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# The build-up construction of quasi self-dual codes over a non-unital ring.

Adel Alahmadi\*, Amani Alkathiry†, Alaa Altassan ‡,  
Alexis Bonnetcaze§, Hatoon Shoaib¶, Patrick Solé||\*\*

## Abstract

There is a local ring  $E$  of order 4, without identity for the multiplication, defined by generators and relations as

$$E = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle.$$

We study a recursive construction of self-orthogonal codes over  $E$ . We classify, up to permutation equivalence, self-orthogonal codes of length  $n$  and size  $2^n$  (called here quasi self-dual codes or QSD) up to the length  $n = 12$ . In particular, we classify Type IV codes (QSD codes with even weights) up to  $n = 12$ .

**Keywords:** non-unital rings, self-orthogonal codes, Type IV codes.

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# 1 Introduction

The build-up method is a powerful technique to construct self-dual codes over fields and rings [8, 11]. Starting from a self-dual code of length  $n$ , it builds a self-dual code of length  $n + 2$  by a simple recursion. Under mild conditions, all codes sought for can be generated in that way.

In this article, we adapt this method to generate quasi self-dual (QSD) codes over the ring  $E$ , a non-unital, non-commutative ring of order 4. While the four unital rings of order four  $\mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2 + u\mathbb{F}_2, \mathbb{F}_2 \times \mathbb{F}_2$  have been used as alphabets in Coding Theory [15], the theory of codes over non-unital rings only started recently, in [1]. The ring  $E$  is one of the few non-unital rings of order 4 to study [5]. The importance of order 4 is the notion of Type IV codes. Over fields and unital rings, these are divisible self-dual codes over an alphabet of size 4. They have been studied over  $\mathbb{F}_4$  since the 1970's [10], and for the last three of the four rings mentioned above in [4]. To define them over  $E$  we need to replace self-dual codes by quasi self-dual codes. These are defined in [1] as self-orthogonal codes of length  $n$ , and size  $2^n$ . As explained in [1] some missing duality properties in non-unital rings make the study of self-dual codes over these rings intractable. Type IV codes can then be introduced as the subclass of QSD codes, all Hamming weights of which are even.

As in [11], only codes with dimension of residue code  $k_1 > 1$  are guaranteed to be obtained by the build-up construction, but QSD codes with dimension of residue code  $k_1 \leq 1$  are easy to construct directly. In particular, this method preserves the Type IV property. As a result, we classify QSD codes of length at most 12. We also classify Type IV codes of length  $n \leq 12$ .

The material is organized as follows. The next section collects some necessary facts and notations about rings, codes and duality. Section 3 derives the main construction. Section 4 contains numerical data. Section 5 concludes the article, and points out some open problems.

## 2 Background material

### 2.1 Binary codes

Denote by  $wt(x)$  the Hamming weight of  $x \in \mathbb{F}_2^n$ . The dual of a binary linear code  $C$  is denoted by  $C^\perp$  and defined as

$$C^\perp = \{y \in \mathbb{F}_2^n \mid \forall x \in C, (x, y) = 0\},$$

where  $(x, y) = \sum_{i=1}^n x_i y_i$ , denotes the standard inner product. A code  $C$  is **self-orthogonal** if it is included in its dual:  $C \subseteq C^\perp$ . Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

### 2.2 Rings

Following [5] we define a ring on two generators  $a, b$  by the relations

$$E = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle.$$

A model for this ring can be obtained by taking  $a, b$  to be matrices over  $\mathbb{F}_2$  defined by

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus,  $E$  has characteristic two, and consists of four elements  $E = \{0, a, b, c\}$ , with  $c = a + b$ . The addition table is immediate from these definitions.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

The multiplication table is as follows.

×	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

From this table, we infer that this ring is not commutative, and without an identity element for the multiplication. It is local with maximal ideal  $J = \{0, c\}$ , and residue field  $\mathbb{F}_2 = \{0, 1\}$ , the finite field of order 2. Thus we have a  **$c$ -adic decomposition** as follows. Every element  $e \in E$  can be written

$$e = as + ct,$$

where  $s, t \in \mathbb{F}_2$  and where we have defined a natural **action** of  $\mathbb{F}_2$  on  $E$  by the rule  $r0 = 0r = 0$  and  $r1 = 1r = r$  for all  $r \in E$ . Thus  $a = 1a, c = 1c$  and  $b = a1 + c1$ . Note that for all  $r \in E$ , this action is “distributive” in the sense that  $r(s \oplus t) = rs + rt$ , where  $\oplus$  denote the addition in  $\mathbb{F}_2$ . On occasion we will use the **inner product notation**  $(x, r)$  for  $x \in \mathbb{F}_2^n, r \in E^n$  to denote

$$(x, r) = \sum_{i=1}^n x_i r_i = \sum_{x_i=1} r_i.$$

Denote by  $\alpha : E \rightarrow E/J \simeq \mathbb{F}_2$  the **map of reduction modulo  $J$** . Thus  $\alpha(0) = \alpha(c) = 0$ , and  $\alpha(a) = \alpha(b) = 1$ . This map is extended in the natural way in a map from  $E^n$  to  $\mathbb{F}_2^n$ .

### 2.3 Modules

A **linear  $E$ -code  $C$**  of length  $n$  is a *one-sided  $E$ -submodule* of  $E^n$ . With that code we associate two binary codes of length  $n$  :

1. the **residue code** defined by  $res(C) = \{\alpha(y) \mid y \in C\}$ ,
2. the **torsion code** defined by  $tor(C) = \{x \in \mathbb{F}_2^n \mid cx \in C\}$ .

It is easy to check that  $res(C) \subseteq tor(C)$  [1]. It is traditional to denote by  $k_1$  the dimension of the residue code and  $k_1 + k_2$  that of the torsion code.

A simple application of the first isomorphism theorem [1], shows that

$$|C| = |res(C)||tor(C)| = 2^{2k_1+k_2}.$$

An **additive code** of length  $n$  over  $\mathbb{F}_4$  is an additive subgroup of  $\mathbb{F}_4^n$ . It is an  $\mathbb{F}_2$  vector space with  $4^k$  elements for some  $k \leq n$  (here  $2k$  is an integer, but  $k$  may be half-integral). Using a **generator matrix  $G$** , such a code can be cast as the  $\mathbb{F}_2$ -span of its rows. To every linear  $E$ -code  $C$  is attached an additive  $\mathbb{F}_4$ -code  $\phi(C)$  by the alphabet substitution

$$0 \rightarrow 0, a \rightarrow \omega, b \rightarrow \omega^2, c \rightarrow 1,$$

where  $\mathbb{F}_4 = \mathbb{F}_2[\omega]$ , extended naturally to  $\mathbb{F}_4^n$ . It can be checked that for all  $x \in E^n$ , we have  $Tr(\phi(x)) = \alpha(x)$ , and thus  $res(C) = Tr(\phi(C))$ , where  $Tr()$  denotes the usual **trace** from  $\mathbb{F}_4$  down to  $\mathbb{F}_2$ . Similarly, we see that  $tor(C)$  is the so-called **subfield subcode** of  $\phi(C)$  that is  $\mathbb{F}_2^n \cap \phi(C)$ .

We use the Magma notation

$$[\langle 0, 1 \rangle, \dots, \langle i, A_i \rangle, \dots, \langle n, A_n \rangle]$$

for the **weight distribution** of a quaternary code, where  $A_i$  is the number of codewords of weight  $i$  [13]. Two  $E$ -codes are **permutation equivalent** if there is a permutation of coordinates that maps one to the other.

## 2.4 Duality

Define an **inner product** on  $E^n$  by  $(x, y) = \sum_{i=1}^n x_i y_i$ .

The **right dual**  $C^{\perp R}$  of  $C$  is the right module defined by

$$C^{\perp R} = \{y \in E^n \mid \forall x \in C, (x, y) = 0\}.$$

The **left dual**  $C^{\perp L}$  of  $C$  is the left module defined by

$$C^{\perp L} = \{y \in E^n \mid \forall x \in C, (y, x) = 0\}.$$

Thus the left (resp. right) dual of a left (resp. right) module is a left (resp. right) module. A code is **left** (resp. **right**) **self-dual** if it is equal to its left (resp. right) dual. A left self-dual code  $C$  satisfies  $C^{\perp L} = C$ . Likewise a right self-dual code  $C$  satisfies  $C^{\perp R} = C$ . A code is **self-dual** if it is equal to both of its duals.

**Remark 1** *The repetition code of length 2 defined by  $R_2 = \{00, aa, bb, cc\}$ , is left self-dual. Its right dual is  $R_2^{\perp R} = \langle aa, bb, ab \rangle$ , a supercode of  $R_2$  of size 8.*

**Observation:** In length one, we have  $J^{\perp R} = J$ . By taking direct sums of  $J$  with itself, we see that (right) self-dual codes over  $E$  exist for all lengths.

Remark 1 shows that the product of the sizes of a code and its dual is not always  $4^n$ . A code  $C$  of length  $n$  is **left nice** (resp. **right nice**) if  $|C||C^{\perp L}| = 4^n$  (resp.  $|C||C^{\perp R}| = 4^n$ ). A code is **nice** if it is both left and right nice.

**Remark 2**  $J$  is a right nice code, but it is not a left nice code since  $n = 1$ , and  $J^{\perp L} = E$ . Similarly,  $R_2$  is not right nice since  $R_2^{\perp R}$  is of size 8.

A code  $C$  is **self-orthogonal** if

$$\forall x, y \in C, (x, y) = 0.$$

Clearly,  $C$  is **self-orthogonal** iff  $C \subseteq C^{\perp L}$ . Likewise,  $C$  is **self-orthogonal** iff  $C \subseteq C^{\perp R}$ . Thus, for a self-orthogonal code  $C$ , we always have  $C \subseteq C^{\perp L} \cap C^{\perp R}$ . A code of length  $n$  is **quasi self-dual** if it is self-orthogonal and of size  $2^n$ .

**Remark 3** Every one-sided nice self-dual code is quasi-self-dual but not conversely, as the next example shows.

**Example 1** The code  $R_2$  as a right module is quasi self-dual but not self-dual as  $R_2 \subsetneq R_2^{\perp R}$ .

Following a terminology from [4], a quasi self-dual code over  $E$  with all weights even is called a **Type IV** code.

### 3 Construction

The following result constructs a quasi self-dual code of length  $n + 2$  from a quasi self-dual code of length  $n$ .

**Theorem 1** Let  $C_0$  denote a quasi self-dual code of length  $n$  over  $E$ , with generating set  $r_1, \dots, r_k$ . Let  $x$  be a fixed vector in  $\mathbb{F}_2^n$  satisfying  $(ax, ax) = a$ . (Thus any  $x$  of odd Hamming weight works). Write  $y_i = (x, r_i)$  for  $1 \leq i \leq k$ . The row-span of the following  $k+1$  vectors is a quasi self-dual code  $C$  of length  $n + 2$ .

$$(a, 0, ax), (y_1, y_1, r_1), \dots, (y_k, y_k, r_k).$$

**Proof.** First, we check that  $C$  is self-orthogonal.

- the first vector is orthogonal to itself by definition of  $x$ , since  $a^2 + a = 0$ .
- the last  $k$  vectors are orthogonal to each other and to themselves by self-orthogonality of  $C_0$ , since  $y_i y_j + y_i y_j + (r_i, r_j) = 0$ .

- the first vector is orthogonal to the last  $k$  vectors by definition of the  $y_i$ 's since  $ay_i + (ax, r_i) = ay_i + ay_i = 0$ .
- Any of the last  $k$  vectors is orthogonal to the first vector since  $y_1a + (r_i, ax) = y_1 + (r_i, x) = y_1 + (x, r_i) = 0$ .

Hence  $C$  is self-orthogonal.

We claim that  $|C| = 4|C_0| = 2^{n+2}$ . Indeed define  $\widehat{C}_0$  as the row span of the last  $k$  generators. Write  $S_y = (y, 0, yx)$ , for all  $y \in E$ . Then it can be seen that the construction in the theorem is equivalent to the following disjoint union of four cosets of  $\widehat{C}_0$  :

$$C = \dot{\cup}_{y \in E} (S_y + \widehat{C}_0).$$

Thus  $C$  is quasi self-dual of length  $n + 2$ . ■

**Remark 4** For a given  $C_0$  by varying  $x$  we can construct  $2^{n-1}$  codes  $C$ .

**Corollary 1** In the notation of the Theorem, if  $C_0$  is Type IV, so is  $C$ .

**Proof.** By [1] it is enough to check that  $res(C)$  contains the all-one vector. Let  $j_m$  denote the all-one vector of length  $m$ . By hypothesis  $res(C_0)$  contains  $j_n$ . Thus  $j_n = \sum_{i=1}^k a_i \alpha(r_i)$ , for some  $a_i \in \mathbb{F}_2$ . Consider the same linear combination of the last  $k$  generators of  $C$ .

$$\sum_{i=1}^k a_i \alpha((y_i, y_i, r_i)) = \left( \sum_{i=1}^k a_i \alpha(y_i), \sum_{i=1}^k a_i \alpha(y_i), j_n \right).$$

To check that this combination equals  $j_{n+2}$ , we note that

$$\sum_{i=1}^k a_i \alpha(y_i) = \left( x, \sum_{i=1}^k a_i \alpha(r_i) \right).$$

But, by the defining equation of the  $a_i$ 's we have

$$\left( x, \sum_{i=1}^k a_i \alpha(r_i) \right) = (x, j_n) = 1,$$

since  $x$  has odd Hamming weight. Thus  $j_{n+2} \in res(C)$ , showing that  $C$  is Type IV. ■

The next result shows that all QSD codes can be obtained from the build up construction, subject to a mild condition.



**Theorem 2** *If  $C$  is quasi self-dual code of length  $n + 2$  with  $k_1 > 1$ , then there is a quasi self-dual code  $C_0$  of length  $n$ , such that the construction of Theorem 1 applied to  $C_0$  produces  $C$ .*

**Proof.** Let  $R_1, R_2, \dots, R_{k+1}$  denote the generators of  $C$  viewed as an  $E$ -module. Let  $R'_i$  be  $R_{i+1}$  with the first two columns removed for  $i = 0, \dots, k$ .

W.L.O.G. we may assume that  $R_1$  contains only zeros and  $a$ 's, by using  $a^2 = ab = ac = a$ . We may also assume since  $k_1 > 1$ , that  $R_1 = (a, 0, R'_0)$ ,  $R_2 = (a, a, R'_1)$  and that, most importantly, that  $R_{i+1} = (0, 0, R'_i)$  for  $i \geq 2$ . The orthogonality relations  $(R_{i+1}, R_{j+1}) = 0$  imply  $(R'_i, R'_j) = 0$  for all  $1 \leq i, j \leq k$ . Thus  $C_0$ , spanned by  $R'_1, \dots, R'_k$ , is self-orthogonal.

It is also easy to check that, as in the proof of Theorem 1, we have  $|C_0| = \frac{|C|}{4} = 2^n$ .

The code obtained by applying Theorem 1 to the code  $C_0$  spanned by  $R'_1, \dots, R'_k$ , with  $x$  defined by  $R'_0 = ax$ , is row-equivalent to  $G$ . Indeed, with this definition of  $x$ , it can be checked that  $y_1 = a$ , and  $y_i = 0$  for  $i = 2, \dots, k$ . Indeed, the two conditions  $(R_1, R_2) = (R_2, R_1) = 0$  imply  $ay_1 = y_1a = a$ . Taken together these two conditions imply  $y_1 = a$ . Likewise, for  $2 \leq i \leq k$  the conditions  $(R_1, R_{i+1}) = (R_{i+1}, R_1) = 0$  imply  $ay_i = y_i a = 0$ . Taken together these two conditions imply  $y_i = 0$ . ■

It is easy to characterize codes where the hypothesis of this Theorem is not satisfied, as the next result shows.

**Theorem 3** 1. *If  $C$  is QSD with  $k_1 = 0$ , then  $C = c\mathbb{F}_2^n$ . Such a code is not Type IV.*

2. *If  $C$  is QSD with  $k_1 = 1$ , then  $C = aB \oplus cB^\perp$ , with  $B$  a self-orthogonal  $[n, 1]$  binary code. The code  $C$  is Type IV iff  $B$  is the repetition code of length  $n$  for some even  $n$ .*

**Proof.** By Theorem 6 of [1], we know that  $C = a \text{res}(C) \oplus c \text{tor}(C)$ , with  $\text{res}(C) \subseteq \text{tor}(C)$ .

1. If  $C$  is QSD with  $k_1 = 0$ , then  $C = c \text{tor}(C)$ . Since  $n$  is both its dimension and its length, the binary code  $\text{tor}(C)$  has to be the  $[n, n, 1]$  "universe" code, denoted here by  $\mathbb{F}_2^n$ . The QSD code  $C$  is not Type IV as it contains codewords of odd weight.
2. If  $C$  is QSD with  $k_1 = 1$ , then  $C = a \text{res}(C) \oplus c \text{tor}(C)$ , with  $\text{res}(C)$  a self-orthogonal code of dimension one. By [1, Th. 4], the code  $C$  is Type

IV iff  $\text{res}(C)$  contains the all-one vector, iff  $\text{res}(C)$  is the repetition code of length  $n$  for some even  $n$ . ■

**Example 2** • We obtain an infinite family of Type IV codes for each even  $n$  by letting  $K_n = aU_n + cU_n^\perp$ , with  $U_n$  denoting the repetition code of parameters  $[n, 1, n]$ . Thus  $K_n$  is the analogue of the Klemm code of [2].

- For  $n = 5$ , if we take  $\text{res}(C) = \langle (1, 1, 0, 0, 0) \rangle$ , we obtain a non-Type IV QSD code with weight distribution:

$$[\langle 0, 1 \rangle, \langle 1, 3 \rangle, \langle 2, 6 \rangle, \langle 3, 10 \rangle, \langle 4, 9 \rangle, \langle 5, 3 \rangle].$$

## 4 Numerical results

All the calculations in this section were performed in Magma [13]. When classifying QSD codes or Type IV codes up to permutation equivalence, the following result will be used.

**Theorem 4** *Two QSD E-codes  $C$  and  $C'$  are permutation equivalent iff their residue codes are.*

**Proof.** The condition is necessary, since the residue of the permuted code is a permutation of the residue code.

The condition is sufficient. If  $A, B$  are binary codes, and  $P$  is a permutation acting on their coordinate places, it is easy to check, by invariance of the inner product, that  $P(A) = B$  iff  $P(A^\perp) = B^\perp$  [10, chap. 1, ex 35]. By Theorem 6 of [1], we have  $C = a\text{res}(C) \oplus c\text{res}(C)^\perp$ . Thus, if  $P(\text{res}(C)) = \text{res}(C')$ , then, by that result with  $A = \text{res}(C)$  and  $B = \text{res}(C')$ , we obtain, by linearity of the action of  $P$ , that  $P(C) = C'$ . ■

In order to obtain all QSD codes of a given length, we recursively apply Theorem 1 from different codes. For example, Theorem 1 applied to the code whose generator matrix is  $cI_2$  (where  $I_2$  is an identity matrix of order 2) yields two QSD-codes of length 4, not Type IV, and up to equivalence,

it yields one QSD-code with  $k_1 = 1$ . Theorem 1 applied to these two codes yields sixteen QSD-codes of length 6 (not Type IV) and two QSD-codes up to equivalence with  $k_1 = 2$ . In this way, we obtain QSD-codes of length  $2n$  with  $k_1 = n - 1$ . The same construction using generator matrix  $cI_4$  (where  $I_4$  is an identity matrix of order 4) yields QSD-codes of length  $2n$  with  $k_1 = n - 2$ , and so on.

To make sure that enough equivalence classes of codes have been generated by our algorithms, we have consulted the tables of number of equivalence classes of self-orthogonal binary codes contained in [9].

The generator matrices of the classified codes up to  $n = 11$ , and the weight distributions of QSD-codes up to  $n = 12$  can be found at: <http://alexis.bonnecaze.perso.luminy.univ-amu.fr/publication/>.

## 4.1 QSD codes of odd length

From one QSD code of length 3, the number of codes of length  $n$  generated by Theorem 1, by the remark following Theorem 1, is

$$2^{\sum_{m=0}^{(n-3)/2} (2m)} = 2^{((n-2)^2-1)/4}.$$

In the following tables, the rows indicate, in order, the length, the dimension of the residue code ( $k_1$ ), the number of equivalence classes of codes in that set, and the largest minimum distance obtained. This convention is maintained for the next tables.

$n$	3		5		7			9				11				
$k_1$	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5	
$\# \sim$	1	2	1	3	3	2	4	6	6	3	5	10	14	12	4	
$d$	2	1	1	1	1	3	1	1	2	2	1	1	2	2	3	

## 4.2 QSD codes of even length not Type IV

From one QSD code of length 2 (Type IV or not), the number of codes of length  $n$  generated by Theorem 1, by the remark following Theorem 1, is

$$2^{\sum_{m=0}^{(n-4)/2} (2m+1)} = 2^{n^2/4-n+1}.$$

For instance, we have for  $n = 2$ , a unique QSD non Type IV code with  $k_1 = 0$ , as described in Theorem 3. As for QSD non Type IV of even length, we have the following table:

$n$	4		6			8				10					12				
$k_1$	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5				
$\# \sim$	1	2	2	3	4	3	4	8	9	6	5	13	21	21	9				
$d$	1	1	2	1	2	2	1	2	2	2	1	2	2	2	3				

### 4.3 Type IV codes of even length

As observed in [1], Type IV codes can only exist in even length as the residue code must be self-orthogonal and contain the all-one vector. There is a unique Type IV code of length 2 namely  $R_2$  defined in Section 2, since for such a code the weights must be either 0 or 2. Theorem 1 applied to  $R_2$  gives two codes, whose images by  $\phi$  are  $\mathbb{F}_4$ -linear, and permutation equivalent to a  $[4, 2, 2]$  with generator matrix  $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ .

The other Type IV code can be constructed in the form

$$a\langle(1, 1, 1, 1)\rangle + c\langle(1, 1, 1, 1)\rangle^\perp.$$

Its image by  $\phi$  is an  $\mathbb{F}_4$ -additive code (not  $\mathbb{F}_4$ -linear) with generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ w & w & w & w \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and weight distribution [ $\langle 0, 1 \rangle$ ,  $\langle 2, 6 \rangle$ ,  $\langle 4, 9 \rangle$ ]. Note that even though they possess the same weight distribution, these two codes are different, one being linear.

Information for lengths 4 to 12 is summarized in the following table:

$n$	4		6			8				10					12					
$k_1$	1	2	1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
$\# \sim$	1	1	1	1	1	1	2	2	2	1	2	3	3	2	1	3	5	7	6	3
$d$	2	2	2	2	2	2	2	4	2	2	2	2	2	2	2	2	2	2	2	4

## 5 Conclusion and open problems

In this article, we have applied the build-up method of construction of self-dual codes to quasi self-dual codes over a non-unital ring of order 4. As a result, we have been able to classify QSD codes, and Type IV QSD codes of length at most 12 up to equivalence.

To study higher lengths, a mass formula similar to what exists already over finite fields [12], or over  $\mathbb{Z}_4$  [6] might be needed. In view of the super-exponential number of codes generated by Theorem 1, more computing power might be needed to extend the numerical results to lengths larger than 12.

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