of length n starting from v with all vertices distinct, is at most $c_{n,v}(tL)^n/n!$. By assumption there exist $C_v, r > 1$ such that $c_{n,v} \leq C_v r^n$ for every n. Therefore

$$\limsup_{n\to\infty} \frac{c_{n,v}(tL)^n}{n!} \leq C_v \limsup_{n\to\infty} \frac{(rtL)^n}{n!} = 0,$$

as required. ∎

Proposition 1. *All of the graphs in Theorem 1 are good, ν-almost surely.*

Proof. For any graph G with maximal degree ∂, the number of walks of length n started from any $x \in V$ is at most ∂^n, so by Lemma 4 G is good. This verifies the claim for $\mathcal{G}$ almost surely having maximal degree at most ∂.

For G-W trees with offspring distribution having mean $\mu \in (0,\infty)$, $M_n := \mu^{-n}K_n$ is a positive martingale (where K_n is the number of individuals in generation n), so it converges almost surely. Thus, we have that $\sup_n \mu^{-n}K_n < C$ for some (random) finite $C > 0$, ν-almost surely. In a tree, the number of self-avoiding walks of length n started from x is precisely the number of vertices of distance n from x. Since x is some distance k_x from the root, every vertex that can be reached from x in n steps can be reached from the root in at most $k_x + n$ steps. Therefore, ν-almost surely on the event that $x \in \mathcal{V}$, there exists a $C > 0$ such that

$$c_{n,x} \leq \sum_{j=0}^{n+k_x} K_j \leq C \sum_{j=0}^{n+k_x} \mu^j \leq C_x \rho^n,$$

and the result follows by Lemma 4.

For the Gilbert spatial graph with intensity μ let

$$\begin{aligned} N'_n = \#\{(X_1,\dots,X_n) \in \Phi^n : X_1 \in [-1/2,1/2]^d, \{X_i\}_{1\leq i\leq n} \text{ all distinct} \\ \text{and } |X_i - X_{i+1}| \leq 1 \text{ for all } i \in [n-1]\}, \end{aligned}$$

denote the number of self-avoiding paths of length n starting from the unit cube. In particular, $c_{x,n} \leq N'_n$ for every $x \in \Phi \cap [-1/2,1/2]^d$, so that by stationarity, it suffices to show that (ν-almost surely) there exists $C > 0$ such that $N'_n \leq Cr^n$ for every n.

Write $B_1(x)$ for the Euclidean ball of radius 1 centered at $x \in \mathbb{R}^d$ and κ_d for the volume of the d-dimensional unit ball in the Euclidean metric. Applying the multivariate Mecke formula [16, Theorem 4.4] to the mapping

$$(x_1, \dots, x_n) \mapsto \mathbb{1}_{\{x_1 \in [-1/2,1/2]^d\}} \mathbb{1}_{\{x_{i+1} \in B_1(x_i) \text{ for every } i \leq n-1\}}.$$

we obtain

$$\begin{aligned} E_\nu[N'_n] &\leq \mu^n \int_{[-1/2,1/2]^d} \int_{B_1(x_1)} \cdots \int_{B_1(x_{n-1})} 1 \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\ &= \mu^n \kappa_d \int_{[-1/2,1/2]^d} \int_{B_1(x_1)} \cdots \int_{B_1(x_{n-2})} 1 \mathrm{d}x_n \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\ &= \mu^n \kappa_d^{n-1}. \end{aligned}$$

In particular, writing $r = (\max\{2, \mu\kappa_d\})^2$, the Markov inequality gives that

$$\nu(N'_n \geq r^n) \leq \frac{1}{\kappa_d} r^{-n/2}.$$

Hence, by the Borel-Cantelli lemma, ν-almost surely there exists a random $C > 0$ such that $N'_n \leq Cr^n$ for every n. ∎

Lemma 4 amounts to a bound on the rate of growth of the number of self-avoiding walks on a graph, started from fixed locations. It would be of interest to consider what happens when Condition 1 is dropped, while still assuming that the process $\mathbf{M}$ on G almost surely does not admit infinite descending chains. The latter condition puts restrictions both on the growth of the graph and the firing rates. In particular it would be of interest to consider what happens when $(\lambda_v)_{v \in V}$ is an (unbounded) i.i.d. sequence, whence the model becomes a reinforcement model in a random firing environment.

We finish this section with two examples that admit infinite descending chains: an unbounded degree graph with bounded firing rates; and a bounded degree graph with unbounded firing rates.

LEMMA 5. *Consider an infinite rooted tree T, for which all the vertices at distance n from the root have $5(n+2)^2$ children. Then almost surely T is not good.*

SKETCH PROOF. Call a vertex v at distance n from the root "nice" if it has fired between moments $t = \frac{1}{n+2}$ and $t' = \frac{1}{n+1}$. It is an interval of time of length $\frac{1}{(n+1)(n+2)} > \frac{1}{(n+2)^2}$. Hence, the expectation of the number of nice children of any vertex is at least 5, and in fact its law converges (as we take vertices farther and farther from the root) to the Poisson law with parameter 5. Hence, almost surely there exists an infinite (Galton-Watson-like) subtree formed by nice vertices. Any infinitely growing branch of such a subtree provides us with an infinite descending chain. ∎

LEMMA 6. *The graph $\mathbb{Z}_+$, with $\lambda_n = (n!)^2$ almost surely admits infinite descending chains.*

PROOF. Let A_n be the event that $n-1$ fires before n. Then $\mathbb{P}(A_n) = 1/(1+n^2)$, so $\mathbb{P}(A_n \text{ infinitely often}) = 0$. ∎

3. Corrupted compass models. In order to prove our main result, we couple our highly dependent WARM process with an independent percolation-type model that we call a *corrupted compass* model on a graph G.

DEFINITION 2 (Corrupted Compass Model on $G = (V, E)$). Let $\{p_{\{x\}}\}_{x \in V}$ be a collection of elements of $[0, 1]$. Every non-isolated vertex $x \in V$ is independently and with probability $p_{\{x\}}$ called *corrupted*. Let $\mathcal{K} \subset V$ denote the set of corrupted vertices. Independently at each non-isolated vertex $y \in V$, choose an edge η_y from E_y uniformly at random, and define

$$\mathbf{C} := \bigcup_{x \in \mathcal{K}} E_x \cup \{\eta_y : y \in V\} \subset E. \tag{1}$$

The uncorrupted compass model is the choice $p_{\{x\}} = 0$ for each x.

Corrupted compass models may be of interest in their own right, as on regular lattices they can be viewed as examples of "degenerate random environments" [12, 13], which are generalisations of directed percolation models. Of particular interest is the case where $p_{\{x\}} = p_{d_x}(\alpha)$, where d_x is the degree of x, $p_0(\alpha) = 0$ and for $d \geq 1$,

$$b_n(d) := \frac{d-1}{d-1+(2 \vee \frac{n}{d})^\alpha} \quad \text{and} \quad p_d(\alpha) := 1 - \prod_{n=1}^{\infty}(1 - b_n(d)).$$

We have the following elementary lemma.

LEMMA 7. *Let $d \geq 1$ and $\alpha > 1$. Then, $p_d(\alpha) < 1$ and $\lim_{\alpha \uparrow \infty} p_d(\alpha) = 0$.*

Proof. Using the inequality $\log(x) \geq 1 - 1/x$ for all $x > 0$ we obtain for each $a > 0$, and each $\{j_n\}_{n\geq 1} \subset (0, \infty)$

$$\prod_{n\geq 1}\left(1 - \frac{a}{a + j_n}\right) = \prod_{n\geq 1}\frac{j_n}{a + j_n} = \exp\left\{\sum_{n\geq 1}\log\left(\frac{j_n}{a + j_n}\right)\right\} \geq \exp\left\{-a\sum_{n\geq 1}\frac{1}{j_n}\right\}.$$

If $j_n = (2 \vee (n/d))^\alpha$ then the above sum is finite for $\alpha > 1$, which proves the first claim. Moreover, for this choice of j_n, the sum is at most

$$\sum_{n=1}^{2d}\frac{1}{2^\alpha} + d^\alpha \sum_{n=2d+1}^{\infty}\frac{1}{n^\alpha} \leq \frac{2d}{2^\alpha} + d^\alpha \int_{2d}^{\infty}\frac{1}{x^\alpha}\mathrm{d}x = \frac{d}{2^{\alpha-1}}[1 + \frac{1}{\alpha - 1}],$$

which approaches 0 as $\alpha \uparrow \infty$. This proves the second claim. ■

For the graphs appearing in Theorem 1 we prove that for large α the (random) set of edges in the corrupted compass model almost surely has finite clusters. Therefore, the following result is fundamental to our analysis.

Proposition 2. *On the probability space of Section 2 one can define a corrupted compass model (with $p_{\{x\}} \equiv p_{d_x}(\alpha)$ as above) such that $\mathcal{N}^c \subset \mathbf{C}$.*

Proof. Recall the probability space of Section 2. Write $(1)_{f\in E_x}$ for $(n(f))_{f\in E_x}$ when all such $n(f)$ are equal to 1. Let $\eta_x = \mathsf{sel}_x\big(U_0(x); (1)_{f\in E_x}\big)$ and

$$\mathcal{S}^* = \{\eta_x : x \in V\} \subset E.$$

Let $\tau_x = \inf\{t > 0 : N_t(e) > 1 \text{ for some } e \in E_x\}$ denote the first time that an edge incident to x is reinforced. A vertex $x \in V$ is *bad* if there exists a firing time $T_n(x) > \tau_x$ at x that reinforces a previously unreinforced edge $e \in E_x$, i.e. $N_{T_n(x)-}(e) = 1$ and $N_{T_n(x)}(e) = 2$ (note that since $T_n(x) > \tau_x$ this means that there was at least one $e' \in E_x$ with $N_{T_n(x)-}(e') \geq 2$). Define

$$\mathbf{C}^* = \mathcal{S}^* \cup \bigcup_{x \text{ is bad}} E_x.$$

We now define a corrupted compass model on the probability space of Section 2. The compass at (non-isolated) $x \in V$ is defined to be the edge $\eta_x = \mathsf{sel}_x\big(U_0(x); (1)_{f\in E_x}\big)$. The vertex x is corrupted if and only if $U_n(x) \leq b_n(d_x)$ for some $n \in \mathbb{N}$. Note that these corruption events are independent over x and are also independent of the compass selection process. Thus, the above does indeed define a corrupted compass model on G with $p_{\{x\}} =$

$p_{d_x}(\alpha)$ for each x. Let $\mathbf{C}$ denote the set of edges of this corrupted compass model, as per (1). Then, to complete the proof it suffices to show that

$$\mathcal{N}^c \subset \mathbf{C}^*, \qquad \text{and} \tag{2}$$

$$\mathbf{C}^* \subset \mathbf{C}. \tag{3}$$

To prove (2), suppose that $e = (x, x') \in \mathcal{N}^c$. Then, without loss of generality, the first time that e was reinforced was at a firing time $T_n(x)$ at x for some n. If $e \in \mathcal{S}^*$ then $e \in \mathbf{C}^*$. Otherwise $e \notin \mathcal{S}^*$ so in particular $e \neq \mathsf{sel}_x\big(U_0(x); (1)_{f\in E_x}\big)$, so there must be some other edge $e' \in E_x$ that was already reinforced before time $T_n(x)$. Thus, x is a bad vertex, so $e \in \mathbf{C}^*$.

To prove (3), note that by construction $\mathcal{S}^* \subset \mathbf{C}$ is trivially true. Suppose that $x \in V$ is bad. Then, at some firing time $T_n(x)$ at x there was an edge e' in E_x such that $N_{T_n(x)-}(e') \geq 2$ but an edge $e \in E_x$ with $N_{T_n(x)-}(e) = 1$ was chosen. Hence, recalling the notation and proof of Lemma 3,

$$U_n(x) \leq \frac{K_n(x)}{K_n(x) + \sum_{e\in E_x\setminus\kappa_n} N_{T_n(x)-}(e)^\alpha} \leq \frac{d_x - 1}{d_x - 1 + \max_{e\in E_x} N_{T_n(x)-}(e)^\alpha}.$$

It also implies that $\max_{e\in E_x} N_{T_n(x)-}(e) \geq 2 \vee (n/d_x)$, since at least one edge is already reinforced, and after n clock rings at x at least one edge in E_x must have count at least n/d_x. Thus,

$$U_n(x) \leq \frac{d_x - 1}{d_x - 1 + (2 \vee (n/d))^\alpha} = b_n(d_x),$$

so x is corrupt. This proves (3). ■

Remark 1. Notice that the construction (and hence the law) of the corrupted compass model in the proof of Proposition 2 does not depend on $\boldsymbol{\lambda}_V$ at all. In particular, the conclusion of the Proposition holds as long as $\boldsymbol{\lambda}_V$ is such that the construction of the WARM process on G in Lemma 2 is valid.

4. Proof of Theorem 1. We let $\mathbf{C}_x \subset V$ denote the connected cluster of $x \in V$ in $\mathbf{C}$. In view of Proposition 2, the proof of Theorem 1 reduces to establishing that connected clusters $\mathbf{C}_x$ with the choice $p_{\{x\}} = p_{d_x}(\alpha)$ for each x, are finite almost surely for α sufficiently large. The latter is the content of the following proposition.

Proposition 3. *Let $\mathcal{G}$ be any of the graphs in Theorem 1. Then there exists $\alpha_0 > 1$ such that for all $\alpha \geq \alpha_0$: ν-almost surely, all connected clusters of $\mathbf{C}(\mathcal{G})$ (with the choice $p_{\{x\}} = p_{d_x}(\alpha)$) are finite, $\mathbb{P}_{\mathcal{G},\alpha,\boldsymbol{\lambda}_V}$-almost surely.*

The following lemma illustrates this approach in the simplest setting.

LEMMA 8. *If $G = \mathbb{Z}$ then for $\alpha > 1$, all components of $G \setminus \mathcal{N}$ are finite, almost surely.*

PROOF. By Proposition 2, it suffices to show that all connected components of $\mathbf{C}$ are finite. Let $J_i = \{\{2i, 2i+1\} \in \mathbf{C}^c\}$. Then J_i is the event that: neither vertex in $V_i' := \{2i, 2i+1\}$ is corrupt, and $\eta_{2i} = (2i-1, 2i)$, and $\eta_{2i+1} = (2i+1, 2i+2)$. Therefore $\mathbb{P}(J_i) = c(\alpha) > 0$ since $\alpha > 1$. However, since the vertex sets V_i' for $i \in \mathbb{Z}$ are disjoint, the events $(J_i)_{i\in\mathbb{Z}}$ are independent. Thus, we encounter an edge in $\{\{2i, 2i+1\} : i \in \mathbb{Z}\} \cap \mathcal{N}$ after examining the status of at most a Geometric$(c(\alpha))$ number of edges in $\{\{2i, 2i+1\} : i \in \mathbb{Z}\}$ to the right of 0 (and similarly to the left). ∎

In the following subsections we will verify that (for every $x \in V$) $\mathbf{C}_x$ is almost surely finite for each of the 3 different settings in the proposition.

4.1. *Galton-Watson trees.* Let $V^* = \{(n_0, n_1, \ldots, n_k) : k, n_0, n_1, \ldots, n_k \in \mathbb{Z}_+\}$. Let $\boldsymbol{\xi} = (\xi_v)_{v\in V^*}$ be i.i.d. random variables taking values in $\mathbb{Z}_+$, with probability mass function f and having finite mean $\mu = \sum_{n\geq 0} nf(n)$. Let $\boldsymbol{U} = (U_v)_{v\in V^*}$ and $\boldsymbol{U}' = (U_v')_{v\in V^*}$, and $\boldsymbol{U}'' = (U_v'')_{v\in V^*}$ be families of mutually independent standard uniform random variables that are also independent of $\boldsymbol{\xi}$.

We label the root (the unique vertex of generation 0) of our tree as (1), and for any vertex $v = (1, n_1 \ldots, n_k)$ of generation k, its children are labelled $(v, 1), (v, 2), \ldots$. If v is not the root then its parent is denoted by v^{-1}. Let $\mathcal{V}_n$ denote the vertices of generation n.

One *could* generate a Galton-Watson tree $\mathcal{G}$ with a corrupted compass model on it iteratively over n by using the variables $\boldsymbol{\xi}$ to generate the children $(v, 1), \ldots, (v, \xi_v)$ of each $v \in \mathcal{V}_n$ and then deciding whether or not each vertex of generation n is corrupted and whether or not its compass points towards the root. Corruption occurs with probability $p_{k+1}(\alpha)$ if the number of children of v is k (unless v is the root in which case corruption occurs with probability $p_k(\alpha)$). Similarly (unless v is the root) the compass at v points towards the root with probability $1/(k+1)$. We do not generate the tree and corrupted compass this way.

Instead we generate the tree $\mathcal{G}$ and the corrupted compass model on it as follows (also iteratively over $n \geq 0$). Given $\mathcal{V}_n$, for each $v \in \mathcal{V}_n$, decide whether v is corrupted, and (if not) decide on whether η_v is the edge to the parent or some other edge. Then generate the number of children of v conditionally on these events.

To be precise, let $n \geq 1$. Given that $v \in \mathcal{V}_n$:

- $v \in \mathcal{K}$ if $U_v \leq q_C$, where
$$q_C := \sum_{n=0}^{\infty} p_{n+1}(\alpha) f(n).$$
- $v \notin \mathcal{K}$ and η_v is the edge to v^{-1} if $U_v > q_C$ and $U'_v < q_B/(1-q_C)$, where
$$q_B := \sum_{n=0}^{\infty} \frac{(1-p_{n+1}(\alpha))}{n+1} f(n),$$
- otherwise $v \notin \mathcal{K}$ and η_v is not the edge to v^{-1}.

The respective probabilities of these events (given that $v \in \mathcal{V}_n$) are q_C, q_B and $q_F := 1 - q_C - q_B$ respectively.

Given that $v \in \mathcal{K}$, v has exactly $k \geq 0$ children if $U''_v \in \big(\sum_{j=0}^{k-1} g_C(j), \sum_{j=0}^{k} g_C(j)\big]$, where
$$g_C(k) := q_C^{-1} p_{k+1}(\alpha) f(k).$$
Similarly, given that $v \notin \mathcal{K}$ and η_v is the edge to v^{-1}, v has exactly $k \geq 0$ children if $U''_v \in (\sum_{j=0}^{k-1} g_B(j), \sum_{j=0}^{k} g_B(j)]$, where
$$g_B(k) := q_B^{-1} \frac{(1-p_{k+1}(\alpha))}{k+1} f(k).$$
Otherwise, v has exactly $k \geq 0$ children if $U''_v \in \big(\sum_{j=0}^{k-1} g_F(j), \sum_{j=0}^{k} g_F(j)\big]$, where
$$g_F(k) := q_F^{-1} \frac{k(1-p_{k+1}(\alpha))}{k+1} f(k), \qquad \text{for } k \geq 1$$
The case $n = 0$ (i.e. the corruption status and number of children etc. of the root) is slightly different. It is a simple but tedious exercise to verify that this construction defines a Galton-Watson tree $\mathcal{G}$ with offspring distribution having probability mass function f, together with a corrupted compass model (with $p_{\{x\}} = p_{d_x}(\alpha)$ for each $x \in \mathcal{V}$) on it.

Proof of Proposition 3 – Galton-Watson trees. Let m_C, m_B, m_F be the means of random variables with probability mass functions g_C, g_B, g_F respectively. Let C_n, B_n, F_n denote the number of vertices in generation n that are (respectively) corrupted, uncorrupted and have their compass pointing to their parent, and uncorrupted with the compass pointing to a child. Fix $L \in (0,\infty)$ and let $\boldsymbol{\lambda}$ be such that $\lambda_v \le L$ for each $v \in V^*$.

Let $\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}$ denote the annealed/averaged measure defined by

$$\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(\mathcal{G} \in A, \boldsymbol{N}_{\mathcal{G}} \in B) := \int_A \mathbb{P}_{\mathcal{G},\alpha,\boldsymbol{\lambda}_{\mathcal{V}}}(\boldsymbol{N}_{\mathcal{G}} \in B)\mathrm{d}\nu(\mathcal{G}).$$

Under the annealed measure, conditional on $v \in \mathcal{V}_n$, and on the compass type of v (C=corrupted, B=uncorrupted with edge to parent, or F=uncorrupted with edge to child), the expected number of children of type $a \in \{C,B,F\}$ when v is of type $\beta \in \{C,B,F\}$ is $m_\beta q_a$. Let c_n, b_n, f_n denote the expected number of vertices of each type in $\mathbf{C}_{(1)} \cap \mathcal{V}_n$, under the annealed measure. Then,

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} m_C q_C & m_B q_C & m_F q_C \\ m_C q_B & m_B q_B & m_F q_B \\ m_C q_F & 0 & q_F \end{pmatrix} \begin{pmatrix} c_n \\ b_n \\ f_n \end{pmatrix}.$$

Most of the entries of the update matrix are obvious, and correspond to cases where every child of a given type is also in $\mathbf{C}_{(1)} \cap \mathcal{V}_{n+1}$. The entry 0 is because if the compass of $v \in \mathcal{V}_n \cap \mathbf{C}_{(1)} \cap \mathcal{K}^c$ points to its parent then any child v' of v that is uncorrupted and whose compass does not point towards v is not in $\mathbf{C}_{(1)}$. Similarly, if $v \in \mathcal{V}_n \cap \mathbf{C}_{(1)} \cap \mathcal{K}^c$ and its compass points to a child v' of v then v can have at most one child (it would have to be v') that is uncorrupted and whose compass points away from v. The probability that this child v' has this property is q_F.

To show that $\boldsymbol{E}_{\alpha,\boldsymbol{\lambda}}[|\mathbf{C}_{(1)}|] < \infty$ for α sufficiently large, it is sufficient to show that the eigenvalues of the update matrix have absolute values strictly less than 1 for α sufficiently large (since then $\boldsymbol{E}_{\alpha,\boldsymbol{\lambda}}[|\mathbf{C}_{(1)} \cap \mathcal{V}_n|]$ is decreasing exponentially in n and is therefore summable). The eigenvalues are the solutions to the cubic equation

$$0 = (\lambda - m_B q_B)(q_F - \lambda)\lambda + m_C q_C \lambda[m_F q_F - (q_F - \lambda)].$$

Now

$$m_C q_C = \sum_{k=0}^{\infty} k p_{k+1}(\alpha) f(k).$$

Since μ is finite, and $p_{k+1}(\alpha) \to 0$ as $\alpha \to \infty$ for each k, we have that $m_C q_C \to 0$ as $\alpha \to \infty$. Similarly, as $\alpha \to \infty$, the quantities $q_B(\alpha), q_F(\alpha), m_B(\alpha), m_F(\alpha)$

converge respectively to

$$q_B(\infty) := \sum_{n=0}^{\infty} \frac{f(n)}{n+1}, \qquad q_F(\infty) := \sum_{n=0}^{\infty} \frac{nf(n)}{n+1},$$
$$m_B(\infty) := \frac{\sum_{k=0}^{\infty} \frac{kf(k)}{k+1}}{\sum_{n=0}^{\infty} \frac{f(n)}{n+1}}, \qquad m_F(\infty) := \frac{\sum_{k=0}^{\infty} \frac{k^2 f(k)}{k+1}}{\sum_{n=0}^{\infty} \frac{nf(n)}{n+1}},$$

which are all finite and positive. It follows that as $\alpha \to \infty$ the eigenvalues of our update matrix approach the solutions to

$$0 = (\lambda - m_B(\infty)q_B(\infty))(q_F(\infty) - \lambda)\lambda.$$

The solutions are $\lambda = 0$, $\lambda = q_F(\infty) \in (0,1)$ and $\lambda = m_B(\infty)q_B(\infty) = q_F(\infty) \in (0,1)$. This proves that there exists $\alpha_0 > 1$ such that $\boldsymbol{E}_{\alpha,\boldsymbol{\lambda}}[|\mathbf{C}_{(1)}|] < \infty$ for all $\alpha > \alpha_0$. From this one can conclude that $\boldsymbol{E}_{\alpha,\boldsymbol{\lambda}}[|\mathbf{C}_x|\mathbb{1}_{\{x\in\mathcal{V}\}}] < \infty$ for $\alpha > \alpha_0$. ■

4.2. *Bounded-degree graphs.* Here we fix a graph G with degrees bounded by ∂, and let $\mathbb{P} = \mathbb{P}_{G,\alpha,\boldsymbol{\lambda}_V}$, and $\mathbb{E}$ the corresponding expectation. Since $p_{d_x}(\alpha) \le p_\partial(\alpha)$ for every $x \in V$, it is sufficient to show that for the corrupted compass model with $p_y = p_\partial(\alpha)$ for every $y \in V$, the expectation of the size of the component of x is finite.

Proof of Proposition 3 – bounded degrees. Set $p = p_y = p_\partial(\alpha)$ (which can be made arbitrarily small by taking α sufficiently large) for every $y \in V$. Fix $x \in V$. Then

$$\mathbb{E}[|\mathbf{C}_x(\alpha)|] \le 1 + \sum_{n\ge1}\sum_{v\in V}\sum_{\gamma: x \xrightarrow{n} v} \mathbb{P}(\gamma \subset \mathbf{C}),$$

where the interior sum is over the simple paths γ from x to v of length n (i.e. containing n edges), and $\gamma \subset \mathbf{C}$ means that the edges of γ are all in $\mathbf{C}$.

For neighbours x, x', write $x \to x'$ if $\eta_x = \{x, x'\}$ or $x \in \mathcal{K}$, and $x \not\to x'$ otherwise. Clearly, for any simple path γ of length $n \ge 1$ from $\gamma_0 = x$ to $\gamma_n = v$ we have

$$\begin{aligned}\mathbb{P}(\gamma \subset \mathbf{C}) =& \mathbb{P}(\gamma \subset \mathbf{C}, v \in \mathcal{K}) \\ &+ \mathbb{P}(\gamma \subset \mathbf{C}, v \notin \mathcal{K}, \gamma_{n-1} \to v) \\ &+ \mathbb{P}(\gamma \subset \mathbf{C}, v \notin \mathcal{K}, v \to \gamma_{n-1}, \gamma_{n-1} \not\to v).\end{aligned}$$

Define

$$c_n(\alpha) := \sum_{v\in V} \sum_{\gamma: x \xrightarrow{n} v} \mathbb{P}(\gamma \subset \mathbf{C}, v \in \mathcal{K}).$$

$$f_n(\alpha) := \sum_{v\in V} \sum_{\gamma: x \xrightarrow{n} v} \mathbb{P}(\gamma \subset \mathbf{C}, v \notin \mathcal{K}, \gamma_{n-1} \to v),$$

$$b_n(\alpha) := \sum_{v\in V} d_v \sum_{\gamma: x \xrightarrow{n} v} \mathbb{P}(\gamma \subset \mathbf{C}, v \notin \mathcal{K}, v \to \gamma_{n-1}, \gamma_{n-1} \not\to v),$$

We claim that all these three sequences are exponentially decreasing for sufficiently large α. We are going to show that each component of the vector $(c_{n+1}, f_{n+1}, b_{n+1})$ does not exceed the corresponding component of the vector

$$\begin{pmatrix} (\partial-1)p & (\partial-1)p & \frac{\partial-1}{\partial}p \\ \partial-1 & \frac{\partial-1}{\partial} & 0 \\ 0 & \partial-1 & \frac{\partial-1}{\partial} \end{pmatrix} \begin{pmatrix} c_n \\ f_n \\ b_n \end{pmatrix}. \tag{4}$$

Note that for p sufficiently small, all the eigenvalues of the above matrix are less than 1 in absolute value, since as $p \to 0$, this matrix tends to

$$\begin{pmatrix} 0 & 0 & 0 \\ \partial-1 & \frac{\partial-1}{\partial} & 0 \\ 0 & \partial-1 & \frac{\partial-1}{\partial} \end{pmatrix},$$

and its eigenvalues are $\frac{\partial-1}{\partial}$, $\frac{\partial-1}{\partial}$ and 0. Thus, iterations of this matrix are exponentially decreasing, and we get the desired upper bound.

We now have to establish the upper bound for $c_{n+1}, f_{n+1}, b_{n+1}$. Note that

$$\begin{aligned} c_{n+1} &= \sum_{v\in V} \sum_{\gamma: x \xrightarrow{n+1} v} \mathbb{P}(\gamma \subset \mathbf{C}, v \in \mathcal{K}) \\ &= \sum_{u\in V} \sum_{\gamma^*: x \xrightarrow{n} u} \sum_{v\sim u} \mathbb{1}_{\{v\notin\gamma^*\}} \mathbb{P}(\gamma^* \subset \mathbf{C}, v \in \mathcal{K}). \end{aligned}$$

Now

$$\mathbb{P}(\gamma^* \subset \mathbf{C}, v \in \mathcal{K}) = p_{d_v}(\alpha)\mathbb{P}(\gamma^* \subset \mathbf{C}) \le p\,\mathbb{P}(\gamma^* \subset \mathbf{C}).$$

Thus,

$$\begin{aligned}
c_{n+1} &\le p \sum_{u\in V} \sum_{\gamma^*: x \xrightarrow{n} u} \mathbb{P}(\gamma^* \subset \mathbf{C}) \sum_{v\sim u} \mathbb{1}_{\{v\notin\gamma^*\}} \\
&\le p \sum_{u\in V} (d_u - 1) \sum_{\gamma^*: x \xrightarrow{n} u} \Big[\mathbb{P}(\gamma^* \subset \mathbf{C}, u \in \mathcal{K}) \\
&\qquad + \mathbb{P}(\gamma^* \subset \mathbf{C}, u \notin \mathcal{K}, \gamma^*_{n-1} \to u) \\
&\qquad + \mathbb{P}(\gamma^* \subset \mathbf{C}, u \notin \mathcal{K}, u \to \gamma^*_{n-1}, \gamma^*_{n-1} \nrightarrow u)\Big] \\
&\le p(\partial - 1)[c_n + f_n] + p\frac{\partial - 1}{\partial} b_n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f_{n+1} &= \sum_{v\in V} \sum_{\gamma: x \xrightarrow{n+1} v} \mathbb{P}(\gamma \subset \mathbf{C}, v \notin \mathcal{K}, \gamma_n \to v) \\
&\le \sum_{u\in V} \sum_{\gamma^*: x \xrightarrow{n} u} \sum_{v\sim u} \mathbb{1}_{\{v\notin\gamma^*\}} \mathbb{P}(\gamma^* \subset \mathbf{C}, u \to v) \\
&= \sum_{u\in V} \sum_{\gamma^*: x \xrightarrow{n} u} \sum_{v\sim u} \mathbb{1}_{\{v\notin\gamma^*\}} [\mathbb{P}(\gamma^* \subset \mathbf{C}, u \in \mathcal{K}) + \mathbb{P}(\gamma^* \subset \mathbf{C}, u \notin \mathcal{K}, u \to v)] \\
&\le \sum_{u\in V} \sum_{\gamma^*: x \xrightarrow{n} u} \sum_{v\sim u} \mathbb{1}_{\{v\notin\gamma^*\}} [\mathbb{P}(\gamma^* \subset \mathbf{C}, u \in \mathcal{K}) + \mathbb{P}(\gamma^* \subset \mathbf{C}, u \notin \mathcal{K}, \gamma^*_{n-1} \to u) d_u^{-1}].
\end{aligned}$$

The first term is bounded by

$$\sum_{u\in V} (d_u - 1) \sum_{\gamma^*: x \xrightarrow{n} u} \mathbb{P}(\gamma^* \subset \mathbf{C}, u \in \mathcal{K}) \le (\partial - 1) c_n.$$

The second term is bounded by

$$\sum_{u\in V} \frac{d_u - 1}{d_u} \sum_{\gamma^*: x \xrightarrow{n} u} \mathbb{P}(\gamma^* \subset \mathbf{C}, \gamma^*_{n-1} \to u, u \notin \mathcal{K}) \le \frac{\partial - 1}{\partial} f_n.$$

Finally,

$$
\begin{aligned}
b_{n+1} &= \sum_{v\in V}\sum_{\gamma:x\xrightarrow{n+1}v} d_v\cdot\mathbb{P}(\gamma\subset\mathbf{C}, v\notin\mathcal{K}, v\to\gamma_n,\gamma_n\nrightarrow v)\\
&= \sum_{u\in V}\sum_{\gamma^*:x\xrightarrow{n}u}\sum_{v\sim u} d_v\mathbb{1}_{\{v\notin\gamma^*\}}\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, u\nrightarrow v)\mathbb{P}(v\notin\mathcal{K}, v\to u)\\
&\le \sum_{u\in V}\sum_{\gamma^*:x\xrightarrow{n}u}\sum_{v\sim u}\mathbb{1}_{\{v\notin\gamma^*\}}\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, u\nrightarrow v).
\end{aligned}
$$

The interior probability is at most

$$
\begin{aligned}
&\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, \gamma^*_{n-1}\to u, u\nrightarrow v)+\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, \gamma^*_{n-1}\nrightarrow u, u\to\gamma^*_{n-1})\\
&\le \mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, \gamma^*_{n-1}\to u)+\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, \gamma^*_{n-1}\nrightarrow u, u\to\gamma^*_{n-1}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
b_{n+1}\le \sum_{u\in V}\sum_{\gamma^*:x\xrightarrow{n}u}(d_u-1)[&\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, \gamma_{n-1}\to u)\\
&+\mathbb{P}(\gamma^*\subset\mathbf{C}, u\notin\mathcal{K}, \gamma_{n-1}\nrightarrow u, u\to\gamma_{n-1})].
\end{aligned}
$$

It follows that

$$
b_{n+1}\le(\partial-1)f_n+\frac{\partial-1}{\partial}b_n.
$$

This gives us exactly the matrix in (4). ■

4.3. *Gilbert spatial graph.* In the Gilbert spatial graph, it is convenient to introduce additional structure to the Poisson point process, so that we consider an augmented state space. All the events of interest in the corrupted compass model will be defined on this particular space.

In our proof we will make use of the Slivnyak-Mecke formula [16, Theorem 4.1] in the following form, where $\{(X_i,U_i,U_i')\}_{i\ge1}$ is a Poisson point process on $\mathbb{R}^d\times[0,1]^2$ with intensity μ under the measure ν:

(5)
$$
E_\nu\left[\sum_{i\ge1} f((X_i,U_i,U_i'),\{(X_j,U_j,U_j')\}_{j\ge1})\right]
$$

(6)
$$
=\mu\int_{\mathbb{R}^d\times[0,1]^2} E_\nu\big[f((x,u,u'),\{(X_i,U_i,U_i')\}_{i\ge1}\cup\{(x,u,u')\})\big]\mathrm{d}x\mathrm{d}u\mathrm{d}u',
$$

where $f : (\mathbb{R}^d \times [0,1]^2) \times (\mathbb{R}^d \times [0,1]^2)^{\mathrm{LF}} \to [0,\infty)$ is a non-negative Borel-measurable function and where $(\mathbb{R}^d \times [0,1]^2)^{\mathrm{LF}}$ denotes the space of locally finite subsets of $\mathbb{R}^d \times [0,1]^2$. Note that when viewed as a space of σ-finite measures (assigning mass 1 to each point in the set) $(\mathbb{R}^d \times [0,1]^2)^{\mathrm{LF}}$ equipped with the metric [5, A.2.6.1] is a metric space.

Given a set of points $H = (x_i, u_i, u_i')_{i\in I} \in (\mathbb{R}^d \times [0,1]^2)^{\mathrm{LF}}$ we let $H_1 = (x_i)_{i\in I}$, and we say that x_i and x_ℓ (with $\ell \neq i$) are neighbours if and only if $|x_\ell - x_i| < 1$. The vertex set H_1 together with the edge set $E_{H_1} = \{\{x_i, x_j\} \in H_1^2 : 0 < |x_i - x_j| < 1\}$ defines a graph. Let n_i denote the number of neighbours of x_i. If $u_i' \le p_{n_i}(\alpha)$ then we declare x_i to be corrupt and we write $x_i \to x_\ell$ for every neighbour x_ℓ of x_i. If $u_i' > p_{n_i}(\alpha)$ then we write $x_i \to x_\ell$ if x_i and x_ℓ are neighbours and $(j-1)/n_i \le u_i < j/n_i$ where x_ℓ is the jth neighbour of x when the neighbours of x_i are enumerated from closest to farthest (with some fixed but arbitrary tie-breaking rule in the case of tied Euclidean distances).

Let $\mathcal{H} = \{(X_i, U_i, U_i')\}_{i\ge1}$ be a Poisson point process on $\mathbb{R}^d \times [0,1]^2$ with intensity μ, and let $\mathcal{H}_1 = \{X_i\}_{i\ge1}$. Let $\mathcal{G} = (\mathcal{H}_1, E_{\mathcal{H}_1})$ be the graph defined as in the previous paragraph. Note that $\mathcal{H}_1 \sim \Phi$, so this graph is a Gilbert spatial graph. The $(U_i)_{i\ge1}$, $(U_i')_{i\ge1}$ are independent standard uniform random variables that are independent of $(X_j)_{j\ge1}$. It follows that the system of arrows (as introduced in the previous paragraph, but applied to $\mathcal{H}$) is a corrupted compass model on $\mathcal{G}$ with $p_{\{x\}} = p_{d_x}(\alpha)$ for each $x \in \mathcal{H}_1$. Let $\mathbf{C}$ denote the set of edges in this corrupted compass model.

To prove Proposition 3, we use a re-normalisation argument to prove absence of percolation in this corrupted compass model for large $\alpha > 1$. To describe the normalisation, for $z \in \mathbb{Z}^d$ and $M > 2$ we say that a cube $Q_M(z) = z + [-M/2, M/2]^d$ is *M-nice* if

1. all connected clusters $\mathbf{C}_x$ starting from vertices x in $Q_M(z)$ are contained in $Q_{1.5M}(z)$, and
2. every vertex in $Q_{2M}(z)$ is uncorrupted.

The cube $Q_M(z)$ is *M-nasty* if it is not *M-nice*. Note that whether $Q_M(z)$ is nice can be determined by $\{(X_i, U_i, U_i') \in \mathcal{H} : X_i \in Q_{2(M+1)}(z)\}$.

To prove Proposition 3, we show that nice cubes occur with high probability.

Fix $L \in (0,\infty)$ and let $\boldsymbol{\lambda} = (\lambda_x)_{x\in\mathbb{R}^d}$ be such that $\lambda_v \le L$ for every $v \in \mathbb{R}^d$. As in Section 4.1, we write $\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}$ for the annealed/averaged measure.

LEMMA 9. *It holds that*

$$\lim_{M\to\infty}\liminf_{\alpha\to\infty} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(Q_M(o) \textit{ is } M\textit{-nice}) = 1.$$

Before establishing Lemma 9, we discuss how it can be used to complete the proof of Proposition 3.

Proof of Proposition 3 – Gilbert spatial graph. Fix $\mathcal{G}$ and $\mathbf{C}$, and suppose that $\mathbf{C}$ contains an unbounded component. Then there is some infinite self-avoiding path $(y_i)_{i\in\mathbb{Z}_+}$ in $\mathbf{C}$ such that $|y_i - y_{i+1}| < 1$ and $|y_i| \to \infty$ as $i \to \infty$. For each y_i we have that $\mathbf{C}_{y_i}$ is unbounded and $y_i \in Q_M(Mz_i)$ for some $z_i \in \mathbb{Z}^d$. Clearly $z_i \to \infty$ as $i \to \infty$. Moreover, since $M \geq 1$ and $|y_i - y_{i+1}| \leq 1$, we see that $Q_M(Mz_i)\cap Q_M(Mz_{i+1}) \neq \varnothing$, i.e., $\|z_i - z_{i+1}\|_\infty \leq 1$, where we write $\|\cdot\|_\infty$ for the ℓ_∞-norm. Therefore for $M \in \mathbb{N}$, the family of M-nasty cubes of the form $\{Q_M(Mz) : z \in \mathbb{Z}^d\}$ percolates in the sense that there exists a sequence $(z_i)_{i\geq 1}$ in $\mathbb{Z}^d$ such that $|z_i| \to \infty$ as $i \to \infty$, $\|z_i - z_{i+1}\|_\infty \leq 1$ for $i \geq 1$ and $Q_M(Mz_i)$ is M-nasty for every $i \geq 1$.

Next, assuming Lemma 9 we show that M-nasty cubes cannot percolate as above when α is large, by using the theory developed in [17]. Note that since $M > 2$, the fact that niceness (or otherwise) of $Q_M(z)$ can be determined by $\{(X_i, U_i, U_i') \in \mathcal{H} : X_i \in Q_{2(M+1)}(z)\}$ means that the niceness of the cube $Q_M(Mz)$ is independent of the niceness of the cubes $\{Q_M(Mz') : \|z - z'\|_\infty > 2, z' \in \mathbb{Z}^d\}$. In other words, the niceness of cubes is 2-*dependent*. Now think of sites $z \in \mathbb{Z}^d$ such that $Q_M(Mz)$ is M-nice as carrying the label 1, whereas sites $z \in \mathbb{Z}^d$ such that $Q_M(Mz)$ is M-nasty carry the label 0. In particular, an application of [17, Theorem 0.0] together with Lemma 9 shows that for M sufficiently large, for all α sufficiently large (depending on M), the family of M-nasty cubes does not percolate [in the setting of [17, Theorem 0.0] this is achieved by ensuring that the complement of the 1-labeled sites does not percolate]. Therefore, for all α sufficiently large, $\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}$-almost surely, all connected clusters $\mathbf{C}(\mathcal{G})$ are finite. In other words, for ν-almost all $\mathcal{G}$, we have that all connected clusters of $\mathbf{C}(\mathcal{G})$ are finite $\mathbb{P}_{\mathcal{G},\alpha,\boldsymbol{\lambda}_\nu}$-almost surely. ■

Proof of Lemma 9. Let $\varepsilon > 0$. We show that for all M sufficiently large

$$\liminf_{\alpha\to\infty} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(Q_M(o) \text{ is } M\text{-nice}) > 1 - \varepsilon. \tag{7}$$

Let

$$E_z(M) = \{\text{every vertex in } \mathcal{H}_1 \cap Q_{2M}(Mz) \text{ is uncorrupted}\}.$$

Since $\mathcal{H}_1$ is almost surely finite and $\lim_{\alpha\to\infty} p_d(\alpha) = 0$ we conclude that for each $z \in \mathbb{Z}^d$, $M \in \mathbb{N}$,

$$\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_z(M)) \to 1, \qquad \text{as } \alpha \to \infty.$$

Now, given a set of points $H = (x_i, u_i, u_i')_{i\in I} \in (\mathbb{R}^d \times [0,1]^2)^{\mathrm{LF}}$, we define for each $x = x_{i_0} \in H_1$ the forward cluster $F_x' = F'_{(x_{i_0},u_{i_0},u'_{i_0})}(H)$ to be the set containing x as well as every $x' \in H_1$ for which there exists $n \in \mathbb{Z}_+$ and $x_{i_1}, \dots, x_{i_n} = x'$ such that (for each $r = 0, \dots, n-1$) x_{i_r}, $x_{i_{r+1}}$ are neighbours and $(j-1)/n_{i_r} \le u_{i_r} < j/n_{i_r}$ where $x_{i_{r+1}}$ is the jth neighbour of x_{i_r}. In particular, the forward clusters F_x' here do not depend on the quantities $(u_i')_{i\in I}$ at all. Let $E_z'(M)$ be the event that for every vertex $x \in \mathcal{H}_1 \cap Q_{2M}(Mz)$ the forward cluster F_x' consists of at most $M/9$ vertices.

It follows that if both $E_z(M)$ and $E_z'(M)$ occur then for every vertex x' in $Q_M(Mz)$, the connected cluster $\mathbf{C}_{x'}$ is contained in $Q_{1.5M}(Mz)$, and $Q_M(Mz)$ is M-nice. Hence, using translation invariance of the Poisson point process, and applying the Slivnyak-Mecke formula (5) with the function

$$f((x,u,u'),H) = \mathbb{1}\{x \in Q_{2M}(Mz)\}\mathbb{1}\{\#F'_{(x,u,u')}(H \cup \{(x,u,u')\}) > M/9\},$$

we see that $1 - \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_z'(M))$ is equal to

$$\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(\#F'_{(X_i,U_i,U_i')} > M/9 \text{ for some } X_i \in Q_{2M}(Mz)) \tag{8}$$

$$\le \boldsymbol{E}_{\alpha,\boldsymbol{\lambda}}\Bigg[\sum_{X_i \in Q_{2M}(o)} \mathbb{1}\{\#F'_{(X_i,U_i,U_i')}(\mathcal{H}) > M/9\}\Bigg] \tag{9}$$

$$= \mu \int_{Q_{2M}(o)} \int_{[0,1]^2} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}\big(\#F'_{(x,u,u')}(\mathcal{H} \cup \{(x,u,u')\}) > M/9\big)\,\mathrm{d}u\,\mathrm{d}u'\mathrm{d}x \tag{10}$$

$$= \mu(2M)^d \int_{[0,1]} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}\big(\#F'_{(o,u,1)}(\mathcal{H} \cup \{(o,u,1)\}) > M/9\big)\mathrm{d}u. \tag{11}$$

Writing $F_u^* = F'_{(o,u,1)}(\mathcal{H} \cup \{(o,u,1)\})$, we will prove that there exist $C, c > 0$ (not depending on α, u) such that,

$$\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(\#F_u^* > M/9) \le Ce^{-c\sqrt{M}}. \tag{12}$$

From this it follows that

$$1 - \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E'_z(M)) \le C' M^d e^{-c\sqrt{M}},$$

hence we may choose M_0 sufficiently large so that $\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E'_z(M)) \ge 1 - \varepsilon/2$ for every $M \ge M_0$. But for each $M \ge M_0$, $\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_z(M)) \to 1$ as $\alpha \to \infty$ so

$$\liminf_{\alpha\to\infty} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(Q_M(o) \text{ is } M\text{-nice}) \ge \liminf_{\alpha\to\infty} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_z(M) \cap E'_z(M)) > 1 - \varepsilon$$

as required.

Thus it remains to prove (12). Let E_M^{deg} denote the event that all vertices in $Q_{3M}(o)$ have degree at most $\sqrt{M}$. Writing κ_d for the volume of the d-dimensional unit Euclidean ball B_1, another application of (5) gives that

$$\begin{aligned} 1 - \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_M^{\mathsf{deg}}) &\le (3M)^d \mu \sum_{k \ge \sqrt{M}} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(\#(\mathcal{H} \cap B_1) = k) \\ &\le (3M)^d \mu \sum_{k \ge \sqrt{M}} e^{-\mu\kappa_d} \frac{(\mu\kappa_d)^k}{k!}. \end{aligned}$$

Hence, by the Stirling formula, $1 - \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_M^{\mathsf{deg}})$ decays exponentially in $\sqrt{M}$. It therefore suffices to show that $\boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_M^{\mathsf{deg}} \cap \{\# F_u^* \ge M/9\})$ decays exponentially in $\sqrt{M}$. Conditioning on the spatial locations $\mathcal{H}_1 \cap Q_{3M}(o)$, at each forward hop in F_u^* there is a chance of at least $M^{-1/2}$ to backtrack. Hence,

$$\begin{aligned} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(E_M^{\mathsf{deg}} \cap \{\# F_u^* > M/9\}) &= \boldsymbol{E}_{\alpha,\boldsymbol{\lambda}}\Big[\mathbb{1}\{E_M^{\mathsf{deg}}\} \boldsymbol{P}_{\alpha,\boldsymbol{\lambda}}(\# F_u^* > M/9 | \mathcal{H}_1 \cap Q_{3M}(o))\Big] \\ &\le \Big(1 - \frac{1}{\sqrt{M}}\Big)^{M/9}, \end{aligned}$$

which decays at exponential speed in $\sqrt{M}$. ■

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