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STEPAanova TYPE $\mu$-PSEUDO ALMOST AUTOMORPHIC MILD SOLUTIONS OF SEMILINEAR FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

J. VANTERLER DA C. SOUSA *, G. M. N'GUIERkATA

Abstract. In this paper, using necessary and sufficient conditions, the new concept of Stepanov $\mu$-pseudo almost automorphic functions and ergodicity results, we investigate the existence of mild bounded solutions for a class of fractional integro-differential equations in the sense of the Weyl fractional derivative in a Banach space.

1. Introduction

The concept of almost automorphic functions was introduced by S. Bochner [3] in the early sixties while working on some problems in differential geometry. It turns out that it generalizes the concept of almost periodic functions in the sense of H. Bohr. Then came the concept of pseudo-almost automorphic function, which is a natural generalization of that of almost automorphic function. In this sense, N’Guèrekata and Pankov [16] introduced the concept of Stepanov-like almost automorphic. Many important concepts and new generalizations over the years have been discussed and consequently, numerous discussions on problems of existence of mild bounded solutions [5, 6, 7, 8, 9, 12, 10, 11, 18].

Blot et al. [4] established a new concept of weighted pseudo almost automorphic functions using the measure theory and investigated many interesting properties of such functions. Weighted pseudo almost automorphic functions have been studied recently and have become an interesting field. In addition, this paper by Blot et al. [4], made clear the new and general concept about automorphic function, that is, the pseudo almost automorphic function is a $\mu$-pseudo almost automorphic function in the particular case where the measure $\mu$ is the Lebesgue measure.

In 2011 Lizama and Ponce [39], investigated the existence, uniqueness and regularity of solutions for the following integro-differential equations given by

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s) \, ds + f(t, u(t))$$

where $\alpha, \beta \in \mathbb{R}$, $A : D(A) \subset X \rightarrow X$ is a closed linear operator defined on a Banach space $X$, and $f$ belongs to a closed subspace of the space of continuous and bounded functions. The results were discussed when $f(\cdot, \cdot)$ and $u$ are almost periodic (resp. almost automorphic), asymptotically periodic (resp. almost periodic), pseudo-almost periodic (resp. almost automorphic).

Em 2015 Chang et al. [5], discussed the new existence results of mild solutions via concept of Stepanov type $\mu$-pseudo almost automorphic functions to a semilinear integro-differential equations given by

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s) \, ds + f(t, u(t))$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, $\alpha \neq 0$ and $\alpha + \beta > 0$, $A : D(A) \subset X \rightarrow X$ is the generator of an immediately norm continuous $C_0$-semigroup defined on the Banach space $X$, and $f : \mathbb{R} \times X \rightarrow X$ belongs to a closed subspace of the space of continuous and bounded functions satisfying some Lipschitz type conditions. In 2018 Ezzinib et al. [40], investigated weak almost periodic solutions for class of integro-differential equations of the form Eq.(1.1) with conditions on $A$, $f$, $u$ and $\alpha, \beta$.

The theory of fractional differential and integro-differential equations has been used to describe physical and biological phenomena [51, 52, 53, 54] and references therein. In addition, investigating the properties of existence, uniqueness, stability and attractivity of solutions (classic, mild and strong), has been gaining increasing prominence in the scientific community [45, 46, 47, 48, 49, 50] and references therein.

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In 2013 Ponce [17] investigated the existence and uniqueness of bounded solutions for the semilinear fractional differential equation

\[ D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s) \, ds + f(t, u(t)), \quad t \in \mathbb{R} \]

where \( A \) is a closed linear operator defined on a Banach space \( X, \alpha > 0, a \in L^1(\mathbb{R}_+) \) is a scalar-valued kernel and \( f : \mathbb{R} \times X \to X \) satisfies some Lipschitz type conditions. In this work, Ponce established sufficient conditions for the existence and uniqueness of an almost periodic, almost automorphic and asymptotically almost periodic solution, among others. Other interesting results can be obtained in the following references \([30, 31, 32, 33, 34, 35, 36, 37, 38, 41, 42, 43]\).

On the other hand, Xia [29] investigated the existence and uniqueness of a pseudo almost periodic \( PC \)-mild solution for the impulsive fractional integro-differential equations involving Caputo fractional derivative in a Banach space given by

\[ ^cD^\alpha u(t) + Au(t) = f(t, u(t)) + (Ku)(t) + \sum_{k=-\infty}^{\infty} G_k(u(t))\delta(t-\tau_k) \]

where \((Ku)(t) = \int_{-\infty}^{t} k(t-s)g(s, u(s)) \, ds. 0 < \alpha \leq 1, -A : D(A) \subset X \to X \) is a linear infinitesimal operator of an analytic semigroup \( S(t) \), \( f, g \) are pseudo almost periodic in \( t \in \mathbb{R} \) uniformly in the second variable, \( G_k : D(G_k) \subset X \to X \) are continuous impulsive operators, \( \delta(\cdot) \) is the Dirac’s delta-function, \( \tau_k \in T \), where \( T \) will be defined later. Here the fractional derivative is understood in Caputo’s sense. Other results on about almost periodic, almost automorphic, asymptotically almost periodic involving fractional differential and integrodifferential equations, for example, can be obtained \([19, 24, 21, 22, 23, 20, 25, 26, 27, 30]\).

Inspired by above questions and works, we consider in this paper the following semilinear fractional integrodifferential equation

\[ W^-_{-\infty}D^\alpha_t u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s) \, ds + f(t, u(t)), \quad t \in \mathbb{R} \]

where \( W^-_{-\infty}D^\alpha_t (\cdot) \) is Weyl fractional derivative of order \( 0 < \alpha < 1 \), \( A : D(A) \subset \Omega \to \Omega \) is the generator of an \( \alpha \)-resolvent family \( \{T_\alpha(t)\}_{t \geq 0} \) which is uniformly integrable on the Banach space \( \Omega \), and \( f : \mathbb{R} \times \Omega \to \Omega \) belongs to a closed subspace of the space of continuous and bounded functions satisfying some Lipschitz type conditions. Here, we will impose that \( f : \mathbb{R} \times X \to X \) is Stepanov type \( \mu \)-pseudo almost automorphic.

Some particular cases of the choice of \( \alpha \) and \( a(t-s) \) are as follows:

1. Taking \( \alpha = 1 \) in Eq.(1.2), we have
   \[ u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s) \, ds + f(t, u(t)). \]

2. For \( a(t-s) = e^{-\beta(t-s)} \) in Eq.(1.2), we have
   \[ W^-_{-\infty}D^\alpha_t u(t) = Au(t) + \int_{-\infty}^{t} e^{-\beta(t-s)}Au(s) \, ds + f(t, u(t)). \]

3. For \( a(t-s) = \lambda e^{-\beta(t-s)} \) in Eq.(1.2), we have
   \[ W^-_{-\infty}D^\alpha_t u(t) = Au(t) + \lambda \int_{-\infty}^{t} e^{-\beta(t-s)}Au(s) \, ds + f(t, u(t)). \]

4. Taking \( \alpha = 1 \) and \( a(t-s) = \lambda e^{-\beta(t-s)} \) in Eq.(1.2), we have
   \[ u'(t) = Au(t) + \lambda \int_{-\infty}^{t} e^{-\beta(t-s)}Au(s) \, ds + f(t, u(t)). \]

The main objective of this paper is to investigate the existence of a new class of bounded mild solutions (namely the concept of Stepanov type \( \mu \)-pseudo almost automorphic functions) to a semilinear fractional integrodifferential equations given by Eq.(1.2), by means of results of ergodicity and composition theorems of Stepanov type \( \mu \)-pseudo almost automorphic functions.

In addition to the particular cases presented above, from the choice of \( a(t-s) \) and \( \alpha = 1 \), as the results are obtained for Stepanov \( \mu \)-pseudo almost automorphic functions, there are also particular cases, for example, when \( \mu \) is a Lebesgue measure.
To prove our results, we will make the following assumptions:

\((T_1)\) Assume that \(A\) generates an \(\alpha\)-resolvent family \(\{T_\alpha(t)\}_{t \geq 0}\) such that \(\|T_\alpha(t)\| \leq \varphi_\alpha(t)\) for all \(t \geq 0\) where \(\varphi_\alpha(\cdot) \in L^1(\mathbb{R}_+)\) is nonincreasing such that \(\varphi_0 := \sum_{n=0}^{\infty} \varphi_n(n) < \infty\).

\((T_2)\) Assume that \(f \in \mathcal{PAA}^p(\mathbb{R} \times \Omega, \Omega, \mu)\) and there exists a positive number \(L_f\) such that

\[
\|f(t, \zeta) - f(t, \eta)\| \leq L_f \|\zeta - \eta\|
\]

for all \(t \in \mathbb{R}\) and each \(\zeta, \eta \in \Omega\).

\((T_3)\) Suppose that \(f \in \mathcal{PAA}^p(\mathbb{R} \times \Omega, \Omega, \mu)\) and there exists a nonnegative function \(L_f(\cdot) \in \mathcal{B}^S^p(\mathbb{R})\), with \(p > 1\) such that

\[
\lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} L_f(t) d\mu(t) < \infty
\]

for all \(t \in \mathbb{R}\) and each \(x, y \in \Omega\).

\((T_4)\) The function \(f = g + h \in \mathcal{PAA}^p(\mathbb{R} \times \Omega, \Omega, \mu)\) with \(g \in \mathcal{AS}^p(\mathbb{R} \times \Omega, \Omega)\), \(h^p \in \mathcal{C}^p(\Omega, L^p(0, 1; \Omega), \mu)\) and there exists nonnegative functions \(L_f(\cdot), L_g(\cdot) \in \mathcal{AS}^{\kappa}(\mathbb{R}, \mathbb{R})\) with \(\kappa \geq \max \left\{p, \frac{p}{p-1}\right\}\) such that for all \(u, v \in \Omega\) and \(t \in \mathbb{R}\)

\[
\|f(s, u) - f(s, v)\| \leq L_f(t) \|u - v\|
\]

\[(1.6)\]

\[
\|g(s, u) - g(s, v)\| \leq L_g(t) \|u - v\|
\]

\[(1.7)\]

\((T_5)\) The functions \(f = g + h \in \mathcal{PAA}^p(\mathbb{R} \times \Omega, \Omega, \mu)\) satisfy \(g \in \mathcal{AS}^p(\mathbb{R} \times \Omega, \Omega)\) uniformly in \(t \in \mathbb{R}\) and \(h^p \in \mathcal{C}^p(\Omega, L^p(0, 1; \Omega), \mu)\) and for every bounded subset \(M \subset \Omega\), \(\{f(\cdot, \zeta) : \zeta \in M\}\) is bounded \(\mathcal{PAA}^p(\mathbb{R} \times \Omega, \Omega, \mu)\).

In the rest, the article is organized as follows. In Section 2, we present some definitions and results that are essential for the development of this paper. In Section 3, the main result of this paper, that is, we investigated the existence of mild bounded solutions for a fractional integro-differential equation class in the sense of the Weyl derivative in the Banach space, by means of necessary and sufficient conditions, of Stepanov concept \(\mu\)-pseudo almost automorphic functions and results ergodicity.

2. PRELIMINARIES

In this section, we will present some essential definitions and results throughout the paper.

Let \((\Omega, \|\cdot\|)\) and \((\Lambda, \|\cdot\|)\) be two Banach spaces and let \(BC(\mathbb{R}, \Omega)\) denote the Banach space of all bounded continuous functions from \(\mathbb{R}\) to \(\Omega\), equipped with the supremum norm \(\|f\|_\infty = \sup_{t \in \mathbb{R}} \|f(t)\|\). The notation \(BC(\Omega, \Lambda)\) stands for the space of bounded linear operator topology, and we abbreviate to \(B(\Omega)\), whenever \(X = \Lambda\). Throughout this work, we denote by \(B\) the Lebesgue \(\sigma\)-field of \(\mathbb{R}\) and by \(M\) the set of all positive measures \(\mu\) on \(B\) satisfying \(\mu(\mathbb{R}) = +\infty\) and \(\mu([a, b]) = a < b\), for all \(a, b \in \mathbb{R}\).

**Definition 2.1.** [15] A continuous function \(f : \mathbb{R} \to \Omega\) is said to be almost automorphic if for every sequence of real numbers \(\{s_n\}_{n \in \mathbb{N}}\) there exists a subsequence \(\{s_n\}_{n \in \mathbb{N}}\) such that \(g(t) := \lim_{n \to \infty} f(t + s_n)\) is well defined for each \(t \in \mathbb{R}\), and

\[
\lim_{n \to \infty} g(t - s_n) = f(t)
\]

for each \(t \in \mathbb{R}\). The collection of all such functions will be denoted by \(\mathcal{AA}(\mathbb{R}, \Omega)\).

**Definition 2.2.** [15, 13] A continuous function \(f : \mathbb{R} \times \Omega \to \Omega\) is said to be almost automorphic if \(f(t, \zeta)\) is almost automorphic for each \(t \in \mathbb{R}\) uniformly for all \(\zeta \in \Omega\), where \(\mathcal{B}\) is any bounded subset of \(\Omega\). The collection of all such functions will be denoted by \(\mathcal{AA}(\mathbb{R} \times \Omega, \Omega)\).

**Definition 2.3.** [14] The set of all bounded continuous functions with vanishing mean value can be defined as

\[
\mathcal{AA}_0(\mathbb{R}, \Omega) = \left\{ \phi \in BC(\mathbb{R}, \Omega) : \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\|\,d\sigma = 0 \right\}.
\]
Similarly, we define by \( \mathcal{A}_0(\mathbb{R} \times \Lambda \times \Lambda, \Omega) \) the set of all continuous functions \( f : \mathbb{R} \times \Lambda \times \Lambda \to \Omega \) which belong to \( \mathcal{B}C(\mathbb{R} \times \Lambda \times \Lambda, \Omega) \) and satisfy

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \| f(\zeta, \eta) \| \, d\sigma = 0
\]

uniformly for \((\zeta, \eta)\) in any bounded subset of \(\Lambda \times \Lambda\).

**Definition 2.4.** [4] Let \( \mu \in M \). A bounded continuous function \( f : \mathbb{R} \to \Omega \) is said to be \( \mu \)-ergodic if

\[
\lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \| f(t) \| \, d\mu(t) = 0.
\]

We denote the space of all such functions by \( \mathcal{E}(\mathbb{R}, \Omega, \mu) \) (or \( \mathcal{E}(\Omega, \mu) \) for abbreviation).

**Definition 2.5.** [4] Let \( \mu \in M \). A continuous function \( f : \mathbb{R} \times \Lambda \to \Omega \) is said to be \( \mu \)-ergodic if \( f(\cdot, \eta) \) is bounded for each \( \eta \in \Lambda \) and

\[
\lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \| f(t, \eta) \| \, d\mu(t) = 0
\]

uniformly in \( \eta \in \Lambda \). We denote the set of all such functions by \( \mathcal{E}(\mathbb{R} \times \Lambda, \Omega, \mu) \) (or \( \mathcal{E}(\Lambda, \Omega, \mu) \) for abbreviation).

**Definition 2.6.** [4] Let \( \mu \in M \). A continuous function \( f : \mathbb{R} \to \Omega \) is said to be \( \mu \)-pseudo almost automorphic if \( f \) is written in the form:

\[
f = g + \phi
\]

where \( g \in \mathcal{A}_0(\mathbb{R}, \Omega) \) and \( \phi \in \mathcal{E}(\mathbb{R}, \Omega, \mu) \). Let \( \mathcal{PAA}(\mathbb{R}, \Omega, \mu) \) denote the space of all such functions.

**Definition 2.7.** [4] Let \( \mu \in M \). A continuous function \( f : \mathbb{R} \times \Lambda \to \Omega \) is said to be \( \mu \)-pseudo almost automorphic if \( f \) is written in the form:

\[
f = g + \phi
\]

where \( g \in \mathcal{A}_0(\mathbb{R} \times \Lambda, \Omega) \) and \( \phi \in \mathcal{E}(\mathbb{R} \times \Lambda, \Omega, \mu) \).

**Definition 2.8.** [4] Let \( \mu \in M \). Then \((\mathcal{E}(\mathbb{R}, \Omega, \mu), \| \cdot \|_\infty)\) is a Banach space.

For \( \mu \in M \) and \( \tau \in \mathbb{R} \), we denote \( \mu_{\tau} \) the positive measure on \((\mathbb{R}, \mathcal{B})\) defined by

\[
\mu_{\tau}(A) = \mu(a + \tau : a \in A)
\]

for \( A \in \mathcal{B} \).

From \( \mu \in M \), we state the following hypothesis.

\( (H_0) \) For all \( \tau \in \mathbb{R} \), there exist \( \gamma > 0 \) and a bounded interval \( I \) such that

\[
\mu_{\tau}(A) \leq \gamma \mu(A)
\]

when \( A \in \mathcal{B} \) satisfies \( A \cap I = \phi \).

**Lemma 2.9.** [4] Let \( \mu \in M \) satisfy \((H_0)\), then \( \mathcal{E}(\mathbb{R}, \Omega, \mu) \) is translation invariant, therefore, \( \mathcal{PAA}(\mathbb{R}, \Omega, \mu) \) is also translation invariant.

**Lemma 2.10.** [4] Let \( \mu \in M \). Assume that \( \mathcal{PAA}(\mathbb{R}, \Omega, \mu) \) is translation invariant. Then the decomposition of a \( \mu \)-pseudo almost automorphic function in the form \( f = g + \phi \) where \( g \in \mathcal{A}_0(\mathbb{R}, \Omega) \) and \( \phi \in \mathcal{E}(\mathbb{R}, \Omega, \mu) \) is unique.

**Lemma 2.11.** [4] Let \( \mu \in M \). Assume that \( \mathcal{PAA}(\mathbb{R}, \Omega, \mu) \) is translation invariant. Then \( \mathcal{PAA}(\mathbb{R}, \Omega, \mu; \| \cdot \|_\infty) \) is a Banach space.

**Definition 2.12.** [9, 16] The Bochner transform \( f^b(t, s), t \in \mathbb{R}, s \in [0, 1] \) of a solution \( f : \mathbb{R} \to \Omega \) is defined by

\[
f^b(t, s) := f(t + s).
\]

**Definition 2.13.** [9, 16] Let \( p \in [1, \infty) \). The space \( BS^p(\Omega) \) of all Stepanov bounded functions, with the exponent \( p \), consists of all measurable functions \( f : \mathbb{R} \to \Omega \) such that \( f^b \in L^\infty(\mathbb{R}, L^p(0, 1; \Omega)) \). This a Banach space with the norm

\[
\| f \|_{BS^p} = \| f^b \|_{L^\infty(\mathbb{R}, L^p(0, 1; \Omega))} = \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \| f(\tau) \|^p \, d\tau \right)^{1/p}.
\]
Definition 2.14. [16] The space $AS^p(\Omega)$ of Stepanov type almost automorphic (or $S^p$-almost automorphic) functions consists of all $f \in BS^p(\Omega)$ such that $f^b \in AA(L^p(0,1;\Omega))$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\Omega)$ is said to be $S^p$-almost automorphic if its Bochner transform $f^b : \mathbb{R} \rightarrow L^p(0,1;\Omega)$ is almost automorphic in the sense that, for every sequence of real numbers $\{s'_n\}_{n \in \mathbb{N}}$ there exist a subsequence $\{s_n\}_{n \in \mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R},\Omega)$ such that

$$
\lim_{n \to \infty} \left( \int_t^{t+1} \|f(s+s_n) - g(s)\|^p \, ds \right)^{1/p} = 0
$$

and

$$
\lim_{n \to \infty} \left( \int_t^{t+1} \|g(s+s_n) - f(s)\|^p \, ds \right)^{1/p} = 0
$$

pointwise on $\mathbb{R}$.

Definition 2.15. [16] A function $f : \mathbb{R} \times \Lambda \rightarrow \Omega$, $(t,u) \mapsto f(t,u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R},\Omega)$, for each $u \in \Lambda$, is said to be $S^p$-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \Lambda$ if $t \mapsto f(t,u)$ is $S^p$-almost automorphic for each $u \in \Lambda$. That means, for every sequence of real numbers $\{s'_n\}_{n \in \mathbb{N}}$ there exists a subsequence $\{s_n\}_{n \in \mathbb{N}}$ and a function $g(\cdot, u) \in L^p_{loc}(\mathbb{R},\Omega)$ such that

$$
\lim_{n \to \infty} \left( \int_t^{t+1} \|f(s+s_n,u) - g(s,u)\|^p \, ds \right)^{1/p} = 0
$$

and

$$
\lim_{n \to \infty} \left( \int_t^{t+1} \|g(s+s_n,u) - f(s,u)\|^p \, ds \right)^{1/p} = 0
$$

pointwise on $\mathbb{R}$ and for each $u \in \Lambda$. We denote by $AS^p(\mathbb{R} \times \Lambda,\Omega)$ the set of all such functions.

Definition 2.16. [7] A function $f \in BS^p(\Omega)$ is said to be Stepanov type pseudo almost automorphic if it can be decomposed as $f = g + \phi$ where $g \in AS^p(\Omega)$ and $\phi \in AA_0(\mathbb{R},L^p(0,1;\Omega))$. Denote by $PAA$ the set of all functions.

Definition 2.17. [6] Let $\mu \in M$. A function $f \in BS^p(\Omega)$ is said to be Stepanov type $\mu$-pseudo almost automorphic (or $S^p - \mu$-pseudo almost automorphic) if it can be expressed as $f = g + \phi$, where $g \in AS^p(\Omega)$ and $\phi \in \mathcal{E}(L^p(0,1;\Omega),\mu)$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\Omega)$ is said to be Stepanov type $\mu$-pseudo almost automorphic relatively to the measure $\mu$, if its Bochner transform $f^b : \mathbb{R} \rightarrow L^p(0,1;\Omega)$ is $\mu$-pseudo almost automorphic in the sense that there exist two functions $g, \phi : \mathbb{R} \rightarrow \Omega$ such that $f = g + \phi$ where $g \in AS^p(\Omega)$ and $\phi \in \mathcal{E}(L^p(0,1;\Omega),\mu)$, that is, $\phi \in BC(L^p(0,1;\Omega))$ and

$$
\lim_{r \to \infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left( \int_t^{t+1} \|\phi(s)\|^p \, ds \right)^{1/p} \, \mu(dt) = 0.
$$

The set of all such functions will be denoted by $PAA^p(\mathbb{R},\Omega,\mu)$.

Definition 2.18. [6] Let $\mu \in M$. A function $f : \mathbb{R} \times \Lambda \rightarrow \Omega$, $(t,u) \mapsto f(t,u)$ with $f(\cdot, u) \in L^p_{loc}(\mathbb{R},\Omega)$ for each $u \in \Lambda$ is said to be Stepanov type $\mu$-pseudo almost automorphic (or $S^p - \mu$-pseudo almost automorphic) if it can be expressed as $f = g + \phi$, where $g \in AS^p(\mathbb{R} \times \Lambda,\Omega)$ and $\phi \in \mathcal{E}(\Lambda,L^p(0,1;\Omega),\mu)$. We denote by $PAA^p(\mathbb{R} \times \Lambda,\Omega,\mu)$ the set of all such functions.

Lemma 2.19. [6] Let $\mu \in M$ and $I$ be a bounded interval (eventually $I = \emptyset$). Assume that $f(\cdot) \in BS^p(\mathbb{R},\Omega)$. Then the following assertions are equivalent:

1. $f^b(\cdot) \in \mathcal{E}(L^p(0,1;\Omega),\mu)$;
2. $$
\lim_{r \to +\infty} \frac{1}{\mu([-r,r]\setminus I)} \int_{[-r,r]\setminus I} \left( \int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} \, \mu(dt) = 0.
$$
3. For any $\varepsilon > 0$,

$$
\mu \left( \left\{ t \in [-r,r]\setminus I : \left( \int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} > \varepsilon \right\} \right) = 0.
$$

Lemma 2.20. [6] Let $\mu \in M$. Satisfy ($H_0$). Then $\mathcal{E}(L^p(0,1;\Omega),\mu)$ is translation invariant, therefore $PAA^p(\mathbb{R},\Omega,\mu)$ is also translation invariant.
Lemma 2.21. [6] Let $\mu \in M$ satisfy $(H_0)$. If $f \in \mathcal{P.AA}(\mathbb{R}, \Omega, \mu)$ then $f \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ for each $1 \leq p < \infty$. In other words $\mathcal{P.AA}(\mathbb{R}, \Omega, \mu) \subseteq \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$. Thus we have $\mathcal{A}(\mathbb{R}, \Omega) \subseteq \mathcal{P.AA}(\mathbb{R}, \Omega, \mu) \subseteq \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$.

Thus, we have $\mathcal{A}(\mathbb{R}, X) \subseteq \mathcal{P.AA}(\mathbb{R}, X, \mu) \subseteq \mathcal{P.AA}^p(\mathbb{R}, X, \mu)$.

Lemma 2.22. [6] Let $\mu \in M$ and $f \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ be such that $f = g + \chi$, where $g \in \mathcal{A}^p(\Omega)$ and $\chi \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$. If $\mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ is translation invariant, then
\[
\{g(t) : t \in \mathbb{R}\} \subseteq \{f(t) : t \in \mathbb{R}\}
\]

(the closure of range $f$).

Lemma 2.23. [6] Let $\mu \in M$. Assume that $\mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ is translation invariant. Then $(\mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu), \|, \|_{S^p})$ is a Banach space.

Lemma 2.24. [6] Let $\mu \in M$. Suppose that $f = g + h \in \mathcal{P.AA}(\mathbb{R} \times \Omega, \Omega, \mu)$ with $g \in \mathcal{A}^p(\mathbb{R} \times \Omega, \Omega)$, $h \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$ and satisfies the following condition:

$(H_1)$ There exists a constant $L > 0$ such that, for all $x, y \in \Omega$ and $t \in \mathbb{R}$,
\[
\|f(t, x) - f(t, y)\| \leq L \|x - y\|.
\]

If $v = v_1 + v_2 \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ with $v_1 \in \mathcal{A}^p(\Omega), v_2 \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$ and $K_1 = \{v_1(t) : t \in \mathbb{R}\}$ is compact. Then $f(\cdot, v(\cdot)) \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$.

Lemma 2.25. [6] Let $\mu \in M$ and $f = g + h \in \mathcal{P.AA}^p(\mathbb{R} \times \Omega, \Omega, \mu)$ with $g \in \mathcal{A}^p(\mathbb{R} \times \Omega, \Omega)$, $h \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$. Assume that the following conditions are satisfied:

1. There exists a nonnegative function $L(\cdot) \in \mathcal{B}^p(\mathbb{R})$ with $p > 1$ such that, for all $\zeta, \eta \in \Omega$ and $t \in \mathbb{R}$,
\[
\left(\int_t^{t+1} \|f(s, \zeta) - f(s, \eta)\| ds\right)^{1/p} < L(t) \|\zeta - \eta\|,
\]

2. $g(t, \cdot)$ is uniformly continuous in any bounded subset $K' \subseteq \Omega$ uniformly for $t \in \mathbb{R}$. If $u = u_1 + u_2 \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ with $u_1 \in \mathcal{A}^p(\Omega), u_2 \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$ and $K_2 = \{u_1(t) : t \in \mathbb{R}\}$ is compact then $f(\cdot, u(\cdot)) \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$.

Lemma 2.26. [6] Let $\mu \in M$ and $f = g + \phi \in \mathcal{P.AA}^p(\mathbb{R} \times \Omega, \Omega, \mu)$ with $g \in \mathcal{A}^p(\mathbb{R} \times \Omega, \Omega)$, $\phi \in \mathcal{E}(\Omega, L^p(0, 1; \Omega), \mu)$. Assume that following conditions hold:

1. $f(t, \cdot)$ is uniformly conditions in any bounded subset $K' \subseteq \Omega$ uniformly for $t \in \mathbb{R}$.

2. $g(t, \cdot)$ is uniformly continuous in any bounded subset $K' \subseteq \Omega$ uniformly for $t \in \mathbb{R}$.

3. For any bounded subset $K' \subseteq \Omega$, $\{f(\cdot, \zeta) : \zeta \in K'\}$ is bounded in $\mathcal{P.AA}^p(\mathbb{R} \times \Omega, \Omega, \mu)$. If $\zeta = v_1 + v_2 \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$, $v_1 \in \mathcal{A}^p(\Omega), v_2 \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$ and $Q = \{\zeta(t) : t \in \mathbb{R}\}$, $Q_1 = \{v_1(t) : t \in \mathbb{R}\}$ are compact then $f(\cdot, \zeta(\cdot)) \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$.

Theorem 2.27. [6] Let $\mu \in M$, $p > 1$ and $f = g + \chi \in \mathcal{P.AA}^p(\mathbb{R} \times \Omega, \Omega, \mu)$ with $g \in \mathcal{A}^p(\mathbb{R} \times \Omega, \Omega), \chi \in \mathcal{E}(\Omega, L^p(0, 1; \Omega), \mu)$. Assume that the following conditions are satisfied:

1. There exists nonnegative functions $L_f(\cdot), L_g(\cdot) \in \mathcal{A}^p(\mathbb{R} \times \mathcal{R})$ with $\kappa \geq \max\{p, \frac{p}{p-1}\}$ such that, for all $u, v \in \Omega$ and $t \in \mathbb{R}$,
\[
\|f(s, u) - f(s, v)\| \leq L_f(t) \|u - v\|,
\]
\[
\|g(s, u) - g(s, v)\| \leq L_g(t) \|u - v\|.
\]

2. $u = u_1 + u_2 \in \mathcal{P.AA}^p(\mathbb{R}, \Omega, \mu)$ with $u_1 \in \mathcal{A}^p(\Omega), u_2 \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$ and $K_3 = \{u_1(t) : t \in \mathbb{R}\}$ is compact in $X$. Then there exists $q \in [1, p)$ such that $F : \mathbb{R} \to \Omega$ defined by $F(\cdot) = (f(\cdot, u(\cdot)) \in \mathcal{P.AA}^q(\mathbb{R}, \Omega, \mu)$.

Given a function $g : \mathbb{R} \to X$, the Weyl fractional integral of order $\alpha > 0$ is defined by $[1, 2, 17]$
\[
I_{-\infty}^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t - s)^{\alpha-1} u(s)ds, \ t \in \mathbb{R}
\]
when this integral is convergent.
On the other hand, the Weyl fractional derivative \( W_{-\infty}D_t^\alpha u \) of order \( \alpha > 0 \) is defined by [1, 2, 17]

\[
W_{-\infty}D_t^\alpha u(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)} u(t), \quad t \in \mathbb{R}
\]

where \( n = \lfloor \alpha \rfloor + 1 \).

**Definition 2.28.** [39] Let \( \Omega \) be a Banach space. A strongly continuous function \( T : \mathbb{R}_+ \to \mathcal{B}(\Omega) \) is said to be immediately norm continuous if \( T : (0, \infty) \to \mathcal{B}(\Omega) \) is continuous.

**Definition 2.29.** [17] Let \( A \) be a closed and linear operator with domain \( D(A) \) defined on a Banach space \( \Omega \) and \( \alpha > 0 \). Given \( a \in L^1_{\text{loc}}(\mathbb{R}_+) \), we say that \( A \) is the generator of an \( \alpha \)-resolvent family, if there exists \( \omega \geq 0 \) and a strongly continuous function \( T_\alpha : [0, \infty) \to \mathcal{B}(\Omega) \) such that \( \left\{ \frac{\lambda^\alpha}{1 + \tilde{a}(\lambda)} : \Re(\lambda) > 0 \right\} \subset \rho(A) \) and for all \( x \in \Omega \)

\[
(\lambda^\alpha - (1 + \tilde{a}(\lambda)) A)^{-1} x = \frac{1}{1 + \tilde{a}(\lambda)} \left( \frac{\lambda^\alpha}{1 + \tilde{a}(\lambda)} - A \right)^{-1} x
\]

\[
= \int_{0}^{\infty} e^{-\lambda T_\alpha(t)} x dt, \quad \Re(\lambda) > \omega
\]

where \( \tilde{a} \) denotes the Laplace transform of \( a \). In this case, \( \{T_\alpha(t)\}_{t \geq 0} \) is called the \( \alpha \)–resolvent family generated by \( A \).

**Remark 2.30.** [17] Observe that if \( b(t) = g_a(t) + (g_a * a)(t) \) \( t \geq 0 \), where \( g_a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) and \( (g_a * a)(t) = \int_{0}^{t} g_a(t-s) a(s) ds \), then we have that the family \( \alpha \)-resolvent \( \{T_\alpha(t)\}_{t \geq 0} \) is a \( (b, g_a) \)-regularized family. In particular, if \( a = 0 \), an 1-resolvent family is the same as a \( C_0 \)-semigroup, whereas a 2-resolvent family corresponds to the concepts of sine family. Therefore, if \( A \) is the generator of an \( \alpha \)-resolvent family \( \{T_\alpha(t)\}_{t \geq 0} \) then have that the family \( \{T_\alpha(t)\}_{t \geq 0} \) verifies the following properties:

1. \( T_\alpha(0) = g_a(0) \);
2. \( T_\alpha(t) \zeta \in D(A) \) and \( T_\alpha(t) A \zeta = AT_\alpha(t) \zeta \) for all \( \zeta \in D(A) \) and \( t \geq 0 \);
3. \( T_\alpha(t) \zeta = g_a(t) \zeta + \int_{0}^{t} b(t-s) AT_\alpha(t) \zeta ds \), for all \( x \in D(A) \) and \( t \geq 0 \);
4. \( \int_{0}^{t} b(t-s) T_\alpha(t) \zeta ds \in D(A) \) and \( T_\alpha(t) \zeta = g_a(t) \zeta + A \int_{0}^{t} b(t-s) T_\alpha(s) \zeta ds \), for all \( \zeta \in \Omega \) and \( t \geq 0 \).

Sufficient conditions implying that \( \{T_\alpha(t)\}_{t \geq 0} \subset \mathcal{B}(\Omega) \) is an \( \alpha \)-resolvent family.

Let \( \varrho(\cdot) : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( \varrho(t) \geq 1 \) for all \( t \in \mathbb{R} \) and \( \varrho(t) \to \infty \) as \( |t| \to \infty \). We consider the space [12]

\[
C_{\varrho}(\Omega) = \left\{ u \in C(\mathbb{R}, \Omega) : \lim_{|t| \to \infty} \frac{u(t)}{\varrho(t)} = 0 \right\}.
\]

Endowed with the norm \( \|u\|_{\varrho} = \sup_{t \in \mathbb{R}} \frac{|u(t)|}{\varrho(t)} \), it is a Banach space.

**Lemma 2.31.** [12] A subset \( E \subseteq C_{\varrho}(\Omega) \) is a relatively compact set if it verifies the following conditions:

(C1) The set \( E(t) = \{u(t) : u \in E\} \) is relatively compact in \( X \) for each \( t \in \mathbb{R} \).

(C2) The set \( E \) is equicontinuous.

(C3) For each \( \varepsilon > 0 \), there exists \( L > 0 \) such that \( \|u(t)\| \leq \varepsilon \varrho(t) \) for all \( u \in E \) and all \( |t| > L \).

**Lemma 2.32.** [5] (Leray-Schauder alternative theorem) Let \( D \) be a closed convex subset of a Banach space \( \Omega \) such that \( 0 \in D \). Let \( \Gamma : D \to D \) be a completely continuous map. Then the set \( \{\zeta \in D : \zeta = \lambda \Gamma(\zeta) , 0 < \lambda < 1\} \) is bounded or the map \( \Gamma \) has a fixed point in \( D \).

3. Main results

In this section, we will attack the main results of this paper, that is, new results of the existence of mild solutions for a class of fractional integrodifferential equations in the sense of the Weyl fractional derivative, through the concept of Stepanov type \( \mu \)-pseudo almost automorphic function.
**Definition 3.1.** A function $u : \mathbb{R} \to \Omega$ is said to be a mild solution to Eq.(1.2) if

\begin{equation}
(3.1) \quad u(t) = \int_{-\infty}^{t} T_{\alpha}(t-s) f(s, u(s)) \, ds
\end{equation}

for all $t \in \mathbb{R}$, where $\{T_{\alpha}(t)\}_{t \geq 0}$ is given by Remark 2.30.

**Lemma 3.2.** Let $\mu \in M$, $0 < \alpha < 1$ and $\beta > 0$, $\delta \neq 0$ with $\delta + \beta > 0$ and condition $(T_1)$ holds. If $f : \mathbb{R} \to \Omega$ is Stepanov type $\mu$-pseudo almost automorphic and $F_{\alpha}(t)$ is given by

\begin{equation}
(3.2) \quad F_{\alpha}(t) = \int_{-\infty}^{t} T_{\alpha}(t-s) f(s) \, ds, \quad t \in \mathbb{R}
\end{equation}

then $F_{\alpha} \in \mathcal{PAA}(\mathbb{R}, \Omega, \mu)$.

**Proof.** Indeed, since $f \in \mathcal{PAA}(\mathbb{R}, \Omega, \mu)$, there exists $g_1 \in \mathcal{AS}^p(\Omega)$ and $g_2 \in \mathcal{E}(L^p(0, 1; \Omega), \mu)$ such that $f = g_1 + g_2$ (see Definition 2.17). So

\[ F_{\alpha}(t) = \int_{-\infty}^{t} T_{\alpha}(t-s) g_1(s) \, ds + \int_{-\infty}^{t} T_{\alpha}(t-s) g_2(s) \, ds = \phi(t) + \psi(t), \]

where $\phi(t) = \int_{-\infty}^{t} T_{\alpha}(t-s) g_1(s) \, ds$ and $\psi(t) = \int_{-\infty}^{t} T_{\alpha}(t-s) g_2(s) \, ds$.

The proof will be discussed in two steps.

**Step 1:** $\phi(t) \in \mathcal{AA}(\Omega)$.

Consider $\phi_n^\alpha(t) = \int_{t-n}^{t-n+1} T_{\alpha}(t-s) g_1(s) \, ds$ for each $t \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$. Using the condition $(T_1)$ and H"{o}lder inequality, yields

\[ \|\phi_n^\alpha(t)\| \leq \int_{t-n}^{t-n+1} \|T_{\alpha}(t-s)\| \|g_1(s)\| \, ds \]
\[ \leq \int_{t-n}^{t-n+1} \varphi_{\alpha}(t-s) \|g_1(s)\| \, ds \]
\[ \leq \varphi_{\alpha}(n-1) \left( \int_{t-n}^{t-n+1} \|g_1(s)\|^p \, ds \right)^{1/p} \]
\[ \leq \varphi_{\alpha}(n-1) \|g_1\|_{\mathcal{S}^p}. \]

Since $\sum_{n=1}^\infty \varphi_{\alpha}(n-1) := \sum_{n=1}^\infty \varphi_{\alpha}(n) < \infty$, we denote that norm the well-known Weierstrass theorem that the series

\[ \phi(t) := \int_{-\infty}^{t} T_{\alpha}(t-s) g_2(s) \, ds = \sum_{n=1}^\infty \phi_n^\alpha(t). \]

Clearly, $x(t) \in C(\mathbb{R}, \Omega)$ and $\|\phi(t)\| \leq \sum_{n=1}^\infty \|\phi_n^\alpha(t)\| \leq \sum_{n=1}^\infty \varphi_{\alpha}(n) \|g_1\|_{\mathcal{S}^p}.$

Since $g \in \mathcal{AS}^p(\mathbb{R}, \Omega)$, then for every sequence $\{s_n\}_{n \in \mathbb{N}}$, there exists a sequence $\{s_n\}_{n \in \mathbb{N}}$ and a function $\bar{g}_1(\cdot) \in L_{loc}^p(\mathbb{R}, \Omega)$ such that for each $t \in \mathbb{R}$

\[ \lim_{m \to \infty} \left( \int_{t}^{t+1} \|g_1(s + s_m) - \bar{g}_1(s)\|^p \, ds \right)^{1/p} = 0 \]

and

\[ \lim_{m \to \infty} \left( \int_{t-n}^{t-n+1} \|\bar{g}_1(s + s_m) - g_1(s)\|^p \, ds \right)^{1/p} = 0. \]

Now, let $\tilde{\phi}_n^\alpha(t) = \int_{t-n}^{t-n+1} T_{\alpha}(s) \bar{g}(t-s) \, ds$. Then using the H"{o}lder inequality, yields

\[ \|\tilde{\phi}_n^\alpha(t + s_m) - \tilde{\phi}_n^\alpha(t)\| \leq \int_{t-n}^{t-n+1} \|T_{\alpha}(s)\| \|g_1(t + s_m - \bar{g}_1(t-s))\| \, ds \]
\[
\varphi_\alpha (n - 1) \left( \int_{t-n}^{t-n+1} \| g_1 (t + s_m - s) - \tilde{g}_1 (t - s) \| \, ds \right)^{1/p}.
\]

Note that, \( \| \phi_n^\alpha (t + s_m) - \tilde{\phi}_n^\alpha (t) \| \to 0 \) as \( m \to \infty \). Analogously, it is proved that \( \| \tilde{\phi}_n^\alpha (t + s_m) - \phi_n^\alpha (t) \| = 0 \). Thus, we conclude that each \( \phi_n^\alpha \in \mathcal{AA} (\Omega) \) and consequently their uniform limit \( \phi \in \mathcal{AA} (\Omega) \).

**Step 2.** \( \psi (t) \in \mathcal{E} (\mathbb{R}, \Omega, \mu) \).

Consider \( \psi_n^\alpha (t) = \int_{t-n}^{t-n+1} T_\alpha (t - s) g_2 (s) \, ds \) for each \( t \in \mathbb{R} \) and \( n = 1, 2, \ldots \). Again, using the condition \((T_1)\) and Holder inequality, yields

\[
\| \psi_n^\alpha (t) \| \leq \int_{t-n}^{t-n+1} \| T_\alpha (t - s) \| \| g_2 (s) \| \, ds
\leq \int_{t-n}^{t-n+1} \varphi_\alpha (t - s) \| g_2 (s) \| \, ds
\leq \varphi_\alpha (n - 1) \left( \int_{t-n}^{t-n+1} \| g_2 (s) \|^p \, ds \right)^{1/p}.
\]

Then, for \( r > 0 \), we have

\[
\frac{1}{\mu ([t-r,t])} \int_{[t-r,t]} \| \psi_n^\alpha (t) \| \, d\mu (t) \leq \frac{\varphi_\alpha (n - 1)}{\mu ([t-r,t])} \int_{[t-r,t]} \left( \int_{t-n}^{t-n+1} \| g_2 (s) \|^p \, ds \right)^{1/p} \, d\mu (t).
\]

Since \( g_2^\alpha \in \mathcal{E} (L^p (0,1; \Omega), \mu) \) the above inequality gives rise to \( \psi_n^\alpha \in \mathcal{E} (\mathbb{R}, \Omega, \mu) \) for \( n = 1, 2, \ldots \) Since \( \| g_2 \|_{L^p} \sum_{n=0}^{\infty} \varphi_\alpha (n) < \infty \), then we deduce from the Weierstrass M-test that the series \( \sum_{n=0}^{\infty} \psi_n^\alpha (t) \) is uniformly convergent on \( \mathbb{R} \) and \( \psi (t) = \int_{-\infty}^{t} T_\alpha (t - s) g_2 (s) \, ds = \sum_{n=1}^{\infty} \psi_n^\alpha (t) \).

Applying \( \psi_n \in \mathcal{E} (\mathbb{R}, \Omega, \mu) \) and the inequality

\[
\frac{1}{\mu ([t-r,t])} \int_{[t-r,t]} \| \psi (t) \| \, d\mu (t) \leq \frac{1}{\mu ([t-r,t])} \int_{[t-r,t]} \| \psi (t) - \sum_{n=1}^{k} \psi_n^\alpha (t) \| \, d\mu (t) + \sum_{n=1}^{k} \frac{1}{\mu ([t-r,t])} \int_{[t-r,t]} \| \psi_n^\alpha (t) \| \, d\mu (t) \to 0,
\]

we obtain that the uniform \( \psi (t) = \sum_{n=1}^{\infty} \psi_n^\alpha (t) \in \mathcal{E} (\mathbb{R}, \Omega, \mu) \). Therefore \( F_\alpha (t) = \phi (t) + \psi (t) \) is \( \mu \)-pseudo almost automorphic.

**Theorem 3.3.** Let \( \mu \in M \). Assume the conditions \((H_0), (T_1), (T_2)\) are satisfied and the function \( f = h_1 + h_2 \in \mathcal{PA} \mathcal{A}^{p} (\mathbb{R} \times \Omega, \Omega, \mu) \) with \( h_1 \in \mathcal{AS} \mathcal{P} (\mathbb{R} \times \Omega, \Omega) \) and \( h_2 \in \mathcal{E} (\mathbb{R}, L^p (0,1; \Omega), \mu) \). Then Eq.(1) has a unique \( \mu \)-pseudo almost automorphic mild solution on \( \mathbb{R} \), provided that \( L_f \| \tilde{\phi}_n \|_{L^1 (\mathbb{R})} < 1 \).

**Proof.** Let \( \tilde{\Theta} : \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \to \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \) be the nonlinear operator defined by

\[
(\tilde{\Theta} \zeta) (t) = \int_{-\infty}^{t} T_\alpha (t - s) f (s, \zeta (s)) \, ds,
\]

\((3.3)\)

\[
\tilde{\Theta} : \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \to \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu).
\]

**Step 1.** \( \tilde{\Theta} (\mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu)) \subseteq \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \).

First, using the fact that \( \zeta \in \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \) is relatively compact with the above Lemma 2.21, Lemma 2.24, follows that \( f (\cdot, \zeta (\cdot)) \in \mathcal{PA} \mathcal{A}^{p} (\mathbb{R}, \Omega, \mu) \). Hence, from Lemma 3.2, we know that \( (\tilde{\Theta} \zeta) (\cdot) \in \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \).

**Step 2.** \( \tilde{\Theta} \) has a unique fixed point.

Let \( t \in \mathbb{R}, \zeta, \eta \in \mathcal{PA} \mathcal{A} (\mathbb{R}, \Omega, \mu) \), yields

\[
\left\| (\tilde{\Theta} \zeta) (t) - (\tilde{\Theta} \eta) (t) \right\| \leq \int_{-\infty}^{t} \| T_\alpha (t - s) \| \| f (s, \zeta (s)) - f (s, \eta (s)) \| \, ds
\leq L_f \int_{-\infty}^{t} \| T_\alpha (t - s) \| \| \zeta (s) - \eta (s) \| \, ds
= L_f \int_{0}^{t} \| T_\alpha (s) \| \| x (t - s) - y (t - s) \| \, ds
\leq L_f \| \zeta - \eta \|_{\infty} \| \varphi_\alpha \|_{L^1 (\mathbb{R})}.
\]
which implies that
\[ \left\| \left( \tilde{\Theta} \zeta \right) (t) - \left( \tilde{\Theta} \eta \right) (t) \right\| \leq L_f \| \zeta - \eta \|_{\infty} \| \varphi_\alpha \|_{L^1 (\mathbb{R})}. \]

Therefore, we concluded that, by means of the Banach fixed point theorem with \( L_f \| \varphi_\alpha \|_{L^1} < 1 \), \( \tilde{\Theta} \) has a unique fixed point \( \zeta \) in \( P.AA (\mathbb{R}, \Omega, \mu) \) which is the \( \mu \)-pseudo almost automorphic solution to Eq.(1.2).

**Theorem 3.4.** Let \( \mu \in M \). Assume that \( (H_0), (T_1), (T_3) \) and \( (T_5) \) holds, then Eq.(1.2) admits a unique \( \mu \)-pseudo almost automorphic mild solution whenever \( \| L_f \|_{\mathcal{S}^r} \varphi_0 < 1 \).

**Proof.** To prove this result, we consider the nonlinear operator \( \Gamma \) given by
\[ \left( \tilde{\Theta} \zeta \right) (t) = \int_{-\infty}^{t} T_\alpha (t - s) f (s, \zeta (s)) \, ds, \quad t \in \mathbb{R}. \]

**Step 1.** \( \tilde{\Theta} \) maps \( P.AA (\mathbb{R}, X, \mu) \) and using Lemma 2.21 and Lemma 2.25 it follows that the function \( s \to f (s, \zeta (s)) \) is in \( \mathcal{P}A.A^{\mathcal{F}} (\mathbb{R}, \Omega, \mu) \). On the other hand, using the Lemma 3.2 we infer that \( \Theta \zeta \in \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \), i.e., \( \tilde{\Theta} \) maps \( \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \) into itself.

**Step 2.** \( \tilde{\Theta} \) has a unique fixed point in \( \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \). Indeed, for each \( t \in \mathbb{R}, \zeta, \eta \in \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \), yields
\[ \left\| \left( \tilde{\Theta} \zeta \right) (t) - \left( \tilde{\Theta} \eta \right) (t) \right\| \leq \int_{-\infty}^{t} \| T_\alpha (t - s) \| \| f (s, \zeta (s)) - f (s, \eta (s)) \| \, ds \]
\[ \leq \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} \varphi_\alpha (t - s) L_f (s) \| x - y \|_{\infty} \, ds \]
\[ \leq \sum_{n=1}^{\infty} \varphi_\alpha (n - 1) \left( \int_{t-n}^{t-n+1} \| L_f (s) \|^p \, ds \right)^{1/p} \| \zeta - \eta \|_{\infty} \]
\[ \leq \varphi_0 \| L_f (s) \|_{\mathcal{S}^r} \| \zeta - \eta \|_{\infty}. \]

In this sense, we have
\[ \left\| \tilde{\Theta} \zeta - \tilde{\Theta} \eta \right\| \leq \varphi_0 \| L_f (s) \|_{\mathcal{S}^r} \| \zeta - \eta \|_{\infty}. \]

Since \( \varphi_0 \| L_f (s) \|_{\mathcal{S}^r} < 1 \), using the Banach fixed point theorem, \( \tilde{\Theta} \) has a unique fixed point \( x \in \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \).

**Theorem 3.5.** Let \( \mu \in M \). Assume that \( (H_0), (T_1) \) and \( (T_4) \) are true. Then there exists a unique \( \mu \)-pseudo almost automorphic mild solution to Eq.(1.2), provided that \( \varphi_0 \| L_f \|_{\mathcal{S}^r} < 1 \).

**Proof.** For proof of this result, we will consider the same operator \( \tilde{\Theta} \) given in Theorem 3.3 Eq.(3.3), given by
\[ \left( \tilde{\Theta} \zeta \right) (t) = \int_{-\infty}^{t} T_\alpha (t - s) f (s, \zeta (s)) \, ds, \quad t \in \mathbb{R}. \]

**Step 1.** \( \tilde{\Theta} \) maps \( P.AA (\mathbb{R}, X, \mu) \) and using the Lemma 2.21 and Theorem 2.27 it follows that the functions \( s \to f (s, \zeta (s)) \) is in \( \mathcal{P}A.A^{\mathcal{F}} (\mathbb{R}, \Omega, \mu) \), \( \theta \in [1, p) \). On the other hand, using the Lemma 3.2 we infer that \( \tilde{\Theta} x \in \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \), i.e., \( \tilde{\Theta} \) maps \( \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \) into itself.

**Step 2.** \( \tilde{\Theta} \) has a unique fixed point in \( \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \). Indeed, for each \( t \in \mathbb{R}, \zeta, \eta \in \mathcal{P}A.A (\mathbb{R}, \Omega, \mu) \), yields
\[ \left\| \left( \tilde{\Theta} \zeta \right) (t) - \left( \tilde{\Theta} \eta \right) (t) \right\| \leq \int_{-\infty}^{t} \| T_\alpha (t - s) \| \| f (s, \zeta (s)) - f (s, \eta (s)) \| \, ds \]
\[ \leq \int_{-\infty}^{t} \varphi_\alpha (t - s) L_f (s) \| \zeta (s) - \eta (s) \| \, ds \]
\[ \leq \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} \varphi_\alpha (t - s) L_f (s) \, ds \| \zeta - \eta \|_{\infty} \]
\[ \leq \sum_{n=1}^{\infty} \varphi_\alpha (n - 1) \left( \int_{t-n}^{t-n+1} \| L_f (s) \|^r \, ds \right)^{1/r} \| \zeta - \eta \|_{\infty}. \]
Theorem 3.6. Let for all $t$ follows conditions:

In view of the inequality $\varphi_0 \| L_f \|_{S^r} < 1$, using the Banach fixed point theorem, $\tilde{\Theta}$ has a unique fixed point $x \in PAA(\mathbb{R}, \Omega, \mu)$. Therefore, we conclude the proof. 

So far, the results of the existence of $\mu$-pseudo almost automorphic solutions to Eq. (1.2), have been obtained using the fact that $f$ satisfies the Lipschitz condition. Now, let’s discuss the next result of this paper, removing this condition and imposing another.

The following existence result is based upon Leray-Schauder nonlinear alternative theorem. Consider the follows conditions:

$\textbf{(T}_7\textbf{)}$ There exists a continuous nondecreasing function $\tilde{\Theta} : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, \theta) \| \leq \tilde{\Theta}(\| \theta \|)$$

for all $t \in \mathbb{R}$ and $\theta \in \Omega$.

**Theorem 3.6.** Let $\mu \in M$. Assume that conditions $H_0, T_1$ are satisfied. Let $f : \mathbb{R} \times \Omega \to \Omega$ be a function which satisfies assumptions $(T_7)$-$\textbf{(T}_7\textbf{)}$ and the following additional conditions:

1. For each $\kappa \geq 0$, the function $t \to \int_{-\infty}^{t} \varphi_\alpha (t - s) \tilde{\Theta}(\kappa \tilde{\varrho}(s)) ds$ belongs to $C_0(\mathbb{R})$. Let

$$\lambda(\kappa) = \left\| \int_{-\infty}^{t} \varphi_\alpha (t - s) \tilde{\Theta}(\kappa \tilde{\varrho}(s)) ds \right\|.$$  

2. For each $\varepsilon > 0$, there exists a $\delta > 0$, such that for every $\zeta, \eta \in C_0(\Omega)$, $\| \zeta - \eta \|_\varrho \leq \delta$ implies that

$$\int_{-\infty}^{t} \varphi_\alpha (t - s) \| f(s, \zeta(s)) - f(s, \eta(s)) \| ds \leq \varepsilon$$

for all $t \in \mathbb{R}$.

3. $\liminf_{\xi \to -\infty} \frac{\xi}{\lambda(\xi)} > 1$.

4. For all $a, b \in \mathbb{R}$, $a < b$ and $\kappa > 0$, the set $\{ f(s, \zeta) \mid a \leq s \leq b, \zeta \in C_0(\Omega), \| \zeta \|_\varrho \leq \kappa \}$ is relatively compact in $\Omega$.

Then Eq. (1.2) has a least one $\mu$-pseudo almost automorphic mild solution on $t \in \mathbb{R}$.

**Proof.** Consider the following operator $\tilde{\Theta} : C_0(\Omega) \to C_0(\Omega)$ given by

$$\left( \tilde{\Theta} \zeta \right)(t) := \int_{-\infty}^{t} T_\alpha(t - s) f(s, \zeta(s)) ds, \quad t \in \mathbb{R}.$$  

The main objective of this test is to ensure that $\tilde{\Theta}$ has a fixed point in $PAA(\mathbb{R}, \Omega, \mu)$. For this, it will be investigated in 5 stages.

**Step 1.** $\tilde{\Theta}$ is well defined. For $\zeta \in C_0(\Omega)$, yields

$$\| \left( \tilde{\Theta} \zeta \right)(t) \| \leq \int_{-\infty}^{t} \| T_\alpha(t - s) \| \| f(s, \zeta(s)) \| ds$$

$$\leq \int_{-\infty}^{t} \varphi_\alpha (t - s) \tilde{\Theta}(\| \zeta \|_\varrho) ds$$

$$\leq \int_{-\infty}^{t} \varphi_\alpha (t - s) \tilde{\Theta}(\| \zeta \|_\varrho \tilde{\varrho}(s)) ds.$$  

In this sense, using the condition (F1), we concluded that $\tilde{\Theta}$ is well defined.

**Step 2.** The operator $\tilde{\Theta}$ is continuous.
Indeed, for any \( \varepsilon > 0 \), we take \( \delta > 0 \) involved in condition (F2). If \( \zeta, \eta \in C_\overline{\varrho} (\Omega) \) and \( \| \zeta - \eta \|_\varrho \leq \delta \), then we have
\[
\left\| \left( \overline{\Theta} \zeta \right) (t) - \left( \overline{\Theta} \eta \right) (t) \right\| \leq \int_{-\infty}^{t} \| T_\alpha (t - s) \| \| f (s, \zeta (s)) - f (s, \eta(s)) \| \, ds
\]
\[
\leq \int_{-\infty}^{t} \varphi_\alpha (t - s) \| f (s, \zeta (s)) - f (s, \eta(s)) \| \, ds
\]
\[
\leq \varepsilon
\]
and we concluded this step.

**Step 3.** \( \overline{\Theta} \) is completely continuous.

Let \( B_\kappa (\Omega) \) is a closed ball with center at 0 and radius \( \kappa \) in the space \( \Omega \). Moreover, let \( V = \overline{\Theta} (B_\kappa (C_\overline{\varrho} (\Omega))) \) and \( \nu = \overline{\Theta} (\zeta) \) for \( \zeta \in B_\kappa (C_\overline{\varrho} (\Omega)) \).

Note that, using the condition (F1), the function \( s \mapsto \varphi_\alpha (s) \overline{\Theta} (\kappa \overline{\varrho} (t - s)) \) is integrable on \([0, \infty)\). Hence, for \( \varepsilon > 0 \) we can choose \( a \geq 0 \) such that
\[
\int_{a}^{\infty} \varphi_\alpha (s) \overline{\Theta} (\kappa \overline{\varrho} (t - s)) \, ds \leq \varepsilon.
\]

Since
\[
\nu (t) = \int_{0}^{a} \| T_\alpha (s) f (t - s, \zeta (t - s)) \| \, ds + \int_{a}^{\infty} \| T_\alpha (s) f (t - s, \zeta (t - s)) \| \, ds
\]
and
\[
\int_{a}^{\infty} \| T_\alpha (s) f (t - s, \zeta (t - s)) \| \, ds \leq \int_{a}^{\infty} \| T_\alpha (s) f (t - s, \zeta (t - s)) \| \, ds
\]
\[
\leq \int_{a}^{\infty} \varphi_\alpha (s) \overline{\Theta} (\kappa \overline{\varrho} (t - s)) \, ds
\]
we have \( \nu (t) \in ac_0 (\mathbb{K}) + B_\varepsilon (\Omega) \), where \( ac_0 (\mathbb{K}) \) denotes the convex hull of \( \mathbb{K} \) and
\[
\mathbb{K} = \left\{ T_\alpha (s) f (\xi, \zeta) : 0 \leq s \leq a, \ t - a \leq \xi \leq t, \ \| \zeta \|_\varrho \leq \kappa \right\}.
\]

Using the strong continuity of \( T_\alpha (\cdot) \) and condition (F4) on \( f \), then \( \mathbb{K} \) is a relatively compact set, and \( V(t) \subseteq ac_0 (\mathbb{K}) + B_\varepsilon (\Omega) \). Therefore \( V(t) \) is a relatively subset of \( \Omega \) for \( t \in \mathbb{R} \).

To conclude this step, we will prove that the set \( V \) is equicontinuous. Indeed, consider the following decomposition
\[
\nu (t + \varepsilon) - \nu (t) = \int_{0}^{s} T_\alpha (\sigma) f (t + \sigma - \zeta (t + \sigma - \sigma)) \, d\sigma
\]
\[
+ \int_{0}^{a} \left( T_\alpha (\sigma + s) - T_\alpha (\sigma) \right) f (t - \sigma, \zeta (t - \sigma)) \, d\sigma
\]
\[
+ \int_{a}^{\infty} \left( T_\alpha (\sigma + s) - T_\alpha (\sigma) \right) f (t - \sigma, \zeta (t - \sigma)) \, d\sigma.
\]

(3.5)

For each \( \varepsilon > 0 \), we can choose \( a > 0 \) and \( \delta_1 > 0 \) such that
\[
\left\| \int_{0}^{s} T_\alpha (\sigma) f (t + s - \sigma, \zeta (t + s - \sigma)) \, d\sigma \right\| + \int_{0}^{a} \left( T_\alpha (\sigma + s) - T_\alpha (\sigma) \right) f (t - \sigma, \zeta (t - \sigma)) \, d\sigma
\]
\[
\leq \int_{0}^{s} \| T_\alpha (\sigma) \| \| f (t + s - \sigma, x (t + s - \sigma)) \| \, d\sigma + \int_{0}^{a} \left( \| T_\alpha (\sigma + s) - T_\alpha (\sigma) \| \right) \| f (t - \sigma, \zeta (t - \sigma)) \| \, d\sigma
\]
\[
\leq \varepsilon \frac{\kappa}{2}
\]
(3.6)

for \( s \leq \delta_1 \). Moreover, since \( \{ f (t - \sigma, \zeta (t - \sigma)) : 0 \leq \sigma \leq a, \ \zeta \in B_\kappa (C_\overline{\varrho} (\Omega)) \} \) is relatively compact and \( T_\alpha (\cdot) \) is strongly continuous, choose \( \delta_2 > 0 \) such that
\[
\left\| \left( T_\alpha (\sigma + s) - T_\alpha (\sigma) \right) f (t - \sigma, \zeta (t - \sigma)) \right\| \leq \varepsilon \frac{\kappa}{2a}
\]
(3.7)
for $s \leq \delta_2$.

Combining the estimates (3.5), (3.6) and (3.7), we get $\|\nu(t+s) - \nu(t)\| \leq \varepsilon$ for $s$ small enough and independent of $\zeta \in B_{\kappa}(C_{\tilde{g}}(\Omega))$.

Finally, using the condition (F1), we have

$$\frac{\|\nu(t)\|}{\tilde{g}(t)} \leq \frac{1}{\tilde{g}(t)} \int_{-\infty}^{t} \varphi_{\alpha}(t-s) \tilde{\Theta}(\kappa \tilde{g}(s)) \, ds \to 0$$

as $|t| \to \infty$ and this converge is independent of $\zeta \in B_{\kappa}(C_{\tilde{g}}(\Omega))$. Therefore, using Lemma 2.31, we concluded that, $V$ is relatively compact set in $C_{\tilde{g}}(\Omega)$.

**Step 4.** $P = \{\zeta^\gamma : \zeta^\gamma = \gamma \tilde{\Theta}(\zeta^\gamma), \gamma \in (0,1)\}$ is bounded.

First, assume that $\zeta^\gamma (\cdot)$ is a solution of equation $\zeta^\gamma = \gamma \tilde{\Theta}(\zeta^\gamma)$ for some $0 < \gamma < 1$. Note that

$$\|\zeta^\gamma(t)\| = \left\| \gamma \tilde{\Theta}(\zeta^\gamma) \right\| = \gamma \left\| \int_{-\infty}^{t} T_{\alpha}(t-s) f(s, \zeta^\gamma(s)) \, ds \right\| \leq \int_{-\infty}^{t} \varphi_{\alpha}(t-s) \tilde{\Theta}(\|\zeta^\gamma(s)\|) \, ds \leq \int_{-\infty}^{t} \varphi_{\alpha}(t-s) \tilde{\Theta}(\|\zeta^\gamma\|_{\tilde{g}}) \, ds \leq \lambda \|\zeta^\gamma\|_{\tilde{g}} \tilde{g}(s) .$$

Hence, yields

$$\frac{\|\zeta^\gamma\|}{\lambda \|x^\gamma\|_{\tilde{g}}} \leq 1 .$$

Using the condition (F3), we conclude that the set $P$ is bounded.

**Step 5.** It follows from Lemma 2.21, (T6-T7) and Lemma 2.26 that the function $t \to f(t, \zeta(t))$ belongs to $\mathcal{PAA}^p(\Omega, \Omega, \mu)$ whenever $\zeta \in \mathcal{PAA}(\Omega)$ and noting that $\mathcal{PAA}(\Omega)$ is a closed subspace of $C_{\tilde{g}}(\Omega)$ consequently we can consider $\tilde{\Theta} : \mathcal{PAA}(\Omega, \Omega, \mu) \to \mathcal{PAA}(\Omega, \Omega, \mu)$. Using conditions (F1)-(F3), we deduce that this map is completely continuous. Applying Lemma 2.32, we infer that $\tilde{\Theta}$ has a fixed point $\zeta \in \mathcal{PAA}(\Omega, \Omega, \mu)$. □

**REFERENCES**


(J. Vanterler da C. Sousa) CENTER FOR MATHEMATICS, COMPUTING AND COGNITION, FEDERAL UNIVERSITY OF ABC, AVENIDA DOS ESTADOS, 5001, BAIRRO BANCU, 09.210-580, SANTO ANDRE, SP - BRAZIL
Email address: jose.vanterler@ufabc.edu.br, vanterlermatematico@hotmail.com

(G. M. N’Guérékata) NEERLAB, DEPARTMENT OF MATHEMATICS, MORGAN STATE UNIVERSITY, BALTIMORE, MD 21251, USA
Email address: Gaston.NGuerekata@morgan.edu