



**HAL**  
open science

# Three-dimensional turbulence without vortex stretching

Wouter J.T. Bos

► **To cite this version:**

Wouter J.T. Bos. Three-dimensional turbulence without vortex stretching. *Journal of Fluid Mechanics*, 2021, 915, pp.A121. 10.1017/jfm.2021.194 . hal-03189871

**HAL Id: hal-03189871**

**<https://hal.science/hal-03189871v1>**

Submitted on 6 Apr 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Three-dimensional turbulence without vortex stretching

By **Wouter J. T. BOS**,

LMFA-CNRS, Ecole Centrale de Lyon, Université de Lyon, Ecully, France

(Received 2020)

We consider three-dimensional turbulence from which vortex stretching is removed. The resulting system conserves enstrophy but does not conserve kinetic energy. Using spectral closure, it is shown that enstrophy is transferred to small scales by a direct cascade. The inviscid truncated systems tends to an equipartition of enstrophy over wave-vectors. No inverse cascade is observed once the scales larger than the forcing scale are in equipartition.

---

## 1. Introduction

One of the fascinating aspects of turbulence is the intricate interplay between vorticity and velocity. Indeed, the vorticity, defined as the curl of the velocity, is both advected and stretched by a turbulent velocity field. Obviously the importance of vortex stretching in the dynamics of turbulence has been recognized early on in turbulence research (see e.g. Taylor, 1938, Betchov, 1956, Ashurst *et al.*, 1987), and textbooks stress its important role in the nonlinear dynamics of turbulence (Tennekes & Lumley, 1972, Tsinober, 2009). Nevertheless, not all facets of vortex stretching are understood and even in recent investigations the precise role of vortex stretching is investigated (Johnson & Meneveau, 2016, Carbone & Bragg, 2020, Buaria *et al.*, 2020) and models are proposed to obtain a better understanding of the effects of vorticity stretching and velocity gradient dynamics in general (Chertkov *et al.*, 1999, Chevillard & Meneveau, 2006).

Different approaches can be used to obtain a better understanding of a particular feature of turbulence. One method is to attempt to disentangle in a simulation or experiment the influence of a particular term or structure from other features. Such an attempt is not without difficulty to investigate vortex stretching in turbulence, since in an instantaneous flow-field it is non-trivial to recognize which flow features are caused by vortex stretching. Indeed, for this, the whole Lagrangian history must be taken into account, which is experimentally quite involved (Guala *et al.*, 2005). Furthermore, separating vortex stretching from other effect often requires identifying regions dominated by either stretching or advection, which depends on thresholds and arbitrary definitions.

Another approach to identify the influence of a certain effect, is to modify the physical system in order to isolate the influence of a particular feature, by deliberately removing this feature from the dynamics and to compare the resulting set-up to the original one, or by altering the spatial dimension of the system. This approach has been used to study turbulence for decades. The resulting system does in general not correspond to a physical system, but allows by comparing to the original set-up, to show what the influence of the additional or missing feature is.

Typical examples of such modifications are the removal of the pressure from the equations as first investigated by Burgers (Burgers, 1950), and later followed by Polyakov,

1995, Boldyrev, 1999. Other examples are the removal of a certain class of the modes on which the flow-fields are projected to focus on certain triadic interactions (Biferale *et al.*, 2012, Alexakis, 2017, Briard *et al.*, 2017, Qu *et al.*, 2018), or the decimation of Fourier-space to change the fractal dimension of space (Frisch *et al.*, 1976, Frisch *et al.*, 2012, Lanotte *et al.*, 2015). The change of the dimension of space can also be directly investigated by reformulating turbulence in more than three dimensions Gotoh *et al.*, 2007, Yamamoto *et al.*, 2012, Berera *et al.*, 2020, or by considering intermediate systems such as axisymmetric turbulence, with properties of both two and three-dimensional systems (Leprovost *et al.*, 2006, Naso *et al.*, 2010, Qu *et al.*, 2017, Qin *et al.*, 2020), or thin layer turbulence (Celani *et al.*, 2010, Benavides & Alexakis, 2017, Favier *et al.*, 2019).

The present work follows this approach by altering the Navier-Stokes equations. The modification is drastic since we will remove the vortex stretching from the system, which changes the nonlinearity of the Navier-Stokes equations. One way to do this is to consider two-dimensional turbulence, since in a 2D velocity field, the vorticity is perpendicular to the velocity (and its gradient) so that the vortex stretching term drops out of the vorticity equation. However, this does not only remove the vortex-stretching, but does also change the dimension of the system. In this investigation we will directly remove the vortex stretching from the three-dimensional Navier-Stokes equations without changing the space dimension, and investigate the statistical properties of the resulting system. The approach can also be seen as extending the case of pure advection of vorticity, as in 2D turbulence, to a system with a higher dimension.

The approach by which we investigate the resulting system is closure theory. Indeed, closure allows to investigate systems which are unphysical, which does not seem possible in experiments. Furthermore, for an exploratory investigation such as the present one, closure allows to attain at low computational cost high Reynolds numbers to test assumptions on asymptotic scaling. The assumptions underlying closure theory do not violate the detailed conservation properties of invariants of the governing equations and do in general correctly capture asymptotic scaling of second and third-order velocity correlations (Sagaut & Cambon, 2008). Higher order moments, reflecting the intermittency properties of the flow can in principle be addressed by closure (Chen *et al.*, 1989, Bos & Rubinstein, 2013), but are not always correctly reproduced. Furthermore, the instantaneous structure of the flow cannot be reproduced by a purely statistical approach. We therefore do think that the present investigation should be consolidated by Direct Numerical Simulations, but this will be left for future work.

In the following section (Sec. 2) we will present the theoretical framework and present the considered modified Navier-Stokes system. Then, in section 3 we derive a closed expression for the evolution-equation of the kinetic energy using an Eddy-Damped Quasi-Normal Markovian (EDQNM) approach. The equations will be integrated numerically and the results are presented in Sec. 4. Sec. 5 contains the conclusions.

## 2. Navier-Stokes equations without vortex stretching

We consider three-dimensional, incompressible, statistically homogeneous turbulence, maintained by an external force term  $\mathbf{f}(\mathbf{x}, t)$ . The velocity  $\mathbf{u}(\mathbf{x}, t)$  of this unmodified flow is then evolving following the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \mathcal{P} + \mathbf{f}, \quad (2.1)$$

with  $\mathcal{P}(\mathbf{x}, t)$  the pressure,  $\nu$  the kinematic viscosity and  $\nabla \cdot \mathbf{u} = 0$ . The time-dependence of the velocity, force and pressure is omitted here and in the following .

In order to remove the vortex-stretching from the dynamics, we consider the curl of Eq. (2.1), yielding the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \underbrace{\mathbf{u} \cdot \nabla \boldsymbol{\omega}}_{\text{Advection}} - \nu \Delta \boldsymbol{\omega} = \underbrace{\boldsymbol{\omega} \cdot \nabla \mathbf{u}}_{\text{Stretching}} + \nabla \times \mathbf{f} \quad (2.2)$$

If the vorticity is only advected and not stretched, we remove the stretching term, leading to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \nu \Delta \boldsymbol{\omega} = \nabla \times \mathbf{f}. \quad (2.3)$$

This equation is similar to the two-dimensional Navier-Stokes equations with this difference that the vorticity is now a three-component vector. The Navier-Stokes equations corresponding to this dynamics read

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \mathcal{P} - \boldsymbol{\phi} + \mathbf{f}, \quad (2.4)$$

with  $\boldsymbol{\phi}$  a force, or damping, applied to the velocity field defined such that

$$\nabla \times \boldsymbol{\phi} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \quad (2.5)$$

Such an artificial forcing term, applied to all scales is somewhat similar in spirit to the helical forcing used in Plunian *et al.*, 2020. It is important to realize that The present investigation will not address the question whether vortex-stretching or strain self-amplification is most important in the process of energy transfer (Johnson & Meneveau, 2016, Carbone & Bragg, 2020), since removing the vortex-stretching term from the vorticity-equation will simultaneously suppress the strain self-amplification from the dynamics. When, in the following, we will discuss the effect of vortex-stretching, we do implicitly mean the combined effect of vortex-stretching and strain self-amplification.

From (2.3) follows for a periodic or statistically homogeneous system

$$\frac{dZ}{dt} = \beta_{\text{in}} - \beta \quad (2.6)$$

with the enstrophy  $Z$  given by

$$Z = \frac{1}{2} \langle \|\boldsymbol{\omega}\|^2 \rangle, \quad (2.7)$$

the enstrophy injection by external forcing

$$\beta_{\text{in}} = \langle \boldsymbol{\omega} \cdot \nabla \times \mathbf{f} \rangle, \quad (2.8)$$

and the enstrophy dissipation

$$\beta = -\nu \langle \boldsymbol{\omega} \cdot \Delta \boldsymbol{\omega} \rangle \quad (2.9)$$

so that the enstrophy of the unforced inviscid system ( $\nu = 0$ ,  $\mathbf{f} = 0$ ) obeys

$$\frac{dZ}{dt} = 0, \quad (2.10)$$

and is thus conserved by the nonlinear interactions of the system.

The kinetic energy balance is

$$\frac{dK}{dt} = \epsilon_{\text{in}} - \epsilon - \Psi \quad (2.11)$$

where the kinetic energy  $K$ , energy input and viscous dissipation are defined, respectively,

by

$$K = \frac{1}{2} \langle \|\mathbf{u}\|^2 \rangle \quad (2.12)$$

$$\epsilon_{\text{in}} = \langle \mathbf{u} \cdot \mathbf{f} \rangle \quad (2.13)$$

$$\epsilon = -\nu \langle \mathbf{u} \cdot \Delta \mathbf{u} \rangle. \quad (2.14)$$

The energy input or destruction due to the absence of the vortex stretching term is

$$\Psi = \langle \phi \cdot \mathbf{u} \rangle. \quad (2.15)$$

In the unforced, inviscid case, the energy balance reads,

$$\frac{dK}{dt} = -\Psi, \quad (2.16)$$

so that the inviscid system [Eq. (2.4) with  $\nu = 0$ ] does not necessarily conserve energy.

From these considerations it follows that  $Z$  (or other moments of the vorticity) are conserved by the nonlinearity of the system. We have not identified other invariants. We will focus on the mirror-symmetric case and whether an invariant, such as the volume averaged helicity, is conserved in the case without mirror-symmetry will be left for future research.

### 3. Fourier representation and closure of the system

In Fourier-space the Biot-Savart operator becomes an algebraic operation, which allows to rewrite the Navier-Stokes equations without vortex stretching in a convenient form. The explicit form of the evolution of the Fourier-modes is derived in Appendix A and can be written in the form

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = -i \int \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \Gamma_{ijm}(\mathbf{k}, \mathbf{p}, \mathbf{q}) u_j(\mathbf{p}) u_m(\mathbf{q}) d\mathbf{p} d\mathbf{q} - \nu k^2 u_i(\mathbf{k}) + f_i(\mathbf{k}). \quad (3.1)$$

where

$$\Gamma_{ijm}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \left[ \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j P_{im}(\mathbf{k}) \right] \quad (3.2)$$

and  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j k^{-2}$ . Note that for  $\lambda = 1$  we have  $\Gamma_{ijm}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = k_j P_{im}(\mathbf{k})$  and we retrieve the unmodified Navier-Stokes equations. The case  $\lambda = 0$  corresponds to the dynamics without vortex stretching. Here and in the following we will distinguish the Fourier-coefficients  $\mathbf{u}(\mathbf{k})$  from the associated velocity field  $\mathbf{u}$  by their dependence on the wavevector.

The energy spectrum, defined as the spherically averaged energy density in Fourier-space, is then governed by the Lin-equation

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k) + P(k). \quad (3.3)$$

where  $P(k)$  represents the energy input,

$$\int P(k) dk = \langle \mathbf{u} \cdot \mathbf{f} \rangle \equiv \epsilon_{\text{in}}. \quad (3.4)$$

The nonlinear transfer term is given by

$$T(k) = -4i\pi k^2 \int \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \Gamma_{ijm}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \langle u_i(-\mathbf{k}) u_j(\mathbf{p}) u_m(\mathbf{q}) \rangle d\mathbf{p} d\mathbf{q}, \quad (3.5)$$

and its integral is

$$\int T(k) dk = -\Psi, \quad (3.6)$$

which is strictly zero in non-modified turbulence. The dissipation spectrum is related to the viscous dissipation by

$$\int 2\nu k^2 E(k) dk = \epsilon. \quad (3.7)$$

The nonlinear transfer contains a triple velocity correlation. For this correlation we derive a closed expression within the framework of the EDQNM theory (Orszag, 1970). The details of the derivation can be found in Appendix B, leading for the case without vortex stretching to

$$T(k) = \frac{1}{2} \int_{\Delta} \frac{dp}{p} \frac{dq}{q} \Theta_{k,p,q} \left\{ \left[ f_{(1)}^{\lambda} X_{(1)} + f_{(2)}^{\lambda} X_{(2)} \right] k^3 E(p) E(q) - \left[ f_{(3)}^{\lambda} X_{(3)} + f_{(4)}^{\lambda} X_{(4)} \right] p^3 E(k) E(q) - \left[ f_{(5)}^{\lambda} X_{(5)} + f_{(6)}^{\lambda} X_{(6)} \right] q^3 E(p) E(k) \right\}, \quad (3.8)$$

where the  $\Delta$  denotes the integration domain in the  $p - q$  plane, where  $k, p, q$  can form the sides of a triangle. The  $X$  terms are

$$X_{(1)} = (1 - z^2)(1 + y^2), \quad X_{(2)} = -xyz - y^2 z^2 \quad (3.9)$$

$$X_{(3)} = xy(1 - z^2), \quad X_{(4)} = z(-y^2 - xyz) \quad (3.10)$$

$$X_{(5)} = y(-x^2 - xyz), \quad X_{(6)} = (y + zx)(1 + y^2) \quad (3.11)$$

and the  $f^{\lambda}$  terms read for  $\lambda = 0$ ,

$$f_{(1)}^{\lambda} = y^2 \left( \frac{q}{k} \right)^2, \quad f_{(2)}^{\lambda} = yz \frac{pq}{k^2} \quad (3.12)$$

$$f_{(3)}^{\lambda} = xy \frac{q^2}{pk}, \quad f_{(4)}^{\lambda} = yz \frac{q}{p} \quad (3.13)$$

$$f_{(5)}^{\lambda} = xy \frac{p}{k}, \quad f_{(6)}^{\lambda} = y^2. \quad (3.14)$$

The triad interaction time is defined as

$$\Theta(k, p, q) = \frac{1 - \exp[-(\theta_k + \theta_p + \theta_q)t]}{\theta_k + \theta_p + \theta_q} \quad (3.15)$$

with

$$\theta_k = \alpha \sqrt{\int_0^k s^2 E(s) ds} + \nu k^2, \quad (3.16)$$

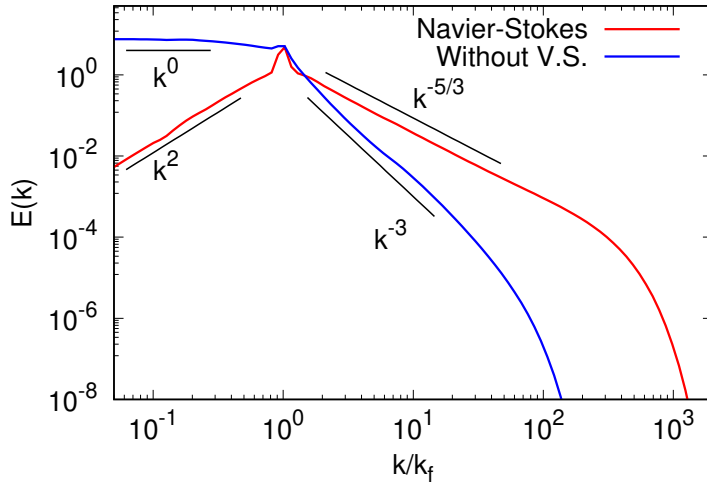
and  $\alpha = 0.5$ . For the classical EDQNM closure we have, using  $\lambda = 1$  in the derivation, that all  $f^{\lambda}$  terms are equal to unity, yielding

$$T(k) = \frac{1}{2} \int_{\Delta} \frac{dp}{p} \frac{dq}{q} \Theta_{k,p,q} \left\{ [X_{(1)} + X_{(2)}] k^3 E(p) E(q) - [X_{(3)} + X_{(4)}] p^3 E(k) E(q) - [X_{(5)} + X_{(6)}] q^3 E(p) E(k) \right\}. \quad (3.17)$$

which can be symmetrized to find the expression in textbooks.

#### 4. Results

In this section we integrate the closure equations. We will first highlight the differences with respect to unmodified three-dimensional turbulence. Subsequently we investigate



(a)

FIGURE 1. Steady-state kinetic energy spectra of forced Navier-Stokes turbulence compared to the energy spectra of the system without vortex stretching for the same parameters.

by dimensional analysis and variation of the Reynolds number the inertial range of the system. Finally we will consider the inviscid system to show the equilibrium properties of the system.

#### 4.1. Numerical set-up and parameters

To integrate the equations we use the same in-house EDQNM code which has been used over the last decades in our laboratory (Sagaut & Cambon, 2008), containing a routine written by Chuck Leith (Leith, 1971). A logarithmic discretization is used and a resolution of minimum 20 points per decade was used in our simulations. Simulations were performed on a domain  $k \in [0.05k_f, 5k_\eta]$ , where  $k_f$  is the forcing frequency ( $k_f = 1$ ) and  $k_\eta = \nu^{-3/4}\epsilon_{\text{in}}^{1/4}$ . The forcing term in Eq. (3.3) is defined by (Briard *et al.*, 2017),

$$P(k) = A \exp(-100 [\ln(k/k_f)]^2) \quad (4.1)$$

with  $A$  determined such that the integral of  $P(k)$  is unity. This ensures an energy input which is fixed at  $\epsilon_{\text{in}} = 1$  and an enstrophy-input close to unity ( $\beta_{\text{in}} = 1.02$ ). All results are reported when a statistically steady state is reached, except the inviscid relaxation simulations in paragraph 4.4.

#### 4.2. Comparison with classical turbulence and scaling ranges

In order to highlight the differences between unmodified high-Reynolds number turbulence and the system without vortex stretching, we first present the results of the integration of the two systems, using the same parameters, defined in the foregoing paragraph, and viscosity  $\nu = 1 \times 10^{-4}$ . This corresponds for the unmodified turbulence to a Taylor-scale Reynolds number  $R_\lambda \approx 10^3$ .

The resulting spectra are shown in Fig. 1. The Navier-Stokes system yields a spectrum with for  $k > k_f$  a Kolmogorov  $k^{-5/3}$  scaling and for  $k < k_f$  a  $k^2$  dependence, reminiscent of energy-equipartition. For large  $k$ , a dissipation range is observed, where the energy spectrum falls-off more rapidly than a powerlaw.

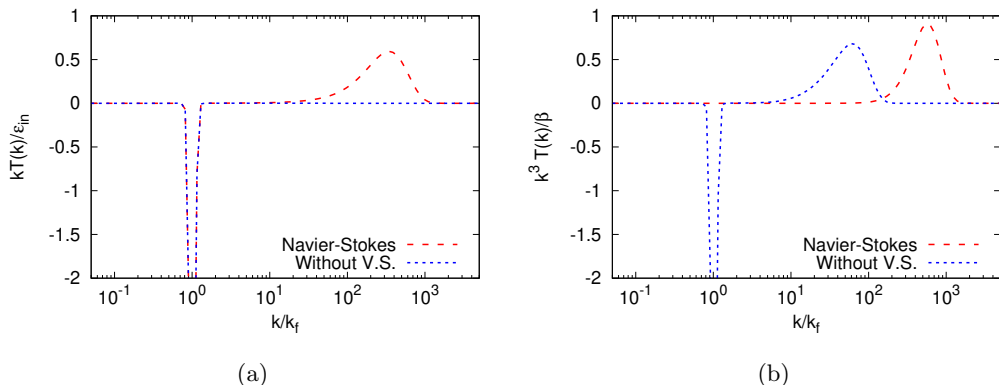


FIGURE 2. Spectra of (a) energy transfer and (b) enstrophy transfer for both systems. The energy transfer is normalized by the energy injection rate  $\epsilon_{in}$ , whereas the enstrophy spectra are normalized by the enstrophy dissipation rate  $\beta$ .

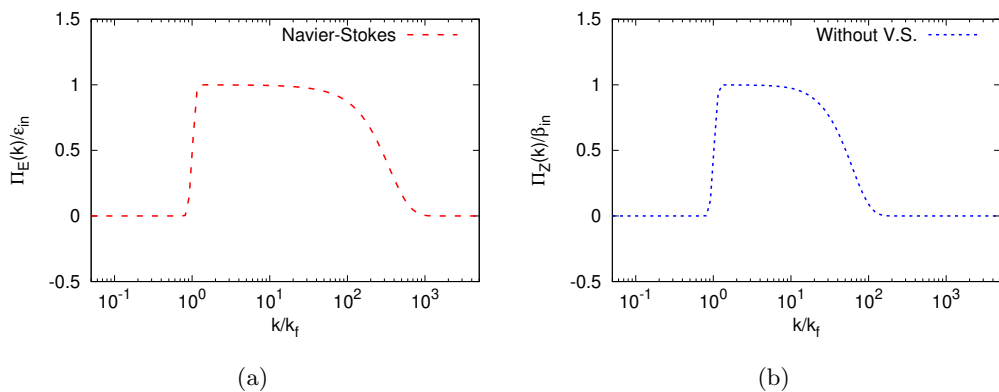


FIGURE 3. Fluxes of (a) energy in Navier-Stokes turbulence (b) enstrophy in the system without vortex-stretching. The fluxes are normalized by the energy injection rate  $\epsilon_{in}$  and enstrophy injection rate  $\beta_{in}$ , respectively.

The system without vortex-stretching shows a peak, representing the forcing-scale, like the Navier-Stokes system. However, for larger and smaller wavenumbers, the spectrum is steeper or shallower, respectively. An inertial range is observed with a wave number dependence close to  $k^{-3}$ . A dissipation range is observed which starts at a wavenumber about ten times smaller than for the reference case. Furthermore, for small  $k$ , an approximately flat spectrum is observed. This range is associated with the statistical equilibrium properties of the new system and we will come back to this in Sec. 4.4.

In Fig. 2, transfer spectra are presented. The multiplication by  $k$  allows to assess in this semi-logarithmic representation the conservation of energy by comparing the positive and negative areas (or lobes) delimited by the spectra [Fig. 2(a)]. In the Navier-Stokes case the energy is transferred towards the small scales. This is illustrated by the negative dip around the forcing frequency, where the nonlinear interaction absorbs the energy, expelling it around the dissipation scale, where energy is dissipated. In between these two scales the energy is conserved. The energy transfer of the turbulent flow without vortex stretching shows that virtually no energy is transferred, and all energy is locally destroyed by the nonlinear interaction. Indeed, no positive lobe in the transfer is observed, illustrating that the energy is absorbed and destroyed by the nonlinearity. The amount



of destroyed energy by nonlinear effects corresponds to the term  $\Psi$  in the energy balance Eq. (2.11).

By multiplying the transfer spectrum by  $k^2$ , the enstrophy-transfer spectrum is obtained. It is observed [Fig. 2(b)] that the system without vortex-stretching conserves enstrophy, which is transferred from the injection scale to the enstrophy dissipation scale. The Navier-Stokes enstrophy balance shows that enstrophy is strongly enhanced throughout the cascade. Indeed, the transfer spectra are normalized by the enstrophy-dissipation  $\beta$ . This shows that the amount of enstrophy at the viscous end of the cascade is so much larger than its injected value that the latter is negligible. Indeed, considering the enstrophy transfer near the injection scales seems to indicate that no enstrophy is injected. However, the injected enstrophy is the same in both systems, but due to the very strong production of enstrophy by vortex-stretching, the normalization by the enstrophy dissipation does not allow to see the injected enstrophy in this representation.

We quantify this by the ratios of injected to dissipated energy and injected to dissipated enstrophy. We obtain for the here considered case of Navier-Stokes turbulence,

$$\frac{\epsilon}{\epsilon_{\text{in}}} = 1, \quad \frac{\beta}{\beta_{\text{in}}} = 1.3 \times 10^4 \quad (4.2)$$

and without vortex stretching,

$$\frac{\epsilon}{\epsilon_{\text{in}}} = 2.5 \times 10^{-3}, \quad \frac{\beta}{\beta_{\text{in}}} = 1. \quad (4.3)$$

Comparing these figures illustrates the enormous amount of enstrophy which is generated by vortex-stretching in high-Reynolds-number turbulence. It also shows how important the vortex-stretching is to ensure energy conservation.

In order to complete the picture, we show in Fig. 3 the fluxes associated with the conserved quantities. In Fig. 3(a,b) we show the energy and enstrophy flux towards small scales, defined, respectively as

$$\Pi_E(k) = - \int_0^k T(k) dk, \quad \Pi_Z(k) = - \int_0^k k^2 T(k) dk. \quad (4.4)$$

A clear inertial range is observed for both quantities, where the flux is approximately constant. In both cases, these fluxes are in the direction of the small scales. No fluxes in the other direction are observed. Indeed, a steady state is observed where scales  $k < k_f$  are in statistical equilibrium. This equipartition state is associated with zero net transfer. Extending the wavenumber domain to smaller  $k$  was observed to extend this equipartition state, where  $E(k) \sim k^2$  for Navier-Stokes turbulence and  $E(k) \sim k^0$  for the system without vortex-stretching. It is for these systems therefore not necessary to add a large-scale friction to the system to allow a steady state, unlike the case of two-dimensional turbulence, where energy piles up in the forced system in the absence of large-scale damping terms. A further analysis of the equipartition range is postponed to Sec. 4.4, where the truncated inviscid system is considered.

#### 4.3. Dimensional analysis and scaling

The foregoing analysis shows that, without vortex-stretching, enstrophy is conserved by the nonlinear interactions and is transferred to small scales, where it is dissipated. No inverse (or direct) cascade of energy is observed. Arguments *à la* Kolmogorov, assuming scale locality will lead to a scaling depending on the enstrophy-flux and the local length-scale (or wavenumber). In a steady state the enstrophy flux is equal to the enstrophy

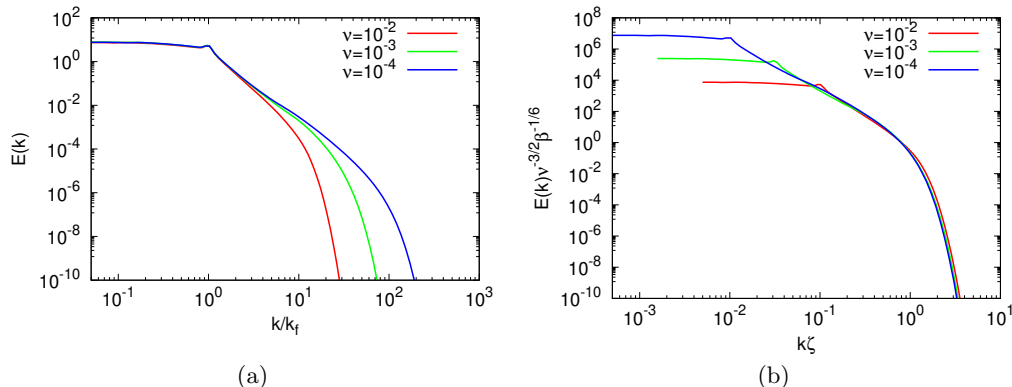


FIGURE 4. Reynolds number scaling of the energy spectra of turbulence without vortex stretching. (a) Non-normalized spectra for three distinct values of the viscosity. (b) The same spectra normalized by enstrophy-dissipation and viscosity.

dissipation  $\beta$ , so that we obtain from dimensional arguments that

$$E(k) \sim \beta^{2/3} k^{-3}. \quad (4.5)$$

The equivalent of the viscous Kolmogorov scale will now become

$$\zeta \equiv \frac{\nu^{1/2}}{\beta^{1/6}}. \quad (4.6)$$

The energy spectra should then collapse in the high wavenumber range, for large Reynolds numbers using the lengthscale  $\zeta$  and the viscosity,

$$E(k) = \nu^{3/2} \beta^{1/6} f(k\zeta) \quad (4.7)$$

where  $f$  is a unique function. This is assessed in Fig. 4 where we show plots of the energy spectrum associated with our turbulence without vortex stretching for  $\nu = 0.01, 0.001, 0.0001$ . In Fig. 4(a) we show the non-normalized spectra, which coincide at the large scales. Normalizing using the above scaling arguments allows to collapse all three cases in the dissipation and inertial ranges [Fig. 4(b)].

We can also explain why in Fig. 1 the viscous cut-off of both systems is different by an order of magnitude. The Kolmogorov-scale in turbulence is of order  $\eta = \nu^{3/4} \epsilon_{\text{in}}^{-1/4}$  and the Kolmogorov-like scale in the modified system is given by expression (4.6). The ratio is then,

$$\frac{\eta}{\zeta} = \frac{\beta_{\text{in}}^{1/6}}{\epsilon_{\text{in}}^{1/4}} \nu^{1/4}. \quad (4.8)$$

Since both  $\epsilon_{\text{in}}$  and  $\beta_{\text{in}}$  are order unity, and  $\nu = 10^{-4}$ , this ratio is around 10 for the spectra shown in Fig. 1.

#### 4.4. Inviscid equilibrium

In the foregoing it was observed that for scales larger than the forcing scale, *i.e.* for wavenumbers  $k < k_f$ , the energy spectrum is flat in turbulence without vortex-stretching. In the current section we will show that this scaling is associated with the inviscid equilibrium state of the system.

The inviscid equilibrium properties of turbulence have received interest in turbulence research since they make a direct connection between thermodynamics and fluid mechan-

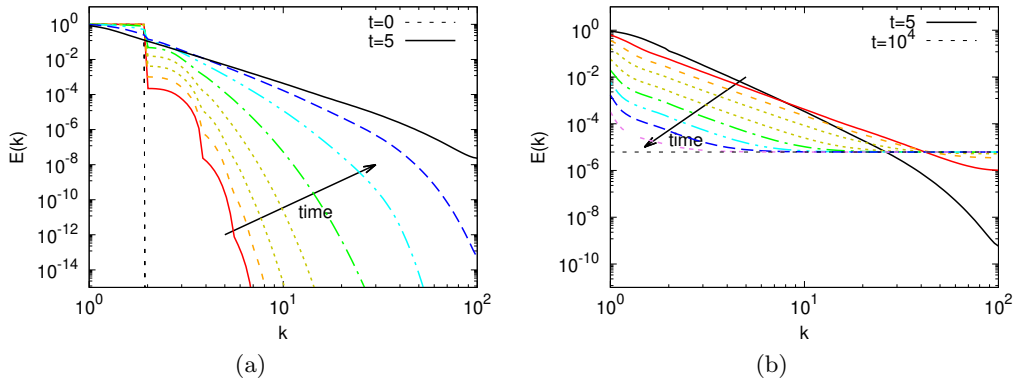


FIGURE 5. Inviscid relaxation to an equilibrium state where enstrophy is equipartitioned over the different modes. (a) Short time behaviour, showing the kinetic energy spectra at  $t = 0$  and  $t = 0.04; 0.08; 0.16; 0.32; 0.64; 1.28; 2.56; 5.12$ . (b) Long time evolution, showing the spectra at  $t_1 = 5.12; 10; 20; 40; 80; 160; 320; 640; 1280$  and  $t = 10^4$

ics. Early investigations showed that a Galerkin-truncation of the Navier-Stokes system, in the absence of viscosity allows an equilibrium solution where all Fourier modes contain, statistically, the same amount of energy (Lee, 1952, Kraichnan, 1973). The resulting system shows then an energy spectrum proportional to  $k^2$ . Kraichnan extended these ideas to two-dimensional turbulence (Kraichnan, 1967). In the present case, where only one particular invariant is present in the system, it is plausible that the equilibrium distribution corresponds to an equidistribution of enstrophy between modes. Since the enstrophy spectrum is related to the energy spectrum by

$$E_Z(k) \sim k^2 E(k), \quad (4.9)$$

we can expect an equilibrium spectrum,

$$E(k) \sim k^0. \quad (4.10)$$

We check this by integrating the inviscid system, starting from a concentrated energy (and enstrophy distribution),

$$E(k, 0) = H(2 - k) \quad (4.11)$$

with  $H$  the Heaviside function. The domain is  $k \in [1, 100]$ . In Fig. 5(a) we show the short-time evolution of the system. At very short times we observe the staircase scaling recently discussed in Fang *et al.*, 2020. Then, at intermediate times, as shown in Fig. 5(b) an enstrophy-cascade coexists with a thermalized part as also observed in the three-dimensional case (Cichowlas *et al.*, 2005, Bos & Bertoglio, 2006). The relaxation towards a  $E(k) \sim k^0$  spectrum shows the equipartition of enstrophy. It also explains the wavenumber dependence of the energy spectra for scales larger than the forcing-scale observed in Fig. 1. Indeed, for such large scales in forced 3D turbulence the modes are shown to be in thermal equilibrium, showing a  $k^2$  equipartition energy spectrum, as observed in Fig. 1 (see for instance Alexakis & Brachet, 2019 for a discussion). Transposing this to the enstrophy-conserving dynamics observed in the present investigation suggests the observed  $k^0$  scaling observed in Fig. 1 for turbulence without vortex stretching.

## 5. Conclusion

Three-dimensional turbulence without vortex-stretching is different from two-dimensional turbulence. Both systems conserve enstrophy and cascade that quantity to the small scales. However, whereas in 2D turbulence energy is transferred towards large scales, displaying thereby a double cascade, the modified 3D system does not conserve energy and a simple cascade is observed, associated with a  $k^{-3}$  inertial range.

This absence of vortex stretching also alters the absolute equilibrium states of the truncated inviscid system. In the 2D case the statistical equilibrium is a function of both invariants, whereas in the 3D turbulence without vortex stretching the equilibrium distribution is a simple equipartition of enstrophy, corresponding to a flat  $k^0$  energy spectrum. This behavior is also observed in the forced system for scales larger than the forcing scale.

What we can therefore safely state, is that vortex-stretching is unseparable from the energy cascade mechanism. Indeed, in its absence, energy is not conserved and the dynamics of the flow is radically changed. In classical turbulence at high Reynolds numbers, vortex stretching amplifies enstrophy by several orders of magnitude between the injection scale and the dissipation scale. In the absence of vortex stretching, the enstrophy becomes scale independent in the inertial range. However, in the same range for the stretching-less system, the energy is destroyed and only a very small fraction survives the cascade towards the dissipation scale. We repeat here that the present investigation does not address the dynamical difference between vortex-stretching and strain self-amplification (Johnson & Meneveau, 2016, Carbone & Bragg, 2020), since both effects are suppressed by removing the vortex-stretching term from the vorticity equation. It does not seem easy to remove only one of the two effects from the Navier-Stokes equations without altering the other one.

The present investigation opens up several perspectives. In particular Direct Numerical Simulations will allow to assess the fine properties of the flow, such as intermittency, and will allow to investigate the physical space structure of this new type of turbulence.

Recently turbulence was investigated in the presence of local surgery, where strongly vortical regions were locally damped (Buzzicotti *et al.*, 2020). A local variant of the present work, where vortex-stretching is not suppressed entirely, but only in sub-domains of space could constitute an interesting direction for further research. A parametrical study, varying  $\lambda$  in Eq. (3.2) between zero and unity would allow to assess how exactly the statistics depend on the strength of the vortex-stretching. Indeed, the procedure of fractal decimation (Lanotte *et al.*, 2015) shows that statistics such as intermittency can be extremely sensitive to such a control parameter around a critical value or dimension.

A final perspective is the mathematical investigation of turbulence without vortex-stretching. Indeed, since the steep spectral energy distribution in the inertial range suggests that the flow is statistically smooth, considerations about existence and uniqueness of solutions of the present system might give a new angle to assess the mathematical properties of Navier-Stokes turbulence.

## Appendix A: Fourier-representation of the Navier-Stokes equation without vortex-stretching

The vorticity equation without vortex stretching reads

$$\frac{\partial \omega_i(\mathbf{k})}{\partial t} = -ik_j \int_{\delta} u_j(\mathbf{p}) \omega_i(\mathbf{q}) \quad (5.1)$$

where we use the short-hand notation

$$\int_{\delta} \equiv \int d\mathbf{p}d\mathbf{q} \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}). \quad (5.2)$$

Here and in the following we omit time-arguments and Fourier-modes are indicated by their dependence on the wave-vector. Forcing and viscous terms can be added afterwards.

Since for a solenoidal field we have  $\nabla \times \nabla \times \mathbf{u} = -\Delta \mathbf{u}$ , and in Fourier-space the Laplacian becomes an algebraic operator, the vorticity equation can be easily uncurled, yielding

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = \frac{i}{k^2} \epsilon_{iab} \epsilon_{bcd} \int_{\delta} k_a k_j q_c u_j(\mathbf{p}) u_d(\mathbf{q}). \quad (5.3)$$

Developing the permutation tensor gives

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = \frac{i}{k^2} \int_{\delta} (k_d k_j q_i u_j(\mathbf{p}) u_d(\mathbf{q}) - k_a q_a k_j u_j(\mathbf{p}) u_i(\mathbf{q})). \quad (5.4)$$

The first term in brackets can be symmetrized

$$\begin{aligned} \frac{\partial u_i(\mathbf{k})}{\partial t} &= \frac{i}{k^2} \int_{\delta} (k_d k_j q_i u_j(\mathbf{p}) u_d(\mathbf{q})/2 + k_d k_j p_i u_j(\mathbf{p}) u_d(\mathbf{q})/2 \\ &\quad - k_a q_a k_j u_j(\mathbf{p}) u_i(\mathbf{q})), \end{aligned} \quad (5.5)$$

and using that  $\mathbf{p} + \mathbf{q} = \mathbf{k}$  gives

$$\begin{aligned} \frac{\partial u_i(\mathbf{k})}{\partial t} &= \frac{i}{k^2} \int_{\delta} (k_d k_j k_i u_j(\mathbf{p}) u_d(\mathbf{q})/2 \\ &\quad - k_a q_a k_j u_j(\mathbf{p}) u_i(\mathbf{q})). \end{aligned} \quad (5.6)$$

Removing the potential part by multiplying both sides with  $P_{im}(\mathbf{k})$  and relabeling gives,

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = -i k_j P_{im}(\mathbf{k}) \int_{\delta} \frac{(k_a q_a)}{k^2} u_j(\mathbf{p}) u_m(\mathbf{q}). \quad (5.7)$$

Comparison with the Navier-Stokes equations,

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = -i k_j P_{im}(\mathbf{k}) \int_{\delta} u_j(\mathbf{p}) u_m(\mathbf{q}), \quad (5.8)$$

allows then to write the general form

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = -i \int_{\delta} \left( \lambda \frac{(k_a p_a)}{k^2} + \frac{(k_a q_a)}{k^2} \right) k_j P_{im}(\mathbf{k}) u_j(\mathbf{p}) u_m(\mathbf{q}), \quad (5.9)$$

which gives the case of turbulence without vortex stretching for  $\lambda = 0$ , and which, for  $\lambda = 1$  reduces to the Navier-Stokes equations, since the term in brackets yields unity.

## Appendix B. EDQNM closure of the nonlinear transfer

We follow the procedure outlined in Bos & Bertoglio, 2013, leading to equations of the EDQNM family. Alternative procedures (Orszag, 1970, Sagaut & Cambon, 2008) should yield the same closure.

We start with

$$\frac{\partial u_i(\mathbf{k})}{\partial t} = \int_{\delta} \Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) [u_j(\mathbf{p}) u_m(\mathbf{q})] - \nu k^2 u_i(\mathbf{k}). \quad (5.1)$$

The velocity correlations obey,

$$\left(\frac{1}{2}\frac{\partial}{\partial t} + \nu k^2\right) \langle u_i(\mathbf{k})u_i(-\mathbf{k}) \rangle = \int_{\delta} \Gamma_{ijm}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \langle u_j(\mathbf{p})u_m(\mathbf{q})u_i(-\mathbf{k}) \rangle, \quad (5.2)$$

where the triple correlations need to be determined. We can formally invert equation (5.1) to obtain expressions for evolution of the three modes constituting the triple correlations,  $u = u^{(0)} + u^{(1)}$ , with  $u^{(0)}$  the Gaussian or independent velocity estimate, and  $u^{(1)}$  the perturbation by nonlinear direct triad interaction within the mode  $\mathbf{k} = \mathbf{p} + \mathbf{q}$ ,

$$\begin{aligned} u_i^{(1)}(-\mathbf{k}) &= \int_0^t ds G'(k) (\Gamma_{iab}(-\mathbf{k}, \mathbf{p}, \mathbf{q}) [u'_a(-\mathbf{p})u'_b(-\mathbf{q})] + \Gamma_{iab}(-\mathbf{k}, \mathbf{q}, \mathbf{p}) [u'_a(-\mathbf{q})u'_b(-\mathbf{p})]), \\ u_j^{(1)}(\mathbf{p}) &= \int_0^t ds G'(p) (\Gamma_{jab}(\mathbf{p}, -\mathbf{k}, \mathbf{q}) [u'_a(\mathbf{k})u'_b(-\mathbf{q})] + \Gamma_{jab}(\mathbf{p}, \mathbf{q}, -\mathbf{k}) [u'_a(-\mathbf{q})u'_b(\mathbf{k})]), \\ u_m^{(1)}(\mathbf{q}) &= \int_0^t ds G'(q) (\Gamma_{mab}(\mathbf{q}, -\mathbf{k}, \mathbf{p}) [u'_a(\mathbf{k})u'_b(-\mathbf{p})] + \Gamma_{mab}(\mathbf{q}, \mathbf{p}, -\mathbf{k}) [u'_a(-\mathbf{p})u'_b(\mathbf{k})]). \end{aligned}$$

The primed quantities depend on the time  $t = s$ . The  $G'$ 's are Green's functions. There is a difference in the treatment of Eulerian and Lagrangian theory here (Kraichnan, 1965), but the final expressions of the single-time closure which we will derive here are insensitive to this difference (see Bos & Bertoglio, 2013). Substituting in the triple correlation the velocity modes  $u = u^{(0)} + u^{(1)}$  and retaining the first order terms, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t} \langle u_i(\mathbf{k})u_i(-\mathbf{k}) \rangle = \sum_{i=1..6} \int_{\delta} \int_0^t ds F_{(i)}(k), \quad (5.3)$$

with

$$F_{(1)} = G'(k)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{iab}(-\mathbf{k}, \mathbf{p}, \mathbf{q}) [u'_a(-\mathbf{p})u'_b(-\mathbf{q})u_j(\mathbf{p})u_m(\mathbf{q})] \quad (5.4)$$

$$F_{(2)} = G'(k)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{iab}(-\mathbf{k}, \mathbf{q}, \mathbf{p}) [u'_a(-\mathbf{q})u'_b(-\mathbf{p})u_j(\mathbf{p})u_m(\mathbf{q})] \quad (5.5)$$

$$F_{(3)} = G'(p)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{jab}(\mathbf{p}, -\mathbf{k}, \mathbf{q}) [u'_a(\mathbf{k})u'_b(-\mathbf{q})u_m(\mathbf{q})u_i(-\mathbf{k})] \quad (5.6)$$

$$F_{(4)} = G'(p)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{jab}(\mathbf{p}, \mathbf{q}, -\mathbf{k}) [u'_a(-\mathbf{q})u'_b(\mathbf{k})u_m(\mathbf{q})u_i(-\mathbf{k})] \quad (5.7)$$

$$F_{(5)} = G'(q)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{mab}(\mathbf{q}, -\mathbf{k}, \mathbf{p}) [u'_a(\mathbf{k})u'_b(-\mathbf{p})u_j(\mathbf{p})u_i(-\mathbf{k})] \quad (5.8)$$

$$F_{(6)} = G'(q)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{mab}(\mathbf{q}, \mathbf{p}, -\mathbf{k}) [u'_a(-\mathbf{p})u'_b(\mathbf{k})u_j(\mathbf{p})u_i(-\mathbf{k})] \quad (5.9)$$

and using definition

$$\langle u'_i(\mathbf{k})u_j(-\mathbf{k}) \rangle = P_{ij}(\mathbf{k})U(k)R'(k) \quad (5.10)$$

with  $R'(k)$  a time-correlation function

$$R'(k) = \frac{\langle u'_i(\mathbf{k})u_i(-\mathbf{k}) \rangle}{\langle u_i(\mathbf{k})u_i(-\mathbf{k}) \rangle}, \quad (5.11)$$

and  $U(k) = E(k)/4\pi k^2$ . This can be written

$$\frac{\partial E(k)}{\partial t} = 4\pi k^2 \sum_{i=1..6} \int_{\delta} \int_0^t ds F_{(i)}(k), \quad (5.12)$$

$$F_{(1)} = G'(k)R'(p)R'(q)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{iab}(-\mathbf{k}, \mathbf{p}, \mathbf{q})P_{aj}(\mathbf{p})P_{bm}(\mathbf{q})U(p)U(q) \quad (5.13)$$

$$F_{(2)} = G'(k)R'(p)R'(q)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{iab}(-\mathbf{k}, \mathbf{q}, \mathbf{p})P_{bj}(\mathbf{p})P_{am}(\mathbf{q})U(p)U(q) \quad (5.14)$$

$$F_{(3)} = G'(p)R'(k)R'(q)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{jab}(\mathbf{p}, -\mathbf{k}, \mathbf{q})P_{ia}(\mathbf{k})P_{bm}(\mathbf{q})U(k)U(q) \quad (5.15)$$

$$F_{(4)} = G'(p)R'(k)R'(q)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{jab}(\mathbf{p}, \mathbf{q}, -\mathbf{k})P_{ib}(\mathbf{k})P_{am}(\mathbf{q})U(k)U(q) \quad (5.16)$$

$$F_{(5)} = G'(q)R'(k)R'(p)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{mab}(\mathbf{q}, -\mathbf{k}, \mathbf{p})P_{ia}(\mathbf{k})P_{jb}(\mathbf{p})U(k)U(p) \quad (5.17)$$

$$F_{(6)} = G'(q)R'(k)R'(p)\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{mab}(\mathbf{q}, \mathbf{p}, -\mathbf{k})P_{ib}(\mathbf{k})P_{ja}(\mathbf{p})U(k)U(p) \quad (5.18)$$

All the two-time dependence is contained in the quantities  $R', G'$ . Markovianization consists here in assuming exponential time-dependence for these quantities. Furthermore, assuming  $G' = R'$  for  $s < t$ , allows to write

$$\int_0^t G'(k)R'(p)R'(q)ds = \Theta(k, p, q), \quad (5.19)$$

resulting in a Markovian closure.

To advance we need to contract and substitute the  $\Gamma$ s. These are defined as (see Appendix A),

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) = -i \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j P_{im}(\mathbf{k}) \quad (5.20)$$

so that

$$\Gamma_{iab}(-\mathbf{k}, \mathbf{p}, \mathbf{q}) = i \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_a P_{ib}(\mathbf{k}) \quad (5.21)$$

$$\Gamma_{iab}(-\mathbf{k}, \mathbf{q}, \mathbf{p}) = i \left( \lambda \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} \right) k_a P_{ib}(\mathbf{k}) \quad (5.22)$$

$$\Gamma_{jab}(\mathbf{p}, -\mathbf{k}, \mathbf{q}) = -i \left( \lambda \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} + \frac{-\mathbf{q} \cdot \mathbf{p}}{p^2} \right) p_a P_{jb}(\mathbf{p}) \quad (5.23)$$

$$\Gamma_{jab}(\mathbf{p}, \mathbf{q}, -\mathbf{k}) = -i \left( \lambda \frac{-\mathbf{q} \cdot \mathbf{p}}{p^2} + \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} \right) p_a P_{jb}(\mathbf{p}) \quad (5.24)$$

$$\Gamma_{mab}(\mathbf{q}, -\mathbf{k}, \mathbf{p}) = -i \left( \lambda \frac{\mathbf{k} \cdot \mathbf{q}}{q^2} + \frac{-\mathbf{p} \cdot \mathbf{q}}{q^2} \right) q_a P_{mb}(\mathbf{q}) \quad (5.25)$$

$$\Gamma_{mab}(\mathbf{q}, \mathbf{p}, -\mathbf{k}) = -i \left( \lambda \frac{-\mathbf{p} \cdot \mathbf{q}}{q^2} + \frac{\mathbf{k} \cdot \mathbf{q}}{q^2} \right) q_a P_{mb}(\mathbf{q}) \quad (5.26)$$

and the product of the  $\Gamma$ s is

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{iab}(-\mathbf{k}, \mathbf{p}, \mathbf{q}) = \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_a P_{ib}(\mathbf{k}) k_j P_{im}(\mathbf{k})$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{iab}(-\mathbf{k}, \mathbf{q}, \mathbf{p}) = \left( \lambda \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} \right) \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j k_a P_{ib}(\mathbf{k}) P_{im}(\mathbf{k})$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{jab}(\mathbf{p}, -\mathbf{k}, \mathbf{q}) = - \left( \lambda \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} + \frac{-\mathbf{q} \cdot \mathbf{p}}{p^2} \right) \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j p_a P_{jb}(\mathbf{p}) P_{im}(\mathbf{k})$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{jab}(\mathbf{p}, \mathbf{q}, -\mathbf{k}) = - \left( \lambda \frac{-\mathbf{q} \cdot \mathbf{p}}{p^2} + \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} \right) \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j p_a P_{jb}(\mathbf{p}) P_{im}(\mathbf{k})$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{mab}(\mathbf{q}, -\mathbf{k}, \mathbf{p}) = - \left( \lambda \frac{\mathbf{k} \cdot \mathbf{q}}{q^2} + \frac{-\mathbf{p} \cdot \mathbf{q}}{q^2} \right) \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j q_a P_{mb}(\mathbf{q}) P_{im}(\mathbf{k})$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q})\Gamma_{mab}(\mathbf{q}, \mathbf{p}, -\mathbf{k}) = - \left( \lambda \frac{-\mathbf{p} \cdot \mathbf{q}}{q^2} + \frac{\mathbf{k} \cdot \mathbf{q}}{q^2} \right) \left( \lambda \frac{\mathbf{p} \cdot \mathbf{k}}{k^2} + \frac{\mathbf{q} \cdot \mathbf{k}}{k^2} \right) k_j q_a P_{mb}(\mathbf{q}) P_{im}(\mathbf{k})$$

The terms in brackets yield all value 1 for  $\lambda = 1$ . For  $\lambda = 0$  we have, using the definitions  $\mathbf{q} \cdot \mathbf{k} = kqy$ ,  $\mathbf{p} \cdot \mathbf{k} = pkz$ ,  $\mathbf{q} \cdot \mathbf{p} = -pqx$ ,

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \times \Gamma_{iab}(-\mathbf{k}, \mathbf{p}, \mathbf{q}) = y^2 \left(\frac{q}{k}\right)^2 k_a P_{ib}(\mathbf{k}) k_j P_{im}(\mathbf{k}) \quad (5.27)$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \times \Gamma_{iab}(-\mathbf{k}, \mathbf{q}, \mathbf{p}) = yz \frac{pq}{k^2} k_a P_{ib}(\mathbf{k}) k_j P_{im}(\mathbf{k}) \quad (5.28)$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \times \Gamma_{jab}(\mathbf{p}, -\mathbf{k}, \mathbf{q}) = -xy \frac{q^2}{pk} p_a P_{jb}(\mathbf{p}) k_j P_{im}(\mathbf{k}) \quad (5.29)$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \times \Gamma_{jab}(\mathbf{p}, \mathbf{q}, -\mathbf{k}) = -yz \frac{q}{p} p_a P_{jb}(\mathbf{p}) k_j P_{im}(\mathbf{k}) \quad (5.30)$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \times \Gamma_{mab}(\mathbf{q}, -\mathbf{k}, \mathbf{p}) = -xy \frac{p}{k} q_a P_{mb}(\mathbf{q}) k_j P_{im}(\mathbf{k}) \quad (5.31)$$

$$\Gamma_{ijm}(\mathbf{k}, -\mathbf{p}, -\mathbf{q}) \times \Gamma_{mab}(\mathbf{q}, \mathbf{p}, -\mathbf{k}) = -y^2 q_a P_{mb}(\mathbf{q}) k_j P_{im}(\mathbf{k}), \quad (5.32)$$

So that

$$F_{(1)} = y^2 \left(\frac{q}{k}\right)^2 k_a P_{ib}(\mathbf{k}) k_j P_{im}(\mathbf{k}) P_{aj}(\mathbf{p}) P_{bm}(\mathbf{q}) \Theta(k, p, q) U(p) U(q) \quad (5.33)$$

$$F_{(2)} = yz \frac{pq}{k^2} k_a P_{ib}(\mathbf{k}) k_j P_{im}(\mathbf{k}) P_{bj}(\mathbf{p}) P_{am}(\mathbf{q}) \Theta(k, p, q) U(p) U(q) \quad (5.34)$$

$$F_{(3)} = -xy \frac{q^2}{pk} p_a P_{jb}(\mathbf{p}) k_j P_{im}(\mathbf{k}) P_{ia}(\mathbf{k}) P_{bm}(\mathbf{q}) \Theta(k, p, q) U(k) U(q) \quad (5.35)$$

$$F_{(4)} = -yz \frac{q}{p} p_a P_{jb}(\mathbf{p}) k_j P_{im}(\mathbf{k}) P_{ib}(\mathbf{k}) P_{am}(\mathbf{q}) \Theta(k, p, q) U(k) U(q) \quad (5.36)$$

$$F_{(5)} = -xy \frac{p}{k} q_a P_{mb}(\mathbf{q}) k_j P_{im}(\mathbf{k}) P_{ia}(\mathbf{k}) P_{jb}(\mathbf{p}) \Theta(k, p, q) U(k) U(p) \quad (5.37)$$

$$F_{(6)} = -y^2 q_a P_{mb}(\mathbf{q}) k_j P_{im}(\mathbf{k}) P_{ib}(\mathbf{k}) P_{ja}(\mathbf{p}) \Theta(k, p, q) U(k) U(p) \quad (5.38)$$

which is written as

$$F_{(1)} = y^2 \left(\frac{q}{k}\right)^2 [k_1 k_2 P_{34}^k P_{12}^p P_{34}^q] \Theta(k, p, q) U(p) U(q) \quad (5.39)$$

$$F_{(2)} = yz \frac{pq}{k^2} [k_1 k_2 P_{34}^k P_{13}^p P_{24}^q] \Theta(k, p, q) U(p) U(q) \quad (5.40)$$

$$F_{(3)} = -xy \frac{q^2}{pk} [k_1 p_2 P_{23}^k P_{14}^p P_{34}^q] \Theta(k, p, q) U(k) U(q) \quad (5.41)$$

$$F_{(4)} = -yz \frac{q}{p} [k_1 p_2 P_{34}^k P_{13}^p P_{24}^q] \Theta(k, p, q) U(k) U(q) \quad (5.42)$$

$$F_{(5)} = -xy \frac{p}{k} [k_1 q_2 P_{23}^k P_{14}^p P_{34}^q] \Theta(k, p, q) U(k) U(p) \quad (5.43)$$

$$F_{(6)} = -y^2 [k_1 q_2 P_{34}^k P_{12}^p P_{34}^q] \Theta(k, p, q) U(k) U(p). \quad (5.44)$$

Note that we prefer for purely technical reasons to replace the indices by numbers, which allows to more easily organize and order the different terms. Summation over repeated indexed is still assumed. A procedure explained in Leslie, 1973 allows to rewrite the integral over wavevectors as a scalar integral over  $p, q$  space. This allows to rewrite the



expression as

$$\frac{\partial E(k)}{\partial t} = \frac{1}{2} \int_{\Delta} \frac{dp}{p} \frac{dq}{q} \Theta(k, p, q) \times \left[ f_{(1)}^{\lambda} X_{(1)} + f_{(2)}^{\lambda} X_{(2)} \right] k^3 E(p) E(q) \quad (5.45)$$

$$- \left[ f_{(3)}^{\lambda} X_{(3)} + f_{(4)}^{\lambda} X_{(4)} \right] p^3 E(k) E(q) \quad (5.46)$$

$$- \left[ f_{(5)}^{\lambda} X_{(5)} + f_{(6)}^{\lambda} X_{(6)} \right] q^3 E(p) E(k) \quad (5.47)$$

where the  $X$  are

$$X_{(1)} = k^{-2} k_1 k_2 P_{34}^k P_{12}^p P_{34}^q = (1 - z^2)(1 + y^2) \quad (5.48)$$

$$X_{(2)} = k^{-2} k_1 k_2 P_{34}^k P_{13}^p P_{24}^q = -xyz - y^2 z^2 \quad (5.49)$$

$$X_{(3)} = (kp)^{-1} k_1 p_2 P_{23}^k P_{14}^p P_{34}^q = xy(1 - z^2) \quad (5.50)$$

$$X_{(4)} = (kp)^{-1} k_1 p_2 P_{34}^k P_{13}^p P_{24}^q = z(-y^2 - xyz) \quad (5.51)$$

$$X_{(5)} = (kq)^{-1} k_1 q_2 P_{23}^k P_{14}^p P_{34}^q = y(-x^2 - xyz) \quad (5.52)$$

$$X_{(6)} = (kq)^{-1} k_1 q_2 P_{34}^k P_{12}^p P_{34}^q = (y + zx)(1 + y^2) \quad (5.53)$$

As a consistency check we can assess the case  $\lambda = 1$  leading to the classical EDQNM closure.

**Declaration of Interests.** The author reports no conflict of interest.

**Acknowledgments.** We thank Tong Wu for useful comments.

## REFERENCES

- ALEXAKIS, ALEXANDROS 2017. Helically decomposed turbulence. *Journal of Fluid Mechanics*, 812, 752.
- ALEXAKIS, ALEXANDROS & BRACHET, MARC-ETIENNE 2019. On the thermal equilibrium state of large-scale flows. *Journal of Fluid Mechanics*, 872, 594–625.
- ASHURST, WM. T. , KERSTEIN, A. R. , KERR, R. M. , & GIBSON, C. H. 1987. Alignment of vorticity and scalar gradient with strain rate in simulated Navier-Stokes turbulence. *Phys. Fluids*, 30.
- BENAVIDES, SANTIAGO JOSE & ALEXAKIS, ALEXANDROS 2017. Critical transitions in thin layer turbulence. *Journal of Fluid Mechanics*, 822, 364–385.
- BERERA, ARJUN , HO, RICHARD DJG , & CLARK, DANIEL 2020. Homogeneous isotropic turbulence in four spatial dimensions. *Physics of Fluids*, 32(8), 085107.
- BETCHOV, R 1956. An inequality concerning the production of vorticity in isotropic turbulence. *Journal of Fluid Mechanics*, 1(5), 497–504.
- BIFERALE, LUCA , MUSACCHIO, STEFANO , & TOSCHI, FEDERICO 2012. Inverse energy cascade in three-dimensional isotropic turbulence. *Physical review letters*, 108(16), 164501.
- BOLDYREV, SA 1999. Turbulence without pressure in d dimensions. *Physical Review E*, 59(3), 2971.
- BOS, W. J. T. & BERTOGLIO, J. P. 2006. Dynamics of spectrally truncated inviscid turbulence. *Phys. Fluids*, 18, 071701.
- BOS, W. J. T. & BERTOGLIO, J. P. 2013. Lagrangian Markovianized Field Approximation for turbulence. *J. Turbul.*, 14, 99.
- BOS, W. J. T. & RUBINSTEIN, R. 2013. On the strength of the nonlinearity in isotropic turbulence. *J. Fluid Mech.*, 733, 158–170.
- BRIARD, ANTOINE , BIFERALE, LUCA , & GOMEZ, THOMAS 2017. Closure theory for the split energy-helicity cascades in homogeneous isotropic homochiral turbulence. *Physical Review Fluids*, 2(10), 102602.

- BUARIA, DHAWAL , BODENSCHATZ, EBERHARD , & PUMIR, ALAIN 2020. Vortex stretching and enstrophy production in high Reynolds number turbulence. *arXiv preprint arXiv:2006.01312*.
- BURGERS, J.M. 1950. Correlation problems in a one dimensional model of turbulence. Parts I-IV. *Verhand. Kon. Ned. Akademie v. Wetenschappen*, 53, 247–260; 393–406; 718–731; 732–742.
- BUZZICOTTI, MICHELE , BIFERALE, LUCA , & TOSCHI, FEDERICO 2020. Statistical properties of turbulence in the presence of a smart small-scale control. *Physical Review Letters*, 124(8), 084504.
- CARBONE, MAURIZIO & BRAGG, ANDREW D 2020. Is vortex stretching the main cause of the turbulent energy cascade? *Journal of Fluid Mechanics*, 883.
- CELANI, A. , MUSACCHIO, S. , & VINCENZI, D. 2010. Turbulence in more than two and less than three dimensions. *Phys. Rev. Lett.*, 104, 184506.
- CHEN, H. , HERRING, J.R. , KERR, R.M. , & KRAICHNAN, R. H. 1989. Non-Gaussian statistics in isotropic turbulence. *Phys. Fluids A*, 1, 1844.
- CHERTKOV, M. , PUMIR, A. , & SHRAIMAN, B. I. 1999. Lagrangian tetrad dynamics and the phenomenology of turbulence. *Phys. Fluids*, 11, 2394.
- CHEVILLARD, L. & MENEVEAU, C. 2006. Lagrangian dynamics and statistical geometric structure of turbulence. *Phys. Rev. Lett.*, 97, 174501.
- CICHOWLAS, C. , BONAÏTI, P. , DEBBASCH, F. , & BRACHET, M. 2005. Effective Dissipation and Turbulence in Spectrally Truncated Euler Flows. *Phys. Rev. Lett.*, 95, 264502.
- FANG, LE , WU, T , & BOS, WOUTER J. T. 2020. Staircase scaling of short-time energy transfer in turbulence. *Journal of Turbulence*, 21(4), 234–242.
- FAVIER, BENJAMIN , GUERVILLY, CÉLINE , & KNOBLOCH, EDGAR 2019. Subcritical turbulent condensate in rapidly rotating Rayleigh–Bénard convection. *Journal of Fluid Mechanics*, 864.
- FRISCH, U , LESIEUR, M , & SULEM, PL 1976. Crossover Dimensions for Fully Developed Turbulence. *Physical Review Letters*, 37(19), 1312.
- FRISCH, U. , POMYALOV, A. , PROCACCIA, I. , & RAY, S. 2012. Turbulence in noninteger dimensions by fractal Fourier decimation. *Phys. Rev. Lett.*, 108(7), 074501.
- GOTOH, TOSHIYUKI , WATANABE, YUSAKU , SHIGA, YOSHITAKA , NAKANO, TOHRU , & SUZUKI, EIJIRO 2007. Statistical properties of four-dimensional turbulence. *Physical Review E*, 75(1), 016310.
- GUALA, MICHELE , LÜTHI, BEAT , LIBERZON, ALEXANDER , TSINOBER, ARKADY , & KINZELBACH, WOLFGANG 2005. On the evolution of material lines and vorticity in homogeneous turbulence. *Journal of Fluid Mechanics*, 533, 339.
- JOHNSON, PERRY L & MENEVEAU, CHARLES 2016. Large-deviation statistics of vorticity stretching in isotropic turbulence. *Physical Review E*, 93(3), 033118.
- KRAICHNAN, R. H. 1965. Lagrangian-History Closure Approximation for Turbulence. *Phys. Fluids*, 8, 575.
- KRAICHNAN, R. H. 1967. Inertial Ranges in Two-Dimensional Turbulence. *Phys. Fluids*, 10, 1417.
- KRAICHNAN, R. H. 1973. Helical turbulence and absolute equilibrium. *J. Fluid Mech.*, 59, 745.
- LANOTTE, ALESSANDRA S , BENZI, ROBERTO , MALAPAKA, SHIVA K , TOSCHI, FEDERICO , & BIFERALE, LUCA 2015. Turbulence on a fractal Fourier set. *Physical review letters*, 115(26), 264502.
- LEE, T.D. 1952. On some statistical properties of hydrodynamical and magnetohydrodynamical fields. *Q. Appl. Math.*, 10, 69.
- LEITH, C.E. 1971. Atmospheric predictability and two-dimensional turbulence. *J. Atmos. Sci.*, 28, 145–161.
- LEPROVOST, N. , DUBRULLE, B. , & CHAVANIS, P.-H. 2006. Dynamics and thermodynamics of axisymmetric flows: Theory. *Physical Review E*, 73, 046308.
- LESLIE, D.C. 1973. *Developments in the theory of turbulence*. Oxford University Press.
- NASO, A. , MONCHAUX, R. , CHAVANIS, P.H. , & DUBRULLE, B. 2010. Statistical mechanics of Beltrami flows in axisymmetric geometry: Theory reexamined. *Phys. Rev. E*, 81, 066318.
- ORSZAG, S. A. 1970. Analytical theories of Turbulence. *J. Fluid Mech.*, 41, 363.

- PLUNIAN, FRANCK , TEIMURAZOV, ANDREI , STEPANOV, RODION , & VERMA, MAHENDRA KUMAR 2020. Inverse cascade of energy in helical turbulence. *Journal of Fluid Mechanics*, 895.
- POLYAKOV, ALEXANDER M 1995. Turbulence without pressure. *Physical Review E*, 52(6), 6183.
- QIN, ZECONG , FALLER, HUGUES , DUBRULLE, BÉRENGÈRE , NASO, AURORE , & BOS, WOUTER JT 2020. Transition from non-swirling to swirling axisymmetric turbulence. *Physical Review Fluids*, 5(6), 064602.
- QU, BO , BOS, WOUTER J. T. , & NASO, AURORE 2017. Direct numerical simulation of axisymmetric turbulence. *Phys. Rev. Fluids*, 2, 094608.
- QU, BO , NASO, AURORE , & BOS, WOUTER JT 2018. Cascades of energy and helicity in axisymmetric turbulence. *Phys. Rev. Fluids*, 3(1), 014607.
- SAGAUT, P. & CAMBON, C. 2008. *Homogeneous Turbulence Dynamics*. Cambridge University Press.
- TAYLOR, GEOFFREY INGRAM 1938. Production and dissipation of vorticity in a turbulent fluid. *Proceedings of the Royal Society of London. Series A-Mathematical and Physical Sciences*, 164(916), 15–23.
- TENNEKES, H. & LUMLEY, J.L. 1972. *A first course in turbulence*. The MIT Press.
- TSINOBER, ARKADY 2009. *An informal conceptual introduction to turbulence*, volume 483. Springer.
- YAMAMOTO, T , SHIMIZU, H , INOSHITA, T , NAKANO, T , & GOTOH, TOSHIYUKI 2012. Local flow structure of turbulence in three, four, and five dimensions. *Physical Review E*, 86(4), 046320.