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The build-up construction of quasi self-dual codes over a non-unital ring

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There is a local ring E of order 4, without identity for the multiplication, defined by generators and relations as

$$E = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle.$$

We study a recursive construction of self-orthogonal codes over E. We classify, up to permutation equivalence, self-orthogonal codes of length n and size 2^n (called here quasi self-dual codes or QSD) up to the length $n = 12$. In particular, we classify Type IV codes (QSD codes with even weights) up to $n = 12$.

Keywords: Non-unital rings; self-orthogonal codes; type IV codes.

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1. Introduction

The build-up method is a powerful technique to construct self-dual codes over fields and rings [7, 10]. Starting from a self-dual code of length n, it builds a self-dual code of length $n+2$ by a simple recursion. Under mild conditions, all codes sought for can be generated in that way.

In this paper, we adapt this method to generate quasi self-dual (QSD) codes over the ring E, a non-unital, non-commutative ring of order 4. While the four unital rings of order four $\mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2 + u\mathbb{F}_2, \mathbb{F}_2 \times \mathbb{F}_2$ have been used as alphabets in Coding Theory [14], the theory of codes over non-unital rings only started recently, in [1]. The ring E is one of the few non-unital rings of order 4 to study [4]. The importance of order 4 is the notion of Type IV codes. Over fields and unital rings, these are divisible self-dual codes over an alphabet of size 4. They have been studied over $\mathbb{F}_4$ since the 1970s [9], and for the last three of the four rings mentioned above in [3]. To define them over E we need to replace self-dual codes by quasi self-dual codes. These are defined in [1] as self-orthogonal codes of length n, and size 2^n. As explained in [1] some missing duality properties in non-unital rings make the study of self-dual codes over these rings intractable. Type IV codes can then be introduced as the subclass of QSD codes, all Hamming weights of which are even.

As in [10], only codes with dimension of residue code $k_1 > 1$ are guaranteed to be obtained by the build-up construction, but QSD codes with dimension of residue code $k_1 \leq 1$ are easy to construct directly. In particular, this method preserves the Type IV property. As a result, we classify QSD codes of length at most 12. We also classify Type IV codes of length $n \leq 12$.

This paper is organized as follows. Section 2 collects some necessary facts and notations about rings, codes and duality. Section 3 derives the main construction. Section 4 contains numerical data. Section 5 concludes the paper, and points out some open problems.

2. Background Material

2.1. *Binary codes*

Denote by $wt(x)$ the Hamming weight of $x \in \mathbb{F}_2^n$. The dual of a binary linear code C is denoted by $C^\perp$ and defined as

$$C^\perp = \{y \in \mathbb{F}_2^n \,|\, \forall\, x \in C, (x, y) = 0\},$$

where $(x, y) = \sum_{i=1}^n x_i y_i$, denotes the standard inner product. A code C is **self-orthogonal** if it is included in its dual: $C \subseteq C^\perp$. Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

2.2. *Rings*

Following [4] we define a ring on two generators a, b by the relations

$$E = \langle a, b \,|\, 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle.$$

A model for this ring can be obtained by taking a, b to be matrices over $\mathbb{F}_2$ defined by

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus, E has characteristic two, and consists of four elements $E = \{0, a, b, c\}$, with $c = a + b$. The addition table is immediate from these definitions.

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The multiplication table is as follows:

$\times$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

From this table, we infer that this ring is not commutative, and without an identity element for the multiplication. It is local with maximal ideal $J = \{0, c\}$, and residue field $\mathbb{F}_2 = \{0, 1\}$, the finite field of order 2. Thus we have a **c-adic decomposition** as follows. Every element $e \in E$ can be written

$$e = as + ct,$$

where $s, t \in \mathbb{F}_2$ and where we have defined a natural **action** of $\mathbb{F}_2$ on E by the rule $r0 = 0r = 0$ and $r1 = 1r = r$ for all $r \in E$. Thus $a = 1a, c = 1c$ and $b = a1 + c1$. Note that for all $r \in E$, this action is "distributive" in the sense that $r(s \oplus t) = rs + rt$, where $\oplus$ denote the addition in $\mathbb{F}_2$. On occasion we will use the **inner product notation** (x, r) for $x \in \mathbb{F}_2^n$, $r \in E^n$ to denote

$$(x, r) = \sum_{i=1}^{n} x_i r_i = \sum_{x_i=1} r_i.$$

Denote by $\alpha : E \to E/J \simeq \mathbb{F}_2$ the **map of reduction modulo** J. Thus $\alpha(0) = \alpha(c) = 0$, and $\alpha(a) = \alpha(b) = 1$. This map is extended in the natural way in a map from E^n to $\mathbb{F}_2^n$.

2.3. *Modules*

A **linear** E-code C of length n is a *one-sided* E-submodule of E^n. With that code we associate two binary codes of length n :

(1) the **residue code** defined by $\mathrm{res}(C) = \{\alpha(y) \mid y \in C\}$,
(2) the **torsion code** defined by $\mathrm{tor}(C) = \{x \in \mathbb{F}_2^n \mid cx \in C\}$.

It is easy to check that $\mathrm{res}(C) \subseteq \mathrm{tor}(C)$ [1]. It is traditional to denote by k_1 the dimension of the residue code and $k_1 + k_2$ that of the torsion code.

A simple application of the first isomorphism theorem [1], shows that

$$|C| = |\mathrm{res}(C)||\mathrm{tor}(C)| = 2^{2k_1+k_2}.$$

An **additive code** of length n over $\mathbb{F}_4$ is an additive subgroup of $\mathbb{F}_4^n$. It is an $\mathbb{F}_2$ vector space with 4^k elements for some $k \leq n$ (here $2k$ is an integer, but k may be half-integral). Using a **generator matrix** G, such a code can be cast as the $\mathbb{F}_2$-span of its rows. To every linear E-code C is attached an additive $\mathbb{F}_4$-code $\phi(C)$ by the alphabet substitution

$$0 \to 0, a \to \omega, b \to \omega^2, c \to 1,$$

where $\mathbb{F}_4 = \mathbb{F}_2[\omega]$, extended naturally to $\mathbb{F}_4^n$. It can be checked that for all $x \in E^n$, we have $\mathrm{Tr}(\phi(x)) = \alpha(x)$, and thus $\mathrm{res}(C) = \mathrm{Tr}(\phi(C))$, where $\mathrm{Tr}()$ denotes the usual **trace** from $\mathbb{F}_4$ down to $\mathbb{F}_2$. Similarly, we see that $\mathrm{tor}(C)$ is the so-called **subfield subcode** of $\phi(C)$ that is $\mathbb{F}_2^n \cap \phi(C)$.

We use the Magma notation

$$[\langle 0, 1\rangle, \ldots, \langle i, A_i\rangle, \ldots, \langle n, A_n\rangle]$$

for the **weight distribution** of a quaternary code, where A_i is the number of codewords of weight i [12]. Two E-codes are **permutation equivalent** if there is a permutation of coordinates that maps one to the other.

2.4. *Duality*

Define an **inner product** on E^n by $(x, y) = \sum_{i=1}^n x_i y_i$.

The **right dual** $C^{\perp_R}$ of C is the right module defined by

$$C^{\perp_R} = \{y \in E^n \,|\, \forall\, x \in C,\ (x, y) = 0\}.$$

The **left dual** $C^{\perp_L}$ of C is the left module defined by

$$C^{\perp_L} = \{y \in E^n \,|\, \forall\, x \in C,\ (y, x) = 0\}.$$

Thus the left (respectively, right) dual of a left (respectively, right) module is a left (respectively, right) module. A code is **left** (respectively, **right**) **self-dual** if it is equal to its left (respectively, right) dual. A left self-dual code C satisfies $C^{\perp_L} = C$. Likewise a right self-dual code C satisfies $C^{\perp_R} = C$. A code is **self-dual** if it is equal to both of its duals.

Remark 1. The repetition code of length 2 defined by $R_2 = \{00, aa, bb, cc\}$, is left self-dual. Its right dual is $R_2^{\perp_R} = \langle aa, bb, ab\rangle$, a supercode of R_2 of size 8.

Observation. In length one, we have $J^{\perp_R} = J$. By taking direct sums of J with itself, we see that (right) self-dual codes over E exist for all lengths.

Remark 1 shows that the product of the sizes of a code and its dual is not always 4^n. A code C of length n is **left nice** (respectively, **right nice**) if $|C||C^{\perp_L}| = 4^n$ (respectively, $|C||C^{\perp_R}| = 4^n$). A code is **nice** if it is both left and right nice.

Remark 2. J is a right nice code, but it is not a left nice code since $n = 1$, and $J^{\perp_L} = E$. Similarly, R_2 is not right nice since $R_2^{\perp_R}$ is of size 8.

A code C is **self-orthogonal** if

$$\forall\, x, y \in C, (x, y) = 0.$$

Clearly, C is **self-orthogonal** iff $C \subseteq C^{\perp_L}$. Likewise, C is **self-orthogonal** iff $C \subseteq C^{\perp_R}$. Thus, for a self-orthogonal code C, we always have $C \subseteq C^{\perp_L} \cap C^{\perp_R}$. A code of length n is **quasi self-dual** if it is self-orthogonal and of size 2^n.

Remark 3. Every one-sided nice self-dual code is quasi-self-dual but not conversely, as the next example shows.

Example 1. The code R_2 as a right module is quasi self-dual but not self-dual as $R_2 \subsetneq R_2^{\perp_R}$.

Following a terminology from [3], a quasi self-dual code over E with all weights even is called a **Type IV** code.

3. Construction

The following result constructs a quasi self-dual code of length $n + 2$ from a quasi self-dual code of length n.

Theorem 1. *Let C_0 denote a quasi self-dual code of length n over E, with generating set $r_1, \ldots, r_k$. Let x be a fixed vector in $\mathbb{F}_2^n$ satisfying $(ax, ax) = a$. (Thus any x of odd Hamming weight works). Write $y_i = (x, r_i)$ for $1 \le i \le k$. The row-span of the following $k + 1$ vectors is a quasi self-dual code C of length $n + 2$.*

$$(a, 0, ax), (y_1, y_1, r_1), \ldots, (y_k, y_k, r_k).$$

Proof. First, we check that C is self-orthogonal.

- the first vector is orthogonal to itself by definition of x, since $a^2 + a = 0$.
- the last k vectors are orthogonal to each other and to themselves by self-orthogonality of C_0, since $y_i y_j + y_i y_j + (r_i, r_j) = 0$.
- the first vector is orthogonal to the last k vectors by definition of the y_i's since $ay_i + (ax, r_i) = ay_i + ay_i = 0$.
- Any of the last k vectors is orthogonal to the first vector since $y_i a + (r_i, ax) = y_i + (r_i, x) = y_i + (x, r_i) = 0$.

Hence C is self-orthogonal.

We claim that $|C| = 4|C_0| = 2^{n+2}$. Indeed define $\widehat{C_0}$ as the row span of the last k generators. Write $S_y = (y, 0, yx)$, for all $y \in E$. Then it can be seen that the construction in the theorem is equivalent to the following disjoint union of four cosets of $\widehat{C_0}$:

$$C = \dot{\cup}_{y \in E}(S_y + \widehat{C_0}).$$

Thus C is quasi self-dual of length $n + 2$. □

Remark 4. For a given C_0 by varying x we can construct 2^{n-1} codes C.

Corollary 1. *In the notation of the Theorem, if C_0 is Type IV, so is C.*

Proof. By [1] it is enough to check that res(C) contains the all-one vector. Let j_m denote the all-one vector of length m. By hypothesis res(C_0) contains j_n. Thus $j_n = \sum_{i=1}^{k} a_i\alpha(r_i)$, for some $a_i \in \mathbb{F}_2$. Consider the same linear combination of the last k generators of C.

$$\sum_{i=1}^{k} a_i\alpha((y_i, y_i, r_i)) = \left(\sum_{i=1}^{k} a_i\alpha(y_i), \sum_{i=1}^{k} a_i\alpha(y_i), j_n\right).$$

To check that this combination equals j_{n+2}, we note that

$$\sum_{i=1}^{k} a_i\alpha(y_i) = \left(x, \sum_{i=1}^{k} a_i\alpha(r_i)\right).$$

But, by the defining equation of the a_i's we have

$$\left(x, \sum_{i=1}^{k} a_i\alpha(r_i)\right) = (x, j_n) = 1,$$

since x has odd Hamming weight. Thus $j_{n+2} \in$ res(C), showing that C is Type IV. □

The next result shows that all QSD codes can be obtained from the build-up construction, subject to a mild condition.

Theorem 2. *If C is quasi self-dual code of length $n+2$ with $k_1 > 1$, then there is a quasi self-dual code C_0 of length n, such that the construction of Theorem 1 applied to C_0 produces C.*

Proof. Let $R_1, R_2, \ldots, R_{k+1}$ denote the generators of C viewed as an E-module. Let R'_i be R_{i+1} with the first two columns removed for $i = 0, \ldots, k$.

W.L.O.G. we may assume that R_1 contains only zeros and a's, by using $a^2 = ab = ac = a$. We may also assume since $k_1 > 1$, that $R_1 = (a, 0, R'_0), R_2 = (a, a, R'_1)$ and that, most importantly, that $R_{i+1} = (0, 0, R'_i)$ for $i \geq 2$. The orthogonality relations $(R_{i+1}, R_{j+1}) = 0$ imply $(R'_i, R'_j) = 0$ for all $1 \leq i, j \leq k$. Thus C_0, spanned by $R'_1, \ldots, R'_k$, is self-orthogonal.

It is also easy to check that, as in the proof of Theorem 1, we have $|C_0| = \frac{|C|}{4} = 2^n$.

The code obtained by applying Theorem 1 to the code C_0 spanned by $R'_1, \ldots, R'_k$, with x defined by $R'_0 = ax$, is row-equivalent to G. Indeed, with this definition of x, it can be checked that $y_1 = a$, and $y_i = 0$ for $i = 2, \ldots, k$. Indeed, the two conditions $(R_1, R_2) = (R_2, R_1) = 0$ imply $ay_1 = y_1a = a$. Taken together these two conditions imply $y_1 = a$. Likewise, for $2 \leq i \leq k$ the conditions $(R_1, R_{i+1}) = (R_{i+1}, R_1) = 0$ imply $ay_i = y_ia = 0$. Taken together these two conditions imply $y_i = 0$. □

It is easy to characterize codes where the hypothesis of this Theorem is not satisfied, as the next result shows.

Theorem 3. (1) *If C is QSD with $k_1 = 0$, then $C = c\mathbb{F}_2^n$. Such a code is not Type* IV.
(2) *If C is QSD with $k_1 = 1$, then $C = aB \oplus cB^{\perp}$, with B a self-orthogonal $[n, 1]$ binary code. The code C is Type IV iff B is the repetition code of length n for some even n.*

Proof. By [1, Theorem 6], we know that $C = a\ \mathrm{res}(C) \oplus c\ \mathrm{tor}(C)$, with $\mathrm{res}(C) \subseteq \mathrm{tor}(C)$.

(1) If C is QSD with $k_1 = 0$, then $C = c\ \mathrm{tor}(C)$. Since n is both its dimension and its length, the binary code $\mathrm{tor}(C)$ has to be the $[n, n, 1]$ "universe" code, denoted here by $\mathbb{F}_2^n$. The QSD code C is not Type IV as it contains codewords of odd weight.
(2) If C is QSD with $k_1 = 1$, then $C = a\ \mathrm{res}(C) \oplus c\ \mathrm{tor}(C)$, with $\mathrm{res}(C)$ a self-orthogonal code of dimension one. By [1, Theorem 4], the code C is Type IV iff $\mathrm{res}(C)$ contains the all-one vector, iff $\mathrm{res}(C)$ is the repetition code of length n for some even n. □

Example 2. • We obtain an infinite family of Type IV codes for each even n byletting $K_n = aU_n + cU_n^{\perp}$, with U_n denoting the repetition code of parameters $[n, 1, n]$. Thus K_n is the analogue of the Klemm code of [2].
• For $n = 5$, if we take $\mathrm{res}(C) = \langle(1, 1, 0, 0, 0)\rangle$, we obtain a non-Type IV QSD code with weight distribution:

$$[\langle 0, 1\rangle, \langle 1, 3\rangle, \langle 2, 6\rangle, \langle 3, 10\rangle, \langle 4, 9\rangle, \langle 5, 3\rangle].$$

4. Numerical Results

All the calculations in this section were performed in Magma [12]. When classifying QSD codes or Type IV codes up to permutation equivalence, the following result will be used.

Theorem 4. *Two QSD E-codes C and C' are permutation equivalent iff their residue codes are.*

Proof. The condition is necessary, since the residue of the permuted code is a permutation of the residue code.

The condition is sufficient. If A, B are binary codes, and P is a permutation acting on their coordinate places, it is easy to check, by invariance of the inner product, that $P(A) = B$ iff $P(A^{\perp}) = B^{\perp}$ [9, Chap. 1, Example 35]. By [1, Theorem 6], we have $C = a\,\mathrm{res}(C) \oplus c\,\mathrm{res}(C)^{\perp}$. Thus, if $P(\mathrm{res}(C)) = \mathrm{res}(C')$, then, by that result with $A = \mathrm{res}(C)$ and $B = \mathrm{res}(C')$, we obtain, by linearity of the action of P, that $P(C) = C'$. □

In order to obtain all QSD codes of a given length, we recursively apply Theorem 1 from different codes. For example, Theorem 1 applied to the code whose generator matrix is cI_2 (where I_2 is an identity matrix of order 2) yields two QSD-codes of length 4, not Type IV, and up to equivalence, it yields one QSD-code with $k_1 = 1$. Theorem 1 applied to these two codes yields sixteen QSD-codes of length 6 (not Type IV) and two QSD-codes up to equivalence with $k_1 = 2$. In this way, we obtain QSD-codes of length $2n$ with $k_1 = n - 1$. The same construction using generator matrix cI_4 (where I_4 is an identity matrix of order 4) yields QSD-codes of length $2n$ with $k_1 = n - 2$, and so on.

To make sure that enough equivalence classes of codes have been generated by our algorithms, we have consulted the tables of number of equivalence classes of self-orthogonal binary codes contained in [8].

The generator matrices of the classified codes up to $n = 11$, and the weight distributions of QSD-codes up to $n = 12$ can be found at:
`http://alexis.bonnecaze.perso.luminy.univ-amu.fr/publication/`.

4.1. *QSD codes of odd length*

From one QSD code of length 3, the number of codes of length n generated by Theorem 1, by the remark following Theorem 1, is

$$2^{\sum_{m=0}^{(n-3)/2}(2m)} = 2^{((n-2)^2-1)/4}.$$

In the following tables, the rows indicate, in order, the length, the dimension of the residue code (k_1), the number of equivalence classes of codes in that set, and the largest minimum distance obtained. This convention is maintained for the next tables.

n	3	5		7			9				11				
k_1	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5
$\# \sim$	1	2	1	3	3	2	4	6	6	3	5	10	14	12	4
d	2	1	1	1	1	3	1	1	2	2	1	1	2	2	3

4.2. *QSD codes of even length not Type* IV

From one QSD code of length 2 (Type IV or not), the number of codes of length n generated by Theorem 1, by the remark following Theorem 1, is

$$2^{\sum_{m=0}^{(n-4)/2}(2m+1)} = 2^{n^2/4-n+1}.$$

For instance, we have for $n = 2$, a unique QSD non Type IV code with $k_1 = 0$, as described in Theorem 3. As for QSD non Type IV of even length, we have the following table:

n	4	6		8			10				12				
k_1	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5
$\# \sim$	1	2	2	3	4	3	4	8	9	6	5	13	21	21	9
d	1	1	2	1	2	2	1	2	2	2	1	2	2	2	3

4.3. *Type IV codes of even length*

As observed in [1], Type IV codes can only exist in even length as the residue code must be self-orthogonal and contain the all-one vector. There is a unique Type IV code of length 2 namely R_2 defined in Sec. 2, since for such a code the weights must be either 0 or 2. Theorem 1 applied to R_2 gives two codes, whose images by ϕ are $\mathbb{F}_4$-linear, and permutation equivalent to a $[4, 2, 2]$ with generator matrix $\left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right)$.

The other Type IV code can be constructed in the form

$$a\langle(1, 1, 1, 1)\rangle + c\langle(1, 1, 1, 1)\rangle^{\perp}.$$

Its image by ϕ is an $\mathbb{F}_4$-additive code (not $\mathbb{F}_4$-linear) with generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ w & w & w & w \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and weight distribution $[\langle 0, 1\rangle, \langle 2, 6\rangle, \langle 4, 9\rangle]$. Note that even though they possess the same weight distribution, these two codes are different, one being linear.

Information for lengths 4 to 12 is summarized in the following table:

n	4		6			8				10					12					
k_1	1	2	1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
$\# \sim$	1	1	1	1	1	1	2	2	2	1	2	3	3	2	1	3	5	7	6	3
d	2	2	2	2	2	2	2	4	2	2	2	2	2	2	2	2	2	2	2	4

5. Conclusion and Open Problems

In this paper, we have applied the build-up method of construction of self-dual codes to quasi self-dual codes over a non-unital ring of order 4. As a result, we have been able to classify QSD codes, and Type IV QSD codes of length at most 12 up to equivalence.

To study higher lengths, a mass formula similar to what exists already over finite fields [11], or over $\mathbb{Z}_4$ [5] might be needed. In view of the super-exponential number of codes generated by Theorem 1, more computing power might be needed to extend the numerical results to lengths larger than 12.

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