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On Coupled system of Nonlinear Ψ -Hilfer Hybrid Fractional Differential Equations

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Abstract

In this paper, we derive the existence of a solution for coupled system for hybrid fractional differential equation involving Ψ -Hilfer derivative for initial value problem and boundary value problem. In the view of an application, we provide an examples to exhibit the effectiveness of our achieved results.

Key words: Ψ -Hilfer fractional derivative; Fractional differential inequalities; Existence of solution; Extremal solutions; Comparison theorems.

2010 Mathematics Subject Classification: 34A38, 26A33, 34A12, 34A40.

1 Introduction

Motivated by the work of [35, 36], in the present paper, we consider the following Ψ -Hilfer hybrid FDEs of the form

$$\begin{cases} {}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi} \left[\frac{y(t) - w(t, y(t))}{u(t, y(t))} \right] = v \left(t, x(t), k \mathcal{I}_{0+}^{\mu; \Psi} x(t) \right), \text{ a.e. } t \in (0, T], \\ \lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} y(t) = y_0 \in \mathbb{R}, \end{cases} \quad (1.1)$$

and

$$\begin{cases} {}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi} \left[\frac{x(t) - w(t, x(t))}{u(t, x(t))} \right] = v \left(t, y(t), k \mathcal{I}_{0+}^{\mu; \Psi} y(t) \right), \text{ a.e. } t \in (0, T], \\ \lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} x(t) = y_0 \in \mathbb{R}, \end{cases} \quad (1.2)$$

where $0 < \mu < 1$, $0 \leq \nu \leq 1$, $\xi = \mu + \nu(1 - \mu)$, ${}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi}(\cdot)$ is the Ψ -Hilfer fractional derivative of order μ and type ν , $u \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $J = [0, T]$ and $w \in C(J \times \mathbb{R}, \mathbb{R})$ and $v \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications

[1, 3, 13, 17, 20, 24, 25]. The area of differential equations (integer and fractional order) dedicated to quadratic perturbations of nonlinear problems, also known as hybrid differential equations, is one of the most important areas and has attracted considerable attention from researchers. Over the past few decades, with the growth and greater demand for the theory of fractional differential equations, the search for discussing properties of solutions of hybrid differential problems, has gained prominence and greater investigation both in the theoretical sense and involving applications [5, 7, 9, 12, 14, 16]. Some fundamental and important papers involving the existence of solutions of fractional hybrid differential problems for further reading, we recommend the following works [14, 32, 35, 36].

In 2011, Zhao et al. [5], investigate the existence of solutions for fractional hybrid differential equations involving Riemann-Liouville differential operators of order $0 < q < 1$

$$\begin{cases} D^q \left[\frac{x(t)}{f(t, x(t))} \right] = g(t, x(t)), \text{ a.e. } t \in J, \\ x(0) = 0, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}/\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$, is proved under mixed Lipschitz and Caratheodory conditions.

Sun et al. [13], discuss the existence of solutions for the boundary value problem of fractional Hybrid differential equations

$$\begin{cases} D_{0+}^{\alpha} \left[\frac{x(t)}{f(t, x(t))} \right] + g(t, x(t)) = 0, \quad 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, $D_{0+}^{\alpha}(\cdot)$ is the Riemann-Liouville fractional derivative. Other interesting works on fractional hybrid differential equations can be obtained in [4, 10, 11]. In 2017 Ali et al. [36], investigated sufficient conditions to discuss the existence of solutions for the study of coupled systems of fractional order hybrid.

We will here highlight the main objective discussed in this article, and its approach to the result, in order to facilitate and be clear, the main contribution.

We obtain, the equivalent fractional integral equation to the Ψ -Hilfer hybrid FDEs Eqs.(2.3)-(2.4) and establish the existence of solution in the weighted space $C_{1-\xi; \Psi}(J, \mathbb{R})$. Our main objective here is to obtain estimates on Ψ -Hilfer derivative and utilize it to develop the hybrid fractional differential inequalities involving Ψ -Hilfer derivative. Using the fractional differential inequalities in the setting of Ψ -Hilfer derivative, we then derive the existence of extremal solutions and comparison results.

The obtained outcomes in the current paper additionally hold for nonlinear hybrid FDEs with well known fractional derivative operators recorded in [23], for example, RL, Caputo, Ψ -RL, Ψ -Caputo, Hadmard, Katugampola, Riesz, Erdély-Kober, Hilfer and so forth.

The plan of the paper is as follows: In the section 2, we give some definitions and results which are useful to prove the main results. In section 3, we derive an equivalent integral equation(IE) to the problem Eqs.(2.3)-(2.4) and establish existence of solution to it. In section 4, we obtain estimates on Ψ -Hilfer derivative. Section 5 deals with hybrid fractional differential inequalities involving Ψ -Hilfer derivative. Section 6, contribute the study of maximal and minimal solutions. Finally, comparison theorems are proved in the section 7.

2 Preliminaries

Let $[a, b]$ ($0 < a < b < \infty$) be a finite interval and $\Psi \in C^1([a, b], \mathbb{R})$ be an increasing function such that $\Psi'(t) \neq 0, \forall t \in [a, b]$. We consider the weighted space

$$C_{1-\xi; \Psi}[a, b] = \left\{ h \mid h : (a, b] \rightarrow \mathbb{R}, h(a+) \text{ exists and } (\Psi(t) - \Psi(a))^{1-\xi} h(t) \in C[a, b] \right\}, \quad 0 < \xi \leq 1,$$

endowed with the norm

$$\|h\|_{C_{1-\xi; \Psi}[a, b]} = \max_{t \in [a, b]} \left| (\Psi(t) - \Psi(a))^{1-\xi} h(t) \right|. \quad (2.1)$$

Definition 2.1 ([17]) *Let h be an integrable function defined on $[a, b]$. Then the Ψ -Riemann-Liouville fractional integral of order $\mu > 0$ ($\mu \in \mathbb{R}$) of the function h is given by*

$$I_{a^+}^{\mu; \Psi} h(t) = \frac{1}{\Gamma(\mu)} \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} h(s) ds. \quad (2.2)$$

Definition 2.2 ([23]) *The Ψ -Hilfer fractional derivative of a function h of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$, is defined by*

$${}^H \mathcal{D}_{a^+}^{\mu, \nu; \Psi} h(t) = I_{a^+}^{\nu(1-\mu); \Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right) I_{a^+}^{(1-\nu)(1-\mu); \Psi} h(t).$$

Lemma 2.1 ([17, 23]) *Let $\chi, \delta > 0$ and $\rho > n$. Then*

- (i) $I_{a^+}^{\mu; \Psi} \mathcal{I}_{a^+}^{\chi; \Psi} h(t) = \mathcal{I}_{a^+}^{\mu+\chi; \Psi} h(t)$.
- (ii) $\mathcal{I}_{a^+}^{\mu; \Psi} (\Psi(t) - \Psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mu + \delta)} (\Psi(t) - \Psi(a))^{\mu+\delta-1}$.
- (iii) ${}^H \mathcal{D}_{a^+}^{\mu, \nu; \Psi} (\Psi(t) - \Psi(a))^{\xi-1} = 0$.
- (iv) ${}^H \mathcal{D}_{a^+}^{\mu, \nu; \Psi} (\Psi(t) - \Psi(a))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho - \mu)} (\Psi(t) - \Psi(a))^{\mu-\rho-1}$.

Lemma 2.2 ([23]) *If $h \in C^n[a, b]$, $n - 1 < \mu < n$ and $0 \leq \nu \leq 1$, then*

- (i) $I_{a^+}^{\mu; \Psi} {}^H \mathcal{D}_{a^+}^{\mu, \nu; \Psi} h(t) = h(t) - \sum_{k=1}^n \frac{(\Psi(t) - \Psi(a))^{\xi-k}}{\Gamma(\xi - k + 1)} h_{\Psi}^{[n-k]} I_{a^+}^{(1-\nu)(n-\mu); \Psi} h(a)$ where, $h_{\Psi}^{[n-k]} h(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^{n-k} h(t)$.
- (ii) ${}^H \mathcal{D}_{a^+}^{\mu, \nu; \Psi} I_{a^+}^{\mu; \Psi} h(t) = h(t)$.

Lemma 2.3 ([34]) *Let $0 < \mu < 1$, $0 \leq \nu \leq 1$, $\xi = \mu + \nu(1 - \mu)$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ is bounded, $J = [0, T]$ and $g \in \mathfrak{C}(J \times \mathbb{R}, \mathbb{R}) = \{h \mid \text{the map } \omega \rightarrow h(\tau, \omega) \text{ is continuous for each } \tau \text{ and the map } \tau \rightarrow h(\tau, \omega) \text{ is measurable for each } \omega\}$. A function $y \in C_{1-\xi; \Psi}(J, \mathbb{R})$ is the solution of hybrid FDEs*

$${}^H \mathcal{D}_{0^+}^{\mu, \nu; \Psi} \left[\frac{y(t)}{f(t, y(t))} \right] = g(t, y(t)), \quad \text{a.e. } t \in (0, T], \quad (2.3)$$

$$(\Psi(t) - \Psi(0))^{1-\xi} y(t)|_{t=0} = y_0 \in \mathbb{R}, \quad (2.4)$$

if and only if it is solution of the following hybrid fractional integral equation (IE)

$$y(t) = f(t, y(t)) \left\{ \frac{y_0}{f(0, y(0+))} (\Psi(t) - \Psi(0))^{\xi-1} + \mathcal{I}_{0+}^{\mu; \Psi} g(t, y(t)) \right\}, \quad t \in (0, T]. \quad (2.5)$$

Definition 2.3 ([37]) *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $T : X \times X \rightarrow X$ if $T(x, y) = x$ and $T(y, x) = y$.*

Lemma 2.4 ([35]) *Let S be a non-empty, closed, convex and bounded subset of the Banach algebra X and $\tilde{S} = S \times S$. Suppose that $E, G : X \rightarrow X$ and $F : S \rightarrow X$ are three operators such that*

- (a) *E and G are Lipschitzian with a Lipschitz constants σ and δ respectively;*
- (b) *F is completely continuous;*
- (c) *$y = EyFx + Gy \implies y \in S$ for all $x \in S$ and*
- (d) *$4\sigma M + \delta < 1$ where $M = \sup \{\|Bx\| : x \in S\}$.*

Then, the operator equation $T(y, x) = EyFx + Gy$ has a at least one coupled fixed point in \tilde{S} .

Lemma 2.5 ([38]) *Let S^* be a non-empty, closed, convex and bounded subset of the Banach space E and let $A, C : E \rightarrow E$ and $B : S^* \rightarrow E$ are three operators such that*

- (a) *A and C are Lipschitzian with a Lipschitz constants K and L respectively;*
- (b) *B is completely continuous;*
- (c) *$y = AyBx + Cy \implies y \in S^*$ for all $x \in S^*$ and*
- (d) *$K M^* + L < 1$ where $M^* = \sup \{\|By\| : y \in S^*\}$.*

Then, the operator equation $AyBy + Cy = y$ has a solution in S^ .*

3 IVP for Coupled system of Hyrid FDEs

An application of the Lemma 2.3 gives the equivalent fractional IE to the FDEs Eq.(1.1), given in the following Lemma.

Lemma 3.1 *A function $y \in C_{1-\xi; \Psi}(J, \mathbb{R})$ is the solution of the Cauchy problem for hybrid FDEs Eq.(1.1) if and only if it is solution of the following hybrid fractional IE*

$$y(t) = u(t, y(t)) \left\{ \frac{y_0}{u(0, y(0+))} (\Psi(t) - \Psi(0))^{\xi-1} + \mathcal{I}_{0+}^{\mu; \Psi} v \left(t, x(t), k \mathcal{I}_{0+}^{\mu; \Psi} x(t) \right) \right\} \\ + w(t, y(t)), \quad t \in (0, T]. \quad (3.1)$$

We list the following assumptions to prove the existence of solution to the coupled system of hybrid FDEs Eqs.(1.1)-(1.2).

(H1) The functions $u \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $w \in C(J \times \mathbb{R}, \mathbb{R})$ are bounded and there exists constants $\sigma, \delta > 0$ such that for all $p, q \in \mathbb{R}$ and $t \in J = [0, T]$ we have

$$|u(t, p) - u(t, q)| \leq \sigma |p - q|$$

and

$$|w(t, p) - w(t, q)| \leq \delta |p - q|.$$

(H2) The function $v \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a function $g \in C_{1-\xi}(J, \mathbb{R})$ such that

$$|v(t, p, q)| \leq (\Psi(t) - \Psi(0))^{1-\xi} g(t), \quad a.e. \ t \in J \text{ and } p, q \in \mathbb{R}.$$

Theorem 3.2 *Assume that the hypotheses (H1)-(H2) hold. Then, the coupled system of nonlinear Ψ -Hilfer hybrid FDEs Eqs.(1.1)-(1.2) has a solution $(y, x) \in C_{1-\xi; \Psi}(J, \mathbb{R}) \times C_{1-\xi; \Psi}(J, \mathbb{R})$ provided*

$$4\sigma \left\{ \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + \delta < 1. \quad (3.2)$$

Proof: Let $X := (C_{1-\xi; \Psi}(J, \mathbb{R}), \|\cdot\|_{C_{1-\xi; \Psi}(J, \mathbb{R})})$. Then X is a Banach algebra with the product of vectors defined by $(xy)(t) = x(t)y(t)$, $t \in (0, T]$. Define,

$$S = \{x \in X : \|x\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq R\},$$

where

$$R = K_1 \left\{ \left| \frac{y_0}{u(0, x(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + K_2 (\Psi(T) - \Psi(0))^{1-\xi}$$

and $K_1 > 0$ and $K_2 > 0$ are the constants such that $|u(t, \cdot)| < K_1$ and $|w(t, \cdot)| < K_2$ for all t .

Clearly, S is non-empty, closed, convex and bounded subset of X . If $(y, x) \in S \times S = \tilde{S}$ is a solution of the coupled system of nonlinear Ψ -Hilfer hybrid FDEs Eqs.(1.1)-(1.2), then

$(y, x) \in S \times S = \tilde{S}$ is a solution of the coupled system of fractional IEs

$$\begin{cases} y(t) = u(t, y(t)) \left\{ \frac{y_0}{u(0, y(0+))} (\Psi(t) - \Psi(0))^{\xi-1} + \mathcal{I}_{0+}^{\mu; \Psi} v \left(t, x(t), k \mathcal{I}_{0+}^{\mu; \Psi} x(t) \right) \right\} \\ \quad + w(t, y(t)), \\ x(t) = u(t, x(t)) \left\{ \frac{y_0}{u(0, x(0+))} (\Psi(t) - \Psi(0))^{\xi-1} + \mathcal{I}_{0+}^{\mu; \Psi} v \left(t, y(t), k \mathcal{I}_{0+}^{\mu; \Psi} y(t) \right) \right\} \\ \quad + w(t, x(t)), \quad t \in (0, T]. \end{cases} \quad (3.3)$$

Define three operators $E, G : X \rightarrow X$ and $F : S \rightarrow X$ by

$$\begin{aligned} Ey(t) &= u(t, y(t)), \quad t \in J; \\ Fy(t) &= \frac{y_0}{u(0, y(0+))} (\Psi(t) - \Psi(0))^{\xi-1} + \mathcal{I}_{0+}^{\mu; \Psi} v \left(t, y(t), k \mathcal{I}_{0+}^{\mu; \Psi} y(t) \right), \quad t \in (0, T]; \\ Gy(t) &= w(t, y(t)), \quad t \in J. \end{aligned}$$

Then, the coupled hybrid IEs Eq.(3.3) transformed into the coupled system of operator equations as

$$\begin{cases} y = Ey Fx + Gy, \quad y \in X \\ x = Ex Fy + Gx, \quad x \in X. \end{cases} \quad (3.4)$$

Consider the mapping $T : \tilde{S} \rightarrow X$, $\tilde{S} = S \times S$ defined by

$$T(y, x) = Ey Fx + Gy, \quad (y, x) \in \tilde{S}.$$

Then the coupled system of operator equations (3.4) can be written as

$$y = T(y, x) \text{ and } x = T(x, y), \quad (y, x), (x, y) \in \tilde{S}.$$

To prove that the mapping T has coupled fixed point, we show that the operators E, F and G satisfies all the conditions of Lemma 2.4. The proof is given in the several steps:

Step 1: $E, G : X \rightarrow X$ are Lipschitz operators.

Using the hypothesis **(H1)**, we obtain

$$\begin{aligned} \left| (\Psi(t) - \Psi(0))^{1-\xi} (Ex(t) - Ey(t)) \right| &= \left| (\Psi(t) - \Psi(0))^{1-\xi} (u(t, x(t)) - u(t, y(t))) \right| \\ &\leq \sigma \left| (\Psi(t) - \Psi(0))^{1-\xi} (x(t) - y(t)) \right| \\ &\leq \sigma \|x - y\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}. \end{aligned}$$

This gives,

$$\|Ex - Ey\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq \sigma \|x - y\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}.$$

Therefore, E is Lipschitz operator with Lipschitz constant σ . On the similar line one can verify that that G is Lipschitz operator. Let δ is Lipschitz constant corresponding to operator G .

Step 2: $F : S \rightarrow X$ is completely continuous.

(i) $F : S \rightarrow X$ is continuous.

Let $\{y_n\}$ be any sequence in S such that $y_n \rightarrow y$ as $n \rightarrow \infty$ in S . We prove that $Fy_n \rightarrow Fy$ as $n \rightarrow \infty$ in S . Consider,

$$\begin{aligned} \|Fy_n - Fy\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} &= \max_{t \in J} \left| (\Psi(t) - \Psi(0))^{1-\xi} (Fy_n(t) - Fy(t)) \right| \\ &\leq \max_{t \in J} \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} \times \\ &\quad \left| v\left(s, y_n(s), k \mathcal{I}_{0+}^{\mu; \Psi} y_n(s)\right) - v\left(s, y(s), k \mathcal{I}_{0+}^{\mu; \Psi} y(s)\right) \right| ds. \end{aligned}$$

By continuity of v and Lebesgue dominated convergence theorem, from the above inequality, we obtain

$$\|Fy_n - Fy\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves $F : S \rightarrow X$ is continuous.

(ii) $F(S) = \{Fy : y \in S\}$ is uniformly bounded.

Using hypothesis **(H2)**, for any $y \in S$ and $t \in J$, we have

$$\begin{aligned} &\left| (\Psi(t) - \Psi(0))^{1-\xi} Fy(t) \right| \\ &\leq \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} \left| v\left(s, y(s), k \mathcal{I}_{0+}^{\mu; \Psi} y(s)\right) \right| ds \\ &\leq \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g(s) ds \\ &\leq \left| \frac{y_0}{u(0, y(0+))} \right| + \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} (\Psi(t) - \Psi(0))^{1-\xi} \frac{(\Psi(t) - \Psi(0))^\mu}{\Gamma(\mu + 1)} \\ &\leq \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu + 1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}. \end{aligned}$$

Therefore,

$$\|Fy\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu + 1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}. \quad (3.5)$$

(iii) $F(S)$ is equicontinuous.

Let any $y \in S$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then using hypothesis **(H2)**, we have

$$\begin{aligned} &\left| (\Psi(t_2) - \Psi(0))^{1-\xi} Fy(t_2) - (\Psi(t_1) - \Psi(0))^{1-\xi} Fy(t_1) \right| \\ &= \left| \left\{ \frac{y_0}{u(0, y(0+))} + \frac{(\Psi(t_2) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s) (\Psi(t_2) - \Psi(s))^{\mu-1} v\left(s, y(s), k \mathcal{I}_{0+}^{\mu; \Psi} y(s)\right) ds \right\} \right. \\ &\quad \left. - \left\{ \frac{y_0}{u(0, y(0+))} + \frac{(\Psi(t_1) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s) (\Psi(t_1) - \Psi(s))^{\mu-1} v\left(s, y(s), k \mathcal{I}_{0+}^{\mu; \Psi} y(s)\right) ds \right\} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} \left| v\left(s, y(s), k \mathcal{I}_{0+}^{\mu; \Psi} y(s)\right) \right| ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} \left| v\left(s, y(s), k \mathcal{I}_{0+}^{\mu; \Psi} y(s)\right) \right| ds \right| \\
&\leq \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g(s) ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g(s) ds \right| \\
&\leq \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} ds \right| \\
&= \frac{\|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}}{\Gamma(\mu+1)} \left\{ (\Psi(t_2) - \Psi(0))^{\mu+1-\xi} - (\Psi(t_1) - \Psi(0))^{\mu+1-\xi} \right\}.
\end{aligned}$$

By the continuity of Ψ , from the above inequality it follows that

$$\text{if } |t_1 - t_2| \rightarrow 0 \text{ then } \left| (\Psi(t_2) - \Psi(0))^{1-\xi} Fy(t_2) - (\Psi(t_1) - \Psi(0))^{1-\xi} Fy(t_1) \right| \rightarrow 0.$$

From the parts (ii) and (iii), it follows that $F(S)$ is uniformly bounded and equicontinuous set in X . Then by Arzelá-Ascoli theorem, $F(S)$ is relatively compact. We have proved that, $F : S \rightarrow X$ is a compact operator. Since $F : S \rightarrow X$ is the continuous and compact operator, it is completely continuous.

Step 3: For $y \in X$, $y = EyFx + Gy \implies y \in S$ for all $x \in S$.

Let any $y \in X$ and $x \in S$ such that $y = EyFx + Gy$. Using the hypothesis **(H2)** and boundedness of u and w , for any $t \in J$, we have

$$\begin{aligned}
&\left| (\Psi(t) - \Psi(0))^{1-\xi} y(t) \right| \\
&= \left| (\Psi(t) - \Psi(0))^{1-\xi} [Ey(t)Fx(t) + Gy(t)] \right| \\
&= \left| (\Psi(t) - \Psi(0))^{1-\xi} \left[u(t, y(t)) \left\{ \frac{y_0}{u(0, x(0+))} (\Psi(t) - \Psi(0))^{\xi-1} + \mathcal{I}_{0+}^{\mu; \Psi} v\left(t, x(t), k \mathcal{I}_{0+}^{\mu; \Psi} x(t)\right) \right\} + w(t, y(t)) \right] \right| \\
&= \left| u(t, y(t)) \left\{ \frac{y_0}{u(0, x(0+))} + (\Psi(t) - \Psi(0))^{1-\xi} \mathcal{I}_{0+}^{\mu; \Psi} v\left(t, x(t), k \mathcal{I}_{0+}^{\mu; \Psi} x(t)\right) \right\} + (\Psi(t) - \Psi(0))^{1-\xi} w(t, y(t)) \right| \\
&\leq |u(t, y(t))| \left\{ \left| \frac{y_0}{u(0, x(0+))} \right| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\mu-1} \left| v\left(s, x(s), k \mathcal{I}_{0+}^{\mu; \Psi} x(s)\right) \right| ds \right\} \\
&\quad + (\Psi(t) - \Psi(0))^{1-\xi} |w(t, y(t))| \\
&\leq K_1 \left\{ \left| \frac{y_0}{u(0, x(0+))} \right| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g(s) ds \right\} \\
&\quad + (\Psi(t) - \Psi(0))^{1-\xi} K_2
\end{aligned}$$

$$\leq K_1 \left\{ \left| \frac{y_0}{u(0, x(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} K_2.$$

This gives

$$\|y\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq K_1 \left\{ \left| \frac{y_0}{u(0, x(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} K_2 = R.$$

This implies, $y \in S$.

Step 4: To prove $4\sigma M + \delta < 1$ where $M = \sup \left\{ \|Fy\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} : y \in S \right\}$.

From inequality (3.5), we have

$$M = \sup \left\{ \|Fy\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} : y \in S \right\} \leq \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}.$$

Now, using the condition (3.2), we have

$$4\sigma M + \delta \leq 4\sigma \left\{ \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + \delta < 1.$$

From **Steps 1 to 4**, it follows that all the conditions of Lemma 2.4 are fulfilled. Consequently, by applying Lemma 2.4, the operator T has a coupled solution in $\tilde{S} = S \times S$. Hence, the coupled system of hybrid FDEs Eqs.(1.1)-(1.2) has a solution in $C_{1-\xi; \Psi}(J, \mathbb{R})$. \square

4 BVPs for Coupled system of Hybrid FDEs

In this section, we are concerned with the following BVPs for coupled system of hybrid FDEs of the form

$$\begin{cases} {}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi} \left[\frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} \right] = v_1(t, y(t), x(t)), \text{ a.e. } t \in (0, T], \\ a \lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} y(t) + b \lim_{t \rightarrow T} (\Psi(t) - \Psi(0))^{1-\xi} y(t) = y_0 \in \mathbb{R}, \end{cases} \quad (4.1)$$

and

$$\begin{cases} {}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi} \left[\frac{x(t) - w_2(t, y(t), x(t))}{u_2(t, y(t), x(t))} \right] = v_2(t, y(t), x(t)), \text{ a.e. } t \in (0, T], \\ a \lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} x(t) + b \lim_{t \rightarrow T} (\Psi(t) - \Psi(0))^{1-\xi} x(t) = y_0 \in \mathbb{R}, \end{cases} \quad (4.2)$$

where $0 < \mu < 1$, $0 \leq \nu \leq 1$, $\xi = \mu + \nu(1 - \mu)$ ($0 < \xi \leq 1$), ${}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi}(\cdot)$ is the Ψ -Hilfer fractional derivative of order μ and type ν , $J = [0, T]$, $a \neq 0$ and $b \neq 0$ are the constants, $u_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ ($i = 1, 2$), $w_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ($i = 1, 2$) and $v_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ($i = 1, 2$).

Consider the product space $E = X \times X$, $X = C_{1-\xi; \Psi}(J, \mathbb{R})$ with

- (i) vector addition: $(p, q)(t) + (\bar{p}, \bar{q})(t) = (p(t) + \bar{p}(t), q(t) + \bar{q}(t))$;
- (ii) scalar multiplication: $k(p, q)(t) = (kp(t), kq(t))$, where, $p, q \in X$ and $k \in \mathbb{R}$.

Then, E is a Banach algebra with the norm

$$\|(p, q)\|_E = \|p\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|q\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \quad (4.3)$$

and the vector multiplication defined by

$$(p, q)(t) \cdot (\bar{p}, \bar{q})(t) = (p(t)\bar{p}(t), q(t)\bar{q}(t)), \text{ for any } (p, q), (\bar{p}, \bar{q}) \in E \text{ and } t \in J.$$

Theorem 4.1 *The BVP for Ψ -Hilfer hybrid FDEs*

$${}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi} \left[\frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} \right] = v_1(t, y(t), x(t)), \text{ a.e. } t \in (0, T], \quad (4.4)$$

$$a \lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} y(t) + b \lim_{t \rightarrow T} (\Psi(t) - \Psi(0))^{1-\xi} y(t) = y_0 \in \mathbb{R}, \quad (4.5)$$

is equivalent to the fractional IE

$$y(t) = w_1(t, y(t), x(t)) + u_1(t, y(t), x(t)) \left[(\Psi(t) - \Psi(0))^{\xi-1} \Omega_1 + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right], \text{ } t \in (0, T], \quad (4.6)$$

where

$$\Omega_1 = \frac{y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_1(T, y(T), x(T)) + u_1(T, y(T), x(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right)}{a u_1(0, y(0+), x(0+)) + b u_1(T, y(T), x(T))}.$$

Proof: Let $y \in C_{1-\xi}(J, \mathbb{R})$ is a solution of the BVP for Ψ -Hilfer hybrid FDEs Eqs.(4.4)-(4.5). Taking $\mathcal{I}_{0+}^{\mu; \Psi}$ on both sides of Eq.(4.4) and using Lemma 2.2 (i), we get

$$\frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} - \frac{(\Psi(t) - \Psi(0))^{\xi-1}}{\Gamma(\xi)} \left[\mathcal{I}_{0+}^{1-\xi; \Psi} \frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} \right]_{t=0} = \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t))$$

Let $C^* = \left[\mathcal{I}_{0+}^{1-\xi; \Psi} \frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} \right]_{t=0}$. Thus, we have

$$\frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} = \frac{(\Psi(t) - \Psi(0))^{\xi-1}}{\Gamma(\xi)} C^* + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)).$$

Therefore,

$$y(t) = w_1(t, y(t), x(t)) + u_1(t, y(t), x(t)) \left[\frac{(\Psi(t) - \Psi(0))^{\xi-1}}{\Gamma(\xi)} C^* + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right]. \quad (4.7)$$

Now, we find the value of C^* using condition (4.5). Multiplying by $(\Psi(t) - \Psi(0))^{1-\xi}$ on both sides of equation (4.7), we get

$$\begin{aligned} (\Psi(t) - \Psi(0))^{1-\xi} y(t) &= (\Psi(t) - \Psi(0))^{1-\xi} w_1(t, y(t), x(t)) \\ &\quad + u_1(t, y(t), x(t)) \left[\frac{C^*}{\Gamma(\xi)} + (\Psi(t) - \Psi(0))^{1-\xi} \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right]. \end{aligned} \quad (4.8)$$

Taking limit as $t \rightarrow 0+$ in Eq.(4.8), we obtain

$$\lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} y(t) = \frac{u_1(0, y(0+), x(0+))}{\Gamma(\xi)} C^*. \quad (4.9)$$

Further, taking limit as $t \rightarrow T$ in Eq.(4.8), we obtain

$$\begin{aligned} \lim_{t \rightarrow T} (\Psi(t) - \Psi(0))^{1-\xi} y(t) &= (\Psi(T) - \Psi(0))^{1-\xi} w_1(T, y(T), x(T)) \\ &\quad + u_1(T, y(T), x(T)) \left[\frac{C^*}{\Gamma(\xi)} + (\Psi(T) - \Psi(0))^{1-\xi} \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right]. \end{aligned} \quad (4.10)$$

Using Eqs.(4.9)-(4.10) in the Eq.(4.5), we get

$$\begin{aligned} y_0 &= a \frac{u_1(0, y(0+), x(0+))}{\Gamma(\xi)} C^* + b (\Psi(T) - \Psi(0))^{1-\xi} w_1(T, y(T), x(T)) \\ &\quad + b u_1(T, y(T), x(T)) \left[\frac{C^*}{\Gamma(\xi)} + (\Psi(T) - \Psi(0))^{1-\xi} \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right] \\ &= C^* \left[a \frac{u_1(0, y(0+), x(0+))}{\Gamma(\xi)} + b \frac{u_1(T, y(T), x(T))}{\Gamma(\xi)} \right] \\ &\quad + b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_1(T, y(T), x(T)) + u_1(T, y(T), x(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right). \end{aligned}$$

This gives,

$$\begin{aligned} C^* &= \frac{\Gamma(\xi)}{a u_1(0, y(0+), x(0+)) + b u_1(T, y(T), x(T))} \times \\ &\quad \left[y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_1(T, y(T), x(T)) + u_1(T, y(T), x(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right) \right]. \end{aligned}$$

Putting value of C^* in the Eq.(4.7), we obtain

$$\begin{aligned} y(t) &= w_1(t, y(t), x(t)) + u_1(t, y(t), x(t)) \left\{ (\Psi(t) - \Psi(0))^{\xi-1} \times \right. \\ &\quad \left. \frac{\left[y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_1(T, y(T), x(T)) + u_1(T, y(T), x(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right) \right]}{a u_1(0, y(0+), x(0+)) + b u_1(T, y(T), x(T))} \right. \\ &\quad \left. + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right\} \end{aligned}$$

$$= w_1(t, y(t), x(t)) + u_1(t, y(t), x(t)) \left\{ (\Psi(t) - \Psi(0))^{\xi-1} \Omega_1 + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right\}, \quad t \in (0, T],$$

which is the fractional IE Eq.(4.6).

Conversely, let $y \in C_{1-\xi}(J, \mathbb{R})$ be a solution of the Volterra integral equation Eq.(4.6). The Eq.(4.6) can be rewritten as

$$\frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} = (\Psi(t) - \Psi(0))^{\xi-1} \Omega_1 + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)).$$

Taking Ψ -Hilfer fractional derivative ${}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi}$ on both sides and using Lemma 2.1 (iv) and Lemma 2.2 (i), we obtain

$${}^H\mathcal{D}_{0+}^{\mu, \nu; \Psi} \left[\frac{y(t) - w_1(t, y(t), x(t))}{u_1(t, y(t), x(t))} \right] = v_1(t, y(t), x(t)), \quad a.e. \quad t \in (0, T],$$

which is Eq.(4.4). Multiplying integral equation (4.6) by $(\Psi(t) - \Psi(0))^{1-\xi}$ we obtain

$$\begin{aligned} & (\Psi(t) - \Psi(0))^{1-\xi} y(t) \\ &= (\Psi(t) - \Psi(0))^{1-\xi} w_1(t, y(t), x(t)) \\ & \quad + u_1(t, y(t), x(t)) \left\{ \Omega_1 + (\Psi(t) - \Psi(0))^{\xi-1} \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right\}, \quad t \in J. \end{aligned} \quad (4.11)$$

Taking limit as $t \rightarrow 0+$, from above Eq.(4.11), we obtain

$$\lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} y(t) = u_1(0, y(0+), x(0+)) \Omega_1. \quad (4.12)$$

Further, taking limit as $t \rightarrow T$, from Eq.(4.11), we obtain

$$\begin{aligned} & \lim_{t \rightarrow T} (\Psi(t) - \Psi(0))^{1-\xi} y(t) \\ &= (\Psi(T) - \Psi(0))^{1-\xi} w_1(T, y(T), x(T)) \\ & \quad + u_1(T, y(T), x(T)) \left\{ \Omega_1 + (\Psi(T) - \Psi(0))^{\xi-1} \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right\}, \quad t \in J. \end{aligned} \quad (4.13)$$

Using the Eqs.(4.12)-(4.13) and the value of Ω_1 , consider

$$\begin{aligned} & a \lim_{t \rightarrow 0+} (\Psi(t) - \Psi(0))^{1-\xi} y(t) + b \lim_{t \rightarrow T} (\Psi(t) - \Psi(0))^{1-\xi} y(t) \\ &= a u_1(0, y(0+), x(0+)) \Omega_1 + b (\Psi(T) - \Psi(0))^{1-\xi} w_1(T, y(T), x(T)) \\ & \quad + b u_1(T, y(T), x(T)) \left\{ \Omega_1 + (\Psi(T) - \Psi(0))^{\xi-1} \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right\} \\ &= [a u_1(0, y(0+), x(0+)) + b u_1(T, y(T), x(T))] \Omega_1 + b (\Psi(T) - \Psi(0))^{1-\xi} w_1(T, y(T), x(T)) \\ & \quad + b u_1(T, y(T), x(T)) (\Psi(T) - \Psi(0))^{\xi-1} \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \\ &= y_0, \end{aligned}$$

which is the condition (4.5). This proves, $y \in C_{1-\xi}(J, \mathbb{R})$ is a solution of the BVP for FDEs involving Ψ -Hilfer fractional derivative (4.4)-(4.5). \square

To prove the existence of solution to the coupled system of hybrid FDEs Eqs.(4.1)-(4.2), we need the following hypotheses on u_i , v_i and w_i ($i = 1, 2$):

(H3) The functions $u_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\}) (i = 1, 2)$, $w_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) (i = 1, 2)$ are bounded and there exists constants $\sigma_i, \delta_i > 0 (i = 1, 2)$ such that for all $p, q, \bar{p}, \bar{q} \in \mathbb{R}$, $i = 1, 2$ and $t \in J = [0, T]$ we have

$$|u_i(t, p, q) - u_i(t, \bar{p}, \bar{q})| \leq \sigma_i (|p - \bar{p}| + |q - \bar{q}|)$$

and

$$|w_i(t, p, q) - w_i(t, \bar{p}, \bar{q})| \leq \delta_i (|p - \bar{p}| + |q - \bar{q}|).$$

(H4) The functions $v_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) (i = 1, 2)$ and there exists a functions $g_i \in C_{1-\xi, \Psi}(J, \mathbb{R})$ such that

$$|v_i(t, p, q)| \leq (\Psi(t) - \Psi(0))^{1-\xi} g_i(t), \quad a.e. t \in J \text{ and } p, q \in \mathbb{R},$$

Theorem 4.2 Assume that the hypotheses **(H3)**-**(H4)** hold. Then, the BVPs for coupled system of Ψ -Hilfer hybrid FDEs Eqs.(4.1)-(4.2) has a solution $(y, x) \in E$ provided

$$\left(\sum_{i=1}^2 \sigma_i \right) \left[\sum_{i=1}^2 |\Omega_i| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \left(\sum_{i=1}^2 \|g_i\|_{C_{1-\xi, \Psi}(J, \mathbb{R})} \right) \right] + \sum_{i=1}^2 \delta_i < 1, \quad (4.14)$$

where

$$\Omega_1 = \frac{y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_1(T, y(T), x(T)) + u_1(T, y(T), x(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, y(T), x(T)) \right)}{a u_1(0, y(0+), x(0+)) + b u_1(T, y(T), x(T))} \quad (4.15)$$

and

$$\Omega_2 = \frac{y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_2(T, y(T), x(T)) + u_2(T, y(T), x(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_2(T, y(T), x(T)) \right)}{a u_2(0, y(0+), x(0+)) + b u_2(T, y(T), x(T))}. \quad (4.16)$$

Proof:

Define,

$$S^* = \{(y, x) \in X \times X : \|(y, x)\|_E \leq R^*\},$$

where

$$R^* = M_1 |\bar{\Omega}_1| + M_2 |\bar{\Omega}_2| + (\Psi(T) - \Psi(0))^{1-\xi} [N_1 + N_2] \\ + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \left[M_1 \|g_1\|_{C_{1-\xi, \Psi}(J, \mathbb{R})} + M_2 \|g_2\|_{C_{1-\xi, \Psi}(J, \mathbb{R})} \right]$$

and $M_i > 0 (i = 1, 2)$ and $N_i > 0$ are the constants such that $|u_i(t, \cdot, \cdot)| < M_i (i = 1, 2)$ and $|w_i(t, \cdot, \cdot)| < N_i (i = 1, 2)$, for all $t \in J$. Clearly, S^* is non-empty, closed, convex and bounded subset of $E = X \times X$.

If $(y, x) \in S^* \subseteq X \times X$ is a solution of the coupled system of nonlinear Ψ -Hilfer hybrid FDEs Eqs.(4.1)-(4.2), then $(y, x) \in S^* \subseteq X \times X$ is a solution of the coupled system of

fractional IEs

$$\begin{cases} y(t) = w_1(t, y(t), x(t)) + u_1(t, y(t), x(t)) \left[(\Psi(t) - \Psi(0))^{\xi-1} \Omega_1 + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)) \right], & t \in (0, T] \\ x(t) = w_2(t, y(t), x(t)) + u_2(t, y(t), x(t)) \left[(\Psi(t) - \Psi(0))^{\xi-1} \Omega_2 + \mathcal{I}_{0+}^{\mu; \Psi} v_2(t, y(t), x(t)) \right], & t \in (0, T], \end{cases} \quad (4.17)$$

where Ω_1 and Ω_2 are defined in Eqs.(4.15)- (4.16) respectively. For $i = 1, 2$, define the operators $A_i : E \rightarrow X$, $B_i : S^* \rightarrow X$ and $C_i : E \rightarrow X$ by

$$\begin{aligned} A_1(y, x)(t) &= u_1(t, y(t), x(t)), & t \in J; \\ A_2(y, x)(t) &= u_2(t, y(t), x(t)), & t \in J; \\ B_1(y, x)(t) &= (\Psi(t) - \Psi(0))^{\xi-1} \Omega_1 + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, y(t), x(t)), & t \in (0, T]; \\ B_2(y, x)(t) &= (\Psi(t) - \Psi(0))^{\xi-1} \Omega_2 + \mathcal{I}_{0+}^{\mu; \Psi} v_2(t, y(t), x(t)), & t \in (0, T]; \\ C_1(y, x)(t) &= w_1(t, y(t), x(t)), & t \in J; \\ C_2(y, x)(t) &= w_2(t, y(t), x(t)), & t \in J. \end{aligned}$$

Then, the coupled system of hybrid IEs Eq.(4.17) transformed into

$$\begin{cases} A_1(y, x)(t) B_1(y, x)(t) + C_1(y, x)(t) = y(t), & t \in (0, T], \\ A_2(y, x)(t) B_2(y, x)(t) + C_2(y, x)(t) = x(t), & t \in (0, T]. \end{cases} \quad (4.18)$$

Consider the operators $A = (A_1, A_2) : E \rightarrow E$, $B = (B_1, B_2) : S^* \rightarrow E$ and $C = (C_1, C_2) : E \rightarrow E$. Then, the operator equations (4.18) can be written as

$$A(y, x)(t) B(y, x)(t) + C(y, x)(t) = (y, x)(t), \quad (y, x) \in E \text{ and } t \in J. \quad (4.19)$$

We prove that the operators A , B and C satisfies all the conditions of Lemma 2.5. The proof is given in the following series of steps.

Step 1: $A = (A_1, A_2) : E \rightarrow E$ and $C = (C_1, C_2) : E \rightarrow E$ are Lipschitz operators.

For any $(y, x), (\bar{y}, \bar{x}) \in E$, $t \in J$, we obtain

$$\begin{aligned} & \|A(y, x) - A(\bar{y}, \bar{x})\|_E \\ &= \|(A_1(y, x), A_2(y, x)) - (A_1(\bar{y}, \bar{x}), A_2(\bar{y}, \bar{x}))\|_E \\ &= \|(A_1(y, x) - A_1(\bar{y}, \bar{x})), (A_2(y, x) - A_2(\bar{y}, \bar{x}))\|_E \\ &= \|A_1(y, x) - A_1(\bar{y}, \bar{x})\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|A_2(y, x) - A_2(\bar{y}, \bar{x})\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}. \end{aligned} \quad (4.20)$$

Now, using the hypothesis **(H3)**, we obtain

$$\begin{aligned} & \left| (\Psi(t) - \Psi(0))^{1-\xi} (A_1(y, x)(t) - A_1(\bar{y}, \bar{x})(t)) \right| \\ &= \left| (\Psi(t) - \Psi(0))^{1-\xi} (u_1(t, y(t), x(t)) - u_1(t, \bar{y}(t), \bar{x}(t))) \right| \\ &\leq \sigma_1 (\Psi(t) - \Psi(0))^{1-\xi} [|y(t) - \bar{y}(t)| + |x(t) - \bar{x}(t)|] \\ &\leq \sigma_1 \left[\|y - \bar{y}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|x - \bar{x}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right]. \end{aligned}$$

This gives,

$$\|A_1(y, x) - A_1(\bar{y}, \bar{x})\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq \sigma_1 \left[\|y - \bar{y}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|x - \bar{x}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right]. \quad (4.21)$$

Similarly, we have

$$\|A_2(y, x) - A_2(\bar{y}, \bar{x})\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq \sigma_2 \left[\|y - \bar{y}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|x - \bar{x}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right]. \quad (4.22)$$

Using the inequalities (4.21) and (4.22), from Eq.(4.20), we have

$$\|A(y, x) - A(\bar{y}, \bar{x})\|_E \leq (\sigma_1 + \sigma_2) \left[\|y - \bar{y}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|x - \bar{x}\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right].$$

Therefore, A is Lipschitz operator with Lipschitz constant $K = \sigma_1 + \sigma_2$. On the similar line, it is easy to prove that C is Lipschitz operator. Let $L = \delta_1 + \delta_2$ is the Lipschitz constant corresponding to the operator C .

Step 2: $B = (B_1, B_2) : S^* \rightarrow E$ is completely continuous.

(a) $B = (B_1, B_2) : S^* \rightarrow E$ is continuous.

Let (y_n, x_n) be any sequence of points in S^* such that $(y_n, x_n) \rightarrow (y, x)$ as $n \rightarrow \infty$ in S^* . We prove that $B(y_n, x_n) \rightarrow B(y, x)$ as $n \rightarrow \infty$ in E .

Consider,

$$\begin{aligned} & \|B_1(y_n, x_n) - B_1(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \\ &= \max_{t \in J} \left| (\Psi(t) - \Psi(0))^{1-\xi} (B_1(y_n, x_n)(t) - B_1(y, x)(t)) \right| \\ &\leq \max_{t \in J} \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} |v_1(s, y_n(s), x_n(s)) - v_1(s, y(s), x(s))| ds. \end{aligned}$$

By continuity of the function v_1 and the Lebesgue dominated convergence theorem, from the above inequality, we obtain

$$\|B_1(y_n, x_n) - B_1(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the similar line one can obtain

$$\|B_2(y_n, x_n) - B_2(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $B(y_n, x_n) = (B_1(y_n, x_n), B_2(y_n, x_n))$ converges to $B(y, x) = (B_1(y, x), B_2(y, x))$ as $n \rightarrow \infty$.

This proves $B : S^* \rightarrow E$ is continuous.

(b) $B(S^*) = \{B(y, x) : (y, x) \in S^*\}$ is uniformly bounded.

Using hypothesis **(H4)**, for any $(y, x) \in S^*$ and $t \in J$, we have

$$\begin{aligned} & \left| (\Psi(t) - \Psi(0))^{1-\xi} B_1(y, x)(t) \right| \\ &\leq |\Omega_1| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} |v_1(s, y(s), x(s))| ds \end{aligned}$$

$$\begin{aligned}
&\leq |\Omega_1| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\mu-1}(\Psi(s) - \Psi(0))^{1-\xi} g_1(s) ds \\
&\leq |\Omega_1| + \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} (\Psi(t) - \Psi(0))^{1-\xi} \frac{(\Psi(t) - \Psi(0))^\mu}{\Gamma(\mu+1)} \\
&\leq |\Omega_1| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}.
\end{aligned}$$

Therefore,

$$\|B_1(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq |\Omega_1| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}, \text{ for all } (y, x) \in S^*. \quad (4.23)$$

Hence, B_1 is uniformly bounded on S^* . On the similar line, one can obtain

$$\|B_2(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq |\Omega_2| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_2\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}, \text{ for all } (y, x) \in S^*. \quad (4.24)$$

This proves B_2 is uniformly bounded on S^* . Hence, the operator B is uniformly bounded on S^* .

(c) $B(S^*) = (B_1(S^*), B_2(S^*))$ is equicontinuous.

Let any $(y, x) \in S^*$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then using hypothesis **(H4)**, we have

$$\begin{aligned}
&\left| (\Psi(t_2) - \Psi(0))^{1-\xi} B_1(y, x)(t_2) - (\Psi(t_1) - \Psi(0))^{1-\xi} B_1(y, x)(t_1) \right| \\
&= \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} v_1(s, y(s), x(s)) ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} v_1(s, y(s), x(s)) ds \right| \\
&\leq \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} |v_1(s, y(s), x(s))| ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} |v_1(s, y(s), x(s))| ds \right| \\
&\leq \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g_1(s) ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g_1(s) ds \right| \\
&\leq \left| \frac{(\Psi(t_2) - \Psi(0))^{1-\xi} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}}{\Gamma(\mu)} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\mu-1} ds \right. \\
&\quad \left. - \frac{(\Psi(t_1) - \Psi(0))^{1-\xi} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}}{\Gamma(\mu)} \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\mu-1} ds \right|
\end{aligned}$$

$$= \frac{\|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})}}{\Gamma(\mu+1)} \left\{ (\Psi(t_2) - \Psi(0))^{\mu+1-\xi} - (\Psi(t_1) - \Psi(0))^{\mu+1-\xi} \right\}.$$

By the continuity of Ψ , from the above inequality it follows that

$$\text{if } |t_1 - t_2| \rightarrow 0 \text{ then } \left| (\Psi(t_2) - \Psi(0))^{1-\xi} B_1(y, x)(t_2) - (\Psi(t_1) - \Psi(0))^{1-\xi} B_1(y, x)(t_1) \right| \rightarrow 0,$$

uniformly for all $(y, x) \in S^*$. Following the similar type of steps, we have

$$\text{if } |t_1 - t_2| \rightarrow 0 \text{ then } \left| (\Psi(t_2) - \Psi(0))^{1-\xi} B_2(y, x)(t_2) - (\Psi(t_1) - \Psi(0))^{1-\xi} B_2(y, x)(t_1) \right| \rightarrow 0,$$

uniformly for all $(y, x) \in S^*$.

From the parts (b) and (c), it follows that $B(S^*)$ is uniformly bounded and equicontinuous set in E . Then by Arzelá-Ascoli theorem, $B(S^*)$ is relatively compact. Therefore, $B : S^* \rightarrow E$ is a compact operator. Since $B : S^* \rightarrow E$ is continuous and compact operator, it is completely continuous.

Step 3: For $(y, x) \in E$, $(y, x) = (A_1(y, x) B_1(\bar{y}, \bar{x}) + C_1(y, x), A_2(y, x) B_2(\bar{y}, \bar{x}) + C_2(y, x)) \implies (y, x) \in S^*$, for all $(\bar{y}, \bar{x}) \in S^*$.

Let any $(y, x) \in E$ and $(\bar{y}, \bar{x}) \in S^*$ such that

$$(y, x) = (A_1(y, x) B_1(\bar{y}, \bar{x}) + C_1(y, x), A_2(y, x) B_2(\bar{y}, \bar{x}) + C_2(y, x)).$$

Using the hypothesis **(H4)** and boundedness of u_1 and w_1 , for any $t \in J$, we have

$$\begin{aligned} & \left| (\Psi(t) - \Psi(0))^{1-\xi} y(t) \right| \\ &= \left| (\Psi(t) - \Psi(0))^{1-\xi} [A_1(y, x)(t) B_1(\bar{y}, \bar{x})(t) + C_1(y, x)(t)] \right| \\ &= \left| (\Psi(t) - \Psi(0))^{1-\xi} \left[u_1(t, y(t), x(t)) \left\{ (\Psi(t) - \Psi(0))^{\xi-1} \bar{\Omega}_1 + \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, \bar{y}(t), \bar{x}(t)) \right\} + w_1(t, y(t), x(t)) \right] \right| \\ &= \left| u_1(t, y(t), x(t)) \left\{ \bar{\Omega}_1 + (\Psi(t) - \Psi(0))^{1-\xi} \mathcal{I}_{0+}^{\mu; \Psi} v_1(t, \bar{y}(t), \bar{x}(t)) \right\} + (\Psi(t) - \Psi(0))^{1-\xi} w_1(t, y(t), x(t)) \right| \\ &\leq |u_1(t, y(t), x(t))| \left\{ |\bar{\Omega}_1| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} |v_1(s, \bar{y}(s), \bar{x}(s))| ds \right\} \\ &\quad + (\Psi(t) - \Psi(0))^{1-\xi} |w_1(t, y(t), x(t))| \\ &\leq M_1 \left\{ |\bar{\Omega}_1| + \frac{(\Psi(t) - \Psi(0))^{1-\xi}}{\Gamma(\mu)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} (\Psi(s) - \Psi(0))^{1-\xi} g_1(s) ds \right\} \\ &\quad + (\Psi(t) - \Psi(0))^{1-\xi} N_1 \\ &\leq M_1 \left\{ |\bar{\Omega}_1| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} N_1, \end{aligned}$$

where

$$\bar{\Omega}_1 = \frac{y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(w_1(T, \bar{y}(T), \bar{x}(T)) - u_1(T, \bar{y}(T), \bar{x}(T)) \mathcal{I}_{0+}^{\mu; \Psi} v_1(T, \bar{y}(T), \bar{x}(T)) \right)}{a u_1(0, \bar{y}(0+), \bar{x}(0+)) + b u_1(T, \bar{y}(T), \bar{x}(T))}.$$

This gives

$$\|y\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq M_1 \left\{ |\bar{\Omega}_1| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} N_1. \quad (4.25)$$

Similarly, we can obtain

$$\|x\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \leq M_2 \left\{ |\bar{\Omega}_2| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_2\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} N_2, \quad (4.26)$$

where

$$\bar{\Omega}_2 = \frac{y_0 - b (\Psi(T) - \Psi(0))^{1-\xi} \left(u_2(T, \bar{y}(T), \bar{x}(T)) - u_2(T, \bar{y}(T), \bar{x}(T)) \mathcal{I}_0^{\mu; \Psi} v_2(T, \bar{y}(T), \bar{x}(T)) \right)}{a u_2(0, \bar{y}(0+), \bar{x}(0+)) + b u_2(T, \bar{y}(T), \bar{x}(T))}.$$

Using definition of norm on E and the inequalities (4.25) and (4.26), we obtain

$$\begin{aligned} \|(y, x)\|_E &= \|y\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|x\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \\ &\leq M_1 \left\{ |\bar{\Omega}_1| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} N_1 \\ &\quad + M_2 \left\{ |\bar{\Omega}_2| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \|g_2\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right\} + (\Psi(T) - \Psi(0))^{1-\xi} N_2 \\ &= M_1 |\bar{\Omega}_1| + M_2 |\bar{\Omega}_2| + (\Psi(T) - \Psi(0))^{1-\xi} [N_1 + N_2] \\ &\quad + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \left[M_1 \|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + M_2 \|g_2\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right] \\ &= R^* \end{aligned}$$

This implies, $(y, x) \in S^*$.

Step 4: To prove $K M^* + L < 1$ where $M^* = \sup \{\|B(y, x)\|_E : (y, x) \in S^*\}$.

Here,

$$\begin{aligned} M^* &= \sup \{\|B(y, x)\|_E : (y, x) \in S^*\} \\ &= \sup \{\|(B_1(y, x), B_2(y, x))\|_E : (y, x) \in S^*\} \\ &= \sup \left\{ \|B_1(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|B_2(y, x)\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} : (y, x) \in S^* \right\} \\ &\leq |\Omega_1| + |\Omega_2| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \left(\|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|g_2\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right). \end{aligned}$$

Using the condition (4.14), we observe that

$$\begin{aligned} &K M^* + L \\ &\leq (\sigma_1 + \sigma_2) \left[|\Omega_1| + |\Omega_2| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \left(\|g_1\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} + \|g_2\|_{C_{1-\xi; \Psi}(J, \mathbb{R})} \right) \right] + (\delta_1 + \delta_2) \end{aligned}$$

< 1 .

From steps 1 to 4, it follows that all the conditions of Lemma 2.5 are fulfilled. Consequently, by applying Lemma 2.5, the operator equation $(y, x) = A(y, x)B(y, x) + C(y, x)$ has a solution in S^* . Hence, the BVPs for coupled system of hybrid FDEs Eqs.(4.1)-(4.2) has a solution in $C_{1-\xi; \Psi}(J, \mathbb{R})$. \square

5 Example

In this section, we will provide an examples to illustrate the results we obtained. We consider the particular case when $\Psi(t) = t$ and $\nu = 1$.

Example 5.1 Consider the coupled hybrid FDEs involving Caputo fractional derivative

$$\left\{ \begin{array}{l} {}^C\mathcal{D}_{0^+}^{\frac{1}{2}} \left[\frac{\frac{7}{97} (y(t) - t [y(t) + 1 - \frac{1}{t}])}{\frac{1}{10} (ty(t) - 1)} \right] = \frac{x^2(t)}{1 + x^2(t)} - \frac{\frac{3\sqrt{\pi}}{4} t^{\frac{1}{2}} \mathcal{I}_{0^+}^{\frac{1}{2}} x(t)}{\frac{3\sqrt{\pi}}{4} t^{\frac{1}{2}} \mathcal{I}_{0^+}^{\frac{1}{2}} x(t) + 1}, \text{ a.e. } t \in (0, 1], \\ y(0) = 0, \end{array} \right. \quad (5.1)$$

$$\left\{ \begin{array}{l} {}^C\mathcal{D}_{0^+}^{\frac{1}{2}} \left[\frac{\frac{7}{97} (x(t) - t^{-\frac{1}{2}} x(t))}{\frac{1}{10} t (1 - t^{-\frac{3}{2}} x(t))} \right] = \frac{y^2(t)}{1 + y^2(t)} - \frac{\frac{3\sqrt{\pi}}{4} t^{\frac{1}{2}} \mathcal{I}_{0^+}^{\frac{1}{2}} y(t)}{\frac{3\sqrt{\pi}}{4} t^{\frac{1}{2}} \mathcal{I}_{0^+}^{\frac{1}{2}} y(t) + 1}, \text{ a.e. } t \in (0, 1], \\ x(0) = 0. \end{array} \right. \quad (5.2)$$

Comparing the problem Eqs.(5.1)-(5.2) with coupled system of hybrid FDEs Eqs.(1.1)-(1.2). Then

$$\mu = \frac{1}{2}, \nu = 1, \xi = 1, \Psi(t) = t, y_0 = 0, J = [0, 1],$$

$$\begin{aligned} u(t, y(t)) &= \frac{1}{10} (ty(t) - 1), \\ v(t, y(t), k \mathcal{I}_{0^+}^{\mu} y(t)) &= \frac{y^2(t)}{1 + y^2(t)} - \frac{\frac{3\sqrt{\pi}}{4} t^{\frac{1}{2}} \mathcal{I}_{0^+}^{\frac{1}{2}} y(t)}{\frac{3\sqrt{\pi}}{4} t^{\frac{1}{2}} \mathcal{I}_{0^+}^{\frac{1}{2}} y(t) + 1}, \end{aligned}$$

and

$$w(t, y(t)) = \frac{7}{97} t \left[y(t) + 1 - \frac{1}{t} \right].$$

For $\mu = \frac{1}{2}$, $\nu = 1$ we have $\xi = 1$. In this case the space $C_{1-\xi; \Psi}(J, \mathbb{R})$ reduces to the space of continuous functions $C(J, \mathbb{R})$. One can verify that $x(t) = y(t) = t$ is a solution of the coupled system of hybrid FDEs Eqs.(5.1)-(5.2). Next, we prove that u, v and w satisfies the hypotheses **(H1)** and **(H2)** of the Theorem 3.2. Consider

$$|u(t, p) - u(t, q)| = \left| \frac{1}{10}(tp - 1) - \frac{1}{10}(tq - 1) \right| = \frac{1}{10}t|p - q| \leq \frac{1}{10}|p - q|,$$

and

$$|v(t, p) - w(t, q)| = \left| \frac{7}{97}t \left[p + 1 - \frac{1}{t} \right] - \frac{7}{97}t \left[q + 1 - \frac{1}{t} \right] \right| = \frac{7}{97}t|p - q| \leq \frac{7}{97}|p - q|.$$

Here, we observe that $\sigma = \frac{1}{10} = 0.1$ and $\delta = \frac{7}{97} = 0.07216$. Next,

$$|v(t, p, q)| = \left| \frac{p^2}{1 + p^2} - \frac{\frac{3\sqrt{\pi}}{4}t^{\frac{1}{2}}q}{\frac{3\sqrt{\pi}}{4}t^{\frac{1}{2}}q + 1} \right| \leq \left| \frac{p^2}{1 + p^2} \right| + \left| \frac{\frac{3\sqrt{\pi}}{4}t^{\frac{1}{2}}q}{\frac{3\sqrt{\pi}}{4}t^{\frac{1}{2}}q + 1} \right| \leq 1 + 1 = 2. \quad (5.3)$$

Here $g(t) = 2$. Now we check for condition (3.2). Further, consider

$$4\sigma \left\{ \left| \frac{y_0}{u(0, y(0+))} \right| + \frac{1}{\Gamma(\mu + 1)} \|g\|_{C(J, \mathbb{R})} \right\} + \delta = 4 \frac{1}{10} \left\{ \left| \frac{0}{u(0, y(0+))} \right| + \frac{1}{\Gamma(\frac{1}{2} + 1)} 2 \right\} + \frac{7}{97} \\ \approx 0.9748 < 1.$$

We observe that all the conditions of Theorem 3.2 are satisfied. Therefore, the coupled system of hybrid FDEs involving Caputo derivative (5.1)-(5.2) has at least one coupled solution in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$.

Further, we consider the particular case of BVPs for coupled hybrid FDEs Eqs.(4.1)-(4.2) for BVPs for coupled hybrid FDEs involving Caputo fractional derivative as follows:

Example 5.2

$${}_C \mathcal{D}_{0+}^{\frac{1}{3}} \left[\frac{\frac{3}{17}(y(t) - \frac{17}{21}[ty(t) + \frac{21}{17}x(t) - 1])}{\frac{1}{99} \left(\frac{ty(t)}{3} + \frac{tx(t)}{2} + \frac{5}{6} \right)}} \right] = \frac{e^{-t^2}}{97} \left[\frac{y(t)}{2 + y(t)} - \frac{x(t)}{2 + x(t)} \right], \quad a.e. t \in (0, 1], \quad (5.4)$$

$${}_C \mathcal{D}_{0+}^{\frac{1}{3}} \left[\frac{x(t) - \left[\frac{t}{10}(y(t) + x(t) + 10) + 2 \right]}{\frac{1}{98} \left[\frac{ty(t)}{5} + tx(t) + 12 \right]} \right] = \frac{2^{-t}}{87} \left[\frac{t^2 - y(t)x(t)}{1 - y(t)x(t)} \right], \quad a.e. t \in (0, 1], \quad (5.5)$$

$$3y(0) + y(1) = 1, \quad 3x(0) + x(1) = 1. \quad (5.6)$$

Comparing the above Eqs.(5.4)-(5.6) to the Eqs.(4.1)-(4.2), we get

$$\mu = \frac{1}{3}, \nu = 1, \xi = 1, \Psi(t) = t, a = 3, b = 1, y_0 = 1, J = [0, 1],$$

$$u_1(t, y(t), x(t)) = \frac{1}{99} \left(\frac{ty(t)}{3} + \frac{tx(t)}{2} + \frac{5}{6} \right), \quad u_2(t, y(t), x(t)) = \frac{1}{98} \left[\frac{ty(t)}{5} + tx(t) + 12 \right],$$

$$v_1(t, y(t), x(t)) = \frac{e^{-t^2}}{97} \left[\frac{x(t)}{2+x(t)} - \frac{y(t)}{2+y(t)} \right], \quad v_2(t, y(t), x(t)) = \frac{2^{-t}}{87} \left[\frac{t^2 - x(t)y(t)}{1 - x(t)y(t)} \right],$$

and

$$w_1(t, y(t), x(t)) = \frac{1}{7} \left[ty(t) + \frac{21}{17}x(t) - 1 \right], \quad w_2(t, y(t), x(t)) = \frac{t}{10} (y(t) + x(t) + 10) + 2.$$

For this problem our solution space $C_{1-\xi; \Psi}(J, \mathbb{R})$ reduces to the space $C(J, \mathbb{R})$. One can verify that $x(t) = y(t) = t$ is a solution of the coupled system of hybrid FDEs Eqs.(5.4)-(5.6). Next, we prove that u_i, v_i and $w_i (i = 1, 2)$ satisfies the hypotheses **(H3)** and **(H4)** of the Theorem 4.2. We find the required values of u_i, v_i and $w_i (i = 1, 2)$ at $t = 0$ and $t = 1$.

$$u_1(0, y(0), x(0)) = \frac{5}{594}, \quad u_2(0, y(0), x(0)) = \frac{6}{49},$$

$$u_1(1, y(1), x(1)) = \frac{5}{297}, \quad u_2(1, y(1), x(1)) = \frac{33}{245},$$

$$v_1(1, y(1), x(1)) = 0, \quad v_2(1, y(1), x(1)) = 0,$$

and

$$w_1(1, y(1), x(1)) = \frac{3}{17}, \quad w_2(1, y(1), x(1)) = \frac{32}{10}.$$

Further, we find the Lipschitz constants for the functions u_i and $w_i (i = 1, 2)$.

$$|u_1(t, p, q) - u_1(t, \bar{p}, \bar{q})| = \left| \frac{1}{99} \left(\frac{tp}{3} + \frac{tq}{2} + \frac{5}{6} \right) - \frac{1}{99} \left(\frac{t\bar{p}}{3} + \frac{t\bar{q}}{2} + \frac{5}{6} \right) \right| \leq \frac{1}{99} \{ |p - \bar{p}| + |q - \bar{q}| \}.$$

and

$$|u_2(t, p, q) - u_2(t, \bar{p}, \bar{q})| = \left| \frac{1}{98} \left[\frac{tp}{5} + tq + 12 \right] - \frac{1}{98} \left[\frac{t\bar{p}}{5} + t\bar{q} + 12 \right] \right| \leq \frac{1}{98} \{ |p - \bar{p}| + |q - \bar{q}| \}.$$

Here, we observe that $\sigma_1 = \frac{1}{99}$ and $\sigma_2 = \frac{1}{98}$. Further,

$$|w_1(t, p, q) - w_1(t, \bar{p}, \bar{q})| = \left| \frac{1}{7} \left[tp + \frac{21}{17}q - 1 \right] - \frac{1}{7} \left[t\bar{p} + \frac{21}{17}\bar{q} - 1 \right] \right| \leq \frac{2}{7} \{ |p - \bar{p}| + |q - \bar{q}| \}.$$

and

$$|w_2(t, p, q) - w_2(t, \bar{p}, \bar{q})| = \left| \left[\frac{t}{10} (p + q + 10) + 2 \right] - \left[\frac{t}{10} (\bar{p} + \bar{q} + 10) + 2 \right] \right| \leq \frac{1}{10} \{ |p - \bar{p}| + |q - \bar{q}| \}.$$

Here, we observe that $\delta_1 = \frac{2}{7}$ and $\delta_2 = \frac{1}{10}$. Next,

$$|v_1(t, p, q)| = \left| \frac{e^{-t^2}}{97} \left[\frac{p}{2+p} - \frac{q}{2+q} \right] \right| \leq \frac{2}{97} \quad (5.7)$$

and

$$|v_2(t, p, q)| = \left| \frac{2^{-t}}{87} \left[\frac{t^2 - pq}{1 - x(t)y(t)pq} \right] \right| \leq \frac{1}{87}. \quad (5.8)$$

Here $g_1(t) = \frac{2}{97}$ and $g_2(t) = \frac{1}{87}$. Using the above calculated values one can find the values of Ω_1 and Ω_2 for this problem as $\Omega_1 = \frac{-2079}{85}$ and $\Omega_2 = \frac{11}{3}$.

Now we check for condition (4.14). Further, consider

$$\begin{aligned} & (\sigma_1 + \sigma_2) \left[|\Omega_1| + |\Omega_2| + \frac{(\Psi(T) - \Psi(0))^{\mu+1-\xi}}{\Gamma(\mu+1)} \left(\|g_1\|_{C(J, \mathbb{R})} + \|g_2\|_{C(J, \mathbb{R})} \right) \right] + (\delta_1 + \delta_2) \\ &= \left(\frac{1}{99} + \frac{1}{98} \right) \left[\frac{2079}{85} + \frac{11}{3} + \frac{1}{\Gamma(\frac{4}{3})} \left(\frac{2}{97} + \frac{1}{87} \right) \right] + \left(\frac{2}{7} + \frac{1}{10} \right) \\ &\approx 0.8727 < 1. \end{aligned}$$

We observe that all the conditions of Theorem 4.2 are satisfied. Therefore, the BVPs for coupled system of hybrid FDEs involving Caputo derivative Eqs.(5.4)-(5.6) has at least one solution in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$.

Conclusion

The existence of solution of IVP for coupled system of Ψ -Hilfer hybrid FDEs is achieved by using fixed point theorem for coupled hybrid FDEs. Further, the existence of solution of BVP for coupled system of Ψ -Hilfer hybrid FDEs using fixed point theorem for three operators. As Ψ -Hilfer fractional derivative covers many important fractional derivatives for different values of the parameters μ , ν and the function Ψ . Further we have provided an examples to illustrate the validity of our outcomes.

As presented in the body of the paper, we have successfully obtained the main result of this paper. However, some open problems that still need to be answered that involve the theory of fractional hybrid differential equations, namely:

1. Would it be possible to discuss the existence of mild solutions to Eqs. (1.1)-(1.2) problems? What are the necessary and sufficient conditions for this to happen?
2. As a consequence of item 1, we can ask about the uniqueness and stability of mild solutions.
3. Is it possible to guarantee solutions involving sectorial and almost-sectorial operators?

There are some questions that need to be answered as outlined above, which will enrich the theory. Other questions about fractional hybrid differential equations, are being discussed and future works are being elaborated, which allowed to answer these questions and others that are still open.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Credit author statement

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