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Formal Ball Monads

Jean Goubault-Larrecq

*LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay
61 avenue du président Wilson, 94230 Cachan, France
goubault@lsv.fr, Phone: +33-1 47 40 22 60, Fax: +33 1 47 40 24 64*

Abstract

The formal ball construction \mathbf{B} is a central tool of quasi-metric space theory. We show that it induces monads on certain natural categories of quasi-metric spaces, with 1-Lipschitz maps as morphisms, or with 1-Lipschitz continuous maps as morphisms. Those are left Kock-Zöberlein monads, and that allows us to characterize their algebras exactly. As an application, we study so-called Lipschitz regular spaces, a natural class of spaces that contain all standard algebraic quasi-metric spaces with relatively compact balls, in particular all metric spaces whose closed balls are compact. There are other Lipschitz regular spaces, as we show, and notably all \mathbf{B} -algebras. That includes all spaces of formal balls, with their d^+ -Scott topology. The value of Lipschitz regularity is that, for a Lipschitz regular standard quasi-metric space X, d , the space $\mathcal{L}X$ of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$, with the Scott topology, retracts onto each of the spaces $\mathcal{L}_\alpha(X, d)$ of α -Lipschitz continuous maps, and that the subspace topology on the latter coincides with the Scott topology.

Keywords: Quasi-metric, formal balls, d -Scott topology, Kock-Zöberlein monad

2000 MSC: 54E99

1. Introduction

Let $\overline{\mathbb{R}}_+$ be the set of extended non-negative reals. A *quasi-metric* on a set X is a map $d: X \times X \rightarrow \overline{\mathbb{R}}_+$ satisfying: $d(x, x) = 0$; $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality); and $d(x, y) = d(y, x) = 0$ implies $x = y$. The pair X, d is then called a *quasi-metric space*.

The space of formal balls $\mathbf{B}(X, d)$ of a quasi-metric space X, d is probably the single most important artifact that has to be considered in the study of quasi-metric spaces [12, 2]. This has a very simple definition: a *formal ball* is syntax for an actual ball, namely a pair (x, r) where $x \in X$ (the *center*) and $r \in \mathbb{R}_+$ (the *radius*). $\mathbf{B}(X, d)$ can itself be equipped with a quasi-metric d^+ , with an ordering \leq^{d^+} , and with various topologies. Our objective is to show that this construction defines monads on natural categories of quasi-metric spaces, and to characterize their algebras.

As an application, we investigate *Lipschitz regular* quasi-metric spaces, which are spaces such that the assignment $U \mapsto \widehat{U}$, where U ranges over d -Scott-open subsets of X , and \widehat{U} is the largest Scott-open subset of $\mathbf{B}(X, d)$ such that $\widehat{U} \cap X = U$, is itself Scott-continuous. This is a desirable property in some situations. That property is satisfied, for example, by standard algebraic quasi-metric spaces with relatively compact balls, as we shall argue, in particular by metric spaces whose closed balls are compact. Our study of formal ball monads will allow us to show that every space of formal balls is Lipschitz regular, although it fails to have relatively compact balls in general.

This work is one of several bricks needed in a study of quasi-metrics on spaces of continuous valuations (a notion close to measures) and non-linear extensions of the latter, which should appear in a series of papers¹.

Outline. We present some preliminaries in Section 2, stressing the peculiarities of quasi-metric spaces, and notably of the so-called d -Scott topology. We show that the formal ball construction induces monads, first on the category of quasi-metric spaces and 1-Lipschitz maps, then in the subtler case of *standard* quasi-metric spaces and 1-Lipschitz *continuous* maps. (Not all Lipschitz maps are continuous with respect to the d -Scott topology, contrarily to what happens with the open ball topology.) This occupies Section 3. Section 4 gives the promised application to Lipschitz regular spaces. The name comes from the fact that those are the spaces in which the function that maps each lower semicontinuous map $f: X \rightarrow \overline{\mathbb{R}}_+$ to the largest α -Lipschitz continuous map below it is itself Scott-continuous, as we show in Section 5.

¹An unpublished version of that work is available on arXiv [6]. The present paper covers Sections 3, 4, and part of Section 6.1 there.

2. Preliminaries

General topology. We refer the reader to [5] for basic notions and theorems of topology, domain theory, and in the theory of quasi-metric spaces. The book [4] is the standard reference on domain theory, and I will assume known the notions of directed complete posets (dcpo), Scott-continuous functions, the way-below relation \ll , and so on. We write $\uparrow x$ for the set of points y such that $x \ll y$. The Scott topology on a poset consists of the Scott-open subsets, the upwards-closed subsets U such that every directed family that has a supremum in U must intersect U . A Scott-continuous map between posets is one that is monotonic and preserves existing directed suprema, and this is equivalent to requiring that it is continuous for the underlying Scott topologies.

The topic of the present paper is on quasi-metric spaces. Chapters 6 and 7 of [5] are a recommended read on that subject. The paper [7] gives additional information on quasi-metric spaces, which we shall also rely on.

As far as topology is concerned, compactness does not imply separation. In other words, we call a subset K of a topological space *compact* if and only if every open cover of K contains a finite subcover. This property is sometimes called quasicompactness.

We shall always write \leq for the specialization preordering of a topological space: $x \leq y$ if and only if every open neighborhood of x is also an open neighborhood of y , if and only if x is in the closure of y . As a result, the closure of a single point y is also its downward closure $\downarrow y$. In general, we write $\downarrow A$ for the downward closure of any set A , $\uparrow A$ for its upward closure, and $\uparrow x = \uparrow\{x\}$.

Quasi-metric spaces. Given a quasi-metric space X, d , the *open ball* $B_{x, < r}^d$ with center $x \in X$ and radius $r \in \mathbb{R}_+$ is $\{y \in X \mid d(x, y) < r\}$. The open ball topology is the coarsest containing all open balls, and is the standard topology on metric spaces.

In the realm of quasi-metric spaces, the *d-Scott topology* is the topology we shall always consider, unless specified otherwise. This is defined as follows. Formal balls are ordered by $(x, r) \leq^{d^+} (y, s)$ iff $d(x, y) \leq r - s$, and form a poset $\mathbf{B}(X, d)$. (Note that $(x, r) \leq^{d^+} (y, s)$ implies $r \geq s$.) We equip $\mathbf{B}(X, d)$ with its Scott topology. There is an injective map $x \mapsto (x, 0)$ from X to $\mathbf{B}(X, d)$, and the *d-Scott topology* is the coarsest that makes it continuous. This allows us to see X as a topological subspace of $\mathbf{B}(X, d)$.

The notation \leq^{d^+} comes from the fact that it is the specialization ordering of $\mathbf{B}(X, d), d^+$, where the quasi-metric d^+ is defined by $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)$.

The d -Scott topology coincides with the open ball topology when d is a metric [5, Proposition 7.4.46], or when X, d is Smyth-complete [5, Proposition 7.4.47]. It coincides with the generalized Scott topology of [1] when X, d is an algebraic Yoneda-complete quasi-metric space [5, Exercise 7.4.69].

We shall define all notions when they are required. For now, let us make clear what we understand by a Yoneda-complete quasi-metric space, and let us leave algebraicity for later. We will not need Smyth-completeness, and will therefore simply say “complete” instead of “Yoneda-complete”. Completeness will mean the usual notion in the case of metric spaces.

A *net* in a space X is a family $(x_i)_{i \in I, \sqsubseteq}$ of elements of X indexed by a set I , preordered by \sqsubseteq in such a way that I is directed. If X is itself preordered, then this net is *monotone* if and only if for all $i, j \in I$, $i \sqsubseteq j$ implies $x_i \leq x_j$. Monotone nets and directed families are essentially the same thing, since every monotone net is a directed family, and every directed family $(x_i)_{i \in I}$ can be seen as a monotone net by defining $i \sqsubseteq j$ if and only if $x_i \leq x_j$.

Recall from Section 7.2.1 of [5] that a *Cauchy-weighted net* $(x_i, r_i)_{i \in I, \sqsubseteq}$ is a monotone net of formal balls on X, d such that $\inf_{i \in I} r_i = 0$. The underlying net $(x_i)_{i \in I, \sqsubseteq}$ is then called *Cauchy-weightable*. A point $x \in X$ is a d -*limit* of the latter net if and only if, for every $y \in X$, $d(x, y) = \limsup_{i \in I, \sqsubseteq} d(x_i, y)$. This is equivalent to: for every $y \in X$, $d(x, y)$ is the supremum of the monotone net $(d(x_i, y) - r_i)_{i \in I, \sqsubseteq}$ [5, Lemma 7.4.9], a formula which we shall prefer for its simplicity. The d -limit is unique if it exists.

X, d is *complete* if and only if every Cauchy-weightable net has a d -limit. (Or: if and only if every Cauchy net has a d -limit; but Cauchy-weighted nets will be easier to work with.) This is also equivalent to requiring that $\mathbf{B}(X, d)$ is a dcpo, and in that case, the least upper bound (x, r) of $(x_i, r_i)_{i \in I, \sqsubseteq}$ is given by $r = \inf_{i \in I} r_i$ and x is the d -limit of $(x_i)_{i \in I, \sqsubseteq}$. This is the Kostanek-Waszkiewicz Theorem [10], see also [5, Theorem 7.4.27].

Example 2.1. $\overline{\mathbb{R}}_+$ comes with a natural quasi-metric $\mathbf{d}_{\mathbb{R}}$, defined by $\mathbf{d}_{\mathbb{R}}(x, y) = 0$ if $x \leq y$, $\mathbf{d}_{\mathbb{R}}(+\infty, y) = +\infty$ if $y \neq +\infty$, $\mathbf{d}_{\mathbb{R}}(x, y) = x - y$ if $x > y$ and $x \neq +\infty$. Then $\leq^{\mathbf{d}_{\mathbb{R}}^+}$ is the usual ordering \leq . We check that the Scott topology on $\overline{\mathbb{R}}_+$ coincides with the $\mathbf{d}_{\mathbb{R}}$ -Scott topology. To this end, observe that $\mathbf{B}(\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}})$ is order-isomorphic to $C = \{(a, b) \in (\mathbb{R} \cup \{+\infty\}) \times]-\infty, 0] \mid a - b \geq 0\}$ through the map $(x, r) \mapsto (x - r, -r)$. Since C is a Scott-closed subset of a

continuous dcpo, it is itself a continuous dcpo. A base of the Scott topology on C is given by open subsets of the form $\uparrow(a, b) = \{(c, d) \mid a < c, b < d\}$, hence a base of the Scott topology on $\mathbf{B}(\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}})$ is given by sets of the form $\{(x, r) \in \mathbf{B}(\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}) \mid a < x - r, b < -r\}$, $(a, b) \in C$. The intersections of the latter with $\overline{\mathbb{R}}_+$ are intervals of the form $\overline{\mathbb{R}}_+ \cap]a, +\infty]$, $a \in \mathbb{R} \cup \{+\infty\}$. Those are exactly the Scott open subsets of $\overline{\mathbb{R}}_+$.

Example 2.2. Any poset X, \leq gives rise to a quasi-metric space in a canonical way, by defining $d_{\leq}(x, y)$ as 0 if $x \leq y$, $+\infty$ otherwise. The d_{\leq} -Scott topology is exactly the Scott topology on X [7, Example 1.8].

To avoid certain pathologies, we shall concentrate on *standard* quasi-metric spaces [7, Section 2]. X, d is standard if and only if, for every directed family of formal balls $(x_i, r_i)_{i \in I}$, for every $s \in \mathbb{R}_+$, $(x_i, r_i)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$ if and only if $(x_i, r_i + s)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$. Writing the supremum of the former as (x, r) , we then have that $r = \inf_{i \in I} r_i$, and that the supremum of the latter is $(x, r + s)$ —this holds not only for $s \in \mathbb{R}_+$, but for every $s \geq -r$. In particular, the radius map $(x, r) \mapsto r$ is Scott-continuous from $\mathbf{B}(X, d)$ to $\overline{\mathbb{R}}_+^{op}$ ($\overline{\mathbb{R}}_+$ with the opposite ordering \geq), and for every $s \in \mathbb{R}_+$, the map $- + s : (x, r) \mapsto (x, r + s)$ is Scott-continuous from $\mathbf{B}(X, d)$ to itself [7, Proposition 2.4].

Most quasi-metric spaces—not all—are standard: all metric spaces, all complete quasi-metric spaces, all posets are standard [7, Proposition 2.2]. $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ is standard, being complete.

Given a map f from a quasi-metric space X, d to a quasi-metric space Y, ∂ , we say that f is α -Lipschitz if and only if $\partial(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. (When $\alpha = 0$ and $d(x, y) = +\infty$, we take the convention that $0 \cdot +\infty = +\infty$.)

For every $\alpha \in \mathbb{R}_+$, and every map $f : X, d \rightarrow Y, \partial$, let $\mathbf{B}^\alpha(f)$ map $(x, r) \in \mathbf{B}(X, d)$ to $(f(x), \alpha r) \in \mathbf{B}(Y, \partial)$. Then f is α -Lipschitz if and only if $\mathbf{B}^\alpha(f)$ is monotonic.

With the open ball topology, every Lipschitz map is continuous, but no longer so if we consider d -Scott topologies instead. There is a notion of Lipschitz *Yoneda-continuous* map, characterized as preserving so-called d -limits. When both X, d and Y, ∂ are standard, f is α -Lipschitz Yoneda-continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous [7, Lemma 6.3]. We take the latter as our definition:

Definition 2.3 (α -Lipschitz continuous). *A map $f: X, d \rightarrow Y, \partial$ between quasi-metric spaces is α -Lipschitz continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous.*

The phrase “ α -Lipschitz continuous” should not be read as “ α -Lipschitz and continuous”, rather as another notion of continuity. The two notions are actually equivalent in the case of standard quasi-metric spaces, as we show in Proposition 2.5 below. The proof is similar to Proposition 7.4.52 of [5], which states a similar result for complete quasi-metric spaces, and relies on the following lemma, similar to Lemma 7.4.48 of *ibid*.

Lemma 2.4. *Let X, d be a standard quasi-metric space. Every open hole $T_{x, > \epsilon}^d$, defined as $\{y \in X \mid d(y, x) > \epsilon\}$, where $\epsilon \in \mathbb{R}_+$, is open in the d -Scott topology: it is the intersection of the Scott-open set $T_{(x, 0), > \epsilon}^{d^+}$ with X .*

PROOF. Let V be the open hole $T_{(x, 0), > \epsilon}^{d^+}$. This is the set of formal balls (y, s) such that $d(y, x) > s + \epsilon$. We claim that V is upwards-closed: for every $(y, s) \in V$ and every (z, t) such that $(y, s) \leq^{d^+} (z, t)$, we have $d(y, x) > s + \epsilon$ and $d(y, z) \leq s - t$; by the triangular inequality $d(y, x) \leq d(y, z) + d(z, x) \leq s - t + d(z, x)$, so $d(z, x) > t + \epsilon$, showing that (z, t) is in V .

Next we claim that V is Scott-open. Let $(y_i, s_i)_{i \in I}$ be a directed family of formal balls that has a supremum (y, s) in V . Since X, d is standard, $(y, s + 2\epsilon)$ is the supremum of the directed family $(y_i, s_i + 2\epsilon)_{i \in I}$. If no (y_i, s_i) were in V , then we would have $d(y_i, x) \leq s_i + \epsilon$, i.e., $(y_i, s_i + 2\epsilon) \leq^{d^+} (x, \epsilon)$ for every $i \in I$. Since $(y, s + 2\epsilon)$ is the least upper bound, $(y, s + 2\epsilon) \leq^{d^+} (x, \epsilon)$, so $d(y, x) \leq s + \epsilon$, contradicting $(y, s) \in V$. Therefore (y_i, s_i) is in V for some $i \in I$, showing that V is Scott-open.

Finally, $V \cap X$ consists of those points y such that $d(y, x) > 0 + \epsilon$, hence is equal to $T_{x, > \epsilon}^d$, whence the claim. \square

Proposition 2.5. *Let X, d and Y, ∂ be two quasi-metric spaces, $\alpha > 0$, and f be a map from X to Y . Consider the following claims:*

1. *f is α -Lipschitz continuous in the sense of Definition 2.3;*
2. *f is α -Lipschitz and continuous, from X with its d -Scott topology to Y with its ∂ -Scott topology.*

Then (1) implies (2), and (2) implies (1) provided that X, d and Y, ∂ are standard.

PROOF. (1) \Rightarrow (2). Assume f is α -Lipschitz continuous. Let V be a ∂ -Scott open subset of Y . By definition, and equating Y with a subspace of $\mathbf{B}(Y, \partial)$, V is the intersection of some Scott-open subset \mathcal{V} of $\mathbf{B}(Y, \partial)$ with Y . Since $\mathbf{B}^\alpha(f)$ is Scott-continuous, $\mathcal{U} = \mathbf{B}^1(f)^{-1}(\mathcal{V})$ is Scott-open in $\mathbf{B}(X, d)$. Look at $U = \mathcal{U} \cap X$, a d -Scott open subset of X . We note that $x \in U$ if and only if $(x, 0) \in \mathcal{U}$ if and only if $\mathbf{B}^1(f)(x, 0) = (f(x), 0)$ is in \mathcal{V} , if and only if $f(x) \in V$, so that $U = f^{-1}(V)$. Hence f is continuous.

(2) \Rightarrow (1), assuming X, d and Y, ∂ standard. Assume f is α -Lipschitz and continuous. Since f is α -Lipschitz, $\mathbf{B}^\alpha(f)$ is monotonic. In order to show that it is Scott-continuous, consider an arbitrary directed family $(x_i, r_i)_{i \in I}$ in $\mathbf{B}(X, d)$, with a supremum (x, r) . We see that family as a monotone net, and let $i \sqsubseteq j$ if and only if $(x_i, r_i) \leq^{d^+} (x_j, r_j)$. Since X, d is standard, $r = \inf_{i \in I} r_i$ and $(x, 0)$ is the supremum of the directed family $(x_i, r_i - r)_{i \in I}$.

$\mathbf{B}^\alpha(f)(x, r) = (f(x), \alpha r)$ is an upper bound of $(f(x_i), \alpha r_i)_{i \in I}$ by monotonicity. Assume that it is not least. Then there is a formal ball (y, s) such that $(f(x_i), \alpha r_i) \leq^{\partial^+} (y, s)$ for every $i \in I$, i.e., $\partial(f(x_i), y) \leq \alpha r_i - s$ for every $i \in I$, and such that $(f(x), \alpha r)$ is not below (y, s) , i.e., $\partial(f(x), y) > \alpha r - s$. Let us note that $\alpha r - s \geq 0$: for every $i \in I$, since $(f(x_i), \alpha r_i) \leq^{\partial^+} (y, s)$ we have $\alpha r_i \geq s$, so $\alpha r = \inf_{i \in I} \alpha r_i \geq s$. Pick a real number η such that $\partial(f(x), y) > \eta > \alpha r - s$. In particular, $\eta > 0$, and $f(x)$ is in the open hole $T_{y > \eta}^\partial$, which is ∂ -Scott open by Lemma 2.4. Since f is continuous, $U = f^{-1}(T_{y > \eta}^\partial)$ is d -Scott open, and contains x by definition. Let \mathcal{U} be a Scott-open subset of $\mathbf{B}(X, d)$ whose intersection with X is equal to U . Since $(x, 0) \in \mathcal{U}$, $(x_i, r_i - r)$ is in \mathcal{U} for all i large enough; in other words, there is an $i_0 \in I$ such that $(x_i, r_i - r) \in \mathcal{U}$ for all $i \in I$ such that $i_0 \sqsubseteq i$. Since \mathcal{U} is upwards-closed, $(x_i, 0)$ is in \mathcal{U} , so x_i is in U , which implies that $f(x_i)$ is in $T_{y > \eta}^\partial$, for every $i \sqsupseteq i_0$. The latter expands to $\partial(f(x_i), y) > \eta$ for every $i \sqsupseteq i_0$. However, $\partial(f(x_i), y) \leq \alpha r_i - s$ for every $i \in I$, and since $r = \inf_{i \in I} r_i$ is also equal to $\inf_{i \in I, i_0 \sqsubseteq i} r_i$ (by directedness of I and the fact that $i \sqsubseteq j$ implies $r_i \geq r_j$), we obtain that $\alpha r - s \geq \eta$. This is impossible since $\eta > \alpha r - s$. \square

The latter has the following nice consequence, which we mention in passing.

Lemma 2.6. *Let X, d be a standard quasi-metric space. For every point $x' \in X$, the function $d(-, x') : x \mapsto d(x, x')$ is 1-Lipschitz continuous from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$.*

PROOF. It is 1-Lipschitz because of the triangular inequality. Relying on Proposition 2.5, and since $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ is standard, we only need to check that

$d(-, x')$ is continuous. By Example 2.1, the $\mathbf{d}_{\mathbb{R}}$ -Scott topology is the Scott topology on $\overline{\mathbb{R}}_+$, hence it suffices to show that the inverse image of the Scott open $]\epsilon, +\infty]$ by $d(-, x')$ is d -Scott open. That inverse image is the open hole $T_{x', > \epsilon}^d$, and we conclude by Lemma 2.4. \square

Of particular interest are the Lipschitz continuous functions from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$. Recall that $f: X, d \rightarrow \overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ is α -Lipschitz continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous. $\mathbf{B}(\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}})$ is order-isomorphic with the Scott-closed set $C = \{(a, b) \in (\mathbb{R} \cup \{+\infty\}) \times (-\infty, 0] \mid a - b \geq 0\}$, through the map $(x, r) \mapsto (x - r, -r)$: see Example 2.1. Every order isomorphism is Scott-continuous. Therefore f is α -Lipschitz continuous if and only if the composition $X \xrightarrow{\mathbf{B}^\alpha(f)} \mathbf{B}(X, d) \xrightarrow{\cong} C$ is Scott-continuous. That composition is $(x, r) \mapsto (f'(x, r), -\alpha r)$, where f' is defined by $f'(x, r) = f(x) - \alpha r$. The map $(x, r) \mapsto -\alpha r$ is Scott-continuous when X, d is standard. Hence we obtain the second part of the following result. The first part is obvious.

Lemma 2.7. *Let X, d be a standard quasi-metric space, $\alpha > 0$, and let f be a map from X to $\overline{\mathbb{R}}_+$. Let $f': \mathbf{B}(X, d) \rightarrow \mathbb{R} \cup \{+\infty\}$ be defined by $f'(x, r) = f(x) - \alpha r$. Then:*

1. *f is α -Lipschitz if and only if f' is monotonic;*
2. *f is α -Lipschitz continuous if and only if f' is Scott-continuous.* \square

Lemma 2.7 is Lemma 6.4 of [7], where Lipschitz Yoneda-continuous maps are used instead of Lipschitz continuous maps. The two notions are equivalent on standard quasi-metric spaces, as we have noticed before Definition 2.3.

The Lipschitz continuous functions to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ are closed under several constructions, which we recapitulate here.

Proposition 2.8. *Let X, d be a standard quasi-metric space, $\alpha, \beta \in \mathbb{R}_+$, and f, g be maps from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$.*

1. *If f is β -Lipschitz continuous, then αf is $\alpha\beta$ -Lipschitz continuous;*
2. *If f is α -Lipschitz continuous and g is β -Lipschitz continuous then $f + g$ is $(\alpha + \beta)$ -Lipschitz continuous;*
3. *If f, g are α -Lipschitz continuous, then so are $\min(f, g)$ and $\max(f, g)$;*
4. *If $(f_i)_{i \in I}$ is any family of α -Lipschitz continuous maps, then the point-wise supremum $\sup_{i \in I} f_i$ is also α -Lipschitz continuous.*

5. If $\alpha \leq \beta$ and f is α -Lipschitz continuous then f is β -Lipschitz continuous.
6. Every constant map is α -Lipschitz continuous.

PROOF. (1–5) were proved in [7, Proposition 6.7 (2)], and are easy consequences of Lemma 2.7 (2). For (6), using the same lemma, we observe that for each constant a , the map $(x, r) \mapsto a - \alpha r$ is Scott-continuous, because in a standard space, the radius map is Scott-continuous from $\mathbf{B}(X, d)$ to \mathbb{R}_+^{op} . \square

We shall also need the following result, which is obvious considering our definition of Lipschitz continuity.

Lemma 2.9. *Let X, d and Y, ∂ and Z, \mathfrak{d} be three quasi-metric spaces. For every α -Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ and every β -Lipschitz continuous map $g: Y, \partial \rightarrow Z, \mathfrak{d}$, $g \circ f$ is $\alpha\beta$ -Lipschitz continuous.*

PROOF. $\mathbf{B}^{\alpha\beta}(g \circ f)$ maps (x, r) to $(g(f(x)), \alpha\beta r) = \mathbf{B}^\beta(g)(f(x), \alpha r) = \mathbf{B}^\beta(g)(\mathbf{B}^\alpha(f)(x, r))$. Since $\mathbf{B}^\alpha(f)$ and $\mathbf{B}^\beta(g)$ are Scott-continuous by assumption, so is their composition $\mathbf{B}^{\alpha\beta}(g \circ f)$. \square

3. Formal Ball Monads

A *monad* on a category is the data of an endofunctor T , two natural transformations $\eta: \text{id} \rightarrow T$ and $\mu: T^2 \rightarrow T$ satisfying: $\mu_X \circ \eta_{TX} = \text{id}_{TX}$, $\mu_X \circ T\eta_X = \text{id}_{TX}$, and $\mu_X \circ \mu_{TX} = \mu_X \circ T(\mu_X)$.

We now examine the space $\mathbf{B}(X, d)$, with its quasi-metric $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)$, and show that that construction yields a monad on two categories of quasi-metric spaces: first, with 1-Lipschitz maps as morphisms, second with 1-Lipschitz *continuous* maps as morphisms. The latter case is more involved than the former.

In each case, we *define* the endofunctor \mathbf{B} as mapping the object X, d to $\mathbf{B}(X, d), d^+$ and every 1-Lipschitz map $f: X, d \rightarrow Y, \partial$ to $\mathbf{B}^1(f)$. The unit η_X will be the embedding $x \mapsto (x, 0)$, and multiplication μ_X will map $((x, r), s)$ to $(x, r + s)$.

3.1. In the category of quasi-metric spaces and 1-Lipschitz maps

Lemma 3.1. *Let X, d be a quasi-metric space. The following relations hold:*

1. $\mu_X \circ \eta_{\mathbf{B}(X,d)} = \text{id}_{\mathbf{B}(X,d)}$;
2. $\mu_X \circ \mathbf{B}^1(\eta_X) = \text{id}_{\mathbf{B}(X,d)}$;
3. $\mu_X \circ \mu_{\mathbf{B}(X,d)} = \mu_X \circ \mathbf{B}^1(\mu_X)$;
4. $\eta_{\mathbf{B}(X,d)} \circ \mu_X \geq \text{id}_{\mathbf{B}(\mathbf{B}(X,d),d^+)}$.

PROOF. (1) $\mu_X(\eta_{\mathbf{B}(X,d)}(x, r)) = \mu_X((x, r), 0) = (x, r + 0) = (x, r)$.

(2) $\mu_X(\mathbf{B}^1(\eta_X)(x, r)) = \mu_X(\eta_X(x), r) = \mu_X((x, 0), r) = (x, 0 + r) = (x, r)$.

(3) $\mu_X(\mu_{\mathbf{B}(X,d)}(((x, r), s), t)) = \mu_X((x, r), s + t) = (x, r + s + t)$, while $\mu_X(\mathbf{B}^1(\mu_X)(((x, r), s), t)) = \mu_X(\mu_X((x, r), s), t) = \mu_X((x, r + s), t) = (x, r + s + t)$.

(4) $\eta_{\mathbf{B}(X,d)}(\mu_X((x, r), s)) = \eta_{\mathbf{B}(X,d)}(x, r + s) = ((x, r + s), 0)$. We must check that this is larger than or equal to $((x, r), s)$, namely that $d^+((x, r), (x, r + s)) \leq s - 0$. Since $d^+((x, r), (x, r + s)) = \max(d(x, x) - r + r + s, 0) = s$, this is clear. \square

Theorem 3.2. *The triple (\mathbf{B}, η, μ) is a monad on the category of quasi-metric spaces and 1-Lipschitz maps.*

PROOF. For every 1-Lipschitz map f , $\mathbf{B}^1(f)$ is not just monotonic, but also 1-Lipschitz. Indeed, $\partial^+(\mathbf{B}^1(f)(x, r), \mathbf{B}^1(f)(x', r')) = \partial^+((f(x), r), (f(x'), r')) = \max(\partial(f(x), f(x')) - r + r', 0) \leq \max(d(x, x') - r + r', 0) = d^+((x, r), (x', r'))$. Hence \mathbf{B} defines an endofunctor.

The unit η is 1-Lipschitz, since $d^+((x, 0), (y, 0)) = d(x, y)$. The multiplication is 1-Lipschitz, too, since $d^+((x, r + s), (y, r' + s')) = \max(d(x, y) - r + r' - s + s', 0)$, while $d^{++}(((x, r), s), ((y, r'), s')) = \max(d^+((x, r), (y, r')) - s + s', 0) = \max(d(x, y) - r + r' - s + s', -s + s', 0)$, which is at least as large. The monad equations are from Lemma 3.1. \square

A *left KZ-monad* [3, Definition 4.1.2, Lemma 4.1.1] (short for *Kock-Zöberlein monad*) is a monad (T, η, μ) on a poset-enriched category such that T is monotonic on homsets, and either one of the following equivalent conditions hold:

1. $T\eta_X \leq \eta_{TX}$ for every object X ;

2. a morphism $\alpha: TX \rightarrow X$ is the structure map of a T -algebra if and only if $\alpha \circ \eta_X = \text{id}_X$ and $\text{id}_{TX} \leq \eta_X \circ \alpha$;
3. $\mu_X \dashv \eta_{TX}$ for every object X ;
4. $T\eta_X \dashv \mu_X$ for every object X .

The notion stems from work by A. Kock on doctrines in 2-categories [9], and the above equivalence is due to Kock, in the more general case of 2-categories. The notation $f \dashv g$ means that the two morphisms f and g are *adjoint*, namely, $f \circ g \leq \text{id}$ and $\text{id} \leq g \circ f$. A T -algebra is an object X together with a morphism $\alpha: TX \rightarrow X$, called its *structure map*, such that $\alpha \circ \eta_X = \text{id}_X$ and $\alpha \circ \mu_X = \alpha \circ T\alpha$. TX is always a T -algebra with structure map μ_X , called the *free T -algebra* on X .

The category of standard quasi-metric spaces and 1-Lipschitz maps is poset-enriched. Each homset is ordered by: for $f, g: X, d \rightarrow Y, \partial$, $f \leq g$ if and only if for every $x \in X$, $f(x) \leq^\partial g(x)$. If $f \leq g$, then $\mathbf{B}^1(f) \leq \mathbf{B}^1(g)$, since for every $(x, r) \in \mathbf{B}(X, d)$, $\mathbf{B}^1(f)(x, r) = (f(x), r) \leq^{\partial^+} (g(x), r) = \mathbf{B}^1(g)(x, r)$.

Condition (3) of a left KZ-monad reads: $\mu_X \circ \eta_{TX} \leq \text{id}_{TX}$ and $\text{id}_X \leq \eta_{TX} \circ \mu_X$. For $T = \mathbf{B}$, those follow from Lemma 3.1 (1) and (4). Hence:

Proposition 3.3. *The triple (\mathbf{B}, η, μ) is a left KZ-monad on the category of quasi-metric spaces and 1-Lipschitz maps. \square*

Kock's theorem between the equivalence of the four conditions defining KZ-monads yields the following immediately. The *closed ball* $B_{x, \leq r}^d$ is the set of points y such that $d(x, y) \leq r$. Despite the name, that is in general not closed, whether in the open ball topology or in the d -Scott topology, unless d is a metric. Indeed, closed balls are upwards-closed, while closed sets must be downwards-closed.

Proposition 3.4. *Let X, d be a quasi-metric space, and $\alpha: \mathbf{B}(X, d), d^+ \rightarrow X, d$ be a 1-Lipschitz map. The following are equivalent:*

1. α is the structure map of a \mathbf{B} -algebra;
2. for every $x \in X$, $\alpha(x, 0) = x$ and for all $r, s \in \mathbb{R}_+$, $\alpha(x, r + s) = \alpha(\alpha(x, r), s)$;
3. for every $x \in X$, and every $r \in \mathbb{R}_+$, $\alpha(x, r)$ is a point in the closed ball $B_{x, \leq r}^d$, which is equal to x if $r = 0$;

4. for every $x \in X$, $\alpha(x, 0) = x$.

PROOF. The equivalence between (1) and (2) is the definition of an algebra of a monad. Look at the second equivalent condition defining left KZ-monads, applied to the left KZ-monad \mathbf{B} (Proposition 3.3). This implies that (1) is equivalent to $\alpha \circ \eta_X = \text{id}_X$ (i.e., $\alpha(x, 0) = x$ for every $x \in X$), and to $\text{id}_X \leq \eta_X \circ \alpha$; the latter means that for every $x \in X$ and every $r \in \mathbb{R}_+$, $(x, r) \leq^{d^+} \eta_X(\alpha(x, r))$, equivalently, $d(x, \alpha(x, r)) \leq r$, i.e., $\alpha(x, r) \in B_{x, \leq r}^d$. Finally, clearly (3) implies (4). In the converse direction, note that since α is 1-Lipschitz, $d(\alpha(x, 0), \alpha(x, r)) \leq d^+((x, 0), (x, r)) = r$. Since $\alpha(x, 0) = x$, this implies that $\alpha(x, r)$ is in $B_{x, \leq r}^d$. \square

Remark 3.5. *It is natural to ask whether (\mathbf{B}, η, μ) is a strong monad, namely whether it has a so-called tensorial strength [8, 11]. (We will not define the concept, and will refer the reader to the latter papers for details.) That is not the case. By [11, Proposition 3.4], the tensorial strength is uniquely determined if the ambient category has finite products and has enough points. The product of quasi-metric spaces X, d and Y, ∂ is understood with the product quasi-metric $(d \times \partial)((x, y), (x', y')) = \max(d(x, x'), \partial(y, y'))$. Hence, if the tensorial strength exists, it must be given by the formula $t_{X, Y}: (x, (y, r)) \in X \times \mathbf{B}(Y, \partial) \mapsto ((x, y), r) \in (X \times Y, d \times \partial)$. However, that map is not 1-Lipschitz. We compute:*

$$\begin{aligned} (d \times \partial)^+(((x, y), r), ((x', y'), r')) &= \max(d(x, x') - r + r', \partial(y, y') - r + r', 0) \\ (d \times \partial^+)((x, (y, r)), (x', (y', r'))) &= \max(d(x, x'), \partial(y, y') - r + r', 0). \end{aligned}$$

The former is not less than or equal to the latter in general: e.g., take $X = Y = \overline{\mathbb{R}}_+$, $d = \partial = \mathbf{d}_{\mathbb{R}}$, $r = 0$, $r' = 1$, $x = 1$, $x' = y = y' = 0$.

3.2. *In the category of standard quasi-metric spaces and 1-Lipschitz continuous maps*

We now turn to the more difficult case of categories of quasi-metric spaces with 1-Lipschitz *continuous* maps as morphisms.

The following is the first part of Exercise 7.4.54 of [5]. It might seem a mistake that this does not require X, d to be standard: to dispel any doubt, we give a complete proof.

Lemma 3.6. *Let X, d be a quasi-metric space. The map $\mu_X: ((x, r), s) \in \mathbf{B}(\mathbf{B}(X, d), d^+) \mapsto (x, r + s) \in \mathbf{B}(X, d)$ is Scott-continuous.*

PROOF. The map μ_X is monotonic: if $((x, r), s) \leq^{d^{++}} ((x', r'), s')$, then $d^+((x, r), (x', r')) \leq s - s'$, meaning that $\max(d(x, x') - r + r', 0) \leq s - s'$, and this implies $d(x, x') \leq r - r' + s - s'$, hence $(x, r + s) \leq^{d^+} (x', r' + s')$.

We claim that μ_X is Scott-continuous. Consider a directed family of formal balls $((x_i, r_i), s_i)_{i \in I}$ in $\mathbf{B}(\mathbf{B}(X, d), d^+)$ with a supremum $((x, r), s)$. We must show that $(x, r + s)$ is the supremum of the directed family $(x_i, r_i + s_i)_{i \in I}$. It is certainly an upper bound, since μ_X is monotonic. Let (y, t) be another upper bound of $(x_i, r_i + s_i)_{i \in I}$. Let $a = \max(t - s, 0)$, $b = t - a = \min(s, t)$.

We claim that $((y, a), b)$ is an upper bound of $((x_i, r_i), s_i)_{i \in I}$. For every $i \in I$, by assumption $(x_i, r_i + s_i) \leq^{d^+} (y, t)$, so $d(x_i, y) \leq r_i + s_i - t$. We must check that $d^+((x_i, r_i), (y, a)) \leq s_i - b$, namely that $\max(d(x_i, y) - r_i + a, 0) \leq s_i - b$, and this decomposes into $d(x_i, y) \leq r_i - a + s_i - b$ and $s_i \geq b$. The latter is proved as follows: since $b = \min(s, t)$, $b \leq s$, and since $((x, r), s)$ is an upper bound of $((x_i, r_i), s_i)_{i \in I}$, $s \leq s_i$ for every $i \in I$. The former condition is equivalent to $d(x_i, y) \leq r_i + s_i - t$, since $a + b = t$, and this is our assumption.

Since $((x, r), s)$ is the least upper bound of $((x_i, r_i), s_i)_{i \in I}$, $((x, r), s) \leq^{d^{++}} ((y, a), b)$, so $\max(d(x, y) - r + a, 0) \leq s - b$. In particular, $d(x, y) \leq r + s - a - b = r + s - t$, so $(x, r + s) \leq^{d^+} (y, t)$. This shows that $(x, r + s)$ is the least upper bound of $(x_i, r_i + s_i)_{i \in I}$, hence that μ_X is Scott-continuous. \square

Lemma 3.7. *For every quasi-metric space X, d :*

1. *the map $\eta_{\mathbf{B}(X, d)}: (x, r) \mapsto ((x, r), 0)$ is Scott-continuous;*
2. *the d^+ -Scott topology on $\mathbf{B}(X, d)$ coincides with the Scott topology.*

PROOF. (1) This is [5, Exercise 7.4.53]. Monotonicity is clear: if $(x, r) \leq^{d^+} (y, s)$, then $d(x, y) \leq r - s$, so $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0) = 0$. For every directed family $(x_i, r_i)_{i \in I}$ in $\mathbf{B}(X, d)$, with supremum (x, r) , by monotonicity $((x, r), 0)$ is an upper bound of $((x_i, r_i), 0)_{i \in I}$. Consider another upper bound $((y, s), t)$. For every $i \in I$, $((x_i, r_i), 0) \leq^{d^{++}} ((y, s), t)$, namely $d^+((x_i, r_i), (y, s)) \leq 0 - t$. That implies $t = 0$, and $d(x_i, y) - r_i + s \leq 0$. The latter means that $(x_i, r_i) \leq^{d^+} (y, s)$, and as this holds for every $i \in I$, $(x, r) \leq^{d^+} (y, s)$. Therefore $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0) = 0$, showing that $((x, r), 0) \leq ((y, s), 0) = ((y, s), t)$.

(2) This is [5, Exercise 7.4.54]. Using (1), every d^+ -Scott open subset V of $\mathbf{B}(X, d)$ is Scott-open: by definition, $V = \eta_{\mathbf{B}(X, d)}^{-1}(\mathcal{V})$ for some Scott-open subset \mathcal{V} of $\mathbf{B}(\mathbf{B}(X, d), d^+)$, and since $\eta_{\mathbf{B}(X, d)}$ is Scott-continuous, V is

Scott-open. To show the converse implication, we observe that $\mu_X \circ \eta_{\mathbf{B}(X,d)}$ is the identity map, by Lemma 3.1 (ii). Then for every Scott-open subset V of $\mathbf{B}(X, d)$, V is equal to $(\mu_X \circ \eta_{\mathbf{B}(X,d)})^{-1}(V)$, hence to $\eta_{\mathbf{B}(X,d)}^{-1}(\mathcal{V})$ where \mathcal{V} is the subset $\mu_X^{-1}(V)$, which is Scott-open by Lemma 3.6. This exhibits V as a d^+ -Scott open subset of $\mathbf{B}(X, d)$. \square

Lemma 3.8. *Let X, d be a standard quasi-metric space. The map $\eta_X: x \mapsto (x, 0)$ is 1-Lipschitz continuous from X, d to $\mathbf{B}(X, d), d^+$.*

PROOF. It is 1-Lipschitz, because $d^+((x, 0), (y, 0)) = d(x, y)$. It is continuous from X with the d -Scott topology to $\mathbf{B}(X, d)$ with the Scott topology by definition, and the latter coincides with the d^+ -Scott topology by Lemma 3.7 (2). Now apply Proposition 2.5. \square

The first three statements of Lemma 3.1 seem to indicate that $T = \mathbf{B}$ gives rise to a monad, where the functor \mathbf{B} maps every quasi-metric space X, d to the quasi-metric space $\mathbf{B}(X, d), d^+$, and every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ to $\mathbf{B}^1(f): (x, r) \mapsto (f(x), r)$.

The devil hides in details, one says. We must work in a category of standard, not arbitrary quasi-metric spaces, for η_X to be a morphism (see Lemma 3.8). As a consequence, we must show that \mathbf{B} maps standard spaces to standard spaces. This is done in several steps.

Lemma 3.9. *Let X, d be a standard quasi-metric space, and let $(x_i, r_i)_{i \in I, \sqsubseteq}$ be a monotone net of formal balls on X, d with supremum (x, r) . Then $r = \inf_{i \in I} r_i$ and x is the d -limit of $(x_i)_{i \in I, \sqsubseteq}$.*

PROOF. This is similar to the proof of [5, Lemma 7.4.26], which assumes that $\mathbf{B}(X, d)$ is a dcpo, whereas we only assume that X, d is standard.

Since X, d is standard, $r = \inf_{i \in I} r_i$. Since $(x_i, r_i) \leq^{d^+} (x, r)$ for each $i \in I$, $d(x_i, x) \leq r_i - r$. For every $y \in X$, $d(x_i, y) \leq d(x_i, x) + d(x, y) \leq r_i - r + d(x, y)$. Taking suprema over $i \in I$, $\sup_{i \in I} (d(x_i, y) - r_i + r) \leq d(x, y)$. Note that $(x_i, r_i - r)_{i \in I, \sqsubseteq}$ is a Cauchy-weighted net, so it only remains to show that the latter inequality is an equality.

Assume that the inequality were strict. Let $s = \sup_{i \in I} (d(x_i, y) - r_i + r) < d(x, y)$. In particular, $s < +\infty$. Note also that $s \geq 0$, because $-r_i + r$ is arbitrarily close to 0. For every $i \in I$, $d(x_i, y) - r_i + r \leq s$, so $d(x_i, y) \leq r_i - r + s$, i.e., $(x_i, r_i - r + s) \leq^{d^+} (y, 0)$ (and $r_i - r + s \geq 0$, so that $(x_i, r_i - r + s)$ is a well-defined formal ball). Since X, d is standard, and (x, r)

is the supremum of $(x_i, r_i)_{i \in I}$, (x, s) is the supremum of $(x_i, r_i - r + s)_{i \in I}$. By the definition of s , $(x_i, r_i - r + s) \leq^{d^+} (y, 0)$ for every $i \in I$. It follows that $(x, s) \leq^{d^+} (y, 0)$, that is, $d(x, y) \leq s$, which is impossible. \square

This has the following interesting consequence (which we shall not use, however). Standardness says that if $(x_i, r_i)_{i \in I, \sqsubseteq}$ and $(x_i, s_i)_{i \in I, \sqsubseteq}$ are two monotone nets of formal balls with the same underlying net $(x_i)_{i \in I, \sqsubseteq}$, then one of them has a supremum if and only if the other one has, *provided* that r_i and s_i differ by a constant. In that case, those suprema are of the form (x, r) and (x, s) for the same point x (and where r and s differ by the same constant). The following proposition shows that this holds *without* any condition on r_i and s_i .

That might be used to (re)define the notion of d -limit x of a net $(x_i)_{i \in I, \sqsubseteq}$, as the center of the supremum of $(x_i, r_i)_{i \in I, \sqsubseteq}$, for some family of radii r_i that make $(x_i, r_i)_{i \in I, \sqsubseteq}$ a monotone net of formal balls. The following proposition shows that that definition is independent of the chosen radii r_i , assuming just standardness.

Proposition 3.10. *Let X, d be a standard quasi-metric space. Let $(x_i, r_i)_{i \in I, \sqsubseteq}$ and $(x_i, s_i)_{i \in I, \sqsubseteq}$ be two monotone nets of formal balls with the same underlying net $(x_i)_{i \in I, \sqsubseteq}$. If $(x_i, r_i)_{i \in I, \sqsubseteq}$ has a supremum (x, r) , then $r = \inf_{i \in I} r_i$ and $(x_i, s_i)_{i \in I, \sqsubseteq}$ also has a supremum, which is equal to (x, s) , where $s = \inf_{i \in I} s_i$.*

PROOF. If $(x_i, r_i)_{i \in I, \sqsubseteq}$ has a supremum (x, r) , then $r = \inf_{i \in I} r_i$ because X, d is standard. By Lemma 3.9, x is the d -limit of $(x_i)_{i \in I, \sqsubseteq}$. Lemma 7.4.25 of [5] states that if $(x_i, s_i)_{i \in I, \sqsubseteq}$ is a monotone net of formal balls and if $(x_i)_{i \in I, \sqsubseteq}$ has a d -limit x , then (x, s) is the supremum of $(x_i, s_i)_{i \in I, \sqsubseteq}$, where $s = \inf_{i \in I} s_i$. \square

Proposition 3.11. *For every standard quasi-metric space X, d , $\mathbf{B}(X, d), d^+$ is standard.*

PROOF. Let $((x_i, r_i), s_i)_{i \in I}$ be a directed family of formal balls on $\mathbf{B}(X, d), d^+$. This is a monotone net, provided we define $i \sqsubseteq j$ if and only if $((x_i, r_i), s_i) \leq^{d^{++}} ((x_j, r_j), s_j)$.

Assume that $((x_i, r_i), s_i)_{i \in I}$ has a supremum $((x, r), s)$. Since μ_X is Scott-continuous (Lemma 3.6), $(x_i, r_i + s_i)_{i \in I}$ is a directed family with supremum $(x, r + s)$. We use the fact that X, d is standard and apply Lemma 3.9 to obtain that $r + s = \inf_{i \in I} (r_i + s_i)$ and that x is the d -limit of $(x_i)_{i \in I, \sqsubseteq}$.

Since $((x_i, r_i), s_i)_{i \in I}$ is directed, $(s_i)_{i \in I}$ is filtered in \mathbb{R}_+ (i.e., directed in \mathbb{R}_+^{op}). Let $s_\infty = \inf_{i \in I} s_i$. Since μ_X is Scott-continuous hence monotonic, $(x_i, r_i + s_i)_{i \in I}$ is also directed, so $(r_i + s_i)_{i \in I}$ is filtered. Its infimum is $r + s$, as we have just seen. Let $r_\infty = \inf_{i \in I} (r_i + s_i) - s_\infty = r + s - s_\infty$.

Consider $((x, r_\infty), s_\infty)$. For every $i \in I$, we claim that $((x_i, r_i), s_i) \leq^{d^{++}} ((x, r_\infty), s_\infty)$. For that, we compute $d^+((x_i, r_i), (x, r_\infty)) = \max(d(x_i, x) - r_i + r_\infty, 0)$, and we check that this is less than or equal to $s_i - s_\infty$. Since $s_i - s_\infty \geq 0$ by definition of s_∞ , it remains to verify that $d(x_i, x) - r_i + r_\infty \leq s_i - s_\infty$. Using the equality $r_\infty + s_\infty = r + s$, obtained as a consequence of the definition of r_∞ , we have to verify the equivalent inequality $d(x_i, x) \leq r_i + s_i - r - s$. That one is obvious, since $(x_i, r_i + s_i)$ is below the supremum $(x, r + s)$.

Since $((x_i, r_i), s_i) \leq^{d^{++}} ((x, r_\infty), s_\infty)$ for every $i \in I$, $((x, r), s) \leq^{d^{++}} ((x, r_\infty), s_\infty)$. However, we claim that the converse inequality also holds. Indeed, we start by observing that $s \leq s_i$ for every $i \in I$, since $((x_i, r_i), s_i) \leq^{d^{++}} ((x, r), s)$. Hence $s \leq s_\infty$. Since $r_\infty = r + s - s_\infty$, $r_\infty \leq r$. Therefore $d^+((x, r_\infty), (x, r)) = \max(d(x, x) - r_\infty + r, 0) = r - r_\infty$, and the latter is equal to, hence less than or equal to $s_\infty - s$. This means that $((x, r_\infty), s_\infty) \leq^{d^{++}} ((x, r), s)$.

Having inequalities in both directions, we conclude that $((x, r), s) = ((x, r_\infty), s_\infty)$. This entails the important fact that $s = s_\infty = \inf_{i \in I} s_i$.

We use that to show that for any $a \geq -s$, $((x, r), s + a)$ is the supremum of $((x_i, r_i), s_i + a)_{i \in I}$. Since $((x_i, r_i), s_i) \leq^{d^{++}} ((x, r), s)$, we have $((x_i, r_i), s_i + a) \leq^{d^{++}} ((x, r), s + a)$. Now consider any other upper bound $((x', r'), s')$ of $((x_i, r_i), s_i + a)_{i \in I}$. We have $s' \leq s_i + a$ for every $i \in I$, whence using the equality $s = \inf_{i \in I} s_i$, $s' \leq s + a$. We wish to check that $((x, r), s + a) \leq^{d^{++}} ((x', r'), s')$, equivalently $d^+((x, r), (x', r')) \leq s + a - s'$, and that reduces to $s + a - s' \geq 0$ (which we have just shown) and $d(x, x') \leq r + s + a - r' - s'$. In order to establish the latter, recall that $(x, r + s)$ is the supremum of $(x_i, r_i + s_i)_{i \in I}$. Since X, d is standard (and since $a \geq -s \geq -r - s$), $(x, r + s + a)$ is also the supremum of $(x_i, r_i + s_i + a)_{i \in I}$. Since μ_X is monotonic, $(x', r' + s')$ is an upper bound of $(x_i, r_i + s_i + a)_{i \in I}$, so $(x, r + s + a) \leq^{d^+} (x', r' + s')$, or equivalently $d(x, x') \leq r + s + a - r' - s'$: that is exactly what we wanted to prove.

Let us recap: for every directed family $((x_i, r_i), s_i)_{i \in I}$ with supremum $((x, r), s)$, then $s = \inf_{i \in I} s_i$ and for every $a \geq -s$, $((x, r), s + a)$ is the supremum of $((x_i, r_i), s_i + a)_{i \in I}$. This certainly implies that if $((x_i, r_i), s_i)_{i \in I}$ has a supremum, then $((x_i, r_i), s_i + a)_{i \in I}$ also has one for every $a \in \mathbb{R}_+$.

Conversely, if $((x_i, r_i), s_i + a)_{i \in I}$ has a supremum for some $a \in \mathbb{R}_+$, then it is of the form $((x, r), s + a)$ where $s = \inf_{i \in I} s_i$, and for every $a' \geq -s - a$, $((x, r), s + a + a')$ is the supremum of $((x_i, r_i), s_i + a + a')_{i \in I}$. In particular, for $a' = -a$, $((x_i, r_i), s_i)_{i \in I}$ has a supremum. \square

Lemma 3.12. *Let X, d be a standard quasi-metric space. The map $\mu_X: ((x, r), s) \mapsto (x, r + s)$ is 1-Lipschitz continuous from $\mathbf{B}(\mathbf{B}(X, d), d^+), d^{++}$ to $\mathbf{B}(X, d), d^+$.*

PROOF. First, μ_X is 1-Lipschitz as part of Theorem 3.2.

Next, μ_X is Scott-continuous from $\mathbf{B}(\mathbf{B}(X, d), d^+)$ to $\mathbf{B}(X, d)$ by Lemma 3.6. The Scott topology on the latter coincides with its d^+ -Scott topology, by Lemma 3.7 (2), owing to the fact that X, d is standard. Similarly, since $\mathbf{B}(X, d), d^+$ is standard (Proposition 3.11), the Scott topology on the former coincides with its d^{++} -Scott topology. Hence μ_X is continuous from $\mathbf{B}(\mathbf{B}(X, d), d^+)$ to $\mathbf{B}(X, d)$, with their d^{++} -Scott, resp. d^+ -Scott topologies.

Since $\mathbf{B}(X, d), d^+$ and $\mathbf{B}(\mathbf{B}(X, d), d^+), d^{++}$ are standard, by Proposition 3.11, we can apply the (2) \Rightarrow (1) direction of Proposition 2.5, and we obtain that μ_X is 1-Lipschitz continuous. \square

Lemma 3.13. *Let X, d and Y, ∂ be two standard quasi-metric spaces, and f be an α -Lipschitz continuous map from X, d to Y, ∂ , with $\alpha > 0$. Then $\mathbf{B}^\alpha(f)$ is α -Lipschitz continuous from $\mathbf{B}(X, d), d^+$ to $\mathbf{B}(Y, \partial), \partial^+$.*

PROOF. We verify that $\mathbf{B}^\alpha(f)$ is α -Lipschitz:

$$\begin{aligned} \partial^+(\mathbf{B}^\alpha(f)(x, r), \mathbf{B}^\alpha(y, s)) &= \partial^+((f(x), \alpha r), (f(y), \alpha s)) \\ &= \max(\partial(f(x), f(y)) - \alpha r + \alpha s, 0) \\ &\leq \max(\alpha d(x, y) - \alpha r + \alpha s, 0) \\ &= \alpha \max(d(x, y) - r + s, 0) = \alpha d^+((x, r), (y, s)). \end{aligned}$$

By definition of α -Lipschitz continuity, $\mathbf{B}^\alpha(f)$ is Scott-continuous. Since the Scott topology on $\mathbf{B}(X, d)$ coincides with the d^+ -Scott topology, and similarly for Y , thanks to Lemma 3.7 (2), $\mathbf{B}^\alpha(f)$ is continuous with respect to the d^+ -Scott and ∂^+ -Scott topologies. Now use that $\mathbf{B}(X, d), d^+$ and $\mathbf{B}(Y, \partial), \partial^+$ are standard, owing to Proposition 3.11, and apply Proposition 2.5 to conclude that $\mathbf{B}^\alpha(f)$ is α -Lipschitz continuous. \square

Theorem 3.14. *The triple (\mathbf{B}, η, μ) is a monad on the category of standard quasi-metric spaces and 1-Lipschitz continuous maps.*

PROOF. We shall show the equivalent claim that $(\mathbf{B}, \eta, -^\dagger)$ is a Kleisli triple, that is: (i) \mathbf{B} maps objects of the category (standard quasi-metric spaces) to objects of the category; (ii) η_X is a morphism from X, d to $\mathbf{B}(X, d), d^+$ (a 1-Lipschitz continuous map); (iii) for every morphism $f: X, d \rightarrow \mathbf{B}(Y, \partial), \partial^+$, f^\dagger is a morphism from $\mathbf{B}(X, d), d^+$ to $\mathbf{B}(Y, \partial), \partial^+$ such that: (a) $\eta_X^\dagger = \text{id}_{\mathbf{B}(X, d)}$; (b) $f^\dagger \circ \eta_X = f$; (c) $f^\dagger \circ g^\dagger = (f^\dagger \circ g)^\dagger$. For that, we define f^\dagger as mapping (x, r) to $(y, r + s)$, where $(y, s) = f(x)$.

Proposition 3.11 gives us (i), and Lemma 3.8 gives us (ii). We devote the rest of this proof to (iii).

We must start by checking that f^\dagger is a morphism for every morphism $f: X, d \rightarrow \mathbf{B}(Y, \partial), \partial^+$. We have defined $f^\dagger(x, r)$ as $(y, r + s)$ where $(y, s) = f(x)$, and we notice that f^\dagger is equal to $\mu_Y \circ \mathbf{B}^1(f)$. This is 1-Lipschitz continuous because μ_Y and $\mathbf{B}^1(f)$ both are, by Lemma 3.12 and Lemma 3.13 respectively.

The equalities (a), (b), (c) are easily checked.

Any Kleisli triple $(T, \eta, -^\dagger)$ gives rise to a monad (T, η, m) by letting $m_X = \text{id}_X^\dagger$. Here m_X maps $((x, r), s)$ to $(x, r + s)$, hence coincides with μ_X , finishing the proof. \square

The category of standard quasi-metric spaces and 1-Lipschitz continuous maps is poset-enriched, as a subcategory whose objects are the standard quasi-metric spaces, and whose morphisms are merely required to be 1-Lipschitz. As for Proposition 3.3, we obtain immediately:

Proposition 3.15. *The triple (\mathbf{B}, η, μ) is a left KZ-monad on the category of standard quasi-metric spaces and 1-Lipschitz continuous maps.* \square

The following is then proved as Proposition 3.4. Alternatively, it can be deduced from it.

Proposition 3.16. *Let X, d be a standard quasi-metric space, and $\alpha: \mathbf{B}(X, d), d^+ \rightarrow X, d$ be a 1-Lipschitz continuous map. The following are equivalent:*

1. α is the structure map of a \mathbf{B} -algebra;
2. for every $x \in X$, $\alpha(x, 0) = x$ and for all $r, s \in \mathbb{R}_+$, $\alpha(x, r + s) = \alpha(\alpha(x, r), s)$;

3. for every $x \in X$, and every $r \in \mathbb{R}_+$, $\alpha(x, r)$ is a point in the closed ball $B_{x, \leq r}^d$, which is equal to x if $r = 0$;
4. for every $x \in X$, $\alpha(x, 0) = x$. □

4. Lipschitz Regular Spaces

Here and in the following, we work in the category of standard quasi-metric spaces and 1-Lipschitz continuous maps.

For every open subset U of X in its d -Scott topology, there is a largest open subset \widehat{U} of $\mathbf{B}(X, d)$ such that $\widehat{U} \cap X \subseteq U$. Then $\widehat{U} \cap X = U$. This was used in [7, Definition 6.10] in order to define the distance $d(x, \overline{U})$ of any point x to the complement \overline{U} of U as $\sup\{r \in \mathbb{R}_+ \mid (x, r) \in \widehat{U}\}$.

We shall write $\mathcal{O}Y$ for the lattice of open subsets of a topological space Y . The mapping $U \mapsto \widehat{U}$ is monotonic, and preserves arbitrary intersections, but it does not preserve unions in general. Notably, it is not Scott-continuous in general, as we shall see in Remark 4.8.

Definition 4.1 (Lipschitz regular). *A quasi-metric space X, d is Lipschitz regular if and only if the map $U \in \mathcal{O}X \mapsto \widehat{U} \in \mathcal{O}\mathbf{B}(X, d)$ is Scott-continuous.*

The name stems from a result that we shall see later, Proposition 5.4.

Lemma 4.2. *The following are equivalent for a standard quasi-metric space X, d :*

1. X, d is Lipschitz regular;
2. for every point $x \in X$, the map $U \in \mathcal{O}X \mapsto d(x, \overline{U})$ is Scott-continuous.

PROOF. (1) \Rightarrow (2). The map $U \mapsto d(x, \overline{U})$ is the composition of $U \mapsto \widehat{U}$ and of the map $\mathcal{U} \in \mathcal{O}\mathbf{B}(X, d) \mapsto \sup\{r \in \mathbb{R}_+ \mid (x, r) \in \mathcal{U}\}$. The latter is easily seen to be Scott-continuous, and the former is Scott-continuous by (1).

(2) \Rightarrow (1). Let U be the union of a directed family of open subsets $(U_i)_{i \in I}$. We only have to show that every $(x, r) \in \widehat{U}$ is in some \widehat{U}_i . By [7, Lemma 3.4], (x, r) is the supremum of the chain of formal balls $(x, r + 1/2^n)$, $n \in \mathbb{N}$, so one of them is in \widehat{U} . This implies that $d(x, \overline{U}) \geq r + 1/2^n > r$. Using (2), $d(x, \overline{U}_i) > r$ for some $i \in I$, and that implies the existence of a real number $s > r$ such that $(x, s) \in \widehat{U}_i$. Since $(x, s) \leq^{d^+} (x, r)$, (x, r) is also in \widehat{U}_i . □

The following Proposition 4.4 gives a further explanation of Lipschitz regularity, in the special case of algebraic quasi-metric spaces.

A point x in a standard quasi-metric space X, d is a *center point* if and only if, for every $\epsilon > 0$, the open ball $B_{(x,0), < \epsilon}^{d^+} = \{(y, s) \in \mathbf{B}(X, d) \mid d(x, y) < \epsilon - s\}$ is Scott-open in $\mathbf{B}(X, d)$. This is equivalent to requiring that x be a finite point in X, d , a notion that has a more complicated definition [7, Lemma 5.7].

X, d itself is called *algebraic* if and only if every point x is the d -limit of a Cauchy (or even Cauchy-weightable, see Section 5, *ibid.*) net of center points, or equivalently, for every $x \in X$, there is a directed family of formal balls (x_i, r_i) , $i \in I$, where every x_i is a center point, such that $\sup_{i \in I} (x_i, r_i) = (x, 0)$ (Lemma 5.15, *ibid.*).

Every metric space is (standard and) algebraic, since in a metric space every point is a center point, as a consequence of results by Edalat and Heckmann [2]. Indeed, the poset of formal balls of a metric space X, d is continuous, and $(x, r) \ll (y, s)$ if and only if $d(x, y) < r - s$ (Proposition 7 and Corollary 10, *ibid.*): then $B_{(x,0), < \epsilon}^{d^+}$ is equal to $\uparrow(x, \epsilon)$, hence is Scott-open.

Every standard algebraic quasi-metric space X, d is *continuous* [7, Proposition 5.18], where a continuous quasi-metric space is a standard quasi-metric space X, d whose space of formal balls $\mathbf{B}(X, d)$ is a continuous poset (Definition 3.10, *ibid.*) Moreover, when X, d is standard algebraic, $\mathbf{B}(X, d)$ has a basis of formal balls whose centers are center points, and for a center point x , $(x, r) \ll (y, s)$ if and only if $d(x, y) < r - s$. This is the same relation as in metric spaces, but beware that we only require it when x is a center point.

In general, we shall call a *strong basis* of a standard quasi-metric space X, d any set \mathcal{B} of center points of X such that, for every $x \in X$, $(x, 0)$ is the supremum of a directed family of formal balls with center points in \mathcal{B} . (Given that X, d is standard, this is equivalent to [5, Definition 7.4.66].) Hence X, d is algebraic if and only if it has a strong basis.

Remark 4.3. *In metric spaces, a strong basis is nothing else than the familiar concept of a dense subset [5, Exercise 7.4.67]. Strong bases are the correct generalization of dense subsets in the realm of quasi-metric spaces.*

Proposition 4.4. *Let X, d be a standard algebraic quasi-metric space. The following are equivalent:*

1. X, d is Lipschitz regular;

2. X, d has relatively compact balls, namely: for every center point x of X , for all $r, s \in \mathbb{R}_+$ with $s < r$, every open cover of the open ball $B_{x, < r}^d$ contains a finite subcover of the closed ball $B_{x, \leq s}^d$;
3. for every center point x of X , for all $r, s \in \mathbb{R}_+$ with $s < r$, for every directed family of open subsets $(U_i)_{i \in I}$ of open subsets of X such that $B_{x, < r}^d \subseteq \bigcup_{i \in I} U_i$, there is an $i \in I$ such that $B_{x, \leq s}^d \subseteq U_i$.

PROOF. The equivalence of (2) and (3) is a standard exercise. In the difficult direction, notice that any union of open sets can be written as a directed union of finite unions.

(3) \Rightarrow (1). It is easy to see that $U \mapsto \widehat{U}$ is monotonic. Let $(U_i)_{i \in I}$ be a directed family of d -Scott open subsets of X , and $U = \bigcup_{i \in I} U_i$. Pick an arbitrary element (y, s) in \widehat{U} . Our task is to show that (y, s) lies in some \widehat{U}_i .

Since X, d is algebraic, (y, s) is the supremum of a directed family of formal balls (x, r) way-below (y, s) , where each x is a center point. Since \widehat{U} is Scott-open, one of them is in \widehat{U} . From $(x, r) \ll (y, s)$ we obtain $d(x, y) < r - s$. Find a real number $\epsilon > 0$ so that $d(x, y) < r - s - \epsilon$.

The open ball $B_{x, < r}^d$ is the intersection of $\uparrow(x, r)$ with X , and $\uparrow(x, r)$ is included in \widehat{U} because (x, r) is in \widehat{U} and \widehat{U} is upwards-closed. Hence $B_{x, < r}^d$ is included in $\widehat{U} \cap X = U = \bigcup_{i \in I} U_i$. By (3), $B_{x, \leq r - \epsilon}^d$ is included in some U_i , $i \in I$.

Consider $\widehat{U}_i \cup \uparrow(x, r - \epsilon)$. This is an open subset of $\mathbf{B}(X, d)$, and its intersection with X is $U_i \cup B_{x, < r - \epsilon}^d = U_i$. By the maximality of \widehat{U}_i , $\widehat{U}_i = \widehat{U}_i \cup \uparrow(x, r - \epsilon)$, meaning that $\uparrow(x, r - \epsilon)$ is included in \widehat{U}_i . Since $d(x, y) < r - s - \epsilon$, $(x, r - \epsilon) \ll (y, s)$. It follows that (y, s) is in \widehat{U}_i .

(1) \Rightarrow (3). Fix a center point x , two real numbers r and s such that $0 < s < r$, and assume that $B_{x, < r}^d$ is included in the union U of some directed family of open subsets U_i of X .

We claim that (x, s) must be in \widehat{U} . The argument is one we have just seen. Indeed, $\widehat{U} \cup \uparrow(x, r)$ is an open subset of $\mathbf{B}(X, d)$ whose intersection with X equals $U \cup B_{x, < r}^d = U$. By maximality $\widehat{U} \cup \uparrow(x, r) = \widehat{U}$. However, since x is a center point and $d(x, x) < r - s$, we have $(x, r) \ll (x, s)$, so (x, s) is in \widehat{U} .

By (1), (x, s) is in some \widehat{U}_i , so $\uparrow(x, s) \subseteq \widehat{U}_i$, hence, taking intersections with X , $B_{x, \leq s}^d$ is included in U_i . \square

Remark 4.5. *As a special case, every metric space in which closed balls are compact is Lipschitz regular. Indeed, recall that every metric space is standard and algebraic, and compactness immediately implies the relatively compact ball property.*

Having relatively compact balls is a pretty strong requirement. Recall that a topological space X is *core-compact* if and only if $\mathcal{O}X$ is a continuous dcpo; equivalently, for every $y \in X$ and every open neighborhood U of y , y is in some open set V that is way-below U in $\mathcal{O}Y$ [5, Section 5.2.1]. In the presence of sobriety, core-compactness is equivalent to local compactness (ibid., Theorem 8.3.10).

Lemma 4.6. *Every standard algebraic quasi-metric space X, d with relatively compact balls is core-compact in its d -Scott topology.*

PROOF. Assume $y \in U$, where U is d -Scott open in X . Since X, d is standard algebraic, y is in some open ball $B_{x, < r}^d$ included in U , where x is a center point and $r > 0$. Hence $d(x, y) < r$, so that $d(x, y) < r - \epsilon$ for some $\epsilon > 0$. Then y is also in the open neighborhood $B_{x, < r - \epsilon}^d$ of x , and $B_{x, < r - \epsilon}^d \subseteq B_{x, \leq r - \epsilon}^d$ is way-below $B_{x, < \epsilon}^d$, using property (3) of Proposition 4.4. \square

Since algebraic (and even continuous) complete quasi-metric spaces are sober in their d -Scott topology [7, Proposition 4.1], we obtain:

Corollary 4.7. *Every algebraic complete quasi-metric space X, d with relatively compact balls is locally compact in its d -Scott topology.* \square

Another argument consists in using the definition of Lipschitz regularity directly: then $\mathcal{O}X$ is a retract of $\mathcal{O}\mathbf{B}(X, d)$, and when $\mathbf{B}(X, d)$ is a continuous poset, $\mathcal{O}\mathbf{B}(X, d)$ is a completely distributive lattice, in particular a continuous lattice; any retract of a continuous lattice is again continuous, so $\mathcal{O}X$ is continuous, meaning that X is core-compact.

Remark 4.8. *Not all standard algebraic quasi-metric spaces have relatively compact balls. For example, \mathbb{Q} with its usual metric is not core-compact, hence does not have relatively compact balls. In particular, \mathbb{Q} is not Lipschitz regular.*

Remark 4.9. *Lipschitz regularity is therefore a pretty strong requirement—in the case of standard algebraic quasi-metric spaces. On the contrary, we shall see below that spaces of formal balls are always Lipschitz regular (Theorem 4.13), even when not core-compact (Remark 4.14).*

The following lemma shows that the construction $U \mapsto \widehat{U}$ admits a particularly simple form on \mathbf{B} -algebras.

Lemma 4.10. *Let X, d be a quasi-metric space, and assume that there is a continuous map $\alpha: \mathbf{B}(X, d) \rightarrow X$ (with respect to the d^+ -Scott and d -Scott topologies) such that $\alpha(x, r) \in B_{x, \leq r}^d$ and $\alpha(x, 0) = x$ for all $x \in X$ and $r \in \mathbb{R}_+$. Then:*

1. *For every d -Scott open subset U of X , \widehat{U} is equal to $\alpha^{-1}(U)$;*
2. *X, d is Lipschitz regular.*

PROOF. Since α is continuous, $\alpha^{-1}(U)$ is d^+ -Scott open in $\mathbf{B}(X, d)$. Its intersection with X is equal to U , since $(x, 0) \in \alpha^{-1}(U)$ is equivalent to $\alpha(x, 0) \in U$, and $\alpha(x, 0) = x$. By the definition of \widehat{U} as largest, $\alpha^{-1}(U)$ is included in \widehat{U} . To show the converse implication, let (x, r) be an arbitrary element of \widehat{U} . Since $\alpha(x, r)$ is an element of $B_{x, \leq r}^d$, $d(x, \alpha(x, r)) \leq r$, so $(x, r) \leq^{d^+} (\alpha(x, r), 0)$. Since \widehat{U} is upwards-closed, $(\alpha(x, r), 0)$ is in \mathcal{U} . It follows that $\alpha(x, r)$ is in U , so that (x, r) is in $\alpha^{-1}(U)$.

(2) follows from (1), since α^{-1} commutes with unions. □

Remark 4.11. *Lemma 4.10 in particular applies when X, d is a (standard) \mathbf{B} -algebra, with structure map α . Indeed, by the (1) \Rightarrow (2) direction of Proposition 2.5, α is continuous, and the remaining assumptions are item (3) of Proposition 3.16.*

Remark 4.12. *By Lemma 4.10 (1), the standard quasi-metric spaces that are \mathbf{B} -algebras are much more than Lipschitz regular: the map $U \mapsto \widehat{U}$ must preserve all unions, not just the directed unions, and all intersections.*

However rare as \mathbf{B} -algebras may appear to be, recall that (when X, d is standard) $\mathbf{B}(X, d), d^+$ is itself a \mathbf{B} -algebra, with structure map μ_X . Hence the following is clear under a standardness assumption. However, this even holds without standardness.

Theorem 4.13. *For every quasi-metric space, the quasi-metric space $\mathbf{B}(X, d), d^+$ is Lipschitz regular. For every d^+ -Scott open subset U of $\mathbf{B}(X, d)$, $\widehat{U} = \mu_X^{-1}(U)$.*

PROOF. This is Lemma 4.10 with $\alpha = \mu_X$. This is a continuous map because it is Scott-continuous by Lemma 3.6 and because the Scott topologies on $\mathbf{B}(X, d)$ and on $\mathbf{B}(\mathbf{B}(X, d), d^+)$ coincide with the d^+ -Scott topology and with the d^{++} -Scott topology respectively, by Lemma 3.7 (2). The other two assumptions are Lemma 3.1, items (i) and (iv). \square

Remark 4.14. *We exhibit a Lipschitz regular, standard quasi-metric space that is not core-compact. Necessarily, that quasi-metric space cannot be algebraic, by Proposition 4.4. In particular, it cannot be metric. We build that quasi-metric space as $\mathbf{B}(X, d)$ for some quasi-metric space X, d , so that Theorem 4.13 will give us Lipschitz regularity for free.*

Every poset X can be turned into a quasi-metric space by letting $d_{\leq}(x, y) = 0$ if $x \leq y$, $+\infty$ otherwise, see Example 2.2. Then $\mathbf{B}(X, d_{\leq})$ is order-isomorphic with the poset $X \times]-\infty, 0]$ [7, Example 1.6].

Consider the dcpo $X = (\mathbb{N} \times \mathbb{N}) \cup \{\omega\}$, with the ordering defined by $(i, n) \leq (i', n')$ iff $i = i'$ and $n \leq n'$, and where ω is larger than any other element. The non-empty upwards-closed subsets of X are the subsets of the form $\{\omega\} \cup \bigcup_{i \in S} \uparrow(i, n_i)$, where $S \subseteq \mathbb{N}$ and for each i , $n_i \in \mathbb{N}$. Those that are compact are exactly those such that S is finite, and those that are Scott-open are exactly those such that $S = \mathbb{N}$. In particular, note that all compact saturated subsets have empty interior. The same happens in $X \times]-\infty, 0]$. Indeed, assume a compact saturated subset Q of $X \times]-\infty, 0]$ with non-empty interior U . Since Q is compact, its first projection $\pi_1[Q]$ is compact, too, and we see that $\pi_1[Q]$ is also upwards-closed, hence of the form $\{\omega\} \cup \bigcup_{i \in S} \uparrow(i, n_i)$, with S finite. Pick some $j \in \mathbb{N}$ outside of S . Since U is non-empty, it must contain $(\omega, 0)$. However, $(\omega, 0)$ is the supremum of the chain of points $((j, n), 0)$, $n \in \mathbb{N}$, so one of them is in U , hence in Q . This is impossible since $j \notin S$. Since all compact saturated subsets of $X \times]-\infty, 0]$ have empty interior, it follows that $X \times]-\infty, 0]$ is not locally compact.

Note that X is sober. Indeed, consider a non-empty closed subset C . If its complement is empty, then $C = \downarrow\omega$. Otherwise, C is the complement of an open set $\{\omega\} \cup \bigcup_{i \in \mathbb{N}} \uparrow(i, n_i)$, hence is equal to $\bigcup_{i \in S} \downarrow(i, n_i - 1)$, where S is the set of indices i such that $n_i \geq 1$. S is non-empty since we have assumed C non-empty. Pick i_0 from S . Then C is included in the union of $\downarrow(i_0, n_{i_0} - 1)$ and $C' = \bigcup_{i \in S \setminus \{i_0\}} \downarrow(i, n_i - 1)$. Note that C is not included in C' , so if C is irreducible, then $C \subseteq \downarrow(i_0, n_{i_0} - 1)$, from which we obtain $C = \downarrow(i_0, n_{i_0} - 1)$. In any case, we have shown that every irreducible closed subset of X is the downward closure of a unique point, hence X is sober.

Since $] - \infty, 0]$ is a continuous dcpo, it is sober in its Scott topology [5, Proposition 8.2.12 (b)]. Products of sober spaces are sober (Theorem 8.4.8, *ibid.*), so $X \times] - \infty, 0]$ is sober. Since every sober core-compact space is locally compact (Theorem 8.3.10, *ibid.*), we conclude that $X \times] - \infty, 0]$ is not core-compact.

We conclude that $\mathbf{B}(X, d_{\leq}) \cong X \times] - \infty, 0]$ is Lipschitz regular but not core-compact. \square

5. Largest α -Lipschitz Continuous Maps

We equip $\overline{\mathbb{R}}_+$ with the Scott topology of its ordering \leq , or equivalently, with the $\mathbf{d}_{\mathbb{R}}$ -Scott topology. For a topological space X , the continuous maps from X to $\overline{\mathbb{R}}_+$ are usually called *lower semicontinuous*. We shall write $\mathcal{L}X$ for the set of lower semicontinuous maps X to $\overline{\mathbb{R}}_+$, and $\mathcal{L}_{\alpha}(X, d)$ for the set of α -Lipschitz continuous maps from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$.

Lemma 5.1. *Let X, d be a standard quasi-metric space, and $\alpha > 0$. Every α -Lipschitz continuous map from X, d to $\overline{\mathbb{R}}_+, \mathbf{d}_{\mathbb{R}}$ is lower semicontinuous.*

PROOF. Recall that f is α -Lipschitz continuous if and only if $f': (x, r) \mapsto f(x) - \alpha r$ is Scott-continuous, by Lemma 2.7. Then $f^{-1}(]t, +\infty]) = X \cap f'^{-1}(]t, +\infty])$ for every $t \in \mathbb{R}$, showing that f itself is lower semicontinuous. \square

Hence $\mathcal{L}_{\alpha}(X, d)$ is a subset of $\mathcal{L}X$. Both are dcpos with the pointwise ordering. Note that there are in principle at least two distinct topologies on $\mathcal{L}_{\alpha}(X, d)$: the Scott topology, and the subspace topology, induced from the Scott topology on $\mathcal{L}X$. We shall see in Proposition 5.6 that the two coincide when X, d is standard and Lipschitz regular.

Assuming X, d standard, for each $\alpha \in \mathbb{R}_+$, there is a largest α -Lipschitz continuous map $f^{(\alpha)}$ below any lower semicontinuous map $f: X \rightarrow \overline{\mathbb{R}}_+$. Moreover, the family $(f^{(\alpha)})_{\alpha \in \mathbb{R}_+}$ is a chain, and $\sup_{\alpha \in \mathbb{R}_+} f^{(\alpha)} = f$, where suprema are taken pointwise [7, Theorem 6.17]. We also know that, for every d -Scott open subset U of X , for all $\alpha, r \in \mathbb{R}_+$, $(r\chi_U)^{(\alpha)}$ is the map $x \mapsto \min(r, \alpha d(x, \overline{U}))$ (Proposition 6.14, *ibid.*). We shall extend that below. Also, for every α -Lipschitz continuous map $g: X \rightarrow \overline{\mathbb{R}}_+$, for every $t \in \mathbb{R}_+$, tg is $t\alpha$ -Lipschitz continuous (Proposition 2.8 (1)).

Let us call *step function* any function from X to $\overline{\mathbb{R}}_+$ of the form $\sup_{i=1}^m a_i \chi_{U_i}$, where $0 < a_1 < \dots < a_m < +\infty$ and $U_1 \supseteq U_2 \supseteq \dots \supseteq U_m$ form a finite antitone family of open subsets of X .

Lemma 5.2. *Let X, d be a standard quasi-metric space. For a step function $f = \sup_{i=1}^m a_i \chi_{U_i}$, and $\alpha > 0$, $f^{(\alpha)}$ is the function that maps every $x \in X$ to $\min(\alpha d(x, \bar{U}_1), a_1 + \alpha d(x, \bar{U}_2), \dots, a_{i-1} + \alpha d(x, \bar{U}_i), \dots, a_{m-1} + \alpha d(x, \bar{U}_m), a_m)$.*

PROOF. Let $g(x) = \min(\alpha d(x, \bar{U}_1), a_1 + \alpha d(x, \bar{U}_2), \dots, a_{i-1} + \alpha d(x, \bar{U}_i), \dots, a_{m-1} + \alpha d(x, \bar{U}_m), a_m)$. Each of the maps $x \mapsto a_{i-1} + \alpha d(x, \bar{U}_i)$ (where, for convenience, we shall assume $a_0 = 0$, so as not to make a special case for $i = 1$) is α -Lipschitz continuous, and therefore g is α -Lipschitz continuous. Indeed, the map $d(\cdot, \bar{U})$ is 1-Lipschitz (Yoneda-)continuous, as shown in Lemma 6.11 (3) of [7]; the rest of the argument relies on Proposition 2.8.

We claim that $g(x) \leq f(x)$ for every $x \in X$. Let $U_0 = X$, so that U_i makes sense also when $i = 0$, and let $U_{m+1} = \emptyset$. The latter allows us to write $g(x)$ as $\min_{i=0}^m (a_i + \alpha d(x, \bar{U}_{i+1}))$, noticing that $d(x, \bar{U}_{m+1}) = 0$. Indeed, by [7, Lemma 6.11 (1)], for every open subset U , $d(x, \bar{U}) = 0$ if and only if $x \notin U$.

There is a unique index j , $0 \leq j \leq m$, such that $x \in U_j$ and $x \notin U_{j+1}$. Then $g(x) \leq a_j + d(x, \bar{U}_{j+1}) = a_j$. Noticing that $f(x) = a_j$, it follows that $g(x) \leq f(x)$.

Now consider any α -Lipschitz continuous map $h \leq f$, and let us show that $h \leq g$. We fix $x \in X$ and i with $0 \leq i \leq m$, and we claim that $h(x) \leq a_i + \alpha d(x, \bar{U}_{i+1})$. Since h is α -Lipschitz continuous, $h': (x, r) \mapsto h(x) - \alpha r$ is Scott-continuous, so $V = h'^{-1}([a_i, +\infty])$ is open in $\mathbf{B}(X, d)$.

For every element of the form $(y, 0)$ in $V \cap X$, $h'(y, 0) = h(y) > a_i$, hence $f(y) \geq h(y) > a_i$, which implies that y is in U_{i+1} . We have just shown that $V \cap X \subseteq U_{i+1}$, and that implies $V \subseteq \widehat{U}_{i+1}$, by maximality of \widehat{U}_{i+1} .

Now, for every $s \in \mathbb{R}_+$ such that $s < (h(x) - a_i)/\alpha$, i.e., such that $h'(x, s) = h(x) - \alpha s$ is strictly larger than a_i , by definition (x, s) is in V , hence in \widehat{U}_{i+1} . By definition, this means that $s \leq d(x, \bar{U}_{i+1})$. Taking suprema over s , we obtain $(h(x) - a_i)/\alpha \leq d(x, \bar{U}_{i+1})$, equivalently $h(x) \leq a_i + \alpha d(x, \bar{U}_{i+1})$. Since that holds for every i , $0 \leq i \leq m$, $h(x) \leq g(x)$. Hence g is the largest α -Lipschitz continuous map below f , in other words, $g = f^{(\alpha)}$. \square

Given any topological space X , every lower semicontinuous function $f: X \rightarrow \overline{\mathbb{R}}_+$ is the pointwise supremum of a chain of step functions:

$$f_K(x) = \frac{1}{2^K} \sup_{k=1}^{K2^K} k \chi_{f^{-1}([k/2^K, +\infty])}(x) \quad (1)$$

where $K \in \mathbb{N}$. If X, d is a standard quasi-metric space, $f_K^{(\alpha)}$ is given by

Lemma 5.2, namely:

$$f_K^{(\alpha)}(x) = \min \left(\min_{k=1}^{K2^K} \left(\frac{k-1}{2^K} + \alpha d(x, \overline{f^{-1}(\lfloor k/2^K, +\infty \rfloor)}) \right), K \right). \quad (2)$$

Proposition 5.3. *Let X, d be a standard quasi-metric space. For every lower semicontinuous map $f: X \rightarrow \overline{\mathbb{R}}_+$, for every $\alpha \in \mathbb{R}_+$, for every $x \in X$,*

$$f^{(\alpha)}(x) = \sup_{K \in \mathbb{N}} f_K^{(\alpha)}(x).$$

PROOF. We first deal with the case where f is already α -Lipschitz continuous. In that case, we claim the equivalent statement: (*) if f is α -Lipschitz continuous, then for every $x \in X$, $f(x) = \sup_{K \in \mathbb{N}} f_K^{(\alpha)}(x)$.

Fix $j \in \mathbb{N}$ and k such that $1 \leq k \leq K2^K$, and note that if $(x, j/(\alpha 2^K)) \in \widehat{U}_k$, where $U_k = f^{-1}(\lfloor k/2^K, +\infty \rfloor)$, then $\alpha d(x, \overline{U}_k) \geq j/2^K$. This is by definition of $d(x, \overline{U}_k)$.

Recall that $f'(x, r) = f(x) - \alpha r$ defines a Scott-continuous map. For every $(y, 0)$ in $X \cap f'^{-1}(\lfloor k/2^K, +\infty \rfloor)$, $f'(y, 0) = f(y) > k/2^K$, so $X \cap f'^{-1}(\lfloor k/2^K, +\infty \rfloor)$ is included in $f^{-1}(\lfloor k/2^K, +\infty \rfloor) = U_k$. By maximality, $f'^{-1}(\lfloor k/2^K, +\infty \rfloor)$ is included in \widehat{U}_k .

Hence if $(x, j/(\alpha 2^K))$ is in $f'^{-1}(\lfloor k/2^K, +\infty \rfloor)$, then $\alpha d(x, \overline{U}_k) \geq j/2^K$. That happens when $f(x) - j/2^K > k/2^K$, i.e., when x is in $f^{-1}(\lfloor (k+j)/2^K, +\infty \rfloor)$. Therefore $\alpha d(x, \overline{U}_k) \geq j/2^K \chi_{f^{-1}(\lfloor (k+j)/2^K, +\infty \rfloor)}(x)$ for all $j \in \mathbb{N}$ and k such that $1 \leq k \leq K2^K$.

Now fix k_0 with $1 \leq k_0 \leq K2^K$. For every k with $1 \leq k \leq k_0$, letting $j = k_0 - k$, we obtain that $(k-1)/2^K + \alpha d(x, \overline{U}_k) \geq (k-1)/2^K + (k_0 - k)/2^K \chi_{f^{-1}(\lfloor k_0/2^K, +\infty \rfloor)}(x) \geq (k_0 - 1)/2^K \chi_{f^{-1}(\lfloor k_0/2^K, +\infty \rfloor)}(x)$. For every k such that $k_0 < k \leq K2^K$, $(k-1)/2^K + \alpha d(x, \overline{U}_k) \geq k_0/2^K$ is larger than the same quantity already, and similarly for K , which is also larger than or equal to $k_0/2^K$. Using Formula (2), we obtain $f_K^{(\alpha)}(x) \geq (k_0 - 1)/2^K \chi_{f^{-1}(\lfloor k_0/2^K, +\infty \rfloor)}(x)$, and therefore $f_K^{(\alpha)}(x) \geq k_0/2^K \chi_{f^{-1}(\lfloor k_0/2^K, +\infty \rfloor)}(x) - 1/2^K$. Since that holds for every k_0 between 1 and $K2^K$, it follows that $f_K^{(\alpha)}(x) \geq f_K(x) - 1/2^K$. Taking suprema over $K \in \mathbb{N}$, we obtain $\sup_{K \in \mathbb{N}} f_K^{(\alpha)}(x) \geq \sup_{K \in \mathbb{N}} (f_K(x) - 1/2^K) = f(x)$, proving (*).

In the general case, where f is only assumed to be lower semicontinuous, we note that $f \geq f^{(\alpha)}$ implies that $f_K \geq (f^{(\alpha)})_K$. Indeed, that follows from formula (1) and the fact that $(f^{(\alpha)})^{-1}(\lfloor k/2^K, +\infty \rfloor)$ is included in

$f^{-1}(\lfloor k/2^K, +\infty])$ for every k . The mapping $g \mapsto g^{(\alpha)}$ is also monotonic, since $g^{(\alpha)}$ is defined as the largest α -Lipschitz continuous map below g . Therefore $f_K^{(\alpha)} \geq (f^{(\alpha)})_K^{(\alpha)}$. Taking suprema, we obtain that $\sup_{K \in \mathbb{N}} f_K^{(\alpha)}(x) \geq \sup_{K \in \mathbb{N}} (f^{(\alpha)})_K^{(\alpha)}(x) = f^{(\alpha)}(x)$ for every $x \in X$, where the last equality follows from statement (*) (first part of the proof), applied to the α -Lipschitz continuous function $f^{(\alpha)}$.

The converse inequality $\sup_{K \in \mathbb{N}} f_K^{(\alpha)} \leq f^{(\alpha)}$ is easy: for every $K \in \mathbb{N}$, $f_K \leq f$, so $f_K^{(\alpha)} \leq f^{(\alpha)}$. \square

Proposition 5.4. *Let X, d be a standard quasi-metric space. The following are equivalent:*

1. X, d is Lipschitz regular;
2. for every $\alpha \in \mathbb{R}_+$, the map $f \in \mathcal{L}X \mapsto f^{(\alpha)} \in \mathcal{L}_\alpha(X, d)$ is Scott-continuous;
3. for some $\alpha > 0$, the map $f \in \mathcal{L}X \mapsto f^{(\alpha)} \in \mathcal{L}_\alpha(X, d)$ is Scott-continuous.

PROOF. (1) \Rightarrow (2). Clearly $f \mapsto f^{(\alpha)}$ is monotonic. Let $(f_i)_{i \in I}$ be a directed family of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$, and f be their (point-wise) supremum. Note that, for every $t \in \mathbb{R}_+$, $f^{-1}(\lfloor t, +\infty])$ is the union of the directed family of open sets $f_i^{-1}(\lfloor t, +\infty])$, $i \in I$. Then, for every $x \in X$, and every $K \in \mathbb{N}$:

$$\begin{aligned}
f_K^{(\alpha)}(x) &= \min \left(\min_{k=1}^{K2^K} \left(\frac{k-1}{2^K} + \alpha d(x, \overline{f^{-1}(\lfloor k/2^K, +\infty])} \right), K \right) \\
&\quad \text{by Formula (2)} \\
&= \min \left(\min_{k=1}^{K2^K} \left(\frac{k-1}{2^K} + \alpha d(x, \bigcup_{i \in I} \overline{f_i^{-1}(\lfloor k/2^K, +\infty])} \right), K \right) \\
&= \min \left(\min_{k=1}^{K2^K} \left(\frac{k-1}{2^K} + \alpha \sup_{i \in I} d(x, \overline{f_i^{-1}(\lfloor k/2^K, +\infty])} \right), K \right) \\
&\quad \text{by Lipschitz-regularity (Lemma 4.2 (2))} \\
&= \sup_{i \in I} \min \left(\min_{k=1}^{K2^K} \left(\frac{k-1}{2^K} + \alpha d(x, \overline{f_i^{-1}(\lfloor k/2^K, +\infty])} \right), K \right) \\
&= \sup_{i \in I} (f_i)_K^{(\alpha)}(x)
\end{aligned}$$

since multiplication by α , addition, and min are Scott-continuous. Using Proposition 5.3, it follows that $f^{(\alpha)}(x) = \sup_{K \in \mathbb{N}} \sup_{i \in I} (f_i)_K^{(\alpha)}(x) = \sup_{i \in I} \sup_{K \in \mathbb{N}} (f_i)_K^{(\alpha)}(x) = \sup_{i \in I} f_i^{(\alpha)}(x)$.

(2) \Rightarrow (3): obvious.

(3) \Rightarrow (1). (3) applies notably to the family of maps $r\chi_{U_i}$, where $(U_i)_{i \in I}$ is an arbitrary directed family of open subsets of X , and $r \in \mathbb{R}_+$. Let $U = \bigcup_{i \in I} U_i$, so that $\sup_{i \in I} r\chi_{U_i} = r\chi_U$. Then (3) entails that $(r\chi_U)^{(\alpha)} = \sup_{i \in I} (r\chi_{U_i})^{(\alpha)}$. This means that for every $x \in X$, $\min(r, \alpha d(x, \bar{U})) = \sup_{i \in I} \min(r, \alpha d(x, \bar{U}_i)) = \min(r, \alpha \sup_{i \in I} d(x, \bar{U}_i))$. Since r is arbitrary, we make it tend to $+\infty$, leaving x fixed. We obtain that $\alpha d(x, \bar{U}) = \alpha \sup_{i \in I} d(x, \bar{U}_i)$, and since $\alpha > 0$, that $d(x, \bar{U}) = \sup_{i \in I} d(x, \bar{U}_i)$. Hence X, d is Lipschitz regular by Lemma 4.2 (2). \square

Equip $\mathcal{L}X$ with its Scott topology, and $\mathcal{L}_\alpha(X, d)$ with the subspace topology (not the Scott topology!). Then:

Corollary 5.5. *Let $\alpha > 0$, and X, d be a Lipschitz regular standard quasi-metric space. Then the canonical injection $i_\alpha: \mathcal{L}_\alpha(X, d) \rightarrow \mathcal{L}X$ and the map $r_\alpha: f \in \mathcal{L}X \mapsto f^{(\alpha)} \in \mathcal{L}_\alpha(X, d)$ form an embedding-projection pair, viz., r_α and i_α are continuous, $r_\alpha \circ i_\alpha = \text{id}_{\mathcal{L}_\alpha X}$ and $i_\alpha \circ r_\alpha \leq \text{id}_{\mathcal{L}X}$.*

PROOF. We know that i_α is continuous (by definition of the subspace topology), the equalities $r_\alpha \circ i_\alpha = \text{id}_{\mathcal{L}_\alpha X}$ and $i_\alpha \circ r_\alpha \leq \text{id}_{\mathcal{L}X}$ are clear, and r_α is Scott-continuous by Proposition 5.4. Recall however that the topology we have taken on $\mathcal{L}_\alpha(X, d)$ is not the Scott topology. In order to show that r_α is continuous, we therefore proceed as follows. Given any open subset V of $\mathcal{L}_\alpha(X, d)$, by definition of the subspace topology there is a Scott-open subset W of $\mathcal{L}X$ such that $V = W \cap \mathcal{L}_\alpha(X, d)$. Then $r_\alpha^{-1}(V) = r_\alpha^{-1}(W)$ is Scott-open in $\mathcal{L}X$, showing that r_α is continuous from $\mathcal{L}X$ to $\mathcal{L}_\alpha(X, d)$. \square

In the proof of Corollary 5.5, we have paid attention to the fact that the subspace topology on $\mathcal{L}_\alpha(X, d)$ might fail to coincide with the Scott topology. However, when X, d is Lipschitz regular and standard, this is unnecessary:

Proposition 5.6. *Let X, d be a Lipschitz regular standard quasi-metric space. Then the subspace topology on $\mathcal{L}_\alpha(X, d)$ induced by the Scott topology on $\mathcal{L}X$ coincides with the Scott topology.*

PROOF. r_α is Scott-continuous by Proposition 5.4 (2), and i_α is also Scott-continuous, since suprema are computed in the same way in $\mathcal{L}_\alpha(X, d)$ and in $\mathcal{L}X$. In a section-retraction pair, the section is a topological embedding, so i_α

is an embedding of $\mathcal{L}_\alpha(X, d)$, with its Scott topology, into $\mathcal{L}X$. That implies that the Scott topology on $\mathcal{L}_\alpha(X, d)$ coincides with the subspace topology. \square

A similar argument allows us to establish the following. Let $\mathcal{L}_\alpha^a(X, d)$ denote the subspace of all α -Lipschitz continuous maps from X, d to $[0, \alpha a], \mathbf{d}_\mathbb{R}$, for $\alpha \in \mathbb{R}_+$, where $a \in \mathbb{R}_+$, $a > 0$.

To show that $\min(\alpha a. \mathbf{1}, f^{(\alpha)}) \in \mathcal{L}_\alpha^a(X, d)$, we use Proposition 2.8 (3) and (6), which state that the pointwise min of two α -Lipschitz continuous maps is α -Lipschitz continuous and that constant maps are α -Lipschitz continuous. We write $\mathbf{1}$ for the constant map equal to 1.

Corollary 5.7. *Let $\alpha > 0$, $a > 0$, and X, d be a Lipschitz regular standard quasi-metric space. Then the canonical injection $i_\alpha^a: \mathcal{L}_\alpha^a(X, d) \rightarrow \mathcal{L}X$ and the map $r_\alpha^a: f \in \mathcal{L}X \mapsto \min(\alpha a. \mathbf{1}, f^{(\alpha)}) \in \mathcal{L}_\alpha^a(X, d)$ form an embedding-projection pair, viz., r_α^a and i_α^a are continuous, $r_\alpha^a \circ i_\alpha^a = \text{id}_{\mathcal{L}_\alpha^a X}$ and $i_\alpha^a \circ r_\alpha^a \leq \text{id}_{\mathcal{L}X}$. \square*

Remark 5.8. *As in Proposition 5.6, this also shows that, when X, d is Lipschitz regular and standard, the subspace topology (induced by the inclusion into $\mathcal{L}X$) coincides with the Scott topology on $\mathcal{L}_\alpha^a(X, d)$.*

At the risk of repeating ourselves, those results hold, in particular: (1) on all standard algebraic quasi-metric spaces with relatively compact balls, a case that includes all metric spaces whose closed balls are compact; (2) on all \mathbf{B} -algebras, and therefore on all spaces of formal balls $\mathbf{B}(X, d)$ with the d^+ quasi-metric.

References

- [1] Bonsangue, M. M., van Breugel, F., Rutten, J. J. M. M., 1998. Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding. *Theoretical Computer Science* 193, 1–51.
- [2] Edalat, A., Heckmann, R., 1998. A computational model for metric spaces. *Theoretical Computer Science* 193, 53–73.
- [3] Escardó, M. H., 1998. Properly injective spaces and function spaces. *Topology and Its Applications* 89 (1–2).

- [4] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., Scott, D. S., 2003. Continuous lattices and domains. In: Encyclopedia of Mathematics and its Applications. Vol. 93. Cambridge University Press.
- [5] Goubault-Larrecq, J., 2013. Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology. Vol. 22 of New Mathematical Monographs. Cambridge University Press.
- [6] Goubault-Larrecq, J., Jul. 2017. Complete quasi-metrics for hyperspaces, continuous valuations, and previsions, arXiv 1707.03784, v2, <https://arxiv.org/abs/1707.03784>.
- [7] Goubault-Larrecq, J., Ng, K. M., Oct. 2017. A few notes on formal balls. Logical Methods in Computer Science Special Issue of Domain Theory and Applications (the Domains XII Workshop), Cork, Ireland, August 2015. Accepted; arXiv:1606.05445 [math.GN], version 5, October 2017.
- [8] Kock, A., 1972. Strong functors and monoidal monads. Archiv der Mathematik XXIII, 113–120.
- [9] Kock, A., 1995. Monads for which structures are adjoint to units. Journal of Pure and Applied Algebra 104, 41–59, version 3.
- [10] Kostanek, M., Waszkiewicz, P., 2010. The formal ball model for \mathcal{Q} -categories. Mathematical Structures in Computer Science 21 (1), 1–24.
- [11] Moggi, E., 1991. Notions of computation and monads. Information and Computation 93, 55–92.
- [12] Weihrauch, K., Schreiber, U., 1981. Embedding metric spaces into cpo's. Theoretical Computer Science 16 (1), 5–24.